Regularity

April 19, 2001

Contents

1 Averages and the theorem of Lebesgue.

1.1 Notation.

We consider the space $\mathbb{R}^n$ with Lebesgue measure m and we also write $dm(x) = dx$. For $x \in \mathbb{R}^n$ and $r > 0$ we define

$$B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$$

and $B_r = B(0,r)$; so $m(B(x,r)) = m(B_r)$. The closure of $B(x,r)$ is $\bar{B}(x,r)$. We denote by ω_n the $(n-1)$-dimensional surface area of the unit sphere ∂B_1; so $m(B_r) = n^{-1}\omega_n r^n$. For $F \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$ we denote by $\operatorname{dist}(x,F)$ the distance of x to F, i.e.,

$$\operatorname{dist}(x,F) = \inf\{|x-y| : y \in F\}.$$

Furthermore, for any non-empty $D \subseteq \mathbb{R}^n$ we denote the diameter by

$$\operatorname{diam}(D) = \sup\{|x-y| : x,y \in D\}.$$

If $\Omega \subseteq \mathbb{R}^n$ is an open subset, then we denote by $L^1_{loc}(\Omega)$ the space of all locally integrable functions on Ω, i.e., $L^1_{loc}(\Omega)$ consists of all measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ for which $\int_K |f|\, dx < \infty$ for all compact subsets $K \subseteq \Omega$ (with identification of functions which are equal a.e.). We denote by $C_c(\Omega)$ the space of all continuous functions on Ω with compact support in Ω and $C_c^\infty(\Omega)$ is the subspace of all C^∞-functions in $C_c(\Omega)$.

Suppose that $f : \mathbb{R}^n \to [0,\infty]$ is a measurable function. The distribution function $d_f : [0,\infty) \to [0,\infty]$ is defined by

$$d_f(\lambda) = m\{x \in \mathbb{R}^n : f(x) > \lambda\}.$$

It is clear that d_f is decreasing and right-continuous. Moreover, if $d_f(\lambda_0) < \infty$ for some $\lambda_0 \geq 0$, then $d_f(\lambda) \to 0$ as $\lambda \to \infty$. If $f \in L^1(\mathbb{R}^n)$ then

$$d_{|f|}(\lambda) \leq \frac{1}{\lambda}\|f\|_1 \tag{1}$$

for all $\lambda > 0$. Indeed, with $E_\lambda = \{x \in \mathbb{R}^n : |f(x)| > \lambda\}$ we have

$$\|f\|_1 \geq \int_{E_\lambda} |f(x)|\, dx \geq \lambda m(E_\lambda) = \lambda d_{|f|}(\lambda).$$

1.2 Averages.

Let $f \in L^1_{loc}(\mathbb{R}^n)$ be given. For $x \in \mathbb{R}^n$ and $r > 0$ we define

$$\bar{f}_{x,r} = \frac{1}{m(B_r)} \int_{B(x,r)} f(y)\, dy. \tag{2}$$

Remark 1.1 *1. For fixed $r > 0$, the function $x \longmapsto f_{x,r}$ is continuous. This follows immediately from the dominated convergence theorem.*

2. Suppose that $\Omega \subseteq \mathbb{R}^n$ is open and $f \in L^1_{loc}(\mathbb{R}^n)$, then $f_{|\Omega} \in C(\Omega)$ if and only if $f_{x,r} \to f(x)$ as $r \downarrow 0$ uniformly (a.e.) on every compact subset of Ω. Indeed, first suppose that $f_{|\Omega} \in C(\Omega)$ and let $K \subseteq \Omega$ be compact. There exists $\delta > 0$ such that the set

$$K_\delta = \{x \in \mathbb{R}^n : \operatorname{dist}(x, K) \leq \delta\}$$

is contained in Ω. For $0 < r < \delta$ we have

$$\begin{aligned}
\sup_{x \in K} \left|\bar{f}_{x,r} - f(x)\right| &\leq \sup_{x \in K} \frac{1}{m(B_r)} \int_{B(x,r)} |f(y) - f(x)|\, dy \\
&\leq \sup_{x \in K} \sup_{y \in B(x,r)} |f(y) - f(x)| \\
&\leq \sup \{|f(y) - f(x)| : x, y \in K_\delta, |y - x| < r\},
\end{aligned}$$

which converges to zero as $r \downarrow 0$ since f is uniformly continuous on the compact set K_δ. The converse statement is clear from the above observation.

