REPLACEMENT SCHEMA IN JECH'S BOOK.

$(\forall x)(\forall y)(\forall z)\,(\varphi(x,y,p) \wedge \varphi(x,z,p) \rightarrow y=z) \rightarrow$
$\rightarrow (\forall A)(\exists B)(\forall y)\,(y \in B \leftrightarrow (\exists x \in A)\varphi(x,y,p))$

THE DIFFERENCE: THE SET B IS EXACTLY THE SET OF y THAT CORRESPOND TO SOME $x \in A$.

NOW LOOK AT EXERCISE 1.14 IN JECH'S BOOK.

OTHER THING IN JECH'S BOOK: CLASSES

GIVEN A FORMULA $\varphi(x, p_1, \dots, p_n)$

$$C = \{x : \varphi(x, p_1, \dots, p_n)\}$$

IS THE CLASS DETERMINED BY φ AND $p_1, \dots, p_n$.

IF $C = \{x : \varphi(x, p_1, \dots, p_n)\}$
$D = \{x : \psi(x, q_1, \dots, q_m)\}$

THEN $C = D$ ABBREVIATES

$$(\forall x)\,(\varphi(x, p_1, \dots, p_n) \leftrightarrow \psi(x, q_1, \dots, q_m))$$

ETC.

$V = \{x : x = x\}$ THE CLASS OF ALL SETS.

$R = \{x : x \notin x\}$ RUSSELL'S CLASS THAT IS NOT A SET.

WE USE THEM INFORMALLY WHEN IT SEEMS EASIER ON THE EYES.

"THERE IS A CLASS ------" IS SOMETIMES EASIER TO DEAL WITH THAN "THERE IS A FORMULA ..."

OTHER (FUTURE) CLASSES

$ON = \{x : x \text{ IS AN ORDINAL}\}$

$CARD = \{x : x \text{ IS A CARDINAL}\}$

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SOME FAMILIAR NOTIONS.

A <u>RELATION</u> IS A SET THAT CONSISTS OF ORDERED PAIRS ONLY.

IF R IS A RELATION THEN

$\operatorname{dom} R = \{x : (\exists y)(\langle x,y\rangle \in R)\}$

$\operatorname{ran} R = \{y : (\exists x)(\langle x,y\rangle \in R)\}$

ARE THESE SETS ?

- BY REPLACEMENT
  $(\forall z \in R)(\exists! x)((\exists y)(z = \langle x,y\rangle))$
  $(\forall z \in R)(\exists! y)((\exists x)(z = \langle x,y\rangle))$
- BY UNION AND SEPARATION
  NOTE THAT $x, y \in \bigcup z$ IF $\langle x,y\rangle = z$
  AND SO $x, y \in \bigcup\bigcup R$ IF $z \in R$
  SO $\operatorname{dom} R$, $\operatorname{ran} R \subseteq \bigcup\bigcup R$
- ALSO $R \subseteq \operatorname{dom} R \times \operatorname{ran} R$.
- $R^{-1} = \{\langle y,x\rangle : \langle x,y\rangle \in R\}$

A FUNCTION (OR MAP (OR MAPPING)) IS A SPECIAL KIND OF RELATION.

f IS A FUNCTION IF f IS A RELATION

<u>AND</u> $(\forall x)(\forall y)(\forall z)(\langle x,y\rangle \in f \wedge \langle x,z\rangle \in f \rightarrow y = z)$

(OR $(\forall x \in \operatorname{dom} f)(\exists! y)(\langle x,y\rangle \in f)$)

SO FUNCTIONS HAVE THEIR OWN DOMAINS AND ARE ALWAYS SURJECTIVE ....

BUT

$f : A \rightarrow B$

ABBREVIATES : f IS A FUNCTION <u>AND</u> $\operatorname{dom} f = A$ <u>AND</u> $\operatorname{ran} f \subseteq B$.

OF COURSE $f(x)$ IS THE UNIQUE y WITH $\langle x,y\rangle \in f$.

