Cardinal exponentiation.

Notation: if A and B are sets then ^{B}A denotes the set of maps from B to A

Existence of ^{B}A follows from Power Set and Separation: $^{B}A \subseteq \mathcal{P}(B \times A)$.

Definition: if κ and λ are cardinals then $\kappa^{\lambda} = |{}^{\lambda}\kappa|$

This requires AC! We need to know that $^{\lambda}\kappa$ can be well-ordered

Without AC we can prove for arbitrary sets A, B and C:

- $^{(B \cup C)}A \approx {}^{B}A \times {}^{C}A$ if $B \cap C = \emptyset$
- $^{C}A \times {}^{C}B \approx {}^{C}(A \times B)$
- $^{C}({}^{B}A) \approx {}^{(C \times B)}A$

Exercise: write down explicit bijections

With AC we obtain well-known formulas

$$\kappa^{(\lambda \oplus \mu)} = \kappa^{\lambda} \otimes \kappa^{\mu}$$
$$\kappa^{\lambda} \otimes \mu^{\lambda} = (\kappa \otimes \mu)^{\lambda}$$
$$(\kappa^{\lambda})^{\mu} = \kappa^{\lambda \otimes \mu}$$

First result

If $\lambda \geq \omega$ and $2 \leq \kappa$ and $\kappa \preceq {}^{\lambda}2$

Then $^{\lambda}\kappa \approx {}^{\lambda}2 \approx \mathcal{P}(\lambda)$ (and so $\kappa^{\lambda} = 2^{\lambda}$)

- $^{\lambda}2 \approx \mathcal{P}(\lambda)$ via characteristic functions
- $^{\lambda}2 \preceq {}^{\lambda}\kappa$ and $^{\lambda}\kappa \preceq {}^{\lambda}({}^{\lambda}2) \approx {}^{\lambda \otimes \lambda}2 \approx {}^{\lambda}2$

Now apply the Dedekind-Cantor-Schroeder-Bernstein Theorem

THEOREM ${}^{\omega}2 \approx [0,1]$
PROOF: IN THE EXERCISES.

COROLLARY $[0,1]^n \approx {}^n({}^{\omega}2) = {}^{(n\otimes\omega)}2 \approx {}^{\omega}2 \approx [0,1]$
${}^{\omega}[0,1] \approx {}^{\omega}({}^{\omega}2) = {}^{(\omega\otimes\omega)}2 \approx {}^{\omega}2 \approx [0,1]$
CANTOR WAS QUITE PROUD OF THIS
[HIS EARLIER PROOFS OF $[0,1]^n \approx [0,1]$ WERE QUITE INVOLVED.]

FROM $x \prec \mathcal{P}(x)$ WE DEDUCE (USING AC) THAT
$$2^{\aleph_\alpha} \geq \aleph_{\alpha+1} \qquad (\text{ALL } \alpha)$$
THE CASE $\alpha = 0$ IS OLDER AND GAVE RISE TO THE CONTINUUM HYPOTHESIS (CH):
$$2^{\aleph_0} = \aleph_1$$
CANTOR'S ORIGINAL FORMULATION MAKES SENSE WITHOUT AC:
IF $X \subseteq \mathbb{R}$ IS INFINITE THEN $X \approx \mathbb{N}$ OR $X \approx \mathbb{R}$.

GENERALIZE CONTINUUM HYPOTHESIS (GCH)
$$(\forall\alpha)\,(2^{\aleph_\alpha} = \aleph_{\alpha+1})$$

DO CH AND GCH MAKE ANY SENSE?
AT THE END OF THE COURSE WE SHALL PROVE THAT ZF + AC + GCH IS CONSISTENT.
SO CH AND GCH ARE NOT FALSE

COHEN (1963): CH AND GCH ARE NOT PROVABLE FROM ZF + AC.

WHAT CAN WE SAY ABOUT $2^{\aleph_1}$ AND $2^{\aleph_\alpha}$ IN GENERAL?

