HOMEWORK SET THEORY (05) 2022-11-14

2022/23

Hand in next week by 23:59 on 2022-11-22, either by hand in class (on 2022-11-21 of course), or by uploading to the course page on `elo.mastermath.nl`.

Collaboration is not forbidden, encouraged even. You may also hand in joint work, provided each contributes equally to the solutions (honour system).

This homework is about stationary sets and partitions.

1. Let S be a stationary subset of ω_1. Prove that for every $\alpha \in \omega_1$ there is a closed subset of ω_1 of order type $\alpha + 1$ that is a subset of S. *Hint*: Prove the following statement by induction on α:

> "for *every* stationary subset S of ω_1 there is closed subset of order type $\alpha + 1$ that is contained in S".

For the limit case let $\langle \alpha_n : n \in \omega \rangle$ be increasing and cofinal in α. Show that there is a sequence $\langle C_\gamma : \gamma \in \omega_1 \rangle$ of countable closed sets such that $C_\gamma \subseteq S$ for all γ; $\max C_\gamma < \min C_\delta$ whenever $\gamma < \delta$ and if $\gamma = \omega \cdot \delta + n$ then C_γ has order type $\alpha_n + 1$. Consider the set of limit points of $\{\max C_\gamma : \gamma \in \omega_1\}$.

2. Some (standard) applications of Ramsey's theorem.

a. Let $\langle L, < \rangle$ be an infinite linearly ordered set. Prove that L has an infinite subset X that is well-ordered by $<$ or an infinite subset Y that is well-ordered by $>$.

b. Prove that every bounded sequence of real numbers has a convergent subsequence (the Bolzano-Weierstraß theorem). *Hint*: Find a monotone subsequence.

c. Let $\langle P, < \rangle$ be an infinite partially ordered set. Prove that P has an infinite subset C that is linearly ordered by $<$ (a chain) or an infinite subset U that is unordered by $<$, which means that if x and y in U are distinct then neither $x < y$ nor $y < x$.

3. Another application of Ramsey's theorem. Here are four well-behaved families of subsets of ω:

(1) $\mathcal{A} = \big\{\{n\} : n \in \omega\big\}$,
(2) $\mathcal{B} = \big\{n : n \in \omega\big\}$,
(3) $\mathcal{C} = \big\{\omega \setminus \{n\} : n \in \omega\big\}$, and
(4) $\mathcal{D} = \big\{\omega \setminus n : n \in \omega\big\}$.

Let X be an infinite set and $\mathcal{S}$ an infinite family of subsets of X. Prove that there is a sequence $\langle x_n : n \in \omega \rangle$ of points in X and there is a sequence $\langle S_n : n \in \omega \rangle$ of members of $\mathcal{S}$ such that

$$\big\{\{m \in \omega : x_m \in S_n\} : n \in \omega\big\}$$

is equal to one of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$. (So every infinite family of sets is well-behaved somewhere.)

a. Construct a sequence $\langle x_n : n \in \omega \rangle$ of points in X and a sequence $\langle \mathcal{S}_n : n \in \omega \rangle$ of infinite subfamilies of $\mathcal{S}$ such that $\mathcal{S}_0 = \mathcal{S}$ and for every n the following hold: either $\mathcal{S}_{n+1} = \{S \in \mathcal{S}_n : x_n \in S\}$ or $\mathcal{S}_{n+1} = \{S \in \mathcal{S}_n : x_n \notin S\}$, and in addition $\mathcal{S}_{n+1}$ is a proper subset of $\mathcal{S}_n$.

b. Choose $S_n \in \mathcal{S}_n \setminus \mathcal{S}_{n+1}$ for every n. Verify that if $x_m \in S_m$ then $x_m \notin S_n$ for all $n > m$ and, conversely, if $x_m \notin S_m$ then $x_m \in S_n$ whenever $n > m$.

c. Now consider the colouring $F : [\omega]^2 \to 4$ given by: if $i < j$ then

$$F\big(\{i,j\}\big) = \begin{cases} 0 & \text{if } x_i \notin S_j \text{ and } x_j \notin S_i \\ 1 & \text{if } x_i \notin S_j \text{ and } x_j \in S_i \\ 2 & \text{if } x_i \in S_j \text{ and } x_j \notin S_i \\ 3 & \text{if } x_i \in S_j \text{ and } x_j \in S_i \end{cases}$$

Date: dinsdag 15-11-2022 at 15:13:33 (cet).