A TREE IS A PARTIALLY ORDERED SET $(T, <)$ WITH THE PROPERTY THAT FOR EVERY $t \in T$ THE SET $\{s : s < t\}$ IS WELL-ORDERED (BY $<$).

EXAMPLES:

- A A SET WITH AT LEAST TWO ELEMENTS AND $\alpha > 0$ AN ORDINAL: $T = A^{<\alpha} - \bigcup_{\beta < \alpha} {}^{\beta}A$.

  ORDERED BY INCLUSION.

  OFTEN USED SPECIAL CASES:

  $2^{<\omega}, 2^{<\omega_1}, \ldots \quad (A = 2)$

  $\omega^{<\omega}, \omega^{<\omega_1}, \ldots \quad (A = \omega)$

$1^{<\alpha}$ IS JUST THE ORDINAL α.
AND $0^{<\alpha}$ IS JUST $\{\emptyset\}$.

LEVELS: $T_\alpha = \{t \in T : \{s : s < t\}$ HAS ORDERTYPE $\alpha\}$

SO, IF $T = A^{<\alpha}$ THEN $T_\beta = {}^{\beta}A$.

- LET S BE STATIONARY IN ω_1,

  $C = \{ c : c$ IS COUNTABLE AND CLOSED IN ω_1, AND $c \subseteq S\}$

  $c < d$ MEANS d IS AN END-EXTENSION OF c, THAT IS $d \cap [0, \max c] = c$.

  THIS IS A TREE.

KÖNIG'S INFINITY LEMMA [THE SON OF ...]

LET T BE AN INFINITE TREE WITH FINITE LEVELS THEN THERE IS A SEQUENCE $\langle x_n : n \in \omega \rangle$ IN T SUCH THAT $x_n \in T_n$ AND $x_n < x_{n+1}$ FOR ALL n.

PROOF T_0 IS FINITE SO THERE IS $x_0 \in T_0$ SUCH THAT $\{t : x_0 < t\}$ IS INFINITE.

WE CONTINUE RECURSIVELY, MAINTAINING THE PROPERTY THAT $\{t : x_n < t\}$ IS INFINITE.

THEN GIVEN $x_n \in T_n$ THE SET $A = \{t \in T_{n+1} : x_n < t\}$ IS FINITE

SO THERE IS $x_{n+1} \in A$ SUCH THAT $\{t : x_{n+1} < t\}$ IS INFINITE.

EXERCISE:

THIS USES A BIT OF CHOICE; IDENTIFY WHAT CHOICE FUNCTION WE MUST HAVE.

A MAXIMAL CHAIN IN A TREE IS CALLED A BRANCH.
SO IN WORDS:

AN INFINITE TREE WITH FINITE LEVELS HAS AN INFINITE BRANCH.

KÖNIG'S LEMMA IMPLIES $\aleph_0 \to (\aleph_0)^n_k$

WE SHOW IT IMPLIES $\aleph_0 \to (\aleph_0)^2_2$.

LET $F : [\omega]^2 \to 2$ BE A COLOURING.

WE DEFINE A SUBTREE OF $2^{<\omega}$.

START WITH $t_0 = \emptyset$

GIVEN $\langle t_i : i < n \rangle$ BUILD t_n AS FOLLOWS

- $t_n \restriction 0 = \emptyset$ (OF COURSE)
- IF $t_n \restriction k$ IS NOT IN $\{t_i : i < n\}$ THEN <u>STOP</u>
- IF $t_n \restriction k$ IS IN $\{t_i : i < n\}$,
  SAY $t_n \restriction k = t_i$
  THEN WE EXTEND t_n BY $t_n(k) = F(\{i, n\})$

SO, FOR EXAMPLE $t_1(0) = F(\{0,1\})$
$t_2(0) = F(\{0,2\})$

IF $F(\{0,1\}) = F(\{0,2\})$ STOP : $t_2 \restriction 1 \notin \{t_0, t_1\}$
OTHERWISE $t_2(1) = F(\{1,2\})$

- BY CONSTRUCTION $\{t_n \restriction k : k \in \operatorname{dom} t_n\} \subseteq \{t_i : i < n\}$
  SO $\{t_n : n \in \omega\}$ IS A SUBTREE
- ALSO BY CONSTRUCTION $t_n \notin \{t_i : i < n\}$
  SO T IS INFINITE

LET B BE AN INFINITE BRANCH

- IF $t_m, t_n \in B$ AND $t_m \subset t_n$
  THEN $m < n$ AND $t_n(|t_m|) = F(\{m, n\})$
- LET $H_i = \{n : t_n \in B$ AND $t_n{}^\frown i \in B\}$ $(i = 0, 1)$
  IF $m < n$ IN H_i THEN, AS WE JUST SAW,
  $t_m{}^\frown i \in B$ SO $t_m{}^\frown i \subseteq t_n$
  AND $F(\{m, n\}) = t_n(|t_m|) = i$
  SO H_i IS HOMOGENEOUS WITH COLOUR i
  ONE OF H_0 AND H_1 IS INFINITE.

