

REPLACEMENT SCHEMA IN JECH'S BOOK.

$$(\forall x)(\forall y)(\forall z) (\varphi(x, y, p) \wedge \varphi(x, z, p) \rightarrow y=z) \rightarrow \\ \rightarrow (\forall A)(\exists B)(\forall y)(y \in B \leftrightarrow (\exists x \in A) \varphi(x, y, p))$$

THE DIFFERENCE: THE SET B IS EXACTLY THE SET OF y THAT CORRESPOND TO SOME $x \in A$.

NOW LOOK AT EXERCISE 1.14 IN JECH'S BOOK.

OTHER THING IN JECH'S BOOK: CLASSES

GIVEN A FORMULA $\varphi(x, p_1, \dots, p_n)$

$$C = \{x : \varphi(x, p_1, \dots, p_n)\}$$

IS THE CLASS DETERMINED BY φ AND p_1, \dots, p_n

IF $C = \{x : \varphi(x, p_1, \dots, p_n)\}$

$$D = \{x : \psi(x, q_1, \dots, q_m)\}$$

THEN $C = D$ ABBREVIATES

$$(\forall x) (\varphi(x, p_1, \dots, p_n) \leftrightarrow \psi(x, q_1, \dots, q_m))$$

ETC

$$V = \{x : x = x\}$$

THE CLASS OF ALL SETS.

$$R = \{x : x \notin x\}$$

RUSSELL'S CLASS THAT

IS NOT A SET.

WE USE THEM INFORMALLY WHEN IT SEEMS EASIER ON THE EYES.

"THERE IS A CLASS ----" IS SOMETIMES EASIER TO DEAL WITH THAN "THERE IS A FORMULA ----"

OTHER (FUTURE) CLASSES

$$ON = \{x : x \text{ IS AN ORDINAL}\}$$

$$CARD = \{x : x \text{ IS A CARDINAL}\}$$



SOME FAMILIAR NOTIONS.

A RELATION IS A SET THAT CONSISTS OF ORDERED PAIRS ONLY.

IF R IS A RELATION THEN

$$\text{DOM } R = \{x : (\exists y)(\langle x, y \rangle \in R)\}$$

$$\text{RAN } R = \{y : (\exists x)(\langle x, y \rangle \in R)\}$$

ARE THESE SETS?

- BY REPLACEMENT

$$(\forall z \in R)(\exists! x)(\exists y)(z = \langle x, y \rangle)$$

$$(\forall z \in R)(\exists! y)(\exists x)(z = \langle x, y \rangle)$$

- BY UNION AND SEPARATION

NOTE THAT $x, y \in \cup Z$ IF $\langle x, y \rangle \in Z$

AND SO $x, y \in \cup UR$ IF $z \in R$

SO $\text{DOM } R, \text{RAN } R \subseteq \cup UR$

- ALSO $R \subseteq \text{DOM } R \times \text{RAN } R$.

- $R^{-1} = \{\langle y, x \rangle : \langle x, y \rangle \in R\}$

A FUNCTION (OR MAP (OR MAPPING))

IS A SPECIAL KIND OF RELATION.

f IS A FUNCTION IF f IS A RELATION AND $(\forall x)(\forall y)(\forall z)(\langle x, y \rangle \in f \wedge \langle x, z \rangle \in f \rightarrow y = z)$

(OR $(\forall x \in \text{DOM } f)(\exists! y)(\langle x, y \rangle \in f)$)

SO FUNCTIONS HAVE THEIR OWN DOMAINS

AND ARE ALWAYS SURJECTIVE ---

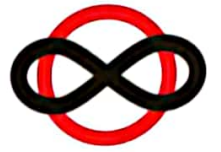
BUT

$$f: A \rightarrow B$$

ABBREVIATES: f IS A FUNCTION AND $\text{DOM } f = A$

AND $\text{RAN } f \subseteq B$.

OF COURSE $f(x)$ IS THE UNIQUE y WITH $\langle x, y \rangle \in f$.



