

WHAT IS A SET? WHAT IS SET THEORY?

BOLZANO, PARADOXIEN DES UNENDLICHEN (1847)
 EINEN INBEGRIFF, DEN WIR EINEM SOLCHEN
 BEGRIFFE UNTERSTELLEN, BEI DEN DIE
 ANORDNUNG SEINER THEILE GLEICHGÜLTIG IST
 (AN DEM SICH ALSO NICHTS FÜR UNS WESENTLICHES
 ÄNDERT, WENN SICH BLOSS DIESE ÄNDERT)
 NENNE ICH EINE MENGE

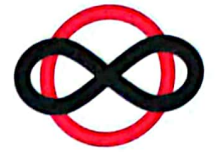
CANTOR, BEITRÄGE ZUR BEGRÜNDUNG DER
 TRANSFINITE MENGENLEHRE (1895)
 UNTER EINE 'MENGE' VERSTEHEN WIR JEDE
 ZUSAMMENFASSUNG Γ VON BESTIMMTEN
 WOHL UNTERSCHIEDENEN OBJECTEN m , UNSERER
 ANSCHAUUNG ODER UNSERES DENKENS,
 (WELCHE DIE ELEMENTE VON Γ GENANT
 WERDEN) ZU EINEM GANZEN.

IN ZEICHEN BRÜCKEN WIR DIES SO AUS
 $\Gamma = \{m\}$.

THIS IS HOW WE THINK ABOUT SETS BUT...
 THERE ARE PROBLEMATIC COLLECTIONS:

- ALL SETS
- ALL ORDINAL NUMBERS [BURALI-FORTI]
- ALL CARDINAL NUMBERS [CANTOR]

SHOULD BE SETS BUT HAVE PROPERTIES
 THAT SETS SHOULD NOT HAVE



PROPOSED SOLUTION [FREGE]

USE FORMULAS TO DESCRIBE SETS

$$\{x : \varphi(x)\}$$

TROUBLE SOME STILL

$$\{x : x = x\}, \{x : \text{ORD}(x)\}, \{x : \text{CARD}(x)\}$$

BUT THESE ARE HUGE 'IMPROPER' SETS

[CANTOR WROTE ABOUT 'PROPER' AND 'IMPROPER' SETS IN CORRESPONDENCE]

$$\text{RUSSELL: } R = \{x : x \notin x\}$$

GIVES SIMPLE LOGICAL CONTRADICTION:

$$R \in R \Leftrightarrow R \notin R$$

SOLUTION: AXIOMS [RULES TO LIVE BY]

ZERMELO: "PRINZIPIEN" OR "AXIOMEN".

[NOT YET A FIRST-ORDER THEORY]

A "DOMAIN" OF OBJECTS THAT WE CALL "THINGS"; THE "SETS" ARE AMONG THESE THINGS.

- Domain \mathcal{O}

- a EXISTS MEANS a BELONGS TO \mathcal{O}

- A "RELATION": $a \in b$ MEANS

" a IS AN ELEMENT OF THE SET b "

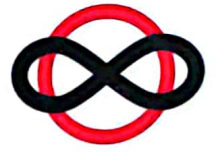
- A THING b THAT CONTAINS AN ELEMENT a CAN BE CALLED A SET AND ONLY THEN, WITH ONE EXCEPTION.

- IF $x \in \Pi$ ALWAYS IMPLIES $x \in N$

THEN Π IS SAID TO BE A SUBSET OF N

- DISJOINT: NO ELEMENT IN COMMON

NO x SATISFIES $x \in \Pi$ AND $x \in N$.



A PROPERTY IS DEFINITE IF WE CAN DECIDE ITS VALIDITY USING THE AXIOMS IN A PURELY LOGICAL WAY -
 SO $a \in b$ IS DEFINITE
 $M \subseteq N$ IS DEFINITE

• AXIOM I: IF $M \subseteq N$ AND $N \subseteq M$ THEN $M = N$
 "A SET IS DETERMINED BY ITS ELEMENTS"
 (AXIOM DER BESTIMMTHEIT, AXIOM OF EXTENSIONALITY)

• AXIOM II: - THERE IS AN (IMPROPER) SET,
 THE "NULL SET" \emptyset THAT CONTAINS
 NO ELEMENTS.

