SOUSLIN'S PROBLEM

REMEMBER: WE WANT A TREE T OF HEIGHT ω_1 SUCH THAT ALL ANTICHAINS ARE COUNTABLE AND ALSO ALL CHAINS.

ASSUME $V=L$ WE BUILD A SOUSLIN TREE AS A SUBTREE OF $2^{<\omega_1} = \bigcup_{\alpha<\omega_1} 2^\alpha$, THE TREE OF ALL COUNTABLE SEQUENCES OF ZEROS AND ONES

$T_0 = \{\emptyset\}$ (EASY START ...)

$T_{\alpha+1} = \{ s^\frown i : s \in T_\alpha, i \in 2\}$

NOTE $s^\frown i = s \cup \{\langle\alpha, i\rangle\}$.

α LIMIT: ASSUME WE HAVE T_β FOR $\beta<\alpha$ AND MAKE T_α, BUT HOW?

WRITE $T\restriction\alpha = \bigcup_{\beta<\alpha} T_\beta$.

WHAT BRANCHES OF $T\restriction\alpha$ SHOULD GO INTO T_α?

[ASSUME T IS FINISHED AND A IS AN UNCOUNTABLE ANTICHAIN; WE KEPT THE LEVELS COUNTABLE. WE CAN ASSUME A IS MAXIMAL

SO IF $s \in T$ THEN THERE IS AN $a \in A$ SUCH THAT $a \le s$ OR $s \le a$.

IF $\alpha < \omega_1$ THEN $T\restriction\alpha$ IS COUNTABLE SO THERE IS AN ORDINAL $f(\alpha) > \alpha$ SUCH THAT FOR EVERY $s \in T\restriction\alpha$ THERE IS AN $a \in A \cap T\restriction f(\alpha)$ SUCH THAT $s \le a$ OR $a \le s$.

LET $C = \{ \gamma : (\forall\alpha\in\gamma)(f(\alpha)<\gamma)\}$.

THEN C IS CUB AND IF $\gamma \in C$ THEN

$A \cap T\restriction\gamma$ IS A MAXIMAL ANTICHAIN IN $T\restriction\gamma$.

IF WE COULD HAVE PREDICTED THIS AND MADE SURE THAT EVERY $s \in T_\gamma$ IS <u>ABOVE</u> SOME $a \in A \cap T\restriction\gamma$ THEN $A \cap T\restriction\gamma$ WOULD BE MAXIMAL IN T AND THAT WOULD BE A CONTRADICTION

HOW CAN WE PREDICT THIS?

WE USE $<_L$ TO TAKE CARE OF THE <u>FIRST</u> SUCH ANTICHAIN, AT EVERY OPPORTUNITY]

So: α is a limit, we have $T\restriction\alpha$

and we let $A_\alpha \subseteq T\restriction\alpha$ be the $<_L$-first maximal antichain in $T\restriction\alpha$ such that $\{\gamma<\alpha : A_\alpha \cap T_\gamma \neq \emptyset\}$ is cofinal in α.

Are there any?

Take a branch $\langle s_\gamma : \gamma<\alpha\rangle$

and look at $B = \{s_\gamma{}^\frown(1-s_{\gamma+1}(\gamma)) : \gamma<\alpha\}$

This is an antichain and can be extended to a maximal one.

So, yes.

For every $s\in T\restriction\alpha$ let b_s be the $<_L$-first branch with $s\in b_s$ and $b_s\cap A_\alpha \neq \emptyset$.

By maximality of A_α this is well-defined

Let $T_\alpha = \{\bigcup b_s : s\in T\restriction\alpha\}$. (countable because $T\restriction\alpha$ is countable).

This whole construction took place in L_{ω_2} and we <u>defined</u> every level using $<_L$.

So T is a <u>definable</u> element of L_{ω_2}

Assume T has an uncountable antichain and let $A\in L_{\omega_2}$ be the $<$-first one.

Also A is definable.

