THE C.U.B. FILTER AND STATIONARY SETS

A FILTER ON A SET A (NON-EMPTY) IS
A FAMILY $\mathcal{F}$ OF SUBSETS SATISFYING
- $A \in \mathcal{F}$ AND $\emptyset \notin \mathcal{F}$
- $X, Y \in \mathcal{F} \rightarrow X \cap Y \in \mathcal{F}$
- $X \in \mathcal{F} \wedge Y \subseteq A \wedge X \subseteq Y \rightarrow Y \in \mathcal{F}$.

AN IDEAL ON A SET A (NON-EMPTY) IS
A FAMILY $\mathcal{I}$ OF SUBSETS SATISFYING
- $\emptyset \in \mathcal{I}$ AND $A \notin \mathcal{I}$
- $X, Y \in \mathcal{I} \rightarrow X \cup Y \in \mathcal{I}$
- $X \in \wedge Y \subseteq A \wedge Y \subseteq X \rightarrow Y \in \mathcal{I}$

THESE ARE DUAL $\mathcal{F}^* = \{X : A \setminus X \in \mathcal{F}\}$ IS AN IDEAL
$\mathcal{I}^* = \{X : A \setminus X \in \mathcal{I}\}$ IS A FILTER

- A INFINITE
  - $[A]^{<\omega} = \{X \subseteq A : X \text{ IS FINITE}\}$ IS AN IDEAL
  - $\{X \subseteq A : A \setminus X \text{ IS FINITE}\}$ IS A FILTER
- $[0,1]$: $\mathcal{I}_0 = \{X \subseteq [0,1] : \lambda(X) = 0\}$ IS AN IDEAL
  $\mathcal{F}_1 = \{X \subseteq [0,1] : \lambda(X) = 1\}$ IS A FILTER

$\mathcal{I}$ IS κ-COMPLETE IF $\mathcal{J} \subseteq \mathcal{I}$ AND $|\mathcal{J}| < \kappa$ IMPLY $\bigcup \mathcal{J} \in \mathcal{I}$
$\mathcal{F}$ IS κ-COMPLETE IF $\mathcal{G} \subseteq \mathcal{F}$ AND $|\mathcal{G}| < \kappa$ IMPLY $\bigcap \mathcal{G} \in \mathcal{F}$.

- $[A]^{<\omega}$ AND $([A]^{<\omega})^*$ ARE $\aleph_0$-COMPLETE
- $\mathcal{I}_0$ AND $\mathcal{F}_1$ ARE $\aleph_1$-COMPLETE (σ-COMPLETE COUNTABLY COMPLETE)

LET μ BE A LIMIT ORDINAL
$C \subseteq \mu$ IS CLOSED IF FOR ALL LIMIT $\delta < \mu$.
IF $\sup(C \cap \delta) = \delta$ THEN $\delta \in C$.
$C \subseteq \mu$ IS C.U.B (CUB) IF IT IS CLOSED AND UNBOUNDED

THIS ONLY INTERESTING IF $\mathrm{cf}(\mu) > \aleph_0$.
IF $\mathrm{cf}(\mu) = \aleph_0$ AND $\langle x_n : n \in \omega \rangle$ IS STRICTLY INCREASING AND COFINAL THEN
$\{x_{2n} : n \in \omega\}$ AND $\{x_{2n+1} : n \in \omega\}$ ARE CUB, AND ALSO DISJOINT.

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WE ASSUME $cf(\mu) > \aleph_0$ FROM NOW ON.

THE CUB SETS GENERATE A $cf(\mu)$-COMPLETE FILTER: $Cub(\mu) = \{X \subseteq \mu : (\exists C)(C \text{ is cub} \wedge C \subseteq X)\}$.

- LET C AND D BE CUB
  - $C \cap D$ IS CLOSED:
    IF δ IS A LIMIT AND $\delta = \sup(C \cap D \cap \delta)$
    THEN ALSO $\delta = \sup(C \cap \delta)$ AND $\delta = \sup(D \cap \delta)$
    AND SO $\delta \in C \cap D$.
  - $C \cap D$ IS UNBOUNDED
    LET $\alpha < \mu$
    LET $\gamma_0 = \min\{\gamma \in C : \gamma > \alpha\}$
    $\delta_0 = \min\{\delta \in D : \delta > \gamma_0\}$
    $\gamma_{n+1} = \min\{\gamma \in C : \gamma > \delta_n\}$
    $\delta_{n+1} = \min\{\delta \in D : \delta > \gamma_{n+1}\}$
    BY RECURSION WE GET $\langle \gamma_n : n \in \omega \rangle$ IN C AND $\langle \delta_n : n \in \omega \rangle$ IN D SUCH THAT
    $\gamma_n < \delta_n < \gamma_{n+1}$ FOR ALL n.
    LET $\beta = \sup_{n \in \omega} \gamma_n = \sup_{n \in \omega} \delta_n$