• f IS INJECTIVE IFF f^{-1} IS A FUNCTION

IN GENERAL:

$$R[A] = \{y : (\exists x \in A)(\langle x, y \rangle \in R)\}$$

SO FOR A FUNCTION f

$$f[A] = \{f(x) : x \in A\}$$

(THIS MAKES FORMAL SENSE EVEN

IF $A \not\subseteq \text{DOM } R$ OR EVEN $A \cap \text{DOM } R = \emptyset$)

LIKEWISE RESTRICTION

$$R \upharpoonright A = \{\langle x, y \rangle \in R : x \in A\}$$

• FOR FUNCTIONS $f: A \rightarrow B$

SURJECTIVE MEANS $\text{RAN } f = B$

BIJECTIVE MEANS INJECTIVE + SURJECTIVE

ORDERINGS

A TOTAL ORDERING IS A PAIR $\langle A, R \rangle$ WHERE
 A IS A SET, R IS A RELATION THAT TOTALLY
 ORDERS A :

- R IS TRANSITIVE ON A

$$(\forall x \in A)(\forall y \in A)(\forall z \in A) (x R y \wedge y R z \rightarrow x R z)$$

- R IS IRREFLEXIVE

$$(\forall x \in A) (\neg (x R x))$$

- TRICHOTOMY

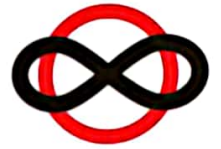
$$(\forall x \in A)(\forall y \in A) (x = y \vee x R y \vee y R x)$$

NOTE - $x R y$ MEANS $\langle x, y \rangle \in R$

- $R \subseteq A \times A$ IS NOT ASSUMED

SO R CAN BE A TOTAL ORDER
 OF MANY SETS

OF COURSE A FUNCTION $f: A \rightarrow B$ IS AN ISOMORPHISM
 OF $\langle A, R \rangle$ AND $\langle B, S \rangle$ IFF IT IS A BIJECTION SUCH
 THAT $x R y \Leftrightarrow f(x) S f(y)$.



R IS A WELL-ORDER OF A ON $\langle A, R \rangle$ IS A WELL-ORDERING
IF $\langle A, R \rangle$ IS A TOTAL ORDERING
AND EVERY NON-EMPTY SUBSET OF A
HAS AN R -LEAST ELEMENT:

$$(\forall B) (B \neq \emptyset \wedge B \subseteq A \rightarrow (\exists x \in B) (\forall y \in A) (y R x \rightarrow y \notin B))$$

IF $x \in A$ THEN $\text{pred}(A, x, R) = \{y \in A : y R x\}$

INDUCTION: FOR $P(x) =$

LET $B \subseteq A \times \{A\}$ BE A SUBSET

IF FOR ALL $x \in A$ THE IMPLICATION
 $\text{pred}(A, x, R) \subseteq B \rightarrow x \in B$
HOLDS

THEN $B = A$.

PROVE IF $A \setminus B \neq \emptyset$ LET $x \in A \setminus B$
BE SUCH THAT $(\forall y \in A) (y R x \rightarrow y \notin A \setminus B)$
OR: $\text{pred}(A, x, R) \subseteq B$
BUT THEN $x \in B$ CONTRADICTION.

RIGIDITY:

IF $f: A \rightarrow A$ IS SUCH THAT
 $y R x \rightarrow f(y) R f(x)$

THEN FOR ALL $x \in A$ WE HAVE $x = f(x)$ OR $x R f(x)$

LET $B = \{x : x = f(x) \text{ OR } x R f(x)\}$

ASSUME $\text{pred}(A, x, R) \subseteq B$

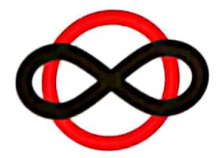
LET $y \in \text{pred}(A, x, R)$

THEN $y = f(y)$ OR $y R f(y)$

SINCE $f(y) R f(x)$ WE GET $y R f(x)$

IT FOLLOWS THAT $f(x) \notin \text{pred}(A, x, R)$

HENCE, BY TRICOTOMY, $x = f(x)$ OR $x R f(x)$.



• IF $\forall x \in A$ THEN $\langle A, R \rangle$ AND $\text{pred}(A, x, R)$ ARE NOT ISOMORPHIC IF $\nexists f: A \rightarrow \text{pred}(A, x, R)$ WERE AN ISOMORPHISM $f(x)$ HAS NO PLACE TO GO: $x = f(x)$ OR $x \in R f(x)$

• IF $\langle A, R \rangle$ AND $\langle B, S \rangle$ ARE ISOMORPHIC THEN THERE IS JUST ONE ISOMORPHISM.

