

CARDINAL EXPONENTIATION.

NOTATION: IF A AND B ARE SETS THEN

${}^B A$ DENOTES THE SET OF MAPS FROM B TO A

EXISTENCE OF ${}^B A$ FOLLOWS FROM POWER SET AND SEPARATION: ${}^B A \in \mathcal{P}(B \times A)$.

DEFINITION: IF κ AND λ ARE CARDINALS THEN $\kappa^\lambda = |{}^\lambda \kappa|$

THIS REQUIRES AC! WE NEED TO KNOW THAT ${}^\lambda \kappa$ CAN BE WELL-ORDERED

WITHOUT AC WE CAN PROVE FOR ARBITRARY SETS A, B AND C :

- $(B \cup C)^A \approx {}^B A \times {}^C A$ IF $B \cap C = \emptyset$
- ${}^C A \times {}^C B \approx {}^C (A \times B)$
- ${}^C ({}^B A) \approx ({}^C \times B)^A$

EXERCISE: WRITE DOWN EXPLICIT BIJECTIONS

WITH AC WE OBTAIN WELL-KNOWN FORMULAS

$$\kappa^{(\lambda \oplus \mu)} = \kappa^\lambda \otimes \kappa^\mu$$

$$\kappa^\lambda \otimes \mu^\lambda = (\kappa \otimes \mu)^\lambda$$

$$(\kappa^\lambda)^\mu = \kappa^{\lambda \otimes \mu}$$

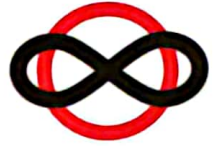
FIRST RESULT

IF $\lambda \geq \omega$ AND $2 \leq \kappa$ AND $\kappa \leq {}^\lambda 2$

THEN ${}^\lambda \kappa \approx {}^\lambda 2 \approx \mathcal{P}(\omega)$ (AND SO $\kappa^\lambda = 2^\lambda$)

- ${}^\lambda 2 \approx \mathcal{P}(\omega)$ VIA CHARACTERISTIC FUNCTIONS
- ${}^\lambda 2 \leq {}^\lambda \kappa$ AND ${}^\lambda \kappa \leq {}^\lambda ({}^\lambda 2) \approx \lambda \otimes \lambda \approx {}^\lambda 2$

NOW APPLY THE DEDERIND-CANTOR-SCHROEDER-BERNSTEIN THEOREM



THEOREM $\omega_2 \approx [0, 1]$

PROOF: IN THE EXERCISES.

COROLLARY $[0, 1]^m \approx^m (\omega_2) = {}^{(m \otimes \omega)}_2 \approx \omega_2 \approx [0, 1]$
 $\omega [0, 1] \approx \omega (\omega_2) = {}^{\omega \otimes \omega}_2 \approx \omega_2 \approx [0, 1]$

CANTOR WAS QUITE PROUD OF THIS

[HIS EARLIER PROOFS OF $[0, 1]^m \approx [0, 1]$ WERE QUITE INVOLVED.]

FROM $\aleph < \aleph(\aleph)$ WE DEDUCE (USING AC) THAT

$$2^{\aleph_\alpha} \geq \aleph_{\alpha+1} \quad (\text{ALL } \alpha)$$

THE CASE $\alpha = 0$ IS OLDER AND GAVE RISE TO THE CONTINUUM HYPOTHESIS (CH):

REAL

$$2^{\aleph_0} = \aleph_1$$

CANTOR'S ORIGINAL FORMULATION MAKES SENSE WITHOUT AC:

IF $X \subseteq \mathbb{R}$ IS INFINITE THEN $X \approx \mathbb{N}$ OR $X \approx \mathbb{R}$.

GENERALIZED CONTINUUM HYPOTHESIS (GCH)

$$(\forall \alpha) (2^{\aleph_\alpha} = \aleph_{\alpha+1})$$

DO CH AND GCH MAKE ANY SENSE?

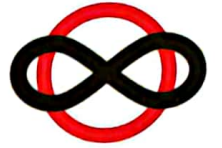
AT THE END OF THE COURSE WE

SHALL PROVE THAT ZF + AC + GCH

IS CONSISTENT.

SO CH AND GCH ARE NOT FALSE

COHEN (1963): CH AND GCH ARE NOT PROVABLE FROM ZF + AC.



WHAT CAN WE SAY ABOUT 2^{S_0} AND 2^{S_α} IN GENERAL?