    THEN $\beta = \sup(C \cap \beta) = \sup(D \cap \beta)$
    AND SO $\beta \in C \cap D$, AND $\beta > \alpha$ OF COURSE.
  - IF $\lambda < cf(\mu)$ AND $\langle C_\eta : \eta < \lambda \rangle$ IS A SEQUENCE OF CUB SETS THEN $\bigcap_{\eta < \lambda} C_\eta$ IS CUB.
    CLOSED: AS BEFORE $\bigcap_{\eta<\lambda} C_\eta \cap \delta \subseteq C_\xi \cap \delta$ ..
    UNBOUNDED START FROM SOME $\alpha < \mu$.
    BUILD λ SEQUENCES $\langle \gamma_{\eta,n} : n \in \omega \rangle$ IN C_η
    - $\gamma_{0,0} = \min\{\gamma \in C_0 : \alpha < \gamma\}$
    - $\eta > 0$: $\gamma_{\eta,n} = \min\{\gamma \in C_\eta : (\forall \xi < \eta)(\gamma_{\xi,n} < \gamma)\}$
    - $\gamma_{0,n+1} = \min\{\gamma \in C_0 : (\forall \xi < \lambda)(\gamma_{\xi,n} < \gamma)\}$

    ALWAYS POSSIBLE BECAUSE $\lambda < cf(\mu)$ AND $\aleph_0 < cf(\mu)$

NOW LET

$$\beta = \sup\{\gamma_{\eta,n} : \eta < \lambda, n \in \omega\}$$

THEN $\beta < \mu$ BECAUSE $cf(\mu) > \lambda \otimes \aleph_0$.
BY CONSTRUCTION

$$\beta = \sup\{\gamma_{\eta,n} : n \in \omega\} \quad \text{FOR ALL } \eta < \lambda$$

AND SO $\beta \in C_\eta$ FOR ALL η.

SO IF μ IS REGULAR THEN
- $Cub(\mu)$ IS μ-COMPLETE
- $[\mu]^{<\mu} \in Cub^*(\mu)$

STILL $cf(\mu) > \aleph_0$:
$X \subseteq \mu$ IS <u>STATIONARY</u> IF $X \notin Cub^*(\mu)$
THAT IS IF $X \cap C \neq \emptyset$ FOR EVERY CUB $C \subseteq \mu$.
X IS NON-STATIONARY IF $X \in Cub^*(\mu)$
$Cub^*(\mu)$ IS CALLED THE <u>NON-STATIONARY IDEAL</u>

EXAMPLE IF λ IS REGULAR AND $cf(\mu) > \lambda$
THEN $S^\mu_\lambda = \{\gamma < \mu : cf\gamma = \lambda\}$ IS STATIONARY
FOR IF C IS CLOSED AND UNBOUNDED
LET $\delta = ot(C)$ AND LET $f : \delta \to C$ BE
THE ISOMORPHISM; THEN $f(\lambda) \in C \cap S^\mu_\lambda$.
FOR f IS STRICTLY INCREASING AND CONTINUOUS.

SO IN ω_2 WE HAVE TWO (EASY) DISJOINT
STATIONARY SUBSETS $S^{\omega_2}_{\omega_0}$ AND $S^{\omega_2}_{\omega_1}$.

HOW IS THAT ON ω_1 ?
THE AXIOM OF DETERMINACY IMPLIES THAT
$Cub(\omega_1)$ IS AN ULTRAFILTER ....
SO THERE : STATIONARY $\equiv$ CUB.
BUT WITH AC WE HAVE MORE VARIETY.

ULAM-MATRIX:

LET κ BE A SUCCESSOR CARDINAL
THEN THERE IS A FAMILY OF κ MANY PAIRWISE DISJOINT STATIONARY SETS.

