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LET M BE A SET.

LET $\gamma: \mathcal{P}(M) \setminus \{\emptyset\} \rightarrow M$ BE A FUNCTION SUCH THAT $\gamma(N) \in N$ FOR ALL N .

[READ ZERMELO'S JUSTIFICATION FOR THIS]

DEFINITION: A γ -SET IS A SUBSET A OF M WITH A WELL-ORDER $<$ SUCH THAT FOR ALL $a \in A$

$$a = \gamma(M \setminus \{x \in A : x < a\})$$

ARE THERE γ -SETS?

- $\{\gamma(M)\}$ IS A γ -SET (WITH $< = \emptyset$)
- $\{m_0, m_1 \mid m_0 = \gamma(M), m_1 = \gamma(M \setminus \{m_0\})\}$ AND $< = \{(m_0, m_1)\}$.

- ETC ...

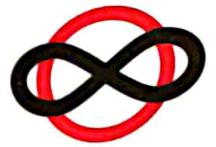
IF $\langle A, <_A \rangle$ AND $\langle B, <_B \rangle$ ARE TWO γ -SETS THEN ONE IS AN INITIAL SEGMENT OF THE OTHER, E.G.,

$$A = \text{pred}(B, b, <_B) \text{ FOR SOME } b \in B$$

AND FOR $\forall x, y \in A: x <_A y \iff x <_B y$



- IF $\langle A, \leq_A \rangle$ AND $\langle B, \leq_B \rangle$ ARE γ -SETS THEN
IF $a \in A \cap B$ THEN $\text{pred}(A, a, \leq_A) = \text{pred}(B, a, \leq_B)$
IF $a, b \in A \cap B$ THEN $a \leq_A b \Leftrightarrow b \leq_B a$
- LET L_γ BE THE SET OF γ -ELEMENTS
[x IS A γ -ELEMENT IF IT BELONGS
TO (THE FIRST COORDINATE OF) A γ -SET]
THEN L_γ HAS AN ORDER $<$ THAT
MAKES IT A γ -SET
 - IF $a, b \in L_\gamma$ THEN $a <_\gamma b$
FOR ALL γ -SETS $\langle A, \leq_A \rangle$ WITH $a, b \in A$
OR $b <_\gamma a$ FOR ALL SUCH γ -SETS.
THIS DETERMINES $<$: IF $a, b \in L_\gamma$
THEN $a, b \in A$ FOR AT LEAST ONE
 γ -SET A .
 - IF $a < b$ AND $b < c$ THEN $a < c$
TAKE $\langle A, \leq_A \rangle$ WITH $c \in A$
THEN ALSO $a, b \in A$ AND
 $a <_\gamma b$ AND $b <_\gamma c$ AND SO $a <_\gamma c$.
 - IF $L \subseteq L_\gamma$ IS NONEMPTY AND $a \in L$
TAKE $\langle A, \leq_A \rangle$ WITH $a \in A$
THEN $\{x \in A: x <_\gamma a\} \in A$
LET m BE THE \leq_A -LEAST ELEMENT
OF $L \cap A$
THEN m IS THE $<$ -LEAST ELEMENT OF L .
 - IF $a \in L$ AND $a \in A$ --
THEN $\{x \in L: x < a\} = \{x \in A: x <_\gamma a\}$
AND SO $a = \gamma(M \setminus \{x \in L: x < a\})$
- $L_\gamma = M$ FOR IF $M \setminus L_\gamma \neq \emptyset$
THEN $\gamma = \gamma(M \setminus L_\gamma)$ YIELDS A γ -SET,
NAMELY $L_\gamma \cup \{\gamma\}$ WITH $x <_\gamma \gamma$ FOR ALL $x \in L_\gamma$
AND THEN $\gamma \in L_\gamma$ CONTRADICTION.