3. Now assume $f \in L^1(\mathbb{R}^n)$ that and write $f_r(x) = \bar{f}_{x,r}$. We claim that $\|f_r - f\|_1 \to 0$ as $r \downarrow 0$. Indeed,

$$\begin{aligned}
\int_{\mathbb{R}^n} |f_r(x) - f(x)|\, dx &\leq \frac{1}{m(B_r)} \int_{\mathbb{R}^n} \left(\int_{B(x,r)} |f(y) - f(x)|\, dy \right) dx \\
&= \frac{1}{m(B_r)} \int_{\mathbb{R}^n} \left(\int_{B_r} |f(y + x) - f(x)|\, dy \right) dx \\
&= \frac{1}{m(B_r)} \int_{B_r} \left(\int_{\mathbb{R}^n} |f(y + x) - f(x)|\, dx \right) dy \\
&\leq \sup_{y \in B_r} \int_{\mathbb{R}^n} |f(y + x) - f(x)|\, dx \\
&= \sup_{y \in B_r} \|\tau_y f - f\|_1 .
\end{aligned}$$

If $f \in C_c(\mathbb{R}^n)$ then it follows by uniform continuity that

$$\sup_{y \in B_r} \|\tau_y f - f\|_1 \to 0 \quad \text{as } r \downarrow 0. \tag{3}$$

Since $C_c(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$ this implies that (3) holds for all $f \in L^1(\mathbb{R}^n)$, which proves the claim.

It follows in particular from this last remark that for every $f \in L^1(\mathbb{R}^n)$ there exists some sequence $r_k \downarrow 0$ such that $f_{x,r_k} \to f(x)$ for almost all $x \in \mathbb{R}^n$. This is however not enough to conclude that $f_{x,r} \to f(x)$ as $r \downarrow 0$ for almost all $x \in \mathbb{R}^n$, which is Lebesgue's theorem. For the proof of this we need some preparations.

1.3 The maximal function.

For $f \in L^1_{loc}(\mathbb{R}^n)$ we define the maximal function $Mf : \mathbb{R}^n \to [0, \infty]$ by

$$Mf(x) = \sup_{r>0} \frac{1}{m(B_r)} \int_{B(x,r)} |f(y)|\, dy. \tag{4}$$

The mapping $M : f \longmapsto Mf$ is called the Hardy-Littlewood maximal operator. Since Mf is the pointwise supremum of continuous functions it follows that Mf is lower semi-continuous. In particular Mf is a measurable function. The following simple properties of the operator M are easily verified:

1. $M(f+g) \leq Mf + Mg$ and $M(\lambda f) = |\lambda| Mf$ for all $f, g \in L^1_{loc}(\mathbb{R}^n)$ and all $\lambda \in \mathbb{C}$;
2. if $|f| \leq |g|$ a.e. in $L^1_{loc}(\mathbb{R}^n)$ then $Mf \leq Mg$;
3. $\|Mf\|_\infty \leq \|f\|_\infty$ for all $f \in L^\infty(\mathbb{R}^n)$.

Observe that $Mf \in L^1(\mathbb{R}^n)$ implies that $f = 0$ a.e. Indeed, if $0 \neq f \in L^1_{loc}(\mathbb{R}^n)$ then there exists a constant $c > 0$ such that $Mf(x) \geq c|x|^{-n}$ as $|x| \to \infty$, so $Mf \notin L^1(\mathbb{R}^n)$. Furthermore, Mf need not be locally integrable in general.