And $(A$ is the $<_L$-first maximal antichain of T such that $\{\gamma : A\cap T_\gamma \neq \emptyset\}$ is cofinal in $\omega_1)^{L_{\omega_2}}$

Let $M \prec L_{\omega_2}$ be countable

Let $\delta = \omega_1\setminus M$, so $\delta = M\cap\omega_1$ (homework 06)

Also $\langle L_\alpha : \alpha\in\omega_1\rangle \in M$ by definability

so if $\alpha<\delta$ then $L_\alpha\in M$ and hence $L_\alpha\subseteq M$ (homew)

so $L_\delta = \bigcup_{\alpha<\delta} L_\alpha \subseteq M$.

But if $x\in M\cap L_{\omega_1}$ then $x\in L_\alpha$ for some $\alpha\in M$

We see $M\cap L_{\omega_1} = L_\delta$.

Also $T\cap M = T\restriction\delta$: if $s\in T\cap M$ then $\operatorname{dom} s\in M$

so $\operatorname{dom} s<\delta$

if $\alpha<\delta$ then $T_\alpha\in M$

so $T_\alpha\subseteq M$ as $|T_\alpha|\leq\aleph_0$

AND ALSO $A \cap M = A \cap T \upharpoonright \delta$

CONDENSATION: $\pi : M \to L_\beta$

- $\pi(x) = x \quad x \in L_\delta$
- $\pi(\omega_1) = \delta$ — $\{\pi(\alpha) : \alpha \in \omega_1 \cap M\}$
- $\pi[T] = T \upharpoonright \delta$ — $\{\pi(s) : s \in T \cap M\}$
- $\pi(A) = A \cap (T \upharpoonright \delta)$ — $\{\pi(a) : a \in A \cap M\}$

SO NOW: BY $M \prec L_{\omega_2}$ AND $M \cong L_\beta$ WE GET
$(A \cap (T \upharpoonright \delta)$ IS THE $<_L$-FIRST MAXIMAL ANTICHAIN
IN $T \upharpoonright \delta$ SUCH THAT $\{\gamma < \delta : (A \cap T \upharpoonright \delta) \cap (T \upharpoonright \delta)_\gamma \neq \emptyset\}$
IS COFINAL IN $\delta)^{L_\beta}$

BUT L_β IS AN INITIAL SEGMENT OF L
SO $A \cap (T \upharpoonright \delta)$ REALLY IS THE FIRST SUCH
MAXIMAL ANTICHAIN IN $T \upharpoonright \delta$

SO $A \cap (T \upharpoonright \delta) = A_\delta$

BUT A_δ WAS MADE MAXIMAL IN ALL OF T.
CONTRADICTION.

WE DID NOT USE THE CUB SET BUT WE WILL NOW.
WE MADE PREDICTIONS, DEALT WITH THE
PREDICTIONS AND WE WERE DONE!

PREDICTION PRINCIPLE:
$\diamondsuit$: THERE IS A SEQUENCE $\langle A_\alpha : \alpha < \omega_1 \rangle$
SUCH THAT $A_\alpha \subseteq \alpha$ FOR ALL α AND
SUCH THAT FOR ALL $A \subseteq \omega_1$ THE
SET $S_A = \{\alpha : A \cap \alpha = A_\alpha\}$ IS STATIONARY.

TWO THINGS: $\diamondsuit$ HOLDS IN L
$\diamondsuit$ IMPLIES THERE IS A SOUSLIN TREE

FIRST THE TREE [JECH 15.26; KUNEN II.7.8]

CONSTRUCT A PARTIAL ORDER $\lhd$ ON ω_1.

PUT $I_\alpha = \{\omega \cdot \alpha + n : n \in \omega\} = [\omega \cdot \alpha, \omega \cdot (\alpha+1))$

FOR INFINITE α WE LET I_α BE THE αTH LEVEL
OF THE TREE.

- FIRST ORDER ω^2 LIKE THE BINARY TREE $2^{<\omega}$.
- WE ALWAYS SPECIFY
  $\omega \cdot \alpha + n \lhd \omega(\alpha+1) + 2n$
  $\omega \cdot \alpha + n \lhd \omega \cdot (\alpha+1) + (2n+1)$

AND NOTHING MORE THE SETS I_α REMAIN
UNORDERED

IF α IS A LIMIT AND $\lhd$ IS DEFINED ON $\omega\cdot\alpha = T|\alpha$ THEN WE LOOK AT $A_\alpha \subseteq \alpha \subseteq \omega\cdot\alpha$.