- IF a IS A THING THEN THERE EXISTS A SET $\{a\}$ THAT CONTAINS JUST a

- IF a AND b ARE TWO THINGS THEN THERE IS ALWAYS A SET $\{a, b\}$ THAT CONTAINS a AND b AND NO OTHER THINGS.

(AXIOM OF ELEMENTARY SETS)

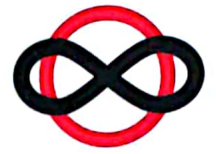
• AXIOM III (AXIOM DER AUSSONDERUNG, SEPARATION)

IF $\mathcal{F}(x)$ IS A PROPERTY DEFINITE FOR THE ELEMENTS OF A SET M

THEN M HAS A SUBSET $M_{\mathcal{F}}$ WHICH CONSISTS OF THE ELEMENTS x OF M

FOR WHICH $\mathcal{F}(x)$ HOLDS: $\{x \in M : \mathcal{F}(x)\}$

[ZERMELO EXPLAINS THAT THIS DOES AWAY WITH THE FIRST THREE PARADOXES: IF WE WANT TO BUILD A NEW SET USING A PROPERTY THEN WE CAN ONLY DO THIS STARTING FROM A GIVEN SET]



NOTATION FOR $\mathcal{P}M$:
 $\{x \in M : \neg(x \in M)\}$

- IF $M_1 \in M$ THEN " $x \notin M_1$ " IS DEFINITE
 SO $M \setminus M_1 = \{x \in M : x \notin M_1\}$
 IS A SET.

- GIVEN M AND N (SETS) " $x \in N$ "
 IS DEFINITE SO $\{x \in M : x \in N\}$ IS A SET.
 NOTATION $M \cap N$ (ZERMELO $[M, N]$)

- GIVEN A SET T OF SETS THE THING
 $\{x : (\forall A \in T)(x \in A)\}$
 IS A SET.

• FOR EACH THING a WE HAVE A SET
 $T_a = \{A \in T : a \in A\}$

• FOR A FIXED $A \in T$ LET
 $D = \{a \in A : T_a = T\}$

NOTATION $D = \bigcap T$ (ZERMELO $\bigcap T$)

[STILL TO PROVE! IF $B \in T$ THEN ALSO
 $D = \{a \in A : T_a = T\}$]

THEOREM: EVERY SET M HAS A SUBSET
 THAT IS NOT AN ELEMENT OF M

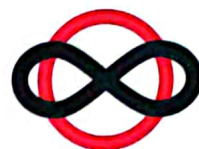
PROOF: EXERCISE \square

SO THERE IS NO SET M IN OUR DOMAIN \mathcal{B}
 THAT CONTAINS ALL THINGS IN \mathcal{B} .

THIS DEFUSES RUSSELL'S PARADOX

• AXIOM IV (AXIOM OF POWER SET)

FOR EVERY SET T THERE IS A SET $\mathcal{P}(T)$
 [THE POWER SET] THAT CONTAINS THE
 SUBSETS OF T AND ONLY THOSE.



AXIOM V (AXIOM OF UNION)

FOR EVERY SET T THERE IS A SET UT
 [THE UNION OF T] THAT CONTAINS
 THE ELEMENTS OF THE ELEMENTS OF T
 AND ONLY THOSE. (ZERMELO $\aleph T$)
 Sum

EXERCISE $M \cup N = N \cup M$

$$M \cup (N \cap R) = (M \cup N) \cap R$$

$$(M \cup N) \cap R = (M \cap R) \cup (N \cap R)$$

$$M \cap (N \cup R) = (M \cap N) \cup (M \cap R)$$

[ZERMELO'S HINT: APPLY AXIOM 2 AND
 SHOW THAT EVERY ELEMENT OF THE
 LEFT HAND SIDE BELONGS TO THE
 RIGHT HAND SIDE AND CONVERSELY]

INTRODUCTION OF THE PRODUCT SET

LET T BE A SET WHOSE ELEMENTS M, N, R, \dots
 ARE ALL (PAIRWISE DISJOINT) SETS AND
 S_i A SUBSET OF UT .