1.4 A covering lemma.

Our next objective is to estimate the size of the maximal function. For this we need a Vitali type covering lemma.

Lemma 1.2 *Suppose that K is a compact subset of $\mathbb{R}^n$ and that $\mathcal{B}$ is a collection of open balls such that $K \subseteq \bigcup \{B : B \in \mathcal{B}\}$. Then there exist finitely many disjoint balls $B_1, \ldots, B_k \in \mathcal{B}$ such that*

$$m(K) \leq 3^n \sum_{j=1}^{k} m(B_j).$$

Proof. Since K is compact we may assume without loss of generality that $\mathcal{B}$ is finite. Let B_1 be a ball in $\mathcal{B}$ with largest radius r_1. Then take $B_2 \in \mathcal{B}$ disjoint from B_1 with largest radius r_2. Let B_3 be the largest ball with radius r_3 in $\mathcal{B}$ disjoint from B_1 and B_2, etc. Since $\mathcal{B}$ is finite this process stops after finitely many, say k steps. By construction the balls $B_1, \ldots, B_k$ are disjoint. Let $\tilde{B}_j$ be the ball with the same center as B_j and with radius $3r_j$. We claim that

$$K \subseteq \bigcup_{j=1}^{k} \tilde{B}_j.$$

Indeed, suppose not. Then there exists $x \in K$ such that $x \notin \tilde{B}_j$ for all $j = 1, \ldots, k$. Since $\mathcal{B}$ covers K, there exists $B \in \mathcal{B}$ with radius r such that $x \in B$. By the choice of B_1 we have $r \leq r_1$. Since $x \notin \tilde{B}_1$ it follows that $B \cap B_1 = \emptyset$. Hence, by the choice of B_2 we have $r \leq r_2$. Since $x \notin \tilde{B}_2$ this implies that $B \cap B_2 = \emptyset$. Continuing this way it follows that B is disjoint with all balls $B_1, \ldots, B_k$ and that $r \leq r_k$. This is clearly a contradiction, which proves the claim. Since $m\left(\tilde{B}_j\right) = 3^n m(B_j)$, we conclude that

$$m(K) \leq \sum_{j=1}^{k} m\left(\tilde{B}_j\right) = 3^n \sum_{j=1}^{k} m(B_j)$$

by which the lemma is proved. ■

1.5 Lebesgue's differentiation theorem.

Using the covering lemma we can now give a weak type estimate for the maximal function. For the case $n = 1$ this result is due to F. Riesz (1932); the general case was proved by N. Wiener (1939).

Theorem 1.3 *If $f \in L^1(\mathbb{R}^n)$ then*

$$m\{x \in \mathbb{R}^n : Mf(x) > \lambda\} \leq \frac{C_n}{\lambda} \|f\|_1 \tag{5}$$

for all $\lambda > 0$ (where C_n is a constant only depending on the dimension n).

Proof. Take $\lambda > 0$ and put $E_\lambda = \{x \in \mathbb{R}^n : Mf(x) > \lambda\}$. Let $K \subseteq E_\lambda$ be compact. For each $x \in K$ there exists a ball $B(x, r_x)$ such that

$$\frac{1}{m(B_{r_x})} \int_{B(x,r_x)} |f(y)|\, dy > \lambda. \tag{6}$$

Now apply Lemma 1.2 to the set K and $\mathcal{B} = \{B(x, r_x) : x \in K\}$. Hence there exist disjoint balls $B(x_1, r_1), \ldots, B(x_k, r_k)$ in $\mathcal{B}$ such that

$$m(K) \le 3^n \sum_{j=1}^{k} m(B(x_j, r_j)).$$

Hence it follows from (6) that

$$\begin{aligned} m(K) &\le 3^n \sum_{j=1}^{k} m\left(B_{r_j}\right) < \frac{3^n}{\lambda} \sum_{j=1}^{k} \int_{B(x_j,r_j)} |f(y)|\, dy \\ &\le \frac{3^n}{\lambda} \int_{\mathbb{R}^n} |f(y)|\, dy = \frac{3^n}{\lambda} \|f\|_1 . \end{aligned}$$

Since $m(E_\lambda) = \sup\{m(K) : K \text{ is compact and } K \subseteq E_\lambda\}$, this implies (5). ■

Using the above weak estimate on Mf we can prove the Lebesgue differentiation theorem.

