HAJNAL'S FREE SET LEMMA

THE SINGULAR CASE.

SO WE HAVE A SINGULAR CARDINAL κ
A CARDINAL λ BELOW κ AND
A MAP $F: \kappa \to \mathcal{P}(\kappa)$ SUCH THAT
$\alpha \notin F(\alpha)$ AND $|F(\alpha)| < \lambda$ FOR ALL α.

LET $\langle \kappa_i : i < \mathrm{cf}\,\kappa \rangle$ BE A STRICTLY INCREASING SEQUENCE OF REGULAR CARDINALS (SUCCESSORS SAY) THAT IS COFINAL IN κ.

WE MAY ASSUME THAT
$$\mathrm{cf}\,\kappa < \lambda < \lambda^+ < \kappa_0 < \cdots\cdots$$
[BECAUSE THE STATEMENT GETS STRONGER WHEN λ GETS LARGER.]

FOR $i < \mathrm{cf}\,\kappa$ LET $X_i = \kappa_i \setminus \sup_{j<i} \kappa_j$, SO $X_0 = \kappa_0$, AND $|X_i| = \kappa_i$ BECAUSE κ_i IS REGULAR AND $i < \mathrm{cf}\,\kappa < \kappa_i$.

NEXT: IF $i < \mathrm{cf}\,\kappa$ THEN
$$\kappa_i \le \left| \bigcup \{ F(\alpha) : \alpha < \kappa_i \} \right| \le \kappa_i \otimes \lambda = \kappa_i.$$
SO WE CAN SHRINK X_i TO
$$Y_i = X_i \setminus \bigcup_{j<i} \bigcup \{ F(\alpha) : \alpha < \kappa_j \}$$
TO GET $|Y_i| = \kappa_i$ AND:
IF $\alpha \in Y_i$, $\beta \in Y_j$ AND $i < j$ THEN $\beta \notin F(\alpha)$.

NEXT APPLY THE REGULAR CASE (AND AC) TO GET $\langle Z_i : i < \mathrm{cf}\,\kappa \rangle$ SUCH THAT
$Z_i \subseteq Y_i$, AND $|Z_i| = \kappa_i$ AND
IF $\alpha, \beta \in Z_i$ AND $\alpha \ne \beta$ THEN
$\alpha \notin F(\beta)$ AND $\beta \notin F(\alpha)$.

SO NOW WE KNOW:
IF $\alpha \in Z_i$, $\beta \in Z_j$, $\alpha \ne \beta$ AND $i \le j$
THEN $\beta \notin F(\alpha)$ (AND $\alpha \notin F(\beta)$ IF $i = j$)

WE NEED TO ENSURE, STILL, THAT
IF $\alpha \in Z_i$, $\beta \in Z_j$ AND $j < i$
THEN $\beta \notin F(\alpha)$.

Here is Hajnal's trick:

- Divide Z_0 into λ^+ subsets of cardinality $\varkappa_0$

$$Z_0 = \bigcup_{\eta<\lambda^+} D_{0,\eta} \quad : \quad \eta \neq \zeta \to D_{0,\eta} \cap D_{0,\zeta} = \emptyset$$
$$|D_{0,\eta}| = \varkappa_0$$

- For $i > 0$ we find $W_i \subseteq Z_i$ of cardinality $\varkappa_i$ and write $W_i = \bigcup_{\eta<\lambda^+} D_{i,\eta}$ such that $\eta \neq \zeta \to D_{i,\eta} \cap D_{i,\zeta} = \emptyset$

$$|D_{i,\eta}| = \varkappa_i$$

As follows:

For $\alpha \in Z_i$ we have $|F(\alpha)| < \lambda$

Hence for each $j < i$ the set $F(\alpha)$ intersects fewer than λ many of the sets $D_{j,\eta}$.

Hence we can make

$$\eta(\alpha, j) = \min\{\eta < \lambda^+ : F(\alpha) \cap W_j \subseteq \bigcup_{\zeta<\eta} D_{j,\zeta}\}$$
$$(\text{so } \eta \geq \eta(\alpha,j) \to F(\alpha) \cap D_{j,\eta} = \emptyset)$$

Because $i < \mathrm{cf}\,\varkappa < \lambda < \lambda^+$ we can find a single $\eta(\alpha) < \lambda^+$ such that for all $j < i$ and all $\eta \geq \eta(\alpha)$ we have $F(\alpha) \cap D_{j,\eta} = \emptyset$.