SAY $\kappa = \lambda^+$.
USE AC TO CHOOSE $\langle f_\alpha : \alpha < \kappa \rangle$ SUCH THAT FOR EACH α THE MAP $f_\alpha : \alpha \to \lambda$ IS INJECTIVE

FOR $\alpha < \kappa$ AND $\xi < \lambda$ PUT

$$X(\alpha, \xi) = \{ \beta > \alpha : f_\beta(\alpha) = \xi \}$$

- $\bigcup \{ X(\alpha,\xi) : \xi < \lambda \} = (\alpha, \kappa)$
  FOR $\beta \in X(\alpha, f_\beta(\alpha))$
- IF $\alpha \neq \beta$ THEN $X(\alpha,\xi) \cap X(\beta,\xi) = \emptyset$
  FOR IF $\gamma > \alpha, \beta$ THEN $f_\gamma(\alpha) \neq f_\gamma(\beta)$
- FOR EVERY α THERE IS A $\xi_\alpha < \lambda$ SUCH THAT $X(\alpha, \xi_\alpha)$ IS STATIONARY
  BY THE κ-COMPLETENESS OF $\mathrm{Cub}^*(\kappa)$.
- THERE IS A ξ SUCH THAT $\{\alpha : \xi_\alpha = \xi\} = A$ HAS CARDINALITY κ.
- THEN $\{ X(\alpha,\xi) : \alpha \in A \}$ IS THE DESIRED FAMILY OF STATIONARY SETS

THE FULL FAMILY $\{ X(\alpha,\xi) : \alpha < \kappa ; \xi < \lambda \}$ IS CALLED AN ULAM MATRIX.

IF κ IS WEAKLY INACCESSIBLE THEN THERE ARE κ MANY REGULAR CARDINALS BELOW κ
SO $\{ S^\kappa_\lambda : \lambda < \kappa ; \lambda \text{ REGULAR} \}$ IS A FAMILY OF κ MANY PAIRWISE DISJOINT STATIONARY SETS

EVEN MORE: IF κ IS REGULAR UNCOUNTABLE AND $S \subseteq \kappa$ IS STATIONARY THEN S CAN BE SPLIT INTO κ MANY STATIONARY SETS.
[SOLOVAY: FOR SUCCESSORS USE ULAM'S MATRIX AGAIN FOR LIMITS (WEAKLY INACCESSIBLE) MORE WORK IS NEEDED]

REMEMBER 2022-10-17 [LÖWENHEIM-SKOLEM]

LET $\kappa > \omega$ BE REGULAR UNCOUNTABLE

LET $\mathcal{F}$ BE A SET OF FINITARY FUNCTIONS

$$f : \kappa^{n_f} \longrightarrow \kappa$$

SUCH THAT $|\mathcal{F}| < \kappa$.

THEN $C = \{\gamma < \kappa : \gamma \text{ IS CLOSED UNDER } \mathcal{F}\}$ IS CUB

- CLOSED: IF $C \cap \delta$ IS COFINAL IN δ

THEN $f[\delta^{n_f}] = \bigcup\{f[\gamma^{n_f}] : \gamma \in C \cap \delta\}$

$$\subseteq \bigcup\{\gamma : \gamma \in C \cap \delta\}$$

$$= \delta.$$

- UNBOUNDED.

LET $\xi < \kappa$ LET $G(\xi)$ BE THE CLOSURE OF ξ UNDER $\mathcal{F}$

BY [2022-10-17] WE HAVE $|G(\xi)| \leq |\xi| \otimes |\mathcal{F}| < \kappa$.

SO $\xi \subseteq G(\xi)$ AND $G(\xi)$ IS BOUNDED IN κ.

LET $g(\xi) < \kappa$ BE SUCH THAT $G(\xi) \subseteq g(\xi)$.

GIVEN $\alpha < \kappa$ LET $\alpha_0 = \alpha$

AND, RECURSIVELY, $\alpha_{m+1} = g(\alpha_m)$

THEN $\sup_{m \in \omega} \alpha_m \in C$.

SO: A MODEL OF A (GENERALIZED) FIRST ORDER THEORY THAT IS OF REGULAR CARDINALITY LARGER THAN THAT OF THE LANGUAGE HAS A CUB OF (ELEMENTARY) SUBMODELS.

DIAGONAL INTERSECTION:

GIVEN $\langle C_\alpha : \alpha < \kappa \rangle$, SUBSETS OF κ DEFINE

$$\Delta_{\alpha < \kappa} C_\alpha = \{\delta < \kappa : (\forall \alpha < \delta)(\delta \in C_\alpha)\}$$

THIS IS THE DIAGONAL INTERSECTION

IF $\kappa > \aleph_0$ IS REGULAR, AND $\langle C_\alpha : \alpha < \kappa \rangle$ IS A SEQUENCE OF CUB SETS THEN

$$C = \Delta_{\alpha < \kappa} C_\alpha$$

IS ALSO CUB.

