2022-09-28 ⑦
2022-10-03 ④

LET M BE A SET.
LET $\gamma : P(M) \setminus \{\emptyset\} \to M$ BE A FUNCTION SUCH THAT $\gamma(N) \in N$ FOR ALL N.
[READ ZERMELO'S JUSTIFICATION FOR THIS]

DEFINITION: A γ-SET IS A SUBSET A OF M WITH A WELL-ORDER $<$ SUCH THAT FOR ALL $a \in A$

$$a = \gamma(M \setminus \{x \in A : x < a\})$$

ARE THERE γ-SETS ?

- $\{\gamma(M)\}$ IS A γ-SET (WITH $< = \emptyset$)
- $\{m_0, m_1\}$ $m_0 = \gamma(M)$, $m_1 = \gamma(M \setminus \{m_0\})$ AND $< = \{\langle m_0, m_1 \rangle\}$.
- ETC ...

- IF $\langle A, <_A \rangle$ AND $\langle B, <_B \rangle$ ARE TWO γ-SETS THEN ONE IS AN INITIAL SEGMENT OF THE OTHER, E.G.,
$A = \mathrm{pred}(B, b, <_B)$ FOR SOME $b \in B$
<u>AND</u> FOR $x, y \in A$: $x <_A y \Leftrightarrow x <_B y$

- IF $\langle A, <_A \rangle$ AND $\langle B, <_B \rangle$ ARE γ-SETS THEN
  IF $a \in A \cap B$ THEN $\mathrm{PRED}(A, a, <_A) = \mathrm{PRED}(B, a, <_B)$
  IF $a, b \in A \cap B$ THEN $a <_A b \Leftrightarrow b <_A a$
- LET L_γ BE THE SET OF γ-ELEMENTS
  [x IS A γ ELEMENT IF IT BELONGS TO (THE FIRST COORDINATE OF) A γ-SET]
  THEN L_γ HAS AN ORDER $<$ THAT MAKES IT A γ-SET
  - IF $a, b \in L_\gamma$ THEN $a <_A b$ FOR ALL γ-SETS $\langle A, <_A \rangle$ WITH $a, b \in A$ OR $b <_A a$ FOR ALL SUCH γ-SETS.
    THIS DETERMINES $<$: IF $a, b \in L_\gamma$ THEN $a, b \in A$ FOR AT LEAST ONE γ-SET A.
  - IF $a < b$ AND $b < c$ THEN $a < c$
    TAKE $\langle A, <_A \rangle$ WITH $c \in A$
    THEN ALSO $b, a \in A$ AND $a <_A b$ AND $b <_A c$ AND SO $a <_A c$.
  - IF $L \subseteq L_\gamma$ IS NONEMPTY AND $a \in L$
    TAKE $\langle A, <_A \rangle$ WITH $a \in A$.
    THEN $\{x : x < a\} \subseteq A$
    LET m BE THE $<_A$-LEAST ELEMENT OF $L \cap A$
    THEN m IS THE $<$-LEAST ELEMENT OF L
  - IF $a \in L$ AND $a \in A$ --
    THEN $\{x \in L : x < a\} = \{x \in A : x <_A a\}$
    AND SO $a = \gamma(M \setminus \{x \in L : x < a\})$
- $L_\gamma = M$ FOR IF $M \setminus L_\gamma \neq \emptyset$
  THEN $y = \gamma(M \setminus L_\gamma)$ YIELDS A γ-SET, NAMELY $L_\gamma \cup \{y\}$ WITH $x <_K y$ FOR ALL $x \in L_\gamma$
  AND THEN $y \in L_\gamma$ CONTRADICTION.

Ordinals (Ordinal numbers)

To repeat: x is <u>transitive</u>
if every element is a subset
that is $(\forall y\in x)(\forall z\in y)(z\in x)$

Examples: $+$: $\emptyset$, $\{\emptyset\}$, $\{\emptyset,\{\emptyset\}\}$, $\{\emptyset,\{\emptyset\},\{\{\emptyset\}\},\{\emptyset,\{\emptyset\}\}\}$
$-$: $\{\{\emptyset\}\}$

A set x is an ordinal (number) if it is
transitive and well-ordered by $\in$
Formally $\in_x = \{\langle y,z\rangle\in x\times x : y\in z\}$
$\langle x,\in_x\rangle$ is a well-order

We usually drop this

(1) If x is an ordinal and $y\in x$ then
y is an ordinal and $y = \mathrm{pred}(x,y,\cancel{\in})$

(2) If x and y are ordinals and $x\cong y$
then $x=y$.
[The unique isomorphism is the identity]

(3) If x and y are ordinals then exactly
one of the following holds: $x\in y$, $x=y$, $y\in x$.
Last week $x\cong \mathrm{pred}(y,z)$ for some $z\in y$
or $x\cong y$
or $y\cong \mathrm{pred}(x,z)$ for some $z\in x$
None simultaneously be $\in$ must be <u>strict</u>
so $x\in x$ does <u>not</u> hold for ordinals.