Theorem 1.4 *If* $f \in L^1(\mathbb{R}^n)$ *then*

$$\lim_{r \downarrow 0} \frac{1}{m(B_r)} \int_{B(x,r)} |f(y) - f(x)|\, dy = 0 \tag{7}$$

for almost all $x \in \mathbb{R}^n$.

Proof. For $f \in L^1(\mathbb{R}^n)$and $x \in \mathbb{R}^n$ we define

$$Nf(x) = \limsup_{r \downarrow 0} \frac{1}{m(B_r)} \int_{B(x,r)} |f(y) - f(x)|\, dy,$$

so $Nf : \mathbb{R}^n \to [0, \infty]$ and a moment's reflection shows that Nf is measurable. We have to show that $Nf = 0$ a.e. on $\mathbb{R}^n$. First we show that Nf satisfies a weak estimate. Take $\lambda > 0$. Since

$$Nf(x) \le Mf(x) + |f(x)|$$

for all x, it follows that

$$\{x \in \mathbb{R}^n : Nf(x) > \lambda\} \subseteq \left\{x \in \mathbb{R}^n : Mf(x) > \frac{\lambda}{2}\right\} \cup \left\{x \in \mathbb{R}^n : |f(x)| > \frac{\lambda}{2}\right\}$$

and so via Theorem 1.3 and (1) we find

$$\begin{aligned} m\{x \in \mathbb{R}^n : Nf(x) > \lambda\} &\leq m\left\{x \in \mathbb{R}^n : Mf(x) > \frac{\lambda}{2}\right\} \\ &\quad + m\left\{x \in \mathbb{R}^n : |f(x)| > \frac{\lambda}{2}\right\} \qquad (8) \\ &\leq \frac{2C_n}{\lambda}\|f\|_1 + \frac{2}{\lambda}\|f\|_1 = \frac{C}{\lambda}\|f\|_1 . \end{aligned}$$

Now fix $f \in L^1(\mathbb{R}^n)$. For every $g \in C_c(\mathbb{R}^n)$ it follows via uniform continuity that $Ng = 0$. Moreover it is clear that N is subadditive and so

$$Nf \leq N(f-g) + Ng = N(f-g).$$

Take $\lambda > 0$. Then it follows from (8)

$$\begin{aligned} m\{x \in \mathbb{R}^n : Nf(x) > \lambda\} &\leq m\{x \in \mathbb{R}^n : N(f-g)(x) > \lambda\} \\ &\leq \frac{C}{\lambda}\|f-g\|_1 . \end{aligned}$$

Since this holds for all $g \in C_c(\mathbb{R}^n)$ and since $C_c(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$ this implies that $m\{x \in \mathbb{R}^n : Nf(x) > \lambda\} = 0$ for all $\lambda > 0$, hence $Nf = 0$ a.e. This completes the proof of the theorem. ■

Corollary 1.5 *Let Ω be an open subset of $\mathbb{R}^n$ and $f \in L^1_{loc}(\Omega)$. Then (7) holds for almost all $x \in \Omega$.*

Proof. Let $\{\Omega_k\}_{k=1}^{\infty}$ be a sequence of open subset of Ω such that $\Omega = \bigcup_{k=1}^{\infty}\Omega_k$, $\overline{\Omega}_k \subseteq \Omega_{k+1}$ and $\overline{\Omega}_k$ is compact for all k. Define $f_k \in L^1(\mathbb{R}^n)$ by $f_k = f$ on $\overline{\Omega}_k$ and $f_k = 0$ on $\mathbb{R}^n \setminus \overline{\Omega}_k$. Applying theorem 1.4 to the function f_k we see that (7) holds a.e. on Ω_k and we are done. ■

The following corollary is now clear.