FOR EACH $\Pi \in T$ IS THE STATEMENT

" $\Pi \cap S_i$ CONSISTS OF EXACTLY ONE POINT"

DEFINITE

SO $T_i = \{\Pi \in T : S_i \cap \Pi \text{ CONSISTS OF ONE POINT}\}$

IS A SET, A SUBSET OF T .

AND " $T_i = T$ " IS AGAIN DEFINITE.

SO THE SUBSETS S_i OF UT THAT HAVE
 EXACTLY ONE ELEMENT IN COMMON WITH
 EACH ELEMENT OF T FORM A SET,
 A SUBSET OF $\mathcal{P}(UT)$

WE CALL THIS SET THE PRODUCT OF T
 AND WRITE $\prod T$



IF $T = \{M, N\}$ OR $T = \{M, N, R\}$ WE WRITE
 $\prod T = M \times N$ OR $\prod T = M \times N \times R$

! IN ORDER TO GET THE THEOREM THAT
A PRODUCT OF SETS IS EMPTY IF AND
ONLY IF (AT LEAST) ONE FACTOR IS EMPTY
WE NEED

• AXIOM VI (AXIOM DER AUSWAHL; AXIOM OF CHOICE)

IF T IS A SET WHOSE ELEMENTS ARE
SETS DIFFERENT FROM \emptyset AND MUTUALLY
DISJOINT THEN $\cup T$ HAS AT LEAST ONE
SUBSET S , THAT HAS EXACTLY ONE
POINT IN COMMON WITH EACH ELEMENT OF T .

"IT IS ALWAYS POSSIBLE TO CHOOSE FROM
EVERY ELEMENT M, N, R, \dots OF T
A SINGLE ELEMENT m, n, r, \dots
AND COLLECT THESE ELEMENTS IN
A SET S ."

"THESE AXIOMS SUFFICE, AS WE SHALL SEE,
TO DERIVE ALL THE ESSENTIAL THEOREMS
OF GENERAL SET THEORY."

"TO BE ABLE TO TALK ABOUT INFINITE
SETS WE NEED ONE MORE AXIOM."

• AXIOM VII (AXIOM DES UNENDLICHEN)

THE DOMAIN CONTAINS AT LEAST ONE SET Z
THAT HAS THE NULL SET \emptyset AS AN
ELEMENT AND IS SUCH THAT WITH EVERY
ELEMENT α ALSO CONTAINS THE
CORRESPONDING SET $\{\alpha\}$.



LET Z BE A SET AS IN AXIOM VII

- FOR EVERY SUBSET Z_1 OF Z IT IS DEFINITE WHETHER IT SATISFIES THE CONDITIONS IN AXIOM VII:

$\emptyset \in Z_1$ AND IF $a \in Z_1$, THEN $\{a\} \in Z_1$,
ARE QUITE DEFINITE

- SO THE SUBSETS OF Z THAT SATISFY THE CONDITIONS FORM A SUBSET OF $\mathcal{P}(Z)$

CONSIDER $Z_0 = \bigcap T$

THEN Z_0 ALSO SATISFIES THE CONDITIONS.
THE SMALLEST SUCH SUBSET OF Z

IF Z' IS ALSO AS IN AXIOM VII

THEN IT HAS A SMALLEST SUBSET Z'_0
THAT SATISFIES THE CONDITIONS

CONSIDER $Z_0 \cap Z'_0$

- IT SATISFIES THE CONDITIONS

$$- Z_0 \cap Z'_0 \in Z_0 \in Z \quad \text{SO } Z_0 \cap Z'_0 \in Z_0$$

$$- Z_0 \cap Z'_0 \in Z'_0 \in Z' \quad \text{SO } Z_0 \cap Z'_0 \in Z'_0$$

CONCLUSION

Z_0 IS THE SMALLEST SET THAT
SATISFIES THE CONDITIONS OF AXIOM VII

IT CONSISTS OF

$\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \dots$

AND MAY SERVE AS THE SEQUENCE
OF NATURAL NUMBERS

IT IS THE SIMPLEST EXAMPLE
OF "COUNTABLY INFINITE" SET.

NEXT TIME: THE MODERN VERSION
OF THIS (WHAT IS 'DEFINITE' ANYWAY?)
(DO WE REALLY HAVE ALL WE NEED?)