ORDINALS (ORDINAL NUMBERS)

To REPEAT: α IS TRANSITIVE

IF EVERY ELEMENT IS A SUBSET

THAT IS $(\forall y \in \alpha)(\forall z \in y)(z \in \alpha)$

EXAMPLES: $+$: $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset, \{\emptyset\}\}\}$

$-$: $\{\{\emptyset\}\}$

A SET α IS AN ORDINAL (NUMBER) IF IT IS

TRANSITIVE AND WELL-ORDERED BY \in

FORMALLY $E_\alpha = \{ \langle y, z \rangle \in \alpha \times \alpha : y \in z \}$

$\langle \alpha, E_\alpha \rangle$ IS A WELL-ORDER

WE USUALLY
DROP THIS

(1) IF α IS AN ORDINAL AND $y \in \alpha$ THEN y IS AN ORDINAL AND $y = \text{pred}(\alpha, y)$

(2) IF α AND β ARE ORDINALS AND $\alpha \cong \beta$ THEN $\alpha = \beta$.

[THE UNIQUE ISOMORPHISM IS THE IDENTITY]

(3) IF α AND β ARE ORDINALS THEN EXACTLY ONE OF THE FOLLOWING HOLDS: $\alpha \in \beta$, $\alpha = \beta$, $\beta \in \alpha$.

LAST WEAK $\alpha \cong \text{pred}(\beta, z)$ FOR SOME $z \in \beta$

OR $\alpha \cong \beta$

OR $\beta \cong \text{pred}(\alpha, z)$ FOR SOME $z \in \alpha$

NONE SIMULTANEOUSLY BE \in MUST BE STRICT

SO $\alpha \in \alpha$ DOES NOT HOLD FOR ORDINALS.

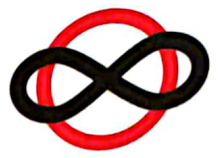
(4) IF $\alpha \in \beta$ AND $\beta \in \alpha$ THEN $\alpha = \beta$

(5) IF C IS A NONEMPTY SET OF ORDINALS THEN $(\exists \alpha \in C)(\forall \beta \in C)(\alpha \in \beta \vee \alpha = \beta)$

TAKE $\alpha \in C$ IF $\alpha \cap C = \emptyset$ DONE

OTHERWISE TAKE $\alpha' = \min(\alpha \cap C)$

THEN $\alpha' \cap C = \emptyset$ DONE



THE ORDINALS DO NOT FORM A SET:

$$\neg (\exists Z) (\forall x \text{ is an ordinal} \rightarrow x \in Z)$$

□ IF $(\exists Z) (\forall x \text{ is an ordinal} \rightarrow x \in Z)$

WE GET $ON = \{x : x \text{ is an ordinal}\}$ BY SEPARATION

- ON IS TRANSITIVE (BY (1))

- ON IS WELL-ORDERED BY \in (BY (3), (4), (5))

SO ON IS AN ORDINAL AND $ON \in ON$.

IF A IS A SET OF ORDINALS THEN

- $\cup A$ IS AN ORDINAL

- IF $(\forall x \in A) (\forall y \in x) (y \in A)$ THEN A IS AN ORDINAL

THEOREM

IF $\langle A, R \rangle$ IS A WELL-ORDER THEN

THERE IS A UNIQUE ORDINAL C

SUCH THAT $\langle A, R \rangle \cong C$

□ UNIQUENESS FROM (2)

EXISTENCE LET $B = \{a \in A : (\exists x) (ORD(x) \wedge \langle pred(A, a, R), R \rangle \cong x)\}$

By (2) $(\forall a \in B) (\exists! x) (ORD(x) \wedge \langle pred(A, a, R), R \rangle \cong x)$

By REPLACEMENT AND SEPARATION WE GET C

SUCH THAT $(\forall a \in B) (\exists x \in C) (\varphi(a, x, A, R))$

AND $(\forall y) (y \in C \Leftrightarrow (\exists a \in B) (\varphi(a, y, A, R)))$

ALSO $f = \{(a, x) \in B \times C : \varphi(a, x, A, R)\}$ IS A FUNCTION

- $DOM f = B$ $RAN f = C$

- f IS AN ISOMORPHISM $a R b \Leftrightarrow f(a) \in f(b)$

- C SATISFIES $(\forall x \in C) (\forall y \in x) (y \in C)$

SO C IS AN ORDINAL.