MORE COMPLICATED TREES GIVE $\aleph_0 \to (\aleph_0)^n_k$
FOR $n > 2$.

$\aleph_0 \to (\aleph_0)^2_2$ IMPLIES KÖNIG'S LEMMA

LET T BE AN INFINITE TREE WITH FINITE LEVELS

WE ASSUME $T = \bigcup_{n\in\omega} T_n$ BECAUSE WE ARE LOOKING FOR AN INFINITE CHAIN.

LET $<_T$ BE THE ORDER OF T

AND WE LET $<$ BE A LINEAR ORDER OF T IN ORDER-TYPE ω, VIA SOME BIJECTION $\omega \leftrightarrow T$

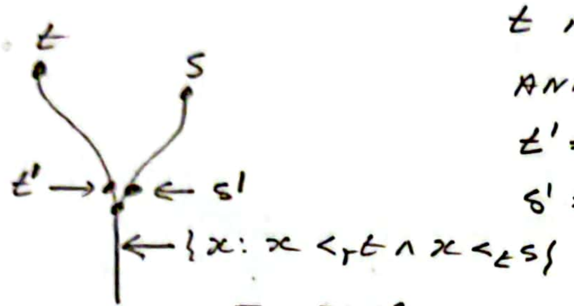
DEFINE A LINEAR ORDER $\prec$ THAT EXTENDS $<_T$:

SO $t \prec s$ IF $t <_T s$ OR

t AND s ARE INCOMPARABLE AND $t' < s'$, WHERE

$t' = \min\{x : x \le_T t \wedge x \not\le_T s\}$

$s' = \min\{x : x \le_T s \wedge x \not\le_T t\}$



DEFINE $F : [T]^2 \to \{0,1\}$ BY

$F(\{s,t\})$ = "TRUTH VALUE OF $t < s \leftrightarrow t \prec s$"

LET H BE INFINITE SUCH THAT F IS CONSTANT ON $[H]^2$

LET $B = \{t \in T : |\{h \in H : t <_T h\}|$ IS INFINITE$\}$.

SINCE EACH T_n IS FINITE WE FIND THAT $B \cap T_n \neq \emptyset$ FOR ALL n.

[$\bigcup_{i\le n} T_i$ IS FINITE AND $H \setminus \bigcup_{i\le n} T_i = \bigcup_{t\in T_n}\{h\in H : t <_T h\}$]

ALSO IF $m \le n$, $t \in B\cap T_n$, $s \in T_m$ AND $s <_T t$ THEN $s \in B \cap T_m$.

ASSUME $t, s \in B \cap T_n$ AND $t \neq s$, SAY $t \prec s$

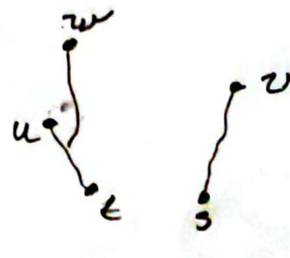
BOTH $\{h\in H : t <_T h\}$ AND $\{h \in H : s <_T h\}$ ARE INFINITE AND $\{h\in H : t, s < h\}$ IS CO-FINITE (ORDER TYPE ω).

TAKE u, v, w IN H SUCH THAT

$t <_T u$, $s <_T v$, $t <_T w$

AND $u < v < w$

AS $t \prec s$ WE ALSO HAVE $u \prec v$ AND $w \prec v$



SO $F(\{u,v\}) = 1$

$F(\{w,v\}) = 0$

CONTRADICTION.

SEE LEMMA 9.26 IN JECH'S BOOK

a) $\kappa \to (\kappa)^2_2$ IMPLIES κ HAS THE TREE PROPERTY

THE TREE PROPERTY IS "KÖNIG'S LEMMA":

IF T IS A TREE OF CARDINALITY κ

WITH ALL LEVELS OF CARDINALITY

LESS THAN κ

THEN T HAS A BRANCH OF CARDINALITY κ.

b) IF κ IS STRONGLY INACCESSIBLE AND HAS THE TREE PROPERTY THEN $\kappa \to (\kappa)^2_2$

[EVEN $\kappa \to (\kappa)^2_m$ FOR $m < \kappa$ AND

EVEN $\kappa \to (\kappa)^n_m$ FOR $n \in \omega$ AND $m < \kappa$]

DOES $\aleph_1$ HAVE THE TREE PROPERTY?