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- f IS INJECTIVE IF f^{-1} IS A FUNCTION

IN GENERAL:

$$R[A] = \{y : (\exists x \in A)(\langle x,y\rangle \in R)\}$$

SO FOR A FUNCTION f

$$f[A] = \{f(x) : x \in A\}$$

(THIS MAKES FORMAL SENSE EVEN IF $A \not\subseteq \mathrm{dom}\,R$ OR EVEN $A \cap \mathrm{dom}\,R = \emptyset$)

LIKEWISE RESTRICTION

$$R \restriction A = \{\langle x,y\rangle \in R : x \in A\}$$

- FOR FUNCTIONS $f : A \to B$

  SURJECTIVE MEANS $\mathrm{ran}\, f = B$

  BIJECTIVE MEANS INJECTIVE + SURJECTIVE

ORDERINGS

A TOTAL ORDERING IS A PAIR $\langle A, R\rangle$ WHERE A IS A SET, R IS A RELATION THAT TOTALLY ORDERS A:

- R IS TRANSITIVE ON A

  $(\forall x \in A)(\forall y \in A)(\forall z \in A)(xRy \wedge yRz \rightarrow xRz)$

- R IS IRREFLEXIVE

  $(\forall x \in A)(\neg(xRx))$

- TRICHOTOMY

  $(\forall x \in A)(\forall y \in A)(x = y \vee xRy \vee yRx)$

NOTE - xRy MEANS $\langle x,y\rangle \in R$

- $R \subseteq A \times A$ IS NOT ASSUMED

  SO R CAN BE A TOTAL ORDER OF MANY SETS

OF COURSE A FUNCTION $f : A \to B$ IS AN ISOMORPHISM OF $\langle A,R\rangle$ AND $\langle B,S\rangle$ IF IT IS A BIJECTION SUCH THAT $xRy \iff f(x)\, S\, f(y)$.

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R IS A WELL-ORDER OF A OR $\langle A,R\rangle$ IS A WELL-ORDERING
IF $\langle A,R\rangle$ IS A TOTAL ORDERING
AND EVERY NON-EMPTY SUBSET OF A
HAS AN R-LEAST ELEMENT:
$(\forall B)(B \neq \emptyset \wedge B \subseteq A \rightarrow (\exists x \in B)(\forall y \in A)(y R x \rightarrow y \notin B))$

IF $x \in A$ THEN $\mathrm{PRED}(A,x,R) = \{y \in A : y R x\}$

INDUCTION:
LET $B \subseteq A$
IF FOR ALL $x \in A$ THE IMPLICATION
$\mathrm{PRED}(A,x,R) \subseteq B \longrightarrow x \in B$
HOLDS
THEN $B = A$.

PROOF IF $A \setminus B \neq \emptyset$ LET $x \in A \setminus B$
BE SUCH THAT $(\forall y \in A)(y R x \rightarrow y \notin A \setminus B)$
OR: $\mathrm{PRED}(A,x,R) \subseteq B$
BUT THEN $x \in B$ CONTRADICTION.

RIGIDITY:
- IF $f : A \rightarrow A$ IS SUCH THAT
$y R x \rightarrow f(y)\, R\, f(x)$
THEN FOR ALL $x \in A$ WE HAVE $x = f(x)$ OR $x R f(x)$

LET $B = \{x : x = f(x) \text{ OR } x R f(x)\}$
ASSUME $\mathrm{PRED}(A,x,R) \subseteq B$
LET $y \in \mathrm{PRED}(A,x,R)$
THEN $y = f(y)$ OR $y R f(y)$
SINCE $f(y) R f(x)$ WE GET $y R f(x)$
IT FOLLOWS THAT $f(x) \notin \mathrm{PRED}(A,x,R)$
HENCE, BY TRICHOTOMY, $x = f(x)$ OR $x R f(x)$.

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- IF $x \in A$ THEN $\langle A,R\rangle$ AND PRED(A,x,R) ARE NOT ISOMORPHIC

  IF $f : A \to$ PRED(A,x,R) WERE AN ISOMORPHISM $f(x)$ HAS NO PLACE TO GO: $x = f(x)$ OR $x \, R \, f(x)$

- IF $\langle A,R\rangle$ AND $\langle B,S\rangle$ ARE ISOMORPHIC THEN THERE IS JUST ONE ISOMORPHISM.

PROOF LET f AND g BE ISOMORPHISMS AND LET $C = \{x \in A : f(x) = g(x)\}$

ASSUME PRED$(A,x,R) \subseteq C$

CONSIDER $f(x)$ AND $g(x)$

IF $f(x) \, S \, g(x)$ THEN THERE MUST BE A $y \in$ PRED(A,x,R) SUCH THAT $g(y) = f(x)$

BUT THEN ALSO $f(y) = g(y) = f(x)$

CONTRADICTION

LIKEWISE $g(x) \, S \, f(x)$ IS FALSE

SO $f(x) = g(x)$.

