HOMEWORK SET THEORY (06) 2022-12-06

2022/23

Hand in next week by 23:59 on 2022-12-13, either by hand in class (on 2022-12-12 of course), or by uploading to the course page on `elo.mastermath.nl`.

Collaboration is not forbidden, encouraged even. You may also hand in joint work, provided each contributes equally to the solutions (honour system).

1. For an infinite cardinal κ let $H(\kappa) = \{x : |\operatorname{trcl} x| < \kappa\}$. Prove the following about $H(\kappa)$.
a. $H(\kappa)$ is transitive.
b. $H(\kappa) \cap \boldsymbol{On} = \kappa$.
c. If $x \in H(\kappa)$ and $y \subseteq x$ then $y \in H(\kappa)$.
d. Show that $H(\kappa)$ is closed under the Gödel operations.
e. [AC] If κ is regular then $x \in H(\kappa)$ if and only if $x \subseteq H(\kappa)$ and $|x| < \kappa$.
f. [AC] If κ is regular and uncountable then $H(\kappa)$ is a model of $\mathsf{ZFC} - \mathsf{P}$.
g. Conclude that $\mathsf{ZFC} - \mathsf{P}$ is consistent with the statement that every set is countable (if ZFC is consistent).

2. This exercise proves that "x is finite" is a Δ_1-property.
a. Verify that the definition of finiteness can be expressed as a Σ_1-formula.
b. Show that finiteness can also be expressed/characterized by a Π_1-formula.
Hint: Look at Homework 03.

3. We work in $H(\aleph_2)$. One can extend the methods used in class to prove the following: if $A \in H(\aleph_2)$ is countable then there is a countable set $M \in H(\aleph_2)$ that contains A and that satisfies the equivalence

$$\phi^M(m_1, \ldots, m_k) \leftrightarrow \phi^{H(\aleph_2)}(m_1, \ldots, m_k)$$

for *all* formulas ϕ and all $m_1, \ldots, m_k \in M$.

Now let $f : \omega_1 \to \omega_1$ be a regressive function and let M be as above for the countable set $\{f\}$.
a. Verify that $\omega_1 \in M$.
b. Prove that $\omega \in M$ *and* $\omega \subseteq M$. *Hint*: ω is the unique first limit ordinal, and $\omega \subseteq M$ can be proven by induction.
c. Prove: if $x \in M$ is countable then $x \subseteq M$. *Hint*: we must have $\big((\exists b)(b : \omega \xrightarrow{onto} x)\big)^M$, take such a $b \in M$ and show that $b(n) \in M$ for all $n \in \omega$.
d. Let $\delta = \min \omega_1 \setminus M$; prove that $\delta = M \cap \omega_1$.
e. Let $\gamma = f(\delta)$ and show that $\{\alpha : f(\alpha) = \gamma\}$ is cofinal. *Hint*: For every $\beta < \delta$ we have, thanks to δ itself: $\big((\exists \alpha \in \omega_1)(\beta < \alpha \wedge f(\alpha) = \gamma)\big)^{H(\aleph_2)}$, hence also $\big((\exists \alpha \in \omega_1)(\beta < \alpha \wedge f(\alpha) = \gamma)\big)^M$. Show that this implies $\big((\forall \beta \in \omega_1)(\exists \alpha \in \omega_1)(\beta < \alpha \wedge f(\alpha) = \gamma)\big)^M$, and hence ....

Date: dinsdag 06-12-2022 at 16:27:57 (cet).