IF A_α IS A MAXIMAL ANTICHAIN IN $T|\alpha$ CHOOSE BRANCHES b_s AS BEFORE:

$$s \in b_s \text{ AND } b_s \cap A_\alpha \neq \emptyset$$

IF A_α IS NOT A MAXIMAL ANTICHAIN DROP THE REQUIREMENT $b_s \cap A_\alpha \neq \emptyset$.

ENUMERATE $\{b_s : s \in T|\alpha\}$ AS $\{c_m : m \in \omega\}$ AND DEFINE $s \lhd \omega\cdot\alpha + m$ IFF $s \in c_m$.

LET A BE A MAXIMAL ANTICHAIN IN THE RESULTING TREE $(\omega_1, \lhd)$

AS BEFORE:

$$C = \{\gamma : \omega\cdot\gamma = \gamma \wedge A\cap\gamma \text{ IS A MAXIMAL AC IN } T|\gamma\}$$

IS CUB.

OUR SEQUENCE $\langle A_\alpha : \alpha < \omega_1\rangle$ IS SUCH THAT THERE IS AN $\alpha \in C$ SUCH THAT

$A \cap \alpha = A_\alpha$

WE MADE SURE THAT A_α STAYED MAXIMAL, SO $A = A_\alpha$ AND A IS COUNTABLE.

$V = L$ IMPLIES $\Diamond$. [JECH 13.21 ; KUNEN VII.5.2]

DEFINE $S_\alpha \subseteq \alpha$ AND $C_\alpha \subseteq \alpha$ FOR ALL α

- $S_0 = C_0 = \emptyset$
- $S_{\alpha+1} = C_{\alpha+1} = \alpha+1$ (NOTHING EXCITING AT SUCCESSOR LEVELS)
- α LIMIT: TAKE THE $<_L$-FIRST PAIR (S_α, C_α) OF SUBSETS OF α SUCH THAT
  - C_α IS CUB IN α
  - $(\forall\gamma \in C_\alpha)(S\cap\gamma \neq S_\gamma)$ ⟵ RECURSION

  IF NO SUCH PAIR EXISTS: $S_\alpha = C_\alpha = \alpha$.

THIS CONSTRUCTION TAKES PLACE IN L_{ω_2} (AS WITH THE SOUSLIN TREE).

CLAIM $\langle S_\alpha : \alpha \in \omega_1\rangle$ IS A $\Diamond$-SEQUENCE

IF NOT THEN THERE IS $S \subseteq \omega_1$ SUCH THAT $\{\alpha \in \omega_1 : S\cap\alpha = S_\alpha\}$ IS NOT STATIONARY, SO THERE IS A CUB SET C SUCH THAT

$(\forall\gamma \in C)(S\cap\gamma \neq S_\gamma)$.

LET (S,C) BE THE $<_L$-FIRST SUCH PAIR (IN L_{ω_2})

LET $M \prec L_{\omega_2}$ BE COUNTABLE

BY ABSOLUTENESS / DEFINABILITY

$\langle S_\alpha : \alpha \in \omega_1 \rangle, \langle C_\alpha : \alpha \in \omega_1 \rangle, \langle S, C \rangle \in M$.

LET $\delta = M \cap \omega_1$ AND LET $\pi : M \longrightarrow L_\beta$ BE THE TRANSITIVE COLLAPSE.

AS BEFORE

- $\pi(x) = x \quad x \in L_\delta$
- $\pi(\omega_1) = \delta$
- $\pi(S) = S \cap \delta$, $\pi(C) = C \cap \delta$
- $\pi(\langle S_\alpha : \alpha \in \omega_1 \rangle) = \langle S_\alpha : \alpha < \delta \rangle$
- $\pi(\langle C_\alpha : \alpha \in \omega_1 \rangle) = \langle C_\alpha : \alpha < \delta \rangle$

$\big((S,C)$ IS THE $<_L$-FIRST PAIR SUCH THAT C IS CUB AND $(\forall \gamma \in C)(S \cap \gamma \neq S_\gamma) \big)^{L_{\omega_2}}$