(4) If $x\in y$ and $y\in z$ then $x\in z$

(5) If C is a nonempty set of ordinals
then $(\exists x\in C)(\forall y\in C)(x\in y \vee x=y)$
Take $x\in C$ if $x\cap C=\emptyset$ done
otherwise take $x' = \min(x\cap C)$
then $x'\cap C=\emptyset$ done

THE ORDINALS DO NOT FORM A SET:

$$\neg(\exists z)(x \text{ is an ordinal} \to x \in z)$$

□ IF $(\exists z)(x \text{ is an ordinal} \to x \in z)$

WE GET $On = \{x : x \text{ is an ordinal}\}$ BY SEPARATION

- On IS TRANSITIVE (BY (1))
- On IS WELL-ORDERED BY $\in$ (BY (3),(4),(5))

SO On IS AN ORDINAL AND $On \in On$.

IF A IS A SET OF ORDINALS THEN

- $\bigcup A$ IS AN ORDINAL
- IF $(\forall x \in A)(\forall y \in x)(y \in A)$ THEN A IS AN ORDINAL

THEOREM

IF $\langle A, R\rangle$ IS A WELL-ORDER THEN THERE IS A UNIQUE ORDINAL C SUCH THAT $\langle A, R\rangle \simeq C$

□ UNIQUENESS FROM (2)

EXISTENCE LET $B = \{a \in A : (\exists x)(\mathrm{ORD}(x) \wedge \langle \mathrm{pred}(A,a,R), R\rangle \simeq x)\}$

BY (2) $(\forall a \in B)(\exists! x)\underbrace{(\mathrm{ORD}(x) \wedge \langle \mathrm{pred}(A,a,R), R\rangle \simeq x)}_{\varphi(a,x,A,R)}$

BY REPLACEMENT AND SEPARATION WE GET C SUCH THAT $(\forall a \in B)(\exists x \in C)(\varphi(a,x,A,R))$ AND $(\forall y)(y \in C \Leftrightarrow (\exists a \in B)(\varphi(a,y,A,R)))$

ALSO $f = \{\langle a, x\rangle \in B \times C : \varphi(a,x,A,R)\}$ IS A FUNCTION

- DOM $f = B$ RAN $f = C$
- f IS AN ISOMORPHISM $aRb \Leftrightarrow f(a) \in f(b)$
- C SATISFIES $(\forall x \in C)(\forall y \in x)(y \in C)$

SO C IS AN ORDINAL.

IF $B = A$ THEN WE ARE DONE

OTHERWISE $B = \mathrm{pred}(A,a,R)$ FOR SOME $a \in A$

BUT THEN $a \in B$ BY DEFINITION — — □

WE CAN NOW DEFINE ORDER-TYPE FOR WELL-ORDERED SETS.

TYPE(A,R) IS THE UNIQUE ORDINAL C SUCH THAT $\langle A,R\rangle \simeq C$.

FROM NOW ON GREEK LETTERS REFER TO ORDINALS.

MANY PEOPLE WRITE $\alpha < \beta$ IF $\alpha \in \beta$ ETC: $\alpha \leq \beta$ IS $\alpha \in \beta \vee \alpha = \beta$

IF X IS A SET OF ORDINALS THEN

$$\sup X = \bigcup X \quad \text{AND} \quad \min X = \bigcap X$$

[DEFINITION OR THEOREM? DEPENDS ....]

WE ALSO HAVE $(\forall\alpha)(\forall\beta)(\alpha \leq \beta \leftrightarrow \alpha \subseteq \beta)$

SUCCESSOR FUNCTION $S(\alpha) = \alpha \cup \{\alpha\}$.

IF α IS AN ORDINAL THEN SO IS $S(\alpha)$.

ALSO $\alpha < S(\alpha)$, AND $(\forall\beta)(\beta < S(\alpha) \to \beta \leq \alpha)$

SUCCESSOR ORDINAL: $(\exists\beta)(\alpha = S(\beta))$

LIMIT ORDINAL: $\alpha \neq 0$ AND α IS NOT A SUCCESSOR

WE PUT $0 = \emptyset$, $1 = S(0)$, $2 = S(1)$, AND SO ON

$0 = \emptyset$, $1 = \{0\}$, $2 = \{0,1\}$, $3 = \{0,1,2\}$, ...