IF T IS A TREE OF CARDINALITY $\aleph_1$

WITH ALL LEVELS COUNTABLE

MUST T HAVE AN UNCOUNTABLE BRANCH?

N.N. ARONSZAJN [1934]: NO.

THERE IS A TREE, AN ARONSZAJN-TREE, OF CARDINALITY $\aleph_1$ WITH ALL LEVELS COUNTABLE AND NO BRANCH OF TYPE ω_1.

VARIOUS CONSTRUCTIONS.

EASIEST(?): START WITH $\{ s \in \mathbb{Q}^{<\omega_1} : s$ IS STRICTLY INCREASING$\}$

THIS TREE HAS NO BRANCHES OF TYPE ω_1.

BUT ITS LEVELS ARE MOSTLY UNCOUNTABLE.

SO WE MAKE A SUBTREE.

$T_0 = \{\emptyset\}$

$T_{\alpha+1} = \{ s^\frown q : s \in T_\alpha \wedge q > \sup \operatorname{ran} s \}$

NOTE $T_0 = \{\emptyset\}$, $T_1 = \{ \langle q \rangle : q \in \mathbb{Q} \}$

$T_2 = \{ \langle p, q \rangle : p, q \in \mathbb{Q},\ p < q \}$

ETC

WHAT TO DO AT LEVEL ω?

WE BUILD $\langle T_\alpha : \alpha < \omega_1 \rangle$ RECURSIVELY
WHILE MAINTAINING THE FOLLOWING PROPERTY:

$(\circledast)$ $\begin{cases} \text{IF } \beta < \gamma,\ s \in T_\beta, \text{ AND } q > \sup \operatorname{ran} s \\ \text{THEN THERE IS A } t \in T_\gamma \text{ SUCH THAT} \\ \quad s \subset t \text{ AND } q > \sup \operatorname{ran} t. \end{cases}$

SO ASSUME WE HAVE $\langle T_\beta : \beta < \alpha \rangle$ WITH $T_\beta \subseteq {}^\beta\mathbb{Q}$ COUNTABLE
THAT SATISFIES $\circledast$ FOR ALL $\beta < \gamma < \alpha$.

- IF $\alpha = \beta + 1$
  LET $T_\alpha = \{ s^\frown q : s \in T_\beta,\ q > \sup \operatorname{ran} s \}$
  THEN $\circledast$ HOLDS WITH $\beta < \gamma < \alpha + 1$
  NOTE ALSO IF T_β IS COUNTABLE THEN SO IS $T_{\beta+1}$
- IF α IS A LIMIT WE SHOULD DEFINE T_α
  BY ASSUMPTION $\circledast$ THE SET $\bigcup_{\beta<\alpha} T_\beta$ IS COUNTABLE.
  FIX A STRICTLY INCREASING AND COFINAL SEQUENCE $\langle \alpha_n : n \in \omega \rangle$ IN α.
  LET $A = \{ \langle s, q \rangle : s \in \bigcup_{\beta<\alpha} T_\beta,\ q \in \mathbb{Q},\ q > \sup \operatorname{ran} s \}$
  WE DEFINE A POINT $t(s,q)$ IN ${}^\alpha\mathbb{Q}$ FOR EACH SUCH PAIR SUCH THAT $s < t(s,q)$
  AND $q > \sup \operatorname{ran} t(s,q)$.
  LET $\langle s, q \rangle \in A$ AND LET m BE SUCH THAT $\operatorname{dom} s < \alpha_m$. LET $\varepsilon = \frac{1}{2}(q - \sup \operatorname{ran} s)$
  RECURSIVELY TAKE $s_n \in T_{\alpha_n}$ FOR $n \geq m$ SUCH THAT $s \subset s_m$, $s_n \subset s_{n+1}$ FOR ALL n
  AND $\sup \operatorname{ran} s_n < \sup \operatorname{ran} s + \varepsilon$.
  THEN LET $t(s,q) = \bigcup_{n \geq m} s_n$.
  BY CONSTRUCTION
  - $t(s,q)$ IS INCREASING
  - $s \subset t(s,q)$
  - $\sup \operatorname{ran} t(s,q) < q$.
- THE TREE $T = \bigcup_{\alpha<\omega_1} T_\alpha$ IS AN ARONSZAJN TREE.