Corollary 1.6 *Let Ω be an open subset of $\mathbb{R}^n$ and $f \in L^1_{loc}(\Omega)$. Then the following statements hold.*

1. *For almost all $x \in \Omega$ we have*

$$\lim_{r \downarrow 0} \frac{1}{m(B_r)} \int_{B(x,r)} f(y)\,dy = f(x). \qquad (9)$$

2. $|f(x)| \leq Mf(x)$ *a.e. on* Ω.

The above results motivate the following definition.

Definition 1.7 *Suppose that* $\Omega \subseteq \mathbb{R}^n$ *is open and that* $f \in L^1_{loc}(\Omega)$*. The set of all points* $x \in \Omega$ *for which*

$$\lim_{r \downarrow 0} \frac{1}{m(B_r)} \int_{B(x,r)} |f(y) - f(x)| \, dy = 0$$

is called the Lebesgue set of the function f*. We denote this set by* $\mathcal{L}_f$*.*

If $f \in L^1_{loc}(\Omega)$ and $x \in \mathcal{L}_f$, then it is clear that (9) holds. In some situations we will need a differentiation formula like (9), but with respect to sets different from balls with center x, e.g. arbitrary balls containing the point x or with respect to cubes. For this purpose we introduce the following concept.

Definition 1.8 *A non-empty family* $\mathcal{Q}_0$ *of measurable subsets of* $\mathbb{R}^n$ *is called regular if*

(i). $m(Q) > 0$ *for all* $Q \in \mathcal{Q}_0$*;*

(ii). *for every* $\varepsilon > 0$ *there exists a set* $Q \in \mathcal{Q}_0$ *such that* $m(Q) < \varepsilon$*;*

(iii). *there exists a constant* $c > 0$ *such that for every* $Q \in \mathcal{Q}_0$ *there exists an open ball* $B_r = B(0,r)$ *such that* $Q \subseteq B_r$ *and* $m(B_r) \leq cm(Q)$*.*

For $x \in \mathbb{R}^n$ *we define* $\mathcal{Q}_x = \{Q + x : Q \in \mathcal{Q}_0\}$*.*

Observe that if $\mathcal{Q}_0$ is a regular family, then for every $r > 0$ there exists $\delta > 0$ such that $Q \in \mathcal{Q}_0$ and $m(Q) < \delta$ imply that $Q \subseteq B_r$. Indeed it follows from (iii) that given $r > 0$ we can take $\delta = c^{-1} m(B_1) r^n$.

Example 1.9 *The following families of subsets of* $\mathbb{R}^n$ *are regular families.*

1. *The collection of all open balls* $\{B(0,r) : r > 0\}$*;*

2. *The collection of all open (or closed) balls containing the point* 0*;*

3. *The family of all open (or closed) cubes containing the point* 0*;*

4. *Take any bounded set* $Q_1 \subseteq \mathbb{R}^n$ *with* $m(Q_1) > 0$ *and define* $\mathcal{Q}_0 = \{\alpha Q_1 : 0 < \alpha \in \mathbb{R}\}$*.*

The collection of all rectangles containing the point 0 *is not a regular family.*

The following proposition indicates the importance of the Lebesgue set.