PROOF :

- C IS CLOSED : ASSUME $\delta = \sup(C \cap \delta)$

  LET $\alpha < \delta$ THEN FOR EVERY $\gamma \in C \cap \delta$

  WITH $\gamma > \alpha$ WE HAVE $\gamma \in C_\alpha$

  HENCE $\delta = \sup(C_\alpha \cap \delta)$

  AND SO $\delta \in C_\alpha$.

- C IS UNBOUNDED

  TAKE $\alpha < \kappa$.

  PUT $\alpha_0 = \alpha$ AND, RECURSIVELY,

  LET $\alpha_{n+1} = \min\{\gamma \in \bigcap_{\beta \leq \alpha_n} C_\beta : \gamma > \alpha_n\}$

  THEN $\langle \alpha_n : n \in \omega \rangle$ IS STRICTLY INCREASING

  AND $\delta = \sup_{n \in \omega} \alpha_n$ IS SMALLER THAN κ.

  LET $\beta < \delta$

  FOR ALL n WITH $\beta < \alpha_n$

  WE HAVE $\alpha_{n+1} \in C_\beta$

  SO $\delta = \sup(C_\beta \cap \delta)$

  AND $\delta \in C_\beta$.

  WE HAVE $\alpha < \delta$ AND $\delta \in C$.

MAKES SOME THINGS EASIER:

SAY $f : \kappa \to \kappa$ IS SOME MAP.

THEN $C = \{\delta < \kappa : (\forall \alpha < \delta)(f(\alpha) < \delta)\}$ IS CUB

LET $C_\alpha = [f(\alpha)+1, \kappa)$

THEN $C = \Delta_{\alpha < \kappa} C_\alpha$.

WHY "DIAGONAL" ?

C IS NOT A TRUE INTERSECTION

BUT ALMOST: $C \setminus C_\alpha$ IS BOUNDED,

FOR ALL α.

THE GENERAL PROCESS IS CALLED DIAGONALIZING.

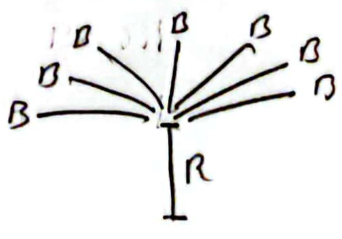
Fodor's Pressing Down Lemma (A workhorse in Set Theory)

Let κ be regular and uncountable,
Let $S \subseteq \kappa$ be stationary, and
Let $f : S \to \kappa$ be regressive
(meaning: if $\alpha > 0$ then $f(\alpha) < \alpha$)
Then there is a stationary set $T \subseteq S$ such that f is constant on T.

So, we must find a $\beta < \kappa$ such that $S_\beta = \{\alpha \in S : f(\alpha) = \beta\}$ is stationary.
Suppose there is no such β, so for every $\beta < \kappa$ we have a cub set C_β such that $C_\beta \cap S_\beta = \emptyset$.
Use AC to obtain a sequence $\langle C_\beta : \beta < \kappa \rangle$ of such sets.
Let $C = \Delta_{\beta < \kappa} C_\beta$
As S is stationary we have $\alpha \in S \cap C$
But $\alpha \in \bigcap_{\beta < \alpha} C_\beta$ and so $\alpha \notin \bigcup_{\beta < \alpha} S_\beta$ which means that $f(\alpha) \nless \alpha$.
Contradiction.

Applications

- If $f : \omega_1 \to \mathbb{R}$ is continuous then there is a $\delta < \omega_1$ such that f is constant on $[\delta, \omega_1)$
- Δ-system Lemma
  Let $\mathcal{A}$ be an uncountable family of finite sets.
  Then there are an uncountable subfamily $\mathcal{B}$ of $\mathcal{A}$ and a fixed finite set R such that if $A, B \in \mathcal{B}$ and $A \neq B$ then $A \cap B = R$.



PROOF: We can assume that $\bigcup\mathcal{A} \subseteq \omega_1$,
and that $\mathcal{A} = \{A_\alpha : \alpha \in \omega_1\}$

Define $f : \omega_1 \to \omega_1$ by $f(\alpha) = \max(\{0\} \cup (A_\alpha \cap \alpha))$

Then f is regressive: if $\alpha > 0$ then $f(\alpha) < \alpha$.