EXERCISE: IF $\langle X_n : n \in \omega \rangle$ IS A SEQUENCE OF SETS SUCH THAT $|X_n| < 2^{\aleph_0}$ FOR ALL n THEN ALSO $|\bigcup_{n\in\omega} X_n| < 2^{\aleph_0}$

CONCLUSION: $2^{\aleph_0} \neq \aleph_\omega$

WE SHALL PROVE MORE LATER, BUT NOW WE CONSIDER

AXIOM 2 (FOUNDATION / REGULARITY)

$$(\forall x)[(\exists y)(y \in x) \rightarrow (\exists y)(y \in x \wedge \neg(\exists z)(z \in x \wedge z \in y))]$$

FIRST A HIERARCHY

BY RECURSION ON On DEFINE

- $V_0 = \emptyset$ (R(α) FOR KUNEN)
- $V_{\alpha+1} = \mathcal{P}(V_\alpha)$
- $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$ IF α IS A LIMIT

WE GET $\langle V_\alpha : \alpha \in \mathrm{On} \rangle$

AND $\underline{WF} = \bigcup_{\alpha\in \mathrm{On}} V_\alpha$ THE WELL-FOUNDED SETS

a) V_α IS TRANSITIVE

b) $(\forall \beta)(\beta < \alpha \rightarrow V_\beta \subseteq V_\alpha)$

BY INDUCTION

- $\alpha = 0$: CLEAR
- α LIMIT b) BY DEFINITION a) IF $x \in V_\alpha$ THEN $x \in V_\beta$ FOR SOME $\beta < \alpha$ AND SO $x \subseteq V_\beta \subseteq V_\alpha$.

$\alpha = \beta + 1$: · V_β is transitive hence so is $\mathcal{P}(V_\beta)$
and $V_\beta \subseteq \mathcal{P}(V_\beta)$
- if $x \in V_\beta$ then $x \subseteq V_\beta$ so $x \in \mathcal{P}(V_\beta)$
- if $x \in \mathcal{P}(V_\beta)$ then $x \subseteq V_\beta \subseteq \mathcal{P}(V_\beta)$

If $x \in \underline{WF}$ then $\text{rank}(x) = \min\{\beta : x \in V_{\beta+1}\}$

note $\min\{\delta : x \in V_\delta\}$ is a successor.

So if $\text{rank}(x) = \beta$ then $x \subseteq V_\beta$, $x \notin V_\beta$
and $\alpha > \beta \rightarrow x \in V_\alpha$.

Almost by definition
- $V_\alpha = \{x \in \underline{WF} : \text{rank}(x) < \alpha\}$
  ($\text{rank}(x) < \alpha \Leftrightarrow (\exists \beta < \alpha)(x \in V_{\beta+1}) \Leftrightarrow x \in V_\alpha$)
- if $y \in WF$ and $x \in y$
  then $x \in WF \wedge \text{rank}(x) < \text{rank}(y)$
  ($\alpha = \text{rank}(y)$: $y \in V_{\alpha+1} = \mathcal{P}(V_\alpha)$ so $y \subseteq V_\alpha$
  and hence $x \in V_\alpha$ and $\text{rank}(x) < \alpha$)
- if $y \in \underline{WF}$ then
  $\text{rank}(y) = \sup\{\text{rank}(x) + 1 : x \in y\}$
  $\geq$: clear from the previous bit
  $\leq$: if $\alpha = \sup\{\text{rank}(x)+1 : x \in y\}$
  then $x \in V_\alpha$ for all $x \in y$, so $y \subseteq V_\alpha$
  and $y \in V_{\alpha+1}$.
- $(\forall \alpha \in On)(\alpha \in \underline{WF} \wedge \text{rank}(\alpha) = \alpha)$
  Induction: if $\beta < \alpha \rightarrow \beta \in V_{\beta+1}$ then $\alpha \subseteq V_\alpha$
  and so $\alpha \in V_{\alpha+1}$.
  $\text{rank}(\alpha) = \sup\{\beta + 1 : \beta \in \alpha\} = \alpha$.
- and so $V_\alpha \cap On = \alpha$ for all $\alpha \in On$.

WF is closed under lots of operations:

- If $x \in WF$ then $\bigcup x$, $\mathcal{P}(x)$, $\{x\} \in WF$
  Exercise calculate their ranks.
- If $x, y \in WF$ then $x \times y$, $x \cup y$, $x \cap y$, $\{x, y\}$, $\langle x, y \rangle$, and ${}^{y}x$ are also in WF
  Exercise calculate their ranks
- There are copies of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ in WF
  Exercise construct these from ω and calculate their ranks.
- Important $(\forall x)(x \in WF \Leftrightarrow x \subseteq WF)$
  $\rightarrow$ transitivity
  $\leftarrow$ $x \in V_{\alpha+1}$, where $\alpha = \sup\{\text{rank}(y)+1 : y \in x\}$
- $(\forall n \in \omega)(|V_n| < \aleph_0)$
- $|V_\omega| = \aleph_0$
- $|V_{\omega+1}| = 2^{\aleph_0}$, $|V_{\omega+2}| = 2^{2^{\aleph_0}}$, ...
- AC $|V_{\omega+\alpha}| = \beth_\alpha$ Beth
  $\beth_0 = \aleph_0$, $\beth_{\alpha+1} = 2^{\beth_\alpha}$, $\beth_\lambda = \bigcup_{\alpha<\lambda} \beth_\alpha$.