IF $B = A$ THEN WE ARE DONE

OTHERWISE $B = pred(A, a, R)$ FOR SOME $a \in A$

BUT THEN $a \in B$ BY DEFINITION - □



WE CAN NOW DEFINE ORDER-TYPE FOR WELL-ORDERED SETS.

TYPE (A, R) IS THE UNIQUE ORDINAL C SUCH THAT $\langle A, R \rangle \cong C$.

FROM NOW ON GREEK LETTERS REFER TO ORDINALS.

MANY PEOPLE WRITE $\alpha < \beta$ IF $\alpha \in \beta$ ETC: $\alpha \leq \beta$ IS $\alpha \in \beta \vee \alpha = \beta$

IF X IS A SET OF ORDINALS THEN

$$\sup X = \cup X \quad \text{AND} \quad \min X = \cap X$$

[DEFINITION OR THEOREM? DEPENDS ----]

WE ALSO HAVE $(\forall \alpha)(\forall \beta) (\alpha \leq \beta \Leftrightarrow \alpha \in \beta)$

SUCCESSOR FUNCTION $S(\alpha) = \alpha \cup \{\alpha\}$.

IF α IS AN ORDINAL THEN SO IS $S(\alpha)$.

ALSO $\alpha < S(\alpha)$, AND $(\forall \beta) (\beta < S(\alpha) \rightarrow \beta \in \alpha)$

SUCCESSOR ORDINAL: $(\exists \beta) (\alpha = S(\beta))$

LIMIT ORDINAL: $\alpha \neq 0$ AND α IS NOT A SUCCESSOR

WE PUT $0 = \emptyset$, $1 = S(0)$, $2 = S(1)$, AND SO ON
 $0 = \emptyset$, $1 = \{0\}$, $2 = \{0, 1\}$, $3 = \{0, 1, 2\}$, ...

α IS A NATURAL NUMBER IF

$$(\forall \beta \leq \alpha) (\beta = 0 \vee (\exists \gamma) (\beta = S(\gamma)))$$



REPEAT FROM 2022-09-19:

7: AXIOM OF INFINITY

$$(\exists x) \left(\underbrace{\alpha \in x \wedge (\forall y \in x) (S(y) \in x)}_{I(x)} \right)$$

TWO APPROACHES:

- AS ZERMELO DID TAKE ONE x_0 SUCH THAT $I(x_0)$
 DEFINE $N = \{ z \in x_0 : (\forall x) (I(x) \rightarrow z \in x) \}$
 AND PROVE THAT N CONSISTS EXACTLY
 THE NATURAL NUMBERS (SEE 2022-09-19)

- TAKE x_0 AND PROVE

$$(\forall \alpha) (\alpha \text{ IS A NATURAL NUMBER} \rightarrow \alpha \in x_0)$$

LET α BE A NATURAL NUMBER

ASSUME $\alpha \notin x_0$.

TAKE β SUCH THAT $\alpha = S(\beta)$

THEN $\beta \notin x_0$ SO $\alpha \cap x_0 \neq \emptyset$

TAKE $\gamma \in \alpha \cap x_0$ SUCH THAT

IF $\beta \in \alpha \cap x_0$ THEN $\gamma = \beta$ OR $\gamma < \beta$

THEN $\gamma = \emptyset$ OR $(\forall \delta) (\delta \in \gamma \rightarrow \delta \in x_0)$

IF $\gamma = \emptyset$ THEN $\gamma \in x_0$; CONTRADICTION

IF $\gamma \neq \emptyset$ THEN TAKE δ SUCH THAT $\gamma = S(\delta)$

THEN $\delta \in x_0$ AND SO $\gamma \in x_0$;

AGAIN A CONTRADICTION

EITHER WAY: WE DEFINE

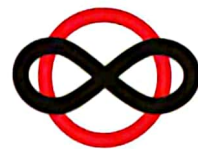
$$\omega = \{ \alpha : \alpha \text{ IS A NATURAL NUMBER} \}.$$

THIS SET EXISTS.