Proposition 1.10 *Let* $\mathcal{Q}_0$ *be a regular family of subsets in* $\mathbb{R}^n$ *and* $\Omega \subseteq \mathbb{R}^n$ *an open subset. If* $f \in L^1_{loc}(\nleq)$ *then for all* $x \in \mathcal{L}_f$ *we have*

$$\lim_{\substack{m(Q)\to 0 \\ Q\in\mathcal{Q}_x}} \frac{1}{m(Q)} \int_Q |f(y) - f(x)| = 0.$$

Consequently,for all $x \in \mathcal{L}_f$.

$$\lim_{\substack{m(Q)\to 0 \\ Q\in\mathcal{Q}_x}} \frac{1}{m(Q)} \int_Q f(y) = f(x)$$

Proof. Take $x \in \mathcal{L}_f$. Given $\varepsilon > 0$ there exists $r_0 > 0$ such that $B(x,r) \subseteq \Omega$ and

$$\frac{1}{m(B_r)} \int_{B(x,r)} |f(y) - f(x)|\, dy < \varepsilon$$

for all $0 < r < r_0$. Define $\delta = c^{-1} m(B_1)\, r_0^n$ and take $Q \in \mathcal{Q}_x$ such that $m(Q) < \delta$. Then there exists $r > 0$ such that $Q \subseteq B(x,r)$ and $m(B_r) \leq cm(Q)$. By the choice of δ this implies that $r < r_0$ and so

$$\begin{aligned} \frac{1}{m(Q)} \int_Q |f(y) - f(x)|\, dy &\leq \frac{1}{m(Q)} \int_{B(x,r)} |f(y) - f(x)|\, dy \\ &\leq \frac{c}{m(B_r)} \int_{B(x,r)} |f(y) - f(x)|\, dy < c\varepsilon. \end{aligned}$$

This suffices to prove the proposition. ■

2 Riesz potentials.

It will be convenient to have some results concerning the so-called Riesz potentials available. We will discuss only some of the elementary properties.

For $0 < \alpha < n$ and measurable function $f : \mathbb{R}^n \to \mathbb{C}$ the Riesz potentials are defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy \tag{10}$$

whenever this integral is absolutely convergent for almost all $x \in \mathbb{R}^n$.

Remark 2.1 *Frequently the Riesz potentials are defined by* $\hat{I}_\alpha f = \gamma(\alpha)^{-1} I_\alpha f$ *with*

$$\gamma(\alpha) = \frac{\pi^{\frac{n}{2}} 2^\alpha \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n}{2} - \frac{\alpha}{2}\right)}.$$

The reason for this normalization is that we then have (at least for very smooth functions)

$$\hat{I}_\alpha\left(\hat{I}_\beta f\right) = \hat{I}_{\alpha+\beta} f$$

whenever $0 < \alpha < n$, $0 < \beta < n$ *with* $\alpha + \beta < n$.

In the proof of the next proposition we will use the following elementary fact. For the sake of completeness we indicate the proof.

Lemma 2.2

1. *If* $f \in L^1(\mathbb{R}^n)$ *and* $g \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, *then the convolution integral*

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)\, g(x-y)\, dy \tag{11}$$

   is absolutely convergent for almost all x.

2. *If* $f \in L^q(\mathbb{R}^n)$ *and* $g \in L^p(\mathbb{R}^n)$, *with* $1 \leq p \leq \infty$ *and* $p^{-1} + q^{-1} = 1$, *then (11) is absolutely convergent for all* x.

Proof.

1. If $p = \infty$ the statement is clear, so we assume that $1 \leq p < \infty$. Let q be the conjugate exponent of p. It follows from Hölder's inequality that

$$\begin{aligned} \int_{\mathbb{R}^n} |f(y)\, g(x-y)|\, dy &= \int_{\mathbb{R}^n} |f(y)|^{\frac{1}{q}} |f(y)|^{\frac{1}{p}} |g(x-y)|\, dy \\ &\leq \|f\|_1^{\frac{1}{q}} \left(\int_{\mathbb{R}^n} |f(y)|\, |g(x-y)|^p\, dy\right)^{\frac{1}{p}}. \end{aligned}$$