AND SO

$\big((S\cap\delta, C\cap\delta)$ IS THE $<_L$-FIRST PAIR --- $(\forall \gamma \in C\cap\delta)(S\cap\delta\cap\gamma \neq S_\gamma) \big)^{L_\beta}$

THIS IS REALLY TRUE BY ABSOLUTENESS

SO WE CHOSE $S_\delta = S \cap \delta$ AND $C_\delta = C \cap \delta$

BUT IN L_β WE HAVE "$C \cap \delta$ IS COFINAL IN δ"

SO $C \cap \delta$ IS COFINAL IN δ

AS C IS CLOSED WE HAVE $\delta \in C$

SO WE ALSO HAVE $S_\delta \neq S \cap \delta$ CONTRADICTION.

THE OTHER EXTREME:

WE HAVE ARONSZAJN TREES:

- CARDINALITY $\aleph_1$
- COUNTABLE LEVELS
- <u>NO</u> UNCOUNTABLE BRANCHES

THINK OF $2^{<\omega}$: COUNTABLE, FINITE LEVELS
$2^{\aleph_0}$ INFINITE BRANCHES

KUREPA: CAN WE HAVE

- CARDINALITY $\aleph_1$
- COUNTABLE LEVELS
- MORE THAN $\aleph_1$ MANY UNCOUNTABLE BRANCHES ?

SUCH A TREE IS A <u>KUREPA TREE</u>

ORIGINAL FORMULATION [KUREPA 1935]

KH (KUREPA HYPOTHESIS)

THERE IS A FAMILY $\mathcal{F} \subseteq \mathcal{P}(\omega_1)$ OF CARDINALITY $\aleph_2$
SUCH THAT FOR ALL $\alpha < \omega_1$ THE FAMILY
$$\{F \cap \alpha : F \in \mathcal{F}\}$$
IS COUNTABLE.

[$\mathcal{F}$ IS CALLED A KUREPA FAMILY]

KUREPA: THERE IS A KUREPA TREE IFF THERE IS A KUREPA FAMILY

TREE $\to$ FAMILY

LET $(T, <_T)$ BE A KUREPA TREE AND LET
$b : \omega_1 \to T$ BE A BIJECTION
SUCH THAT $b(\alpha) <_T b(\beta)$ IMPLIES $\alpha < \beta$.
(SAY LEVEL BY LEVEL.)

LET $\mathcal{F} = \{ X \subseteq \omega_1 : b[X] \text{ IS A BRANCH}\}$.
THEN $\mathcal{F}$ HAS CARDINALITY $\aleph_2$
AND IF $\alpha < \omega_1$ AND $b[\alpha] \subseteq T \restriction \delta$
THEN $\{F \cap \alpha : F \in \mathcal{F}\}$
$\subseteq \{ b^{-1}[\hat{s}] : s \in T_\delta \}$ $\quad \hat{s} = \{t : t < s\}$
IS COUNTABLE.

FAMILY $\to$ TREE

GIVEN A KUREPA FAMILY $\mathcal{K} \subseteq \mathcal{P}(\omega_1)$
LET $\mathcal{F} = \{\chi_K : K \in \mathcal{K}\}$ BE THE SET OF CHARACTERISTIC FUNCTIONS.

LET $T = \{ f \restriction \alpha : f \in \mathcal{F}, \alpha < \omega_1 \}$
- EACH χ_K DETERMINES A BRANCH $\{\chi_K \restriction \alpha : \alpha < \omega_1\}$
- $T_\alpha = \{\chi_K \restriction \alpha : K \in \mathcal{K}\}$ IS COUNTABLE

WE SHALL SEE HOW TO BUILD A KUREPA FAMILY
BUT FIRST A REMARK ABOUT ELEMENTARY
SUBSTRUCTURES OF L_δ FOR LIMIT δ.

IF δ IS A LIMIT AND $X \subseteq L_\delta$

THEN THERE IS A SMALLEST M SUCH THAT $X \subseteq M$ AND $M \prec L_\delta$.

