What is a set? What is Set Theory?

Bolzano, Paradoxien des Unendlichen (1847)

Einen Inbegriff, den wir einem solchen Begriffe unterstellen, bei dem die Anordnung seiner Theile gleichgültig ist (an dem sich also nichts für uns Wesentliches ändert, wenn sich bloss diese ändert) nenne ich eine <u>Menge</u>

Cantor, Beiträge zur Begründung der transfinite Mengenlehre (1895)

Unter eine ‚Menge' verstehen wir jede Zusammenfassung M von bestimmten wohlunterschiedenen Objecten m, unserer Anschauung oder unseres Denkens (welche die ‚Elemente' von M genannt werden) zu einem Ganzen.
In Zeichen drücken wir dies so aus

$$M = \{m\}.$$

This is how we think about sets but...
There are problematic collections:
- All sets
- All ordinal numbers [Burali-Forti]
- All cardinal numbers [Cantor]

Should be sets but have properties that sets should not have

PROPOSED SOLUTION [FREGE]
USE FORMULAS TO DESCRIBE SETS
$\{x : \varphi(x)\}$
TROUBLESOME STILL
$\{x : x = x\}$, $\{x : ORD(x)\}$, $\{x : CARD(x)\}$
BUT THESE ARE HUGE 'IMPROPER' SETS
[CANTOR WROTE ABOUT 'PROPER' AND 'IMPROPER' SETS IN CORRESPONDENCE]
RUSSELL : $R = \{x : x \notin x\}$
GIVES SIMPLE LOGICAL CONTRADICTION:
$$R \in R \leftrightarrow R \notin R$$

SOLUTION : AXIOMS [RULES TO LIVE BY]
ZERMELO : "PRINZIPIEN" OR "AXIOMEN".
[NOT YET A FIRST-ORDER THEORY]
A "DOMAIN" OF OBJECTS THAT WE CALL "THINGS"; THE "SETS" ARE AMONG THESE THINGS.
- DOMAIN $\mathfrak{B}$
- a EXISTS MEANS a BELONGS TO $\mathfrak{B}$
- A 'RELATION' : $a \in b$ MEANS "a IS AN ELEMENT OF THE SET b"
- A THING b THAT CONTAINS AN ELEMENT a CAN BE CALLED A SET AND ONLY THEN, WITH ONE EXCEPTION.
- IF $x \in M$ ALWAYS IMPLIES $x \in N$ THEN M IS SAID TO BE A SUBSET OF N
- DISJOINT : NO ELEMENT IN COMMON
  NO x SATISFIES $x \in M$ AND $x \in N$.

A PROPERTY IS DEFINITE IF WE CAN DECIDE ITS VALIDITY USING THE AXIOMS IN A PURELY LOGICAL WAY.

SO $a \in b$ IS DEFINITE

$M \subseteq N$ IS DEFINITE

- AXIOM I: IF $M \subseteq N$ AND $N \subseteq M$ THEN $M = N$.
  "A SET IS DETERMINED BY ITS ELEMENTS"
  (AXIOM DER BESTIMMTHEIT, AXIOM OF EXTENSIONALITY)
- AXIOM II: THERE IS AN (IMPROPER) SET, THE "NULL SET" $\emptyset$ THAT CONTAINS NO ELEMENTS.
  - IF a IS A THING THEN THERE EXISTS A SET $\{a\}$ THAT CONTAINS JUST a
  - IF a AND b ARE TWO THINGS THEN THERE IS ALWAYS A SET $\{a,b\}$ THAT CONTAINS a AND b AND NO OTHER THINGS.

  (AXIOM OF ELEMENTARY SETS)
- AXIOM III (AXIOM DER AUSSONDERUNG, SEPARATION)
  IF $\mathfrak{F}(x)$ IS A PROPERTY DEFINITE FOR THE ELEMENTS OF A SET M THEN M HAS A SUBSET $M_{\mathfrak{F}}$ WHICH CONSISTS OF THE ELEMENTS x OF M FOR WHICH $\mathfrak{F}(x)$ HOLDS: $\underline{\{x \in M : \mathfrak{F}(x)\}}$

[ZERMELO EXPLAINS THAT THIS DOES AWAY WITH THE FIRST THREE PARADOXES: IF WE WANT TO BUILD A NEW SET USING A PROPERTY THEN WE CAN ONLY DO THIS STARTING FROM A GIVEN SET]

NOTATION FOR $M_{\mathcal{F}}$:
$\{x \in M : \mathcal{F}(x)\}$

- IF $M_1 \subseteq M$ THEN "$x \notin M_1$" IS DEFINITE
  SO $M \setminus M_1 = \{x \in M : x \notin M_1\}$
  IS A SET.
- GIVEN M AND N (SETS) "$x \in N$"
  IS DEFINITE SO $\{x \in M : x \in N\}$ IS A SET
  NOTATION $M \cap N$ (ZERMELO $[M, N]$)
- GIVEN A SET T OF SETS THE THING
  $\{x : (\forall A \in T)(x \in A)\}$
  IS A SET.
  - FOR EACH THING a WE HAVE A SET
    $T_a = \{A \in T : a \in A\}$
  - FOR A FIXED $A \in T$ LET
    $D = \{a \in A : T_a = T\}$
    NOTATION $D = \bigcap T$ (ZERMELO $\mathfrak{D}T$)
    [STILL TO PROVE: IF $B \in T$ THEN ALSO
    $D = \{a \in B : T_a = T\}$]

THEOREM: EVERY SET M HAS A SUBSET
THAT IS NOT AN <u>ELEMENT</u> OF M

PROOF EXERCISE $\square$

SO THERE IS NO SET M IN OUR DOMAIN $\mathfrak{B}$
THAT CONTAINS ALL THINGS IN $\mathfrak{B}$.
THIS DEFUSES RUSSELL'S PARADOX

- AXIOM IV (AXIOM OF POWER SET)
  FOR EVERY SET T THERE IS A SET $\mathcal{P}(T)$
  [THE POWERSET] THAT CONTAINS THE
  SUBSETS OF T AND ONLY THOSE.