Finally, as $\varkappa_i$ is regular there ~~are~~ is a subset W_i of Z_i of cardinality $\varkappa_i$ and a single $\eta_i < \lambda^+$ such that $\eta(\alpha) = \eta_i$ for $\alpha \in W_i$.

Let $\{D_{i,\eta} : \eta < \lambda^+\}$ be any partition of W_i into sets of cardinality $\varkappa_i$.

- Consider $\langle \eta_i : 0 < i < \mathrm{cf}\,\varkappa \rangle$

Because $\mathrm{cf}\,\varkappa < \lambda^+$ this sequence is bounded in λ^+ so we have an η such that $\eta_i < \eta$ for all $i < \mathrm{cf}\,\varkappa$.

So if $\alpha \in W_i$, $j < i$, and $\zeta \geq \eta$ then $F(\alpha) \cap D_{j,\zeta} = \emptyset$

Let $M = \bigcup_{i<\mathrm{cf}\,\varkappa} \bigcup_{\zeta \geq \eta} D_{i,\zeta}$. Then M is free.

Partition Calculus.

Some notation: $[X]^n = \{A \subseteq X : |A| = n\}$
and variations $[X]^{<n}$, $[X]^{\leq n}$.. (usually new but not always)

Symbol: $\kappa \longrightarrow (\lambda)^n_\mu$ means
if $F : [\kappa]^n \to \mu$ is a map (colouring)
then there is a subset H of κ
of cardinality λ such that
F is constant on $[H]^n$.

- κ, λ, μ, and n are cardinals, but
- κ and λ can also be ordinals and "cardinality λ" can be replaced by "order type λ".

Examples

$6 \longrightarrow (3)^2_2$ but $5 \not\longrightarrow (3)^2_2$

or

For $n = 2$ we speak of graphs and colourings of edges
The H stands for <u>Homogeneous</u>:
all edges between points of H
have the same colour

First Big Theorem: Ramsey's Theorem
for all $k, n < \omega$ we have
$$\aleph_0 \longrightarrow (\aleph_0)^n_k.$$

1. It suffices to do the case $k = 2$:
Induction: given $F : [\omega]^n \longrightarrow k+1$
find H such that $\operatorname{ran} F \restriction [H]^n \subseteq k$
or $\operatorname{ran} F \restriction [H]^n \subseteq \{k\}$

② INDUCTION ON n

$n = 1$: $F: \omega \to \{0,1\}$ IS CONSTANT ON AN INFINITE SET.

$n = 2$: LET $F: [\omega]^2 \to \{0,1\}$

$x_0 = 0$ $\quad A_0 \subseteq \omega \setminus [0, x_0]$ INFINITE
AND $i_0 \in \{0,1\}$
SUCH THAT $F(x_0, a) = i_0$ $(a \in A_0)$

$x_1 = \min A_0$, $A_1 \subseteq A_0 \setminus [0, x_1]$ INFINITE
AND $i_1 \in \{0,1\}$
SUCH THAT $F(x_1, a) = i_1$ $(a \in A_1)$

$\vdots$

$x_{j+1} = \min A_j$, $A_{j+1} = A_j \setminus [0, x_{j+1}]$ INFINITE
AND $i_{j+1} \in \{0,1\}$
SUCH THAT $F(x_{j+1}, a) = i_{j+1}$ $(a \in A_{j+1})$

WE GET $X = \{x_0, x_1, x_2, \ldots\} \subseteq \omega$

- $j_1 < j_2 \to x_{j_1} < x_{j_2}$ AND $F(x_{j_1}, x_{j_2}) = \underline{\underline{i_{j_1}}}$
- WE CALL X PRE-HOMOGENEOUS

ONCE MORE THE CASE $n = 1$:
TAKE $i \in \{0,1\}$ SUCH THAT $J = \{j \in \omega : i_j = i\}$ IS INFINITE
FOR $j_1 < j_2$ IN J WE HAVE $F(x_{j_1}, x_{j_2}) = i$
SO $H = \{x_j : j \in J\}$ IS INFINITE AND HOMOGENEOUS.

$n \to n+1$ THE SAME:
$F: [\omega]^{n+1} \to \{0,1\}$ GIVEN

$x_0 = 0$; $A_0 \subseteq \omega \setminus [0, x_0]$ INFINITE AND $i_0 \in \{0,1\}$
SUCH THAT $F(\{x_0\} \cup a) = i_0$ IF $a \in [A_0]^n$

$x_{j+1} = \min A_j$; $A_{j+1} \subseteq A_j \setminus [0, x_{j+1}]$ INFINITE AND
$i_{j+1} \in \{0,1\}$ SUCH THAT $F(\{x_{j+1}\} \cup a) = i_{j+1}$
FOR $a \in [A_{j+1}]^n$.