THE REASON IS THAT WE CAN ALWAYS TAKE THE $<_L$-FIRST ELEMENT OF L_δ WITH A CERTAIN PROPERTY WHEN WE BUILD M.

IN FACT ONE CAN WRITE DOWN WHAT M IS: M CONSISTS OF THE MEMBERS OF L_δ THAT ARE DEFINABLE IN L_δ FROM ELEMENTS OF X.

SO $a \in M$ IFF THERE IS A FORMULA $\varphi(u, v_1, \ldots, v_k)$ AND THERE ARE $x_1, \ldots, x_k$ IN X SUCH THAT a IS THE UNIQUE ELEMENT OF L_δ SUCH THAT $\varphi^{L_\delta}(a, x_1, \ldots, x_k)$ HOLDS.

TO SEE $M \prec L_\delta$

WE HAVE TO CHECK: GIVEN $m_1, \ldots, m_k \in M$

IF $(\exists u \in L_\delta)\varphi^{L_\delta}(u, m_1, \ldots, m_k)$ THEN $(\exists u \in M)\varphi^{L_\delta}(u, m_1, \ldots, m_k)$

REPLACE φ BY ψ:

$\psi(u, v_1, \ldots, v_k)$ IS

$\varphi(u, v_1, \ldots, v_k) \wedge (\forall v)(v <_L u \to \neg\varphi(v, v_1, \ldots, v_k))$

THERE MAY BE MANY a WITH $\varphi^{L_\delta}(a, m_1, \ldots, m_k)$

BUT THERE IS EXACTLY ONE a WITH $\psi^{L_\delta}(a, m_1, \ldots, m_k)$

THAT a IS DEFINABLE FROM $m_1, \ldots, m_k$

IT IS ALSO DEFINABLE FROM MEMBERS OF X

IF $\sigma_i(u, v_1, \ldots, v_{k_i})$ DEFINES m_i FROM $x^i_1, \ldots, x^i_{k_i}$

THEN

$(\exists w_1)(\exists w_2)\ldots(\exists w_k)\left[\psi(u, w_1, \ldots, w_k) \wedge \bigwedge_{i=1}^{k} \sigma_i(w_i, x^i_1, \ldots, x^i_{k_i})\right]$

DEFINES a FROM $x^1_1, \ldots, x^1_{k_1}, \ldots, x^k_1, \ldots, x^k_{k_k} \ldots$

IF $N \prec L_\delta$ AND $X \subseteq N$

THEN THE UNIQUENESS FORCES ALL $m \in M$ TO BE IN N.

ALSO NOTE THAT $|M| \leq |X| \cdot \aleph_0 + \aleph_0$ ← IF $X = \emptyset$

WE NEED A BETTER $\diamondsuit$.
WE STRENGTHEN STATIONARY TO CUB
AND WEAKEN "ONE SET" TO "COUNTABLY MANY"

$\diamondsuit^*$ THERE IS A SEQUENCE $\langle S_\alpha : \alpha < \omega_1 \rangle$
SUCH THAT $S_\alpha \subseteq \mathcal{P}(\alpha)$ AND S_α IS COUNTABLE
FOR ALL α AND FOR EVERY $X \subseteq \omega_1$ THERE
IS A CUB SET $C \subseteq \omega_1$ SUCH THAT
$X \cap \alpha \in S_\alpha$ FOR ALL $\alpha \in C$.

$\diamondsuit^+$ THERE IS A SEQUENCE $\langle S_\alpha : \alpha < \omega_1 \rangle$
SUCH THAT $S_\alpha \subseteq \mathcal{P}(\alpha)$ AND S_α IS COUNTABLE
FOR ALL α AND FOR EVERY $X \subseteq \omega_1$ THERE
IS A CUB SET $C \subseteq \omega_1$ SUCH THAT
$X \cap \alpha \in S_\alpha$ <u>AND $C \cap \alpha \in S_\alpha$</u> FOR ALL $\alpha \in C$.

CLEARLY $\diamondsuit^+$ IMPLIES $\diamondsuit^*$;
LESS CLEARLY, BUT TRUE $\diamondsuit^*$ IMPLIES $\diamondsuit$ (EXERCISE)

TWO IMPLICATIONS: [KUNEN VI.5.2 AND II.7.10]
$V = L$ IMPLIES $\diamondsuit^+$ AND $\diamondsuit^+$ IMPLIES KH.