α IS A NATURAL NUMBER IF

$$(\forall\beta \leq \alpha)(\beta = 0 \vee (\exists\gamma)(\beta = S(\gamma)))$$

REPEAT FROM 2022-09-19:

7: AXIOM OF INFINITY

$$(\exists x)(\underbrace{\emptyset \in x \wedge (\forall y \in x)(S(y) \in x)}_{I(x)})$$

TWO APPROACHES:

- AS ZERMELO DID TAKE ONE x_0 SUCH THAT $I(x_0)$
  DEFINE $N = \{ z \in x_0 : (\forall x)(I(x) \to z \in x)\}$
  AND PROVE THAT N CONSISTS EXACTLY
  THE NATURAL NUMBERS (SEE 2022-09-19).
- TAKE x_0 AND PROVE
  $(\forall \alpha)(\alpha$ IS A NATURAL NUMBER $\to \alpha \in x_0)$
  LET α BE A NATURAL NUMBER
  ASSUME $\alpha \notin x_0$.
  TAKE β SUCH THAT $\alpha = S(\beta)$
  THEN $\beta \notin x_0$ SO $\alpha \cap x_0 \neq \emptyset$
  TAKE $\gamma \in \alpha$ SUCH THAT
  IF $\beta \in \alpha \cap x_0$ THEN $\gamma = \beta$ OR $\gamma < \beta$
  THEN $\gamma = \emptyset$ OR $(\forall \delta)(\delta \in \gamma \to \delta \in x_0)$
  IF $\gamma = \emptyset$ THEN $\gamma \in x_0$; CONTRADICTION
  IF $\gamma \neq 0$ THEN TAKE δ SUCH THAT $\gamma = S(\delta)$
  THEN $\delta \in x_0$ AND SO $\gamma \in x_0$;
  AGAIN A CONTRADICTION

EITHER WAY: WE DEFINE
$\omega = \{\alpha : \alpha$ IS A NATURAL NUMBER$\}$.
THIS SET EXISTS.

ARE THESE THE NATURAL NUMBERS?
[ASK A PHILOSOPHER]
FOR US: YES.

APPENDIX A proper definition of $\langle A_x : x \in Z_0 \rangle$.

The function $\{\langle x, A_x\rangle : x \in Z_0\}$ is (if it exists) a subset of $Z_0 \times \mathcal{P}(Z_0)$ so we work there.

Call $A \subseteq Z_0 \times \mathcal{P}(Z_0)$ an approximation (of f)

if • $\langle \emptyset, \emptyset \rangle \in A$, and (we want $A_\emptyset = \emptyset$)

• if $\langle x, y\rangle \in A$ then $\langle \{x\}, y \cup \{x\}\rangle \in A$.

Let $\mathcal{A} = \{A \in \mathcal{P}(Z_0 \times \mathcal{P}(Z_0)) :$ A is an approximation$\}$

Certainly $Z_0 \times \mathcal{P}(X_0) \in \mathcal{A}$.

Exercise if $\mathcal{B} \subseteq \mathcal{A}$ then $\bigcap \mathcal{B} \in \mathcal{A}$.

Let $A = \bigcap \mathcal{A}$.

Claim $(\forall x \in Z_0)(\exists! y)(\langle x, y\rangle \in A)$

Proof - Let $X = \{\langle \emptyset, \emptyset\rangle\} \cup ((Z_0 \setminus \{\emptyset\}) \times \mathcal{P}(Z_0))$

then $X \in \mathcal{A}$ and if $\langle \emptyset, y\rangle \in X$ then $y = \emptyset$

As $A \subseteq X$ we find $(\exists! y)(\langle \emptyset, y\rangle \in A)$

- Let $W = \{x \in Z_0 : (\exists! y)(\langle x, y\rangle \in A)\}$

• So we know $\emptyset \in W$.

• Let $x \in W$ we show $\{x\} \in W$

Let y be the unique element of $\mathcal{P}(Z_0)$ such that $\langle x, y\rangle \in A$.

We prove if $\langle \{x\}, z\rangle \in A$ then $z = y \cup \{x\}$

Let $A' = A \setminus \{\langle \{x\}, p\rangle : p \neq y \cup \{x\}\}$.

Let $\langle u, v\rangle \in A'$

• If $u \neq x$ then $\langle \{u\}, v \cup \{u\}\rangle \in A'$

for $\{u\} \neq \{x\}$ so $\langle \{u\}, v \cup \{u\}\rangle$

is not in $\{\langle \{x\}, p\rangle : p \in \mathcal{P}(Z_0)\}$

• If $u = x$ then $v = y$

and $\langle \{u\}, v \cup \{v\}\rangle = \langle \{x\}, y \cup \{x\}\rangle \in A$

but $\langle \{x\}, y \cup \{x\}\rangle \notin \{\langle \{x\}, p\rangle : p \neq y \cup \{x\}\}$

so again $\langle \{u\}, v \cup \{v\}\rangle \in A'$.

Conclusion $W = Z_0$ and A is a function (even the) that satisfies $A_\emptyset = \emptyset$ and $A_{\{x\}} = A_x \cup \{x\}$.