NOW $X = \{x_j : j \in \omega\}$ IS SUCH THAT
IF $x \in [X]^{n+1}$ THEN $F(x) = i_j$ WHERE
$x_j = \min x$.

TAKE $i \in \{0,1\}$ SUCH THAT $J = \{j \in \omega : i_j = i\}$
IS INFINITE AND LET $H = \{x_j : j \in J\}$
THEN $F(x) = i$ FOR ALL $x \in [H]^{n+1}$.

QUESTION: WHAT ABOUT $\kappa > \aleph_0$?

NEGATIVE RESULTS FIRST

- $2^{\aleph_0} \not\to (\aleph_1)^2_2$ SO NO DIRECT GENERALIZATION ($2^\kappa \not\to (\kappa^+)^2_2$ --- THIS PERSISTS).
- $2^{\aleph_0} \not\to (3)^2_{\aleph_0}$ — NO DIRECT GENERALIZATION ($2^\kappa \not\to (3)^2_\kappa$
- $\kappa \not\to (\aleph_0)^{\aleph_0}_2$ FOR ALL κ.

EXAMPLES:

- SIERPINSKI LET $\prec$ WELL-ORDER $\mathbb{R}$,

  DEFINE $F: [\mathbb{R}]^2 \to \{0,1\}$ BY

  $F(\{x,y\})$ = TRUTH VALUE OF $x \prec y \Leftrightarrow x < y$

  NORMAL ORDER
- ERDŐS DEFINE $F: [{}^{\omega}2]^2 \to \omega$ BY

  $F(\{f,g\}) = \min\{n : f(n) \neq g(n)\}$.
- ERDŐS-RADO TAKE AN INFINITE κ

  ON $[\kappa]^{\aleph_0}$ DEFINE $x \equiv y$ IFF $x \Delta y$ IS FINITE

  LET C BE A CHOICE SET FOR THE SET OF EQUIVALENCE CLASSES

  FOR $x \in [\kappa]^{\aleph_0}$ LET C_x BE THE ELEMENT OF C IN THE EQUIVALENCE CLASS OF x

  DEFINE $F: [\kappa]^{\aleph_0} \to 2$ BY

  $F(x) = |x \Delta C_x| \bmod 2$

AN UNCOUNTABLE κ THAT SATISFIES $\kappa \to (\kappa)^2_2$ IS REALLY LARGE

IT NOT A SUCCESSOR, IT IS EVEN A STRONG LIMIT, BY $2^\kappa \not\to (\kappa^+)^2_2$.

IF κ IS SINGULAR THEN $\kappa \not\to (\kappa)^2_2$.

WRITE $\kappa = \bigcup_{i < \mathrm{cf}\,\kappa} X_i$; $i \neq j \to X_i \cap X_j = \emptyset$

$|X_i| < \kappa$.

$$F(\{x,y\}) = \begin{cases} 1 & \exists i\ \{x,y\} \subseteq X_i \\ 0 & \exists i \exists j\ i \neq j \wedge x \in X_i \wedge y \in X_j. \end{cases}$$

H HOMOGENEOUS: $H \subseteq X_i$ FOR SOME i

OR $|H| \leq \mathrm{cf}\,\kappa$.

ERDŐS-RADO THEOREM

FOR ALL INFINITE κ WE HAVE

$$(2^\kappa)^+ \longrightarrow (\kappa^+)^2_\kappa$$

LET $\lambda = (2^\kappa)^+$

LET $F : [\lambda]^2 \longrightarrow \kappa$ BE A COLOURING

WE DO A CLOSING-OFF ARGUMENT.

LET $\mathcal{C} = \{ f : f$ IS A FUNCTION, $|f| \leq \kappa$, $\operatorname{dom} f \subseteq \lambda$, AND $\operatorname{ran} f \subseteq \kappa \}$

NOTE IF $C \subseteq \lambda$ AND $|C| \leq \kappa$

THEN $\{ f \in \mathcal{C} : \operatorname{dom} f = C \} = {}^{C}\kappa$

AND $|{}^{C}\kappa| \leq \kappa^\kappa = 2^\kappa$.