EXERCISE: IF $\langle X_m : m \in \omega \rangle$ IS A SEQUENCE OF SETS SUCH THAT $|X_m| < 2^{S_0}$ FOR ALL m THEN ALSO $|\bigcup_{m \in \omega} X_m| < 2^{S_0}$

CONCLUSION: $2^{S_0} \neq S_\omega$

WE SHALL PROVE MORE LATER, BUT NOW WE CONSIDER

AXIOM 2 (FOUNDATION / REGULARITY)

$$(\forall x) [(\exists y)(y \in x) \rightarrow (\exists z)(y \in x \wedge \neg(\exists z)(z \in x \wedge z \in y))]$$

FIRST A HIERARCHY

By RECURSION ON ON DEFINE

- $V_0 = \emptyset$ (RCM FOR KURAN)
- $V_{\alpha+1} = P(V_\alpha)$
- $V_\alpha = \bigcup_{\beta < \alpha} V_\beta$ IF α IS A LIMIT

WE GET $\langle V_\alpha : \alpha \in ON \rangle$

AND $WF = \bigcup_{\alpha \in ON} V_\alpha$ THE WELL-FOUNDED SETS

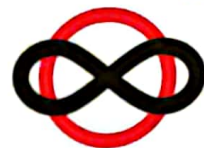
a) V_α IS TRANSITIVE

b) $(\forall \beta)(\beta < \alpha \rightarrow V_\beta \subseteq V_\alpha)$

By INDUCTION

• $\alpha = 0$: CLEAR

• α LIMIT b) BY DEFINITION a) IF $x \in V_\alpha$ THEN $x \in V_\beta$ FOR SOME $\beta < \alpha$ AND SO $x \subseteq V_\beta \subseteq V_\alpha$.



$\alpha = \beta + 1$: V_β IS TRANSITIVE HENCE SO IS $\mathcal{P}(V_\beta)$
 AND $V_\beta \subseteq \mathcal{P}(V_\beta)$
 - IF $x \in V_\beta$ THEN $x \in V_\beta$ SO $x \in \mathcal{P}(V_\beta)$
 - IF $x \in \mathcal{P}(V_\beta)$ THEN $x \subseteq V_\beta \subseteq \mathcal{P}(V_\beta)$

IF $x \in \underline{WF}$ THEN $\text{RANK}(x) = \min\{\beta : x \in V_{\beta+1}\}$
 NOTE $\min\{\beta : x \in V_\beta\}$ IS IT SUCCESSOR.
 SO IF $\text{RANK}(x) = \beta$ THEN $x \in V_\beta, x \notin V_\beta$
 AND $\alpha > \beta \rightarrow x \in V_\alpha$.

ALMOST BY DEFINITION

- $V_\alpha = \{x \in \underline{WF} : \text{RANK}(x) < \alpha\}$
 $(\text{RANK}(x) < \alpha \Leftrightarrow (\exists \beta < \alpha) (x \in V_{\beta+1}) \Leftrightarrow x \in V_\alpha)$
- IF $y \in \underline{WF}$ AND $x \in y$
 THEN $x \in \underline{WF}$ AND $\text{RANK}(x) < \text{RANK}(y)$
 $(\alpha = \text{RANK}(y) : y \in V_{\alpha+1} = \mathcal{P}(V_\alpha)$ SO $y \subseteq V_\alpha$
 AND HENCE $x \in V_\alpha$ AND $\text{RANK}(x) < \alpha$)
- IF $y \in \underline{WF}$ THEN
 $\text{RANK}(y) = \sup\{\text{RANK}(x) + 1 : x \in y\}$
 \geq CLEAR FROM THE PREVIOUS BIT
 \leq IF $\alpha = \sup\{\text{RANK}(x) + 1 : x \in y\}$
 THEN $x \in V_\alpha$ FOR ALL $x \in y$, SO $y \subseteq V_\alpha$
 AND $y \in V_{\alpha+1}$
- $(\forall \alpha \in \text{ON}) (\alpha \in \underline{WF} \wedge \text{RANK}(\alpha) = \alpha)$
 INDUCTION: - IF $\beta \in \alpha \rightarrow \beta \in V_{\beta+1}$ THEN $\alpha \subseteq V_\alpha$
 AND SO $\alpha \in V_{\alpha+1}$
 $\text{RANK}(\alpha) = \sup\{\beta + 1 : \beta \in \alpha\} = \alpha$.
- AND SO $V_\alpha \cap \text{ON} = \alpha$ FOR ALL $\alpha \in \text{ON}$.



WF IS CLOSED UNDER LOTS OF OPERATIONS:

- IF $x \in WF$ THEN $Ux, P(x), |x| \in WF$

EXERCISE CALCULATE THEIR RANKS.

- IF $x, y \in WF$ THEN $xxy, xyx, xxy, \langle x, y \rangle$, AND y^x ARE ALSO IN WF

EXERCISE CALCULATE THEIR RANKS

- THERE ARE COPIES OF $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, AND \mathbb{C} IN WF

EXERCISE CONSTRUCT THESE FROM ω AND CALCULATE THEIR RANKS.