PROOF LET f AND g BE ISOMORPHISMS AND LET $C = \{x \in A: f(x) = g(x)\}$

ASSUME $\text{pred}(A, x, R) \in C$ CONSIDER $f(x)$ AND $g(x)$

IF $f(x) \neq g(x)$ THEN THERE MUST BE A $y \in \text{pred}(A, x, R)$ SUCH THAT $g(y) = f(x)$

BUT THEN ALSO $f(y) = g(y) = f(x)$ CONTRADICTION

LIKEWISE $g(x) \neq f(x)$ IS FALSE SO $f(x) = g(x)$

• IF $\langle A, R \rangle$ AND $\langle B, S \rangle$ ARE TWO WELL-ORDERS THEN EITHER $\langle A, R \rangle \cong \langle B, S \rangle$

OR $\langle A, R \rangle \cong \langle \text{pred}(B, y, S), S \rangle$ FOR SOME $y \in B$
OR $\langle \text{pred}(A, x, R), R \rangle \cong \langle B, S \rangle$ FOR SOME $x \in A$

PROOF: LET $f = \{ \langle x, y \rangle \in A \times B: \langle \text{pred}(A, x, R), R \rangle \cong \langle \text{pred}(B, y, S), S \rangle \}$

THEN f SATISFIES $x_1, x_2 \in \text{dom } f \Leftrightarrow f(x_1) \neq f(x_2)$

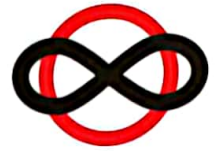
- $\text{dom } f = A$ OR $\text{dom } f = \text{pred}(A, x, R)$ FOR SOME $x \in A$

SAY x IS R -MINIMAL IN $A \setminus \text{dom } f$

THEN $\text{pred}(A, x, R) \in \text{dom } f$

ALSO IF $z \in \text{dom } f$ THEN $\text{pred}(A, z, R) \in \text{dom } f$

IF $h: \text{pred}(A, z, R) \rightarrow \text{pred}(B, w, S)$ IS AN ISOMORPHISM THEN -----



----- R IS ALSO AN ISOMORPHISM
BETWEEN $\text{PREP}(A, u, R)$ AND $\text{PREP}(B, h(u), S)$
FOR ALL $u \in \text{PREP}(A, z, R)$.

So $\text{DOM } R \in \text{PREP}(A, z, R)$ AS WELL:

$x \notin \text{DOM } R$ SO IF $x R y$ THEN $y \notin \text{DOM } R$.

- LIKEWISE $\text{RAN } f = \text{DOM } f^{-1} = B$

OR $\text{DOM } f^{-1} = \text{PREP}(B, y, S)$

FOR SOME y .

So • $\text{DOM } f = A$ AND $\text{DOM } f^{-1} = B$: ISOMORPHISM

• $\text{DOM } f = A$ AND $\text{DOM } f^{-1} = \text{PREP}(B, y, S)$: ISOMORPHISM

• $\text{DOM } f = \text{PREP}(A, z, R)$ AND $\text{DOM } f^{-1} = B$: ISOMORPHISM

• $\text{DOM } f = \text{PREP}(A, z, R)$ AND $\text{DOM } f^{-1} = \text{PREP}(B, y, S)$:

NOT POSSIBLE FOR THEM $(x, y) \in f$

AND $x \in \text{PREP}(A, z, R)$ AND $y \in \text{PREP}(B, y, S)$

[CONTRADICTION]

Axiom 9 AXIOM OF CHOICE

EVERY SET ADMITS A WELL-ORDER

$(\forall A) (\exists R) (\{A, R\} \text{ IS A WELL-ORDER})$

ZERMELO PROVED THAT HIS AXIOM OF CHOICE

IMPLIES THIS VERSION: "THE WELL-ORDERING THEOREM"

THE CONVERSE IS (RELATIVELY EASY).

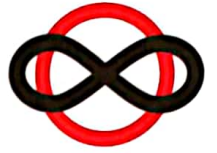
ORIGINAL AC IMPLIES COMMON AC:

IF T IS A FAMILY OF NONEMPTY SETS

THEN THERE IS A FUNCTION

$f: T \rightarrow \cup T$

SUCH THAT $f(t) \in t$ FOR ALL $t \in T$.



LET T BE SUCH A FAMILY.

START WITH $T \times U$ --- HOMEWORK OF
MAKE $\mathcal{P}(T \times U)$

AND THEN

$$\mathcal{J} = \{A \in \mathcal{P}(T \times U) : (\exists t \in T)(\forall a)(a \in A \Leftrightarrow (\exists x \in U)(a = \langle t, x \rangle))\}$$

THIS IS $\{\{t\} \times U : t \in T\}$.

WE MAKE A DISJOINT FAMILY \mathcal{J} OUT OF T

NOW LET $S \in \cup \mathcal{J}$ BE SUCH THAT

$$S \cap (\{t\} \times U)$$

CONSISTS OF ONE ELEMENT, FOR EACH t .

THEN S IS A CHOICE FUNCTION.