Axiom V (Axiom of Union)

For every set T there is a set $\cup T$ [the union of T] that contains the elements of the elements of T and only those. (Zermelo $\mathfrak{S}T$)
Sum

Exercise $M \cup N = N \cup M$

$M \cup (N \cup R) = (M \cup N) \cup R$

$(M \cup N) \cap R = (M \cap R) \cup (N \cap R)$

$(M \cap N) \cup R = (M \cup R) \cap (N \cup R)$

[Zermelo's Hint: Apply Axiom 1 and show that every element of the left hand side belongs to the right hand side and conversely]

Introduction of the product (set)

Let T be a set whose elements $M, N, R, \ldots$ are all (pairwise disjoint) sets and S_1 a subset of $\cup T$.

For each $M \in T$ is the statement "$M \cap S_1$ consists of exactly one point" definite.

So $T_1 = \{M \in T : S_1 \cap M$ consists of one point$\}$ is a set, a subset of T.

And "$T_1 = T$" is again definite.

So the subsets S_1 of $\cup T$ that have exactly one element in common with each element of T form a set, a subset of $\mathfrak{P}(\cup T)$.

We call this set the product of T and write $\prod T$

IF $T = \{M, N\}$ OR $T = \{M, N, R\}$ WE WRITE
$\Pi T = M \times N$ OR $\Pi T = M \times N \times R$.

! IN ORDER TO GET THE THEOREM THAT A PRODUCT OF SETS IS EMPTY IF AND ONLY IF (AT LEAST) ON FACTOR IS EMPTY WE NEED

- AXIOM VI (AXIOM DER AUSWAHL; AXIOM OF CHOICE)
  IF T IS A SET WHOSE ELEMENTS ARE SETS DIFFERENT FROM $\emptyset$ AND MUTUALLY DISJOINT THEN $\cup T$ HAS AT LEAST ONE SUBSET S_1 THAT HAS EXACTLY ONE POINT IN COMMON WITH EACH ELEMENT OF T.
  "IT IS ALWAY POSSIBLE TO CHOOSE FROM EVERY ELEMENT $M, N, R, \ldots$ OF T A SINGLE ELEMENT $m, n, r, \ldots$ AND COLLECT THESE ELEMENTS IN A SET S_1."
  "THESE AXIOMS SUFFICE, AS WE SHALL SEE, TO DERIVE ALL THE ESSENTIAL THEOREMS OF GENERAL SET THEORY."
  "TO BE ABLE TO TALK ABOUT INFINITE SETS WE NEED ONE MORE AXIOM."

- AXIOM VII (AXIOM DES UNENDLICHEN)
  THE DOMAIN CONTAINS AT LEAST ONE SET Z THAT HAS THE NULL SET $\emptyset$ AS AN ELEMENT AND IS SUCH THAT WITH EVERY ELEMENT a ALSO CONTAINS THE CORRESPONDING SET $\{a\}$.

LET Z BE A SET AS IN AXIOM VII

- FOR EVERY SUBSET Z_1 OF Z IT IS DEFINITE WHETHER IT SATISFIES THE CONDITIONS IN AXIOM VII:

  $\emptyset \in Z_1$ AND IF $a \in Z_1$ THEN $\{a\} \in Z_1$ ARE QUITE DEFINITE

- SO THE SUBSETS OF Z THAT SATISFY THE CONDITIONS FORM A SUBSET OF $\mathcal{P}(Z)$
- CONSIDER $Z_0 = \bigcap T$

  THEN Z_0 ALSO SATISFIES THE CONDITIONS

  THE SMALLEST SUCH SUBSET OF Z

IF Z' IS ALSO AS IN AXIOM VII

THEN IT HAS A SMALLEST SUBSET Z_0' THAT SATISFIES THE CONDITIONS

CONSIDER $Z_0 \cap Z_0'$

- IT SATISFIES THE CONDITIONS
- $Z_0 \cap Z_0' \subseteq Z_0 \subseteq Z$ SO $Z_0 \cap Z_0' = Z_0$
- $Z_0 \cap Z_0' \subseteq Z_0' \subseteq Z'$ SO $Z_0 \cap Z_0' = Z_0'$

CONCLUSION

Z_0 IS <u>THE</u> SMALLEST SET THAT SATISFIES THE CONDITIONS OF AXIOM VII

IT CONSISTS OF

$\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \ldots$

AND MAY SERVE AS THE SEQUENCE OF NATURAL NUMBERS

IT IS THE SIMPLEST EXAMPLE OF "COUNTABLY INFINITE" SET.

NEXT TIME: THE MODERN VERSION OF THIS (WHAT IS 'DEFINITE' ANYWAY!)
(DO WE REALLY HAVE ALL WE NEED?)