ASSUME $\diamondsuit^+$. WE CONSTRUCT A KUREPA FAMILY.
WE CONSTRUCT $\mathcal{F} \subseteq \mathcal{P}(\omega_1)$ SUCH THAT
- $(\forall \beta < \omega_1)(|\{F \cap \beta : F \in \mathcal{F}\}| \leq \aleph_0)$
- $(\forall A \subseteq \omega_1)(|A| = \aleph_1 \to (\exists F \in \mathcal{F})(|F| = \aleph_1 \wedge F \subseteq A))$

FOR $C \subseteq \omega_1$ AND $\xi < \omega_1$ LET
$$s(C, \xi) = \sup((\xi+1) \cap (C \cup \{0\}))$$
IF C IS CUB THEN $s(C,\xi) = \max\{\gamma \in C \cup \{0\} : \gamma \leq \xi\}$.
FOR $A \subseteq \omega_1$ LET
$$F(A, C) = \{\xi \in A : A \cap [s(C,\xi), \xi) = \emptyset\}$$
SO IF $\beta \in C$ THEN $F(A,C)$ CONTAINS THE MINIMUM $\xi \in A$
SUCH THAT $s(C,\xi) = \beta$, IF ANY.
- CLEARLY $F(A,C) \subseteq A$
- IF $|A| = \omega_1$ AND C IS CUB THEN $|F(A,C)| = \aleph_1$

LET $\langle S_\alpha : \alpha < \omega_1 \rangle$ BE A $\diamondsuit^+$-SEQUENCE

LET $\mathcal{F}$ BE THE SET OF $F(A,C)$

FOR WHICH – $|A| = \aleph_1$, C IS CUB, $A, C \subseteq \omega_1$

– $(\forall \alpha \in C)(A \cap \alpha, C \cap \alpha \in S_\alpha)$.

IF $A \subseteq \omega_1$ AND $|A| = \aleph_1$ THEN THERE IS A CUB SET C SUCH THAT $A \cap \alpha, C \cap \alpha \in S_\alpha$ FOR ALL $\alpha \in C$.

THEN $F(A,C) \in \mathcal{F}$, $|F(A,C)| = \aleph_1$ AND $F(A,C) \subseteq A$.

NOW LET $\beta < \omega_1$ WE MUST SHOW THAT $\{F \cap \beta : F \in \mathcal{F}\}$ IS COUNTABLE.

LET $F(A,C) \in \mathcal{F}$ AND LET $\beta < \omega_1$

THEN $|F(A,C) \cap \beta| \leq 1$ OR

THERE ARE $\alpha \leq \beta$ AND $B, D \in S_\alpha$

SUCH THAT $F(A,C) \cap \beta = F(B,D) \cup \{x\}$

WHERE $|x| \leq 1$.

LET $\alpha = s(C, \beta)$

IF $\alpha > 0$ THEN LET $B = A \cap \alpha$ AND $D = C \cap \alpha$,

AND $\xi = \min A \setminus \xi$.

- THEN $[\alpha, \xi) \cap A = \emptyset$ $([\xi, \xi) = \emptyset)$

SO $\xi \in F(A,C)$

IF $\xi > \beta$ THEN $F(A,C) \cap \beta \subseteq A \cap \alpha = B$

BUT THEN $F(A,C) \cap \beta = F(B,D)$.

IF $\xi < \beta$ THEN $F(A,C) \cap \beta = (F(A,C) \cap \alpha) \cup \{\xi\}$

$= F(B,D) \cup \{\xi\}$.

IF $\alpha = 0$ THEN $F(A,C) = \emptyset$ OR $F(A,C) = \{\min A\}$

SO $\{F \cap \beta : F \in \mathcal{F}\} \subseteq \bigcup_{\alpha \leq \beta} \{F(B,D) : B, D \in S_\alpha\}$

$\cup \bigcup_{\alpha \leq \beta} \{F(B,D) \cup \{\xi\} : B, D \in S_\alpha, \xi \leq \beta\}$

$\cup \{\{\xi\} : \xi \leq \beta\}$

WHICH IS COUNTABLE.