Let $C = \{\delta \in \omega_1 : (\forall \alpha < \delta)(\max A_\alpha < \delta)\}$

Apply Fodor's Lemma: take $S \subseteq C$ stationary and $\beta \in \omega_1$ such that $f(\alpha) = \beta$ if $\alpha \in S$

Note if $\alpha \in S$ then $A_\alpha \subseteq [0, \beta] \cup [\alpha, \omega_1)$

so if $\alpha_1 < \alpha_2$ in S

then $A_{\alpha_1} \cap A_{\alpha_2} \subseteq [0, \beta] \cup [\alpha_2, \omega_1)$

But $\max A_{\alpha_1} < \alpha_2$ and so

$$A_{\alpha_1} \cap A_{\alpha_2} \subseteq [0, \beta]$$

Next $[0, \beta]$ has only countably many finite subsets so there are an uncountable subset T of S (stationary even) and one finite set $R \subseteq [0, \beta]$ such that $A_\alpha \cap [0, \beta] = R$ for all $\alpha \in T$.

Then $\mathcal{B} = \{A_\alpha : \alpha \in T\}$ is as required.

EXTENSION: Let κ be infinite, let $\lambda > \kappa$ be regular such that $\mu^{<\kappa} < \lambda$ whenever $\mu < \lambda$.

Assume $\mathcal{A}$ is a family of sets such that $|\mathcal{A}| \geq \lambda$ and $(\forall A \in \mathcal{A})(|A| < \kappa)$

Then there are $\mathcal{B} \subseteq \mathcal{A}$ and a set R such that $(\forall A, B \in \mathcal{B})(A \neq B \to A \cap B = R)$

- If κ is regular use $f(\alpha) = \sup(A_\alpha \cap \alpha)$ with domain $E^\lambda_\kappa = \{\alpha : \mathrm{cf}\,\alpha = \kappa\}$
- If κ is singular then there is a regular $\nu < \kappa$ such that $|\{A \in \mathcal{A} : |A| < \nu\}| \geq \lambda$

Hajnal's Free Set Lemma.

Consider maps $F: X \to \mathcal{P}(X)$ with the property that $x \notin F(x)$ for all x.

$M \subseteq X$ is free (under F) if

$$(\forall x, y \in M)(x \neq y \to x \notin F(y) \wedge y \notin F(x))$$

When can we expect a free set M such that $|M| = |X|$?

- $X = \mathbb{R}$; each $F(x)$ closed.
  For $x \in \mathbb{R}$ take $p_x, q_x \in \mathbb{Q}$ such that
  $$x \in (p_x, q_x) \text{ and } (p_x, q_x) \cap F(x) = \emptyset.$$
  Then for some p and q the set
  $M = \{x : p_x = p \wedge q_x = q\}$
  has cardinality $2^{\aleph_0}$ and is free.
- $X = \omega_1$; each $F(\alpha)$ finite.
  Recycle the proof of the Δ-system lemma
  you can stop with S: if $\alpha_1 < \alpha_2$
  then $\max A_{\alpha_1} < \alpha_2$ and $\alpha_2 \notin [0, \rho] \cup [\alpha_2, \omega_1)$.
- $X = \kappa$: some cardinal $F(\alpha) = \alpha$
  then $|F(\alpha)| < |X|$ always but there is no 2-element free set.

Theorem: If κ and λ are infinite with $\lambda < \kappa$ and $F: \kappa \to \mathcal{P}(\kappa)$ is such that for all $\alpha \in \kappa$ we have $\alpha \notin F(\alpha) \wedge |F(\alpha)| < \lambda$. Then there is a free subset of cardinality κ.

Case 1 κ is regular

- If λ is also regular: use the PDL to find $S \subseteq E^{\kappa}_{\lambda}$ stationary and $\beta < \kappa$ such that if $\alpha \in S$ then $\sup(F(\alpha) \cap \alpha) < \beta$
  if $\alpha_1 < \alpha_2$ in S then $\sup F(\alpha_1) < \alpha_2$
- If λ is singular then there is a regular $\nu < \lambda$ such that $S = \{\alpha : |F(\alpha)| < \nu\}$ has cardinality κ. Use the previous step after re-ordering $S \cup \bigcup\{F(\alpha) : \alpha \in S\}$ in order type κ.

Case 2: κ is singular

Will come next week.