Well-founded relations

A relation R is well-founded on A if
$(\forall X \subseteq A)(X \neq \emptyset \rightarrow (\exists y \in X)(\neg(\exists z \in X)(z R y)))$
"Every nonempty subset has an R-minimal element"

Note: we assume nothing about R, just that it is a relation

- well-order = well-founded linear order
- the R-minimal element(s) need not be unique

If $A \in WF$ then $\in$ is well-founded on A.
If $X \subseteq A$ is nonempty let $\alpha = \min\{\text{rank}(y) : y \in X\}$ and pick $y \in X$ with $\text{rank}(y) = \alpha$
If $z \in y$ then $\text{rank}(z) < \text{rank}(y)$ and so $z \notin X$.

CONVERSE:

IF A IS TRANSITIVE AND $\in$ IS WELL-FOUNDED ON A THEN $A \in WF$

TO SHOW $A \subseteq WF$ ASSUME $X = A \setminus WF$ IS NOT EMPTY.
LET $y \in X$ BE $\in$-MINIMAL IN X.
IF $x \in y$ THEN $x \in A \setminus X$, HENCE $x \in WF$
SO $y \subseteq WF$ AND HENCE $y \in WF$ CONTRADICTION.

NEXT: TRANSITIVE CLOSURE

DEFINE
- $U^0 A = A$
- $U^{n+1} A = U(U^n A)$

$TRCL(A) = \bigcup_{n \in \omega} U^n A$.

$TRCL(A)$ IS THE SMALLEST TRANSITIVE SET THAT CONTAINS A.

- $A \subseteq TRCL(A)$
- $TRCL(A)$ IS TRANSITIVE
- IF $A \subseteq T$ AND T IS TRANSITIVE THEN $TRCL(A) \subseteq T$
- IF A IS TRANSITIVE THEN $A = TRCL(A)$
- $x \in A \to TRCL(x) \subseteq TRCL(A)$
- $TRCL(A) = A \cup \bigcup\{TRCL(x) : x \in A\}$.

THE FOLLOWING ARE EQUIVALENT

a) $A \in WF$
b) $TRCL(A) \in WF$
c) $\in$ IS WELL-FOUNDED ON $TRCL(A)$.

NOTE: c) CAN BE USED AS A DEFINITION OF WF THAT DOES NOT USE THE POWER SET AXIOM.

WE GET THE FOLLOWING EQUIVALENCES

a) THE AXIOM OF FOUNDATION
b) $(\forall A)(\in$ IS WELL-FOUNDED ON $A)$
c) $V = WF$

FOUNDATION MAKES LIFE EASIER, SOMETIMES

FOR EXAMPLE: THE FOLLOWING ARE EQUIVALENT

- A IS AN ORDINAL
- A IS TRANSITIVE AND TOTALLY ORDERED BY $\in$
- A IS TRANSITIVE AND EVERY ELEMENT OF A IS TRANSITIVE.

FROM NOW ON WE WORK IN THE FULL ZFC
(WHEN WE TURN TO THE CONSTRUCTIBLE SETS WE SHALL DROP AC FOR A WHILE)
(OR IF AN EXERCISE WARRANTS IT)

SOME APPLICATIONS OF FOUNDATION

- IN ZF: AC IS EQUIVALENT TO
$(\forall\alpha)(P(\alpha)$ CAN BE WELL-ORDERED$)$
- REPLACEMENT WITHOUT UNIQUENESS
IN ZF: $((\forall x\in A)(\exists y)\varphi) \to ((\exists B)(\forall x\in A)(\exists y\in B)\varphi)$
USE $\psi(x,\alpha)$: $\alpha = \min\{\beta : (\exists y\in V_\beta)(\varphi(x,y))\}$
APPLY REPLACEMENT TO ψ
GIVEN $(\forall x\in A)(\exists y)\varphi$
WE GET $(\exists\alpha)(\forall x\in A)(\exists y\in V_\alpha)(\varphi(x,y))$