AS $\diamondsuit^+ \to \diamondsuit^* \to \diamondsuit \to CH$ WE HAVE $|2^{<\omega_1}| = \aleph_1$

TAKE A BIJECTION $h : \omega_1 \to 2^{<\omega_1}$

AND FOR EVERY BRANCH b OF $2^{<\omega_1}$ LET

$F_b \in \mathcal{F}$ BE SUCH THAT $|F_b| = \aleph_1$ AND $h[F_b] \subseteq b$.

THEN $F_b \cap F_c$ IS COUNTABLE IF $b \neq c$

SO $|\mathcal{F}| \geq |2^{\omega_1}| = 2^{\aleph_1} \geq \aleph_2$.

SO FROM $\mathcal{F}$ WE CAN GET A KUREPA TREE WITH $2^{\aleph_1}$ BRANCHES.

$V = L$ IMPLIES $\diamondsuit^+$

DEFINE $f : \omega_1 \to \omega_1$ BY

$f(\alpha) = \min\{\beta : \beta > \alpha \wedge L_\beta \prec L_{\omega_1}\}$

AND LET $S_\alpha = \mathcal{P}(\alpha) \cap L_{f(\alpha)}$.

AS BEFORE f AND $\langle S_\alpha : \alpha < \omega_1 \rangle$ ARE DEFINABLE IN L_{ω_2}.

WE SHOW $\langle S_\alpha : \alpha < \omega_1 \rangle$ IS A $\diamondsuit^+$-SEQUENCE.

IF NOT LET A BE THE $<_L$-FIRST SET SUCH THAT FOR EVERY CUB $C \subseteq \omega_1$

THERE IS AN $\alpha \in C$ SUCH THAT

$A \cap \alpha \notin S_\alpha \vee C \cap \alpha \notin S_\alpha$

AGAIN A WOULD THEN BE DEFINABLE IN L_{ω_2}.

LET $N_0 \prec L_{\omega_2}$ BE THE SMALLEST ELEMENTARY SUBSTR. OF L_{ω_2}

SO $f, \langle S_\alpha : \alpha < \omega_1 \rangle, A \in N_0$

LET $N_{\nu+1} \prec L_{\omega_2}$ BE THE SMALLEST WITH $N_\nu \cup \{N_\nu\} \subseteq N_{\nu+1}$

LET $N_\nu = \bigcup_{\mu < \nu} N_\mu$ IF ν IS A LIMIT.

WE KNOW

- $N_\nu \cap L_{\omega_1} = L_{\alpha_\nu}$ FOR SOME $\alpha_\nu < \omega_1$

  $\langle \alpha_\nu : \nu < \omega_1 \rangle$ IS NORMAL $\quad \alpha_\nu = N_\nu \cap \omega_1$

- $\pi_\nu : N_\nu \cong L_{\beta(\nu)}$ TRANSITIVE COLLAPSE

  $\pi_\nu(x) = x \quad x \in L_{\alpha_\nu}$

  $\pi_\nu(\omega_1) = \alpha_\nu$

  $\pi_\nu(A) = A \cap \alpha_\nu$

- SO $\alpha_\nu < \beta(\nu)$

  NOW: IN $L_{\beta(\nu)}$ WE HAVE "α_ν IS UNCOUNTABLE"

  BECAUSE $\pi_\nu(\omega_1) = \alpha_\nu$

  IN $N_{\nu+1}$ WE HAVE "N_ν IS COUNTABLE"

  BECAUSE $N_\nu \in N_{\nu+1} \prec L_{\omega_2}$.

  SO $\alpha_\nu = N_\nu \cap \omega_1 \in N_{\nu+1}$

  AND $L_{\beta(\nu)} = \pi_\nu[N_\nu] \in N_{\nu+1}$ AS WELL

WE FIND $\beta(\nu) \in N_{\nu+1}$

AND SO $\alpha_\nu < \beta(\nu) < \alpha_{\nu+1}$ ALWAYS.