OTHER POSSIBILITIES

- KUREPA: $\{C \subseteq \mathbb{Q} : C$ IS WELL-ORDERED WITH A MAXIMUM$\}$

  ORDER $C \lhd D$ IF $C \subset D$ AND

  $C = D \cap (\leftarrow, \max C]$

  αTH LEVEL $\{C : \max C = \alpha\}$ (STILL UNCTBLE)

  THIN OUT TO GET COUNTABLE LEVELS AS BEFORE.

- THERE IS A SEQUENCE $\langle s_\alpha : \alpha < \omega_1 \rangle$ OF FUNCTIONS SUCH THAT $s_\alpha : \alpha \to \omega$ IS INJECTIVE AND IF $\beta < \alpha$ THEN $\{\gamma < \beta : s_\beta(\gamma) \neq s_\alpha(\gamma)\}$ IS FINITE.

  DEFINE $T_\alpha = \{ s \in {}^\alpha\omega : \{\gamma < \alpha : s(\gamma) \neq s_\alpha(\gamma)\}$ IS FINITE$\}$

  THEN $\bigcup_{\alpha < \omega_1} T_\alpha$ IS AN ARONSZAJN TREE.

SO $\aleph_0$ HAS THE TREE PROPERTY

$\aleph_1$ DOES NOT HAVE THE TREE PROPERTY.

WHAT ABOUT $\aleph_2$?

THE CONTINUUM HYPOTHESIS IMPLIES NO:

THERE IS AN $\aleph_2$-ARONSZAJN TREE

BUT: . "$\aleph_2$ HAS THE TREE PROPERTY" IMPLIES $\aleph_2$ IS WEAKLY COMPACT IN L

. IF THERE IS A WEAKLY COMPACT CARDINAL THEN THERE IS A MODEL OF ZFC IN WHICH $\aleph_2$ HAS THE TREE PROPERTY.

EXERCISE: (a) GO THROUGH THE CONSTRUCTION OF OUR ARONSZAJN TREE AND SHOW THAT WE CAN ENSURE

$$\forall s \in T : \sup \operatorname{ran} s \in \mathbb{Q}$$

(b) IN THAT NEW SITUATION:

IF $s, t \in T$ AND $\sup \operatorname{ran} s = \sup \operatorname{ran} t$ THEN $s = t$ OR s AND t ARE INCOMPARABLE

(c) IN THAT CASE: T IS THE UNION OF COUNTABLY MANY ANTICHAINS

A IS AN <u>ANTICHAIN</u> IF ANY TWO DISTINCT POINTS OF A ARE INCOMPARABLE.

So, we have an Aronszajn tree such that
- it is of cardinality $\aleph_1$
- its levels are countable
- it is the union of countably many antichains

(so lots of antichains are uncountable)

Can we improve this?

Can we make an Aronszajn tree in which all antichains are countable?

Such a tree is called a Souslin Tree

Why?

Souslin's Problem

If $(X, <)$ is a linearly ordered set without jumps (→][←) or gaps (→ ←) in which every pairwise disjoint family of non-trivial intervals is countable must X be (a subinterval of) $\mathbb{R}$?

Equivalently: must X have a countable dense set?

[Exercise: prove this equivalence.]

From $(X, <)$ we can build a tree.

Take a choice function M for the set of intervals: $M : \{(x,y) \in X^2 : x < y\} \to X$ such that $x < M(x,y) < y$ for all x, y.

Let $T_0 = X$ Assume $X = [a, b]$

$T_1 = \{(a, M(a,b)), (M(a,b), b)\}$

In general $T_{\alpha+1} = \{(x, M(x,y)) : (x,y) \in T_\alpha\} \cup \{(M(x,y), y) : (x,y) \in T_\alpha\}$

α limit let A_α be the set of all endpoints of all intervals in $\bigcup_{\beta<\alpha} T_\beta$.

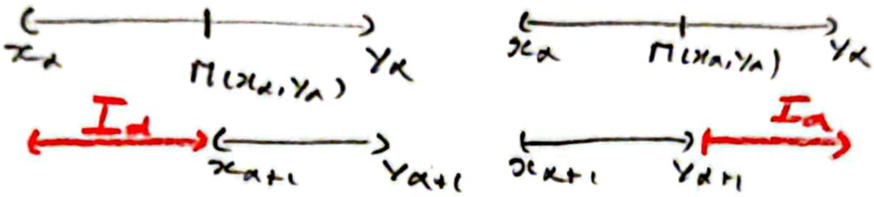
A_α is countable, so $\overline{A_\alpha} \neq X$.

T_α is the family of maximal open intervals of $X \setminus \overline{A_\alpha}$.