   By Fubini's theorem we have

$$\begin{aligned} &\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(y)\, g(x-y)|\, dy\right)^p dx \\ \leq\ & \|f\|_1^{\frac{p}{q}} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(y)|\, |g(x-y)|^p\, dy\right) dx \\ =\ & \|f\|_1^{\frac{p}{q}} \int_{\mathbb{R}^n} |f(y)| \left(\int_{\mathbb{R}^n} |g(x-y)|^p\, dx\right) dy \\ =\ & \|f\|_1^{\frac{p}{q}} \int_{\mathbb{R}^n} |f(y)| \left(\int_{\mathbb{R}^n} |g(x)|^p\, dx\right) dy \\ =\ & \|f\|_1^p \|g\|_p^p < \infty. \end{aligned}$$

This shows that $\int_{\mathbb{R}^n} |f(y) g(x-y)| \, dy < \infty$ for almost all $x \in \mathbb{R}^n$. Observe that this proof also shows that $f * g \in L^p(\mathbb{R}^n)$ and $\|f * g\|_p \leq \|f\|_1 \|g\|_p$.

2. This is an immediate consequence of Hölder's inequality.

■

Proposition 2.3 *If $0 < \alpha < n$ and $1 \leq p < \frac{n}{\alpha}$, then for every $f \in L^p(\mathbb{R}^n)$ the integral (10) is absolutely convergent for almost all $x \in \mathbb{R}^n$. Consequently, for $f \in L^p(\mathbb{R}^n)$ the Riesz potential $I_\alpha f$ is a well defined measurable function.*

Proof. Define

$$K(x) = \frac{1}{|x|^{n-\alpha}}.$$

Then we can write

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(x-y)}{|y|^{n-\alpha}} dy = (K * f)(x)$$

and we have to show the a.e. absolute convergence of this integral. We split K in two parts defined by

$$K_1(y) = \begin{cases} K(y) & \text{if } |y| \leq 1 \\ 0 & \text{if } |y| > 1 \end{cases}$$

and

$$K_2(y) = \begin{cases} K(y) & \text{if } |y| > 1 \\ 0 & \text{if } |y| \leq 1 \end{cases}.$$

Since $K = K_1 + K_2$ it is sufficient to show that both integrals $(K_1 * f)(x)$ and $(K_2 * f)(x)$ are absolutely convergent for almost all $x \in \mathbb{R}^n$. Since $n - \alpha < n$ it follows that $K_1 \in L^1(\mathbb{R}^n)$, which implies that the integral $(K_1 * f)(x)$ is absolutely convergent a.e. for all $f \in L^p(\mathbb{R}^n)$.

Let q be the conjugate exponent of p, i.e., $p^{-1} + q^{-1} = 1$. We claim that $K_2 \in L^q(\mathbb{R}^n)$. Indeed, if $p = 1$ then $q = \infty$ and it is clear that $K_2 \in L^\infty(\mathbb{R}^n)$. Now assume that $p > 1$. Then

$$\int_{\mathbb{R}^n} K_2(y)^q \, dy = \int_{|y|>1} \frac{1}{|y|^{(n-\alpha)q}} dy = \omega_n \int_1^\infty \frac{1}{r^{(n-\alpha)q-n+1}} dr.$$

Since $q^{-1} = 1 - p^{-1} < 1 - \frac{\alpha}{n}$, it follows that $(n-\alpha) q > n$, which implies that this integral is finite. Hence $K_2 \in L^q(\mathbb{R}^n)$. Now we may conclude that for every $f \in L^p(\mathbb{R}^n)$ the integral $(K_2 * f)(x)$ is convergent for almost all x, and the proof is complete. ■

Remark 2.4 *If $0 < \alpha < n$ and $1 < p < q < \infty$, with $q^{-1} = p^{-1} - \frac{\alpha}{n}$ (which implies that $p < \frac{n}{\alpha}$), then it can be shown that $I_\alpha f \in L^q(\mathbb{R}^n)$ for all $f \in L^p(\mathbb{R}^n)$.*

The following estimate will be useful.