LET C BE THE SET OF LIMIT POINTS OF $\{\beta(\nu) : \nu < \omega_1\}$.

WE SHOW $A \cap \alpha$, $C \cap \alpha \in S_\alpha$ FOR ALL $\alpha \in C$.

LET $\alpha \in C$, SAY $\alpha = \sup_{\nu < \lambda} \beta(\nu)$

BECAUSE $\alpha_\nu < \beta(\nu) < \alpha_{\nu+1}$ FOR ALL ν

WE FIND $\alpha = \alpha_\lambda$ TOO.

ALSO $\beta(\alpha) < f(\alpha)$

- IN $L_{f(\alpha)}$ WE HAVE "α IS COUNTABLE"
- IN $L_{\beta(\alpha)}$ WE HAVE "$\alpha = \omega_1$"

  BECAUSE $\alpha = \pi_\lambda(\omega_1) = \omega_1^{L_{\beta(\alpha)}}$

  SO $L_{f(\alpha)}$ HAS A BIJECTION $\sigma : \omega \to \alpha$

  BUT $L_{\beta(\alpha)}$ DOES NOT.

- $A \cap \alpha_\lambda = \pi_\lambda(A) \in L_{\beta(\lambda)} \in L_{f(\alpha)}$

  SO $A \cap \alpha \in S_\alpha$

- NOW $C \cap \alpha \in L_{f(\alpha)}$

  IT SUFFICES TO SHOW $\langle \beta(\nu) : \nu < \lambda \rangle \in L_{f(\alpha)}$.

  AS $C \cap \alpha$ IS THE SET OF ITS LIMIT POINTS

  LET $\beta = \beta(\lambda)$

  WE HAVE SEEN $\beta < f(\alpha)$

  IN $L_{f(\alpha)}$ DEFINE A SEQUENCE $\langle M_\nu : \nu < \lambda' \rangle$

  - M_0 IS THE SMALLEST $M \prec L_\beta$
  - $M_{\nu+1}$ IS THE SMALLEST $M \prec L_\beta$ WITH $M_\nu \cup \{M_\nu\} \subseteq M$.
  - $M_\nu = \bigcup_{\mu < \nu} M_\mu$ ν LIMIT

  λ' IS SIMPLY THE FIRST ORDINAL WHERE THIS ENDS.

  IN $L_{f(\alpha)}$ WE HAVE $\pi'_\nu : M_\nu \cong L_{\beta'(\nu)}$ $(\nu < \lambda')$

  AND WE GET $\langle \beta'(\nu) : \nu < \lambda' \rangle \in L_{f(\alpha)}$.

  COMPARE THIS WITH $\langle N_\nu : \nu < \omega_1 \rangle$ IN L_{ω_2}.

  WE HAVE $N_\nu \prec N_\xi \prec L_{\omega_2}$ WHEN $\nu < \xi$

  IN PARTICULAR $N_\nu \prec N_\lambda \prec L_{\omega_2}$

  WHICH MEANS THAT WE GET THE SAME SEQUENCE $\langle N_\nu : \nu < \lambda \rangle$

IF WE HAD SPECIFIED

N_0: THE SMALLEST $N \prec N_\lambda$

$N_{\nu+1}$: THE SMALLEST $N \prec N_\lambda$

WITH $N_\nu \cup \{N_\nu\} \subseteq N$

$N_\nu = \bigcup_{\mu<\nu} N_\mu$ IF ν IS A LIMIT

WE HAD

$\pi_\lambda : N_\lambda \cong L_\beta$ $\quad (\beta = \beta(\lambda))$

BY INDUCTION

$\pi_\lambda : N_\nu \cong M_\nu$ $\quad \nu < \lambda$

EASY AT LIMITS:

SUCCESSOR USE THE DEFINABILITY

WE FIND $\lambda = \lambda'$ AND FOR $\nu < \lambda$

N_ν AND M_ν HAVE THE SAME

TRANSITIVE COLLAPSE $L_{\beta(\nu)}$

SO $\langle \beta(\nu) : \nu < \lambda \rangle = \langle \beta'(\nu) : \nu < \lambda' \rangle \in L_{f(\kappa)}$.

DONE!