Zeros of Bessel functions

The Bessel function $J_\nu(z)$ of the first kind of order $\nu \in \mathbb{R}$ can be written as

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\nu+k+1)\,k!} \left(\frac{z}{2}\right)^{2k}. \tag{1}$$

This is a solution of the Bessel differential equation which can be written as

$$z^2 y''(z) + z y'(z) + (z^2 - \nu^2) y(z) = 0, \quad \nu \in \mathbb{R}. \tag{2}$$

We will derive some basic facts about the zeros of the Bessel function $J_\nu(z)$ and its derivative $J_\nu'(z)$. We have

Theorem 1. *All zeros of $J_\nu(z)$, except $z = 0$ possibly, are simple.*

Proof. If $z_0 \neq 0$ is a multiple zero of $J_\nu(z)$, then we have at least that $J_\nu(z_0) = 0$ and $J_\nu'(z_0) = 0$. Since $z_0 \neq 0$ it follows from the differential equation (2) that also $J_\nu''(z_0) = 0$. Iteration then leads to $J_\nu^{(n)}(z_0) = 0$ for all $n \in \{0, 1, 2, \ldots\}$, which implies that $J_\nu(z)$ is identically zero. This is a trivial contradiction.

Theorem 2. *All zeros of $J_\nu'(z)$, except $z = 0$ or $z = \pm\nu$ possibly, are simple.*

Proof. If z_0 is a multiple zero of $J_\nu'(z)$, then we have at least that $J_\nu'(z_0) = 0$ and $J_\nu''(z_0) = 0$. For $z_0 \neq 0$ and $z_0 \neq \pm\nu$ it then follows from the differential equation (2) that also $J_\nu(z_0) = 0$. Again this leads to $J_\nu(z)$ being identically zero which is clearly not true.

Theorem 3. *If $z_0 \in \mathbb{C}$ is a zero of $J_\nu(z)$, then also $-z_0$ and $\pm\overline{z_0}$.*

Proof. Since this is trivial for $z_0 = 0$ we now assume that $z_0 \neq 0$. Then it follows from (1) that z_0 is a zero of

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\nu+k+1)\,k!} \left(\frac{z}{2}\right)^{2k}.$$

This series is even and has real coefficients. This implies that $-z_0$ and $\pm\overline{z_0}$ are zeros too.

Theorem 4. *If $z_0 \in \mathbb{C}$ is a zero of $J_\nu'(z)$, then also $-z_0$ and $\pm\overline{z_0}$.*

Proof. From (1) it follows that

$$J_\nu'(z) = \left(\frac{z}{2}\right)^{\nu-1} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{\nu}{2}+k\right)}{\Gamma(\nu+k+1)\,k!} \left(\frac{z}{2}\right)^{2k}.$$

Hence, if $z_0 \neq 0$ is a zero of $J_\nu'(z)$ it must be a zero of

$$\sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{\nu}{2}+k\right)}{\Gamma(\nu+k+1)\,k!} \left(\frac{z}{2}\right)^{2k},$$

which is even and also has real coefficients. This implies that $-z_0$ and $\pm\overline{z_0}$ are zeros too.

Lemma 1. *For $\nu > -1$ we have*

$$(a^2 - b^2) \int_0^z t J_\nu(at) J_\nu(bt)\, dt = z \left[b J_\nu(az) J_\nu'(bz) - a J_\nu'(az) J_\nu(bz) \right]. \tag{3}$$

Proof. The differential equation (2) implies that

$$c^2 z^2 J_\nu''(cz) + cz J_\nu'(cz) + \left(c^2 z^2 - \nu^2\right) J_\nu(cz) = 0, \quad c \in \mathbb{C}.$$