- IMPORTANT $(\forall x) (x \in WF \Leftrightarrow x \in WF)$

→ TRANSITIVITY

← $x \in V_{\alpha+1}$, WHERE $\alpha = \sup\{\text{RANK}(y) + 1 : y \in x\}$

- $(\forall n \in \omega) (|V_n| < S_0^1)$

• $|V_n| = S_0^1$
 • $|V_{n+1}| = 2^{S_0^1}, |V_{n+2}| = 2^{2^{S_0^1}}, \dots$

• AC $|V_{\alpha+1}| = J_\alpha$ BETH
 $J_0 = S_0^1, J_{\alpha+1} = 2^{J_\alpha}, J_\alpha = \bigcup_{x \in \alpha} J_x$

WELL-FOUNDED RELATIONS

A RELATION R IS WELL-FOUNDED ON A IF

$$(\forall X \subseteq A) (X \neq \emptyset \rightarrow (\exists y \in X) (\neg (\exists z \in X) (z R y)))$$

"EVERY NONEMPTY SUBSET HAS AN R -MINIMAL ELEMENT"

NOTE: WE ASSUME NOTHING ABOUT R , JUST THAT IT IS A RELATION

- WELL-ORDER \equiv WELL-FOUNDED LINEAR ORDER
- THE R -MINIMAL ELEMENT(S) NEED NOT BE UNIQUE

IF $A \in WF$ THEN \in IS WELL-FOUNDED ON A .

IF $X \subseteq A$ IS NONEMPTY LET $\alpha = \min\{\text{RANK}(y) : y \in X\}$
 AND PICK $y \in X$ WITH $\text{RANK}(y) = \alpha$
 IF $z \in y$ THEN $\text{RANK}(z) < \text{RANK}(y)$ AND SO $z \notin X$.

CONVERSE:

IF A IS TRANSITIVE AND \in IS WELL-FOUNDED ON A THEN $A \in \underline{WF}$

TO SHOW $A \in \underline{WF}$ ASSUME $X = A \setminus \underline{WF}$ IS NOT EMPTY.

LET $y \in X$ BE \in -MINIMAL IN X .

IF $x \in y$ THEN $x \in A \setminus X$, HENCE $x \in \underline{WF}$

SO $y \in \underline{WF}$ AND HENCE $y \in \underline{WF}$ CONTRADICTION.

NEXT: TRANSITIVE CLOSURE

DEFINE - $U^0 A = A$

- $U^{n+1} A = U(U^n A)$

$TRCL(A) = \bigcup_{new} U^n A$

$TRCL(A)$ IS THE SMALLEST TRANSITIVE SET THAT CONTAINS A .

- $A \in TRCL(A)$
- $TRCL(A)$ IS TRANSITIVE
- IF $A \in T$ AND T IS TRANSITIVE THEN $TRCL(A) \in T$
- IF A IS TRANSITIVE THEN $A = TRCL(A)$
- $x \in A \rightarrow TRCL(x) \in TRCL(A)$
- $TRCL(A) = A \cup U\{TRCL(x) : x \in A\}$

THE FOLLOWING ARE EQUIVALENT

- a) $A \in \underline{WF}$
- b) $TRCL(A) \in \underline{WF}$
- c) \in IS WELL-FOUNDED ON $TRCL(A)$.

NOTE: c) CAN BE USED AS A DEFINITION OF \underline{WF} THAT DOES NOT USE THE POWER SET AXIOM.

WE GET THE FOLLOWING EQUIVALENCES

- a) THE AXIOM OF FOUNDATION
- b) $(\forall A) (\in \text{ IS WELL-FOUNDED ON } A)$
- c) $\underline{V} = \underline{WF}$

FOUNDATION MAKES LIFE EASIER, SOMETIMES

FOR EXAMPLE: THE FOLLOWING ARE EQUIVALENT

- A IS AN ORDINAL
- A IS TRANSITIVE AND TOTALLY ORDERED BY \in
- A IS TRANSITIVE AND EVERY ELEMENT OF A IS TRANSITIVE

FROM NOW ON WE WORK IN THE FULL ZFC
 (WHEN WE TURN TO THE CONSTRUCTIBLE
 SETS WE SHALL DROP AC FOR A WHILE)
 (OR IF AN EXERCISE WARRANTS IT)

SOME APPLICATIONS OF FOUNDATION

- IN ZF: AC IS EQUIVALENT TO
 $(\forall \alpha) (P(\alpha) \text{ CAN BE WELL-ORDERED})$

- REPLACEMENT WITHOUT UNIQUENESS

$$\text{IN ZF: } ((\forall x \in A)(\exists y)\varphi) \rightarrow ((\exists B)(\forall x \in A)(\exists y \in B)\varphi)$$

$$\text{USE } \psi(x, \alpha): \alpha = \pi \cap \{\beta : (\exists \gamma \in V_\beta)(\varphi(x, \gamma))\}$$

APPLY REPLACEMENT TO ψ

$$\text{GIVEN } (\forall x \in A)(\exists y)\varphi$$

$$\text{WE GET } (\exists \alpha)(\forall x \in A)(\exists y \in V_\alpha)(\varphi(x, y))$$