- IF $\langle A,R\rangle$ AND $\langle B,S\rangle$ ARE TWO WELL-ORDERS THEN EITHER $\langle A,R\rangle \simeq \langle B,S\rangle$

  OR $\langle A,R\rangle \simeq \langle \text{PRED}(B,y,S), S\rangle$ FOR SOME $y \in B$

  OR $\langle \text{PRED}(A,x,R), R\rangle \simeq \langle B,S\rangle$ FOR SOME $x \in A$

PROOF: LET $f = \{\langle x,y\rangle \in A \times B : \langle \text{PRED}(A,x,R), R\rangle \simeq \langle \text{PRED}(B,y,S), S\rangle\}$

THEN f SATISFIES $x_1 \, R \, x_2 \Leftrightarrow f(x_1) \, S \, f(x_2)$

– DOM $f = A$ OR DOM $f =$ PRED(A,x,R) FOR SOME x

SAY x IS R-MINIMAL IN $A \setminus$ DOM f

THEN PRED$(A,x,R) \subseteq$ DOM f

ALSO IF $z \in$ DOM f THEN PRED$(A,z,R) \subseteq$ DOM f

IF h: PRED$(A,z,R) \to$ PRED(B,w,R) IS AN ISOMORPHISM THEN -------

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---- h IS ALSO AN ISOMORPHISM BETWEEN $\mathrm{PRED}(A,u,R)$ AND $\mathrm{PRED}(B,h(u),S)$ FOR ALL $u \in \mathrm{PRED}(A,z,R)$.

SO $\mathrm{dom}\, h \in \mathrm{PRED}(A,x,R)$ AS WELL:
$x \notin \mathrm{dom}\, h$ SO IF xRy THEN $y \notin \mathrm{dom}\, h$.

- LIKEWISE $\mathrm{RAN}\, f = \mathrm{DOM}\, f^{-1} = B$
  OR $\mathrm{DOM}\, f^{-1} = \mathrm{PRED}(B,y,S)$
  FOR SOME y.

SO
- $\mathrm{DOM}\, f = A$ AND $\mathrm{DOM}\, f^{-1} = B$: ISOMORPHISM
- $\mathrm{DOM}\, f = A$ AND $\mathrm{DOM}\, f^{-1} = \mathrm{PRED}(B,y,S)$: ISOMORPHISM
- $\mathrm{DOM}\, f = \mathrm{PRED}(A,x,R)$ AND $\mathrm{DOM}\, f^{-1} = B$: ISOMORPHISM
- $\mathrm{DOM}\, f = \mathrm{PRED}(A,x,R)$ AND $\mathrm{DOM}\, f^{-1} = \mathrm{PRED}(B,y,S)$: NOT POSSIBLE FOR THEN $\langle x,y\rangle \in f$ AND $x \in \mathrm{PRED}(A,x,R)$ AND $y \in \mathrm{PRED}(B,y,S)$ [CONTRADICTION]

AXIOM 9 AXIOM OF CHOICE
EVERY SET ADMITS A WELL-ORDER
$(\forall A)(\exists R)(\langle A,R\rangle$ IS A WELL-ORDER$)$

ZERMELO PROVED THAT HIS AXIOM OF CHOICE IMPLIES THIS VERSION: "THE WELL-ORDERING THEOREM"
THE CONVERSE IS (RELATIVELY EASY).

ORIGINAL AC IMPLIES COMMON AC:
IF T IS A FAMILY OF NONEMPTY SETS THEN THERE IS A FUNCTION
$$f : T \longrightarrow \bigcup T$$
SUCH THAT $f(t) \in t$ FOR ALL $t \in T$.

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LET T BE SUCH A FAMILY.

START WITH $T \times \bigcup T$ --- HOMEWORK 01

MAKE $\mathcal{P}(T \times \bigcup T)$

AND THEN

$$J = \{A \in \mathcal{P}(T \times \bigcup T) : (\exists t \in T)(\forall a)(a \in A \leftrightarrow (\exists x \in t)(a = \langle t, x \rangle))\}$$

THIS IS $\{\{t\} \times t : t \in T\}$.

WE MADE A DISJOINT FAMILY J OUT OF T

NOW LET $S \subseteq \bigcup J$ BE SUCH THAT

$S \cap (\{t\} \times t)$

CONSISTS OF ONE ELEMENT, FOR EACH t.

THEN S IS A CHOICE FUNCTION.

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