Then $T = \bigcup_{\alpha<\omega_1} T_\alpha$ is a tree
when ordered by $I < J$ iff $J \subset I$.
And T is a Souslin tree.
Antichain = pairwise disjoint family of intervals.

No ω_1-chain:
Let $\langle (x_\alpha, y_\alpha) : \alpha < \delta \rangle$ be a chain
with $(x_\alpha, y_\alpha) \in T_\alpha$ for all α.
For each α define

$$I_\alpha = \begin{cases} (y_{\alpha+1}, y_\alpha) & \text{if } y_{\alpha+1} = m(x_\alpha, y_\alpha) \\ (x_\alpha, x_{\alpha+1}) & \text{if } x_{\alpha+1} = m(x_\alpha, y_\alpha) \end{cases}$$



The family $\{I_\alpha : \alpha < \delta\}$ is pairwise
disjoint: if $\alpha < \beta$ then
$(x_\beta, y_\beta) \subseteq (x_{\alpha+1}, y_{\alpha+1}) \subseteq (x_\alpha, y_\alpha) \setminus I_\alpha$.
So $\delta < \omega_1$.

Conversely from a Souslin tree you
can make a counterexample, a Souslin line.
Let T be a Souslin tree with tree order $<_T$
Let $\prec$ be a linear order of T
First we prune T.
Let $C = \{t \in T : \{x : t \le x\}$ is countable$\}$
$C_0 = \{t \in C : (\forall s)(s < t \to s \notin C)\}$
Let $T' = T \setminus C$.
Because C_0 is an antichain, hence countable
and $C = \bigcup_{t \in C_0} \{x : t \le x\}$
we see that C is countable
So if $t \in T'$ then $\{s \in T' : t < s\}$ is uncountable
for it is $\{s \in T : t < s\} \setminus C$.
And $t \in T'$ means $\{s \in T : t < s\}$ is uncountable

LET S BE THE SET OF MAXIMAL CHAINS IN T AND (WE RESET T TO T').

IF $C \in L$ THEN C DOES NOT HAVE A MAXIMUM FOR $\{x : \max C < x\}$ WOULD BE EMPTY SO THE ORDER-TYPE OF C IS A LIMIT, SAY $\ell(C)$

ALSO $C \cap T_\alpha \neq \emptyset$ IFF $\alpha < \ell(C)$

AND OF COURSE $|C \cap T_\alpha| = 1$ IF $\alpha < \ell(C)$

WRITE $C = \langle C_\alpha : \alpha < \ell(C) \rangle$ WHERE $\{C_\alpha\} = C \cap T_\alpha$.

LET $\angle$ BE A LINEAR ORDER OF T

IF $C, D \in L$ AND $C \neq D$

LET IF $\alpha = \min\{\gamma : C_\gamma \neq D_\gamma\}$ ($\Delta(C, D)$)

LET $C \lhd D$ IF $C_\alpha \angle D_\alpha$

- LET $K \subseteq L$ BE COUNTABLE

  LET α BE LARGER THAN $\sup\{\ell(C) : C \in K\}$

  TAKE $t \in T_\alpha$ AND THREE INCOMPARABLE POINTS u, v, AND w ABOVE t

  AND C_u, C_v, AND C_w IN L

  WITH $u \in C_u$, $v \in C_v$, AND $w \in C_w$

  SAY $C_u \lhd C_v \lhd C_w$

  THEN (C_u, C_w) IS A NONEMPTY OPEN INTERVAL IN L AND $(C_u, C_w) \cap K = \emptyset$.

  [EVERY $C \in (C_u, C_w)$ CONTAINS t].

- LET $\{(C_i, D_i) : i \in I\}$ BE A PAIRWISE DISJOINT FAMILY OF OPEN INTERVALS, SAY $E_i \in (C_i, D_i)$

  SO $C_i \lhd E_i \lhd D_i$ FOR ALL i

  THEN $\Delta(C_i, E_i), \Delta(E_i, D_i) < \ell(E_i)$

  TAKE α_i IN THAT INTERVAL

  AND LET $t_i = E_{i, \alpha_i}$

  IF $i \neq j$ THEN t_i AND t_j ARE INCOMPARABLE

  IF, SAY $t_i < t_j$ THEN $t_i \in E_j$

  AND THEN $\Delta(E_j, C_i) = \Delta(E_i, C_i)$ AND $\Delta(E_j, D_i) = \Delta(E_i, D_i)$

  AND IT FOLLOWS THAT $C_i \lhd E_j \lhd D_i$ AS WELL

  SO $\{t_i : i \in I\}$ IS AN ANTICHAIN AND HENCE COUNTABLE.