FOR $f \in \mathcal{C}$ DEFINE $\alpha_f \in \lambda$ AS FOLLOWS

- IF THERE IS AN α SUCH THAT

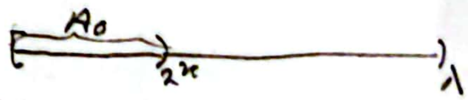
  $\operatorname{dom} f \subseteq \alpha$ AND

  $f(\beta) = F(\{\beta, \alpha\})$ FOR $\beta \in \operatorname{dom} f$

  THEN α_f IS THE FIRST SUCH α.
- OTHERWISE $\alpha_f = 0$.

BY RECURSION DEFINE $\langle \eta_i : i < \kappa^+ \rangle$ IN λ AS FOLLOWS

- $\eta_0 = 2^\kappa$ (THE ORDINAL)



- GIVEN η_i LET

  $A_i = \{ \alpha_f : f \in \mathcal{C}$ AND $\operatorname{dom} f \subseteq \eta_i \}$

  TAKE $\eta_{i+1} > \eta_i$ (MINIMAL) SUCH THAT $A_i \subseteq \eta_{i+1}$.
- IF i IS A LIMIT LET $\eta_i = \sup_{j<i} \eta_j$.

BY INDUCTION WE HAVE $\eta_i < \lambda$ FOR ALL i

AND AS $\kappa^+ \leq 2^\kappa < \lambda$ WE FIND THAT

$$\alpha = \sup_{i<\kappa^+} \eta_i$$

IS SMALLER THAN λ.

NOW CONSTRUCT $\langle \beta_i : i < \kappa^+ \rangle$ BELOW α SUCH THAT $i < j \rightarrow \beta_i < \beta_j$ AS FOLLOWS

LET $\beta_0 = 0$

THEN $f_0 = \{ \langle \beta_0, F(\{\beta_0, \alpha\}) \rangle \}$ BELONGS TO $\mathcal{C}$

AND $\operatorname{dom} f_0 \subseteq \alpha$ AND $f_0(\beta_0) = F(\{\beta_0, \alpha\})$

SO ---

-- THERE IS AN ORDINAL ABOVE β_0, NAMELY α SUCH THAT .....
WE LET $\beta_1 = \alpha_{f_0}$ AND OBSERVE THAT, BY CONSTRUCTION, $\beta_1 \in A_0 \subseteq \eta_1$
GIVEN $\langle \beta_j : j < i \rangle$ WITH $\beta_j < \alpha$ FOR $j < i$
LET $f_i = \{ \langle \beta_j, F(\{\beta_j, \alpha\}) \rangle : j < i \}$
THEN $f_i \in \mathcal{C}$ AND $\{\beta_j : j < i\} \subseteq \eta_k$ FOR SOME $k < \kappa^+$ BECAUSE $cf\,\alpha = \kappa^+$.
AND α_{f_i} IS NON-ZERO BECAUSE OF α
LET $\beta_i = \alpha_{f_i}$ (NOTE: $\beta_i < \eta_{k+1}$)
THUS WE GET $\langle \beta_i : i < \kappa^+ \rangle$ AND BY CONSTRUCTION WE HAVE
IF $i < j$ THEN $F(\{\beta_i, \beta_j\}) = F(\{\beta_i, \alpha\})$
NOW TAKE $\gamma \in \kappa$ SUCH THAT
$I = \{ i < \kappa^+ : F(\{\beta_i, \alpha\}) = \gamma \}$
HAS CARDINALITY κ^+.
NOW $H = \{\beta_i : i \in I\} \cup \{\alpha\}$ IS HOMOGENEOUS.

NOTE $|H| = \kappa^+$ BUT ALSO: ITS ORDER-TYPE IS $\underline{\underline{\kappa^+ + 1}}$

OTHER RESULTS:

- IF $\kappa \geq \aleph_0$ THEN $\kappa \longrightarrow (\kappa, \aleph_0)^2$
  IF $F : [\kappa]^2 \longrightarrow \{0,1\}$ THEN
  EITHER THERE IS A H WITH $|H| = \kappa$ SUCH THAT $F \restriction [H]^2 = 0$
  OR THERE IS A H WITH $|H| = \aleph_0$ SUCH THAT $F \restriction [H]^2 \equiv 1$.
- $\aleph_1 \longrightarrow (\aleph_1)^2_2$ FAILS VERY BADLY
  THERE IS $F : [\omega_1]^2 \longrightarrow \omega_1$ SUCH THAT $F[[A]^2] = \omega_1$
  FOR <u>ALL</u> UNCOUNTABLE SUBSETS A OF ω_1.