Hence we have

$$\begin{aligned} z \frac{d}{dz} & \left[bz J_\nu(az) J_\nu'(bz) - az J_\nu'(az) J_\nu(bz) \right] \\ &= bz J_\nu(az) J_\nu'(bz) + abz^2 J_\nu'(az) J_\nu'(bz) + b^2 z^2 J_\nu(az) J_\nu''(bz) \\ &\quad - az J_\nu'(az) J_\nu(bz) - abz^2 J_\nu'(az) J_\nu'(bz) - a^2 z^2 J_\nu''(az) J_\nu(bz) \\ &= \left(a^2 z^2 - \nu^2\right) J_\nu(az) J_\nu(bz) - \left(b^2 z^2 - \nu^2\right) J_\nu(az) J_\nu(bz) \\ &= (a^2 - b^2) z^2 J_\nu(az) J_\nu(bz). \end{aligned}$$

This implies that

$$\frac{d}{dz} \left[bz J_\nu(az) J_\nu'(bz) - az J_\nu'(az) J_\nu(bz) \right] = (a^2 - b^2) z J_\nu(az) J_\nu(bz),$$

which proves the lemma.

Theorem 5. *For $\nu \geq -1$ the Bessel function $J_\nu(z)$ only has real zeros.*

Proof. Since $\nu \in \mathbb{R}$ we have: if $z_0 \in \mathbb{C}$ is a zero of $J_\nu(z)$, so is $\overline{z_0}$. Now we apply (3) with $z = 1$, $a = z_0$ and $b = \overline{z_0}$ to find that

$$0 = (z_0^2 - \overline{z_0}^2) \int_0^1 t J_\nu(z_0 t) J_\nu(\overline{z_0} t)\, dt = (z_0^2 - \overline{z_0}^2) \int_0^1 t \left| J_\nu(z_0 t) \right|^2 dt.$$

This implies that $z_0^2 = \overline{z_0}^2$, which can only be true if $z_0 = x \in \mathbb{R}$ or $z_0 = iy$ with $y \in \mathbb{R}$. Note that for $z = iy$ with $y \in \mathbb{R}$ we have

$$\sum_{k=0}^\infty \frac{(-1)^k}{\Gamma(\nu + k + 1)\, k!} \left(\frac{z}{2}\right)^{2k} = \sum_{k=0}^\infty \frac{1}{\Gamma(\nu + k + 1)\, k!} \left(\frac{y}{2}\right)^{2k} > 0$$

for $\nu > -1$. This implies that $J_\nu(z)$ only has real zeros for $\nu > -1$. For $\nu = -1$ we have $J_{-1}(z) = -J_1(z)$, which implies that the theorem also holds for $\nu = -1$.

Theorem 6. *For $\nu \geq 0$ the derivative of the Bessel function $J_\nu'(z)$ only has real zeros.*

Proof. Since $\nu \in \mathbb{R}$ we have: if $z_0 \in \mathbb{C}$ is a zero of $J_\nu'(z)$, so is $\overline{z_0}$. As before (3) implies that $z_0 = x \in \mathbb{R}$ or $z_0 = iy$ with $y \in \mathbb{R}$. Note that for $z = iy$ with $y \in \mathbb{R}$ we have

$$\sum_{k=0}^\infty \frac{(-1)^k \left(\frac{\nu}{2} + k\right)}{\Gamma(\nu + k + 1)\, k!} \left(\frac{z}{2}\right)^{2k} = \sum_{k=0}^\infty \frac{\frac{\nu}{2} + k}{\Gamma(\nu + k + 1)\, k!} \left(\frac{y}{2}\right)^{2k} > 0$$

for $\nu \geq 0$.

Theorem 7. *Both $J_\nu(z)$ and $J_\nu'(z)$ have infinitely many positive zeros.*

Proof. Since all positive zeros of $J_\nu(z)$ are simple, this follows from the asymptotic behavior

$$J_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi \nu}{2} - \frac{\pi}{4}\right), \quad |z| \to \infty \quad \text{with} \quad |\arg(z)| \leq \pi - \delta < \pi.$$