ARE THESE THE NATURAL NUMBERS?

[ASK A PHILOSOPHER]

FOR US: YES.



APPENDIX A PROPER DEFINITION OF $\langle A_x : x \in Z_0 \rangle$.

THE FUNCTION $\{ \langle x, A_x \rangle : x \in Z_0 \}$ IS ^{IF IT EXISTS} A SUBSET OF $Z_0 \times \mathcal{P}(Z_0)$ SO WE WORK THERE.

CALL $A \in Z_0 \times \mathcal{P}(Z_0)$ AN APPROXIMATION (OF f) IF

- $\langle \emptyset, \emptyset \rangle \in A$, AND (WE WANT $A_\emptyset = \emptyset$)
- IF $\langle x, y \rangle \in A$ THEN $\langle \{x\}, y \cup \{x\} \rangle \in A$.

LET $\mathcal{A} = \{ A \in \mathcal{P}(Z_0 \times \mathcal{P}(Z_0)) : A \text{ IS AN APPROXIMATION} \}$
CERTAINLY $Z_0 \times \mathcal{P}(Z_0) \in \mathcal{A}$.

EXERCISE IF $\mathcal{B} \in \mathcal{A}$ THEN $\bigcap \mathcal{B} \in \mathcal{A}$.

LET $A = \bigcap \mathcal{A}$.

CLAIM $(\forall x \in Z_0) (\exists! y) (\langle x, y \rangle \in A)$

PROOF - LET $X = \{ \langle \emptyset, \emptyset \rangle \} \cup (Z_0 \setminus \{ \emptyset \} \times \mathcal{P}(Z_0))$

THEN $X \in \mathcal{A}$ AND IF $\langle \emptyset, y \rangle \in X$ THEN $y = \emptyset$

AS $A \in X$ WE FIND $(\exists! y) (\langle \emptyset, y \rangle \in A)$

- LET $W = \{ x \in Z_0 : (\exists! y) (\langle x, y \rangle \in A) \}$

• SO WE KNOW $\emptyset \in W$.

• LET $x \in W$ WE SHOW $\{x\} \in W$

LET y BE THE UNIQUE ELEMENT OF $\mathcal{P}(Z_0)$ SUCH THAT $\langle x, y \rangle \in A$.

WE PROVE IF $\langle \{x\}, z \rangle \in A$ THEN $z = y \cup \{x\}$

LET $A' = A \setminus \{ \langle \{x\}, p \rangle : p \neq y \cup \{x\} \}$.

LET $\langle u, v \rangle \in A'$

• IF $u \neq x$ THEN $\langle \{u\}, v \cup \{u\} \rangle \in A'$

FOR $\{u\} \neq \{x\}$ SO $\langle \{u\}, v \cup \{u\} \rangle$

IS NOT IN $\{ \langle \{x\}, p \rangle : p \in \mathcal{P}(Z_0) \}$

• IF $u = x$ THEN $v = y$

AND $\langle \{u\}, v \cup \{v\} \rangle = \langle \{x\}, y \cup \{x\} \rangle \in A$

BUT $\langle \{x\}, y \cup \{x\} \rangle \notin \{ \langle \{x\}, p \rangle : p \neq y \cup \{x\} \}$

SO AGAIN $\langle \{u\}, v \cup \{v\} \rangle \in A'$.

CONCLUSION $W = Z_0$ AND A IS A FUNCTION (EVEN THE)
THAT SATISFIES $A_\emptyset = \emptyset$ AND $A_{\{x\}} = A_x \cup \{x\}$.