Lemma 2.5 *Let $E \subseteq \mathbb{R}^n$ be a measurable set with $0 < m(E) < \infty$ and suppose that $0 < \alpha < n$. There is a constant $C(\alpha, n) > 0$ such that*

$$\int_E \frac{1}{|y-x|^{n-\alpha}} dy \leq C(\alpha, n)\, m(E)^{\frac{\alpha}{n}}$$

for all $x \in \mathbb{R}^n$. Actually we can take

$$C(\alpha, n) = \alpha^{-1} n^{\frac{\alpha}{n}} \omega_n^{1-\frac{\alpha}{n}}.$$

Proof. Fix $x \in \mathbb{R}^n$. Take $r > 0$ such that $m(B_r) = m(E)$, i.e., $r^n = n\omega_n^{-1} m(E)$. Note that

$$m(E \setminus B(x,r)) = m(B(x,r) \setminus E).$$

Now

$$\begin{aligned} \int_{E\setminus B(x,r)} \frac{1}{|y-x|^{n-\alpha}} dy &\leq r^{\alpha-n} m(E \setminus B(x,r)) = r^{\alpha-n} m(B(x,r) \setminus E) \\ &\leq \int_{m(B(x,r)\setminus E)} \frac{1}{|y-x|^{n-\alpha}} dy. \end{aligned}$$

This implies that

$$\begin{aligned} \int_E \frac{1}{|y-x|^{n-\alpha}} dy &= \int_{E\setminus B(x,r)} \frac{1}{|y-x|^{n-\alpha}} dy + \int_{E\cap B(x,r)} \frac{1}{|y-x|^{n-\alpha}} dy \\ &\leq \int_{m(B(x,r)\setminus E)} \frac{1}{|y-x|^{n-\alpha}} dy + \int_{E\cap B(x,r)} \frac{1}{|y-x|^{n-\alpha}} dy \\ &= \int_{B(x,r)} \frac{1}{|y-x|^{n-\alpha}} dy = \alpha^{-1} \omega_n r^\alpha \\ &= \alpha^{-1} \omega_n \left(n^{-1} \omega_n^{-1} m(E)\right)^{\frac{\alpha}{n}} = C(\alpha, n)\, m(E)^{\frac{\alpha}{n}}. \end{aligned}$$

■

If E is a measurable subset of $\mathbb{R}^n$ then any $f \in L^p(E)$ can be considered as an element of $L^p(\mathbb{R}^n)$, extending f identically equal to zero on the complement of E. In particular if $m(E) < \infty$, then every $f \in L^p(E)$ belongs to $L^1(E)$ and hence is an element of $L^1(\mathbb{R}^n)$. Therefore, by Proposition 2.3 the Riesz potential $I_\alpha f$ is an a.e. well defined measurable function (on $\mathbb{R}^n$) for all $0 < \alpha < n$. The following proposition is only of interest if $n > 1$.

Proposition 2.6 *Let $E \subseteq \mathbb{R}^n$ be measurable with $0 < m(E) < \infty$. Then there exists a constant $C_n > 0$, only depending on n, such that*

$$\int_E |I_1 f|^p \, dx \leq C_n^p m(E)^{\frac{p}{n}} \int_E |f|^p \, dx$$

for all $f \in L^p(E)$ and all $1 \leq p < \infty$.

Proof. First we assume that $1 < p < \infty$. Let q be the conjugate exponent of p. For $f \in L^p(E)$ it follows from Hölder's inequality that

$$\begin{aligned} |I_1 f(x)| &\leq \int_E \frac{|f(y)|}{|x-y|^{n-1}} dy = \int_E \frac{|f(y)|}{|x-y|^{\frac{n-1}{p}}} \frac{1}{|x-y|^{\frac{n-1}{q}}} dy \\ &\leq \left(\int_E \frac{|f(y)|^p}{|x-y|^{n-1}} dy \right)^{\frac{1}{p}} \left(\int_E \frac{1}{|x-y|^{n-1}} dy \right)^{\frac{1}{q}} \end{aligned}$$

for almost every $x \in \mathbb{R}^n$. By the above lemma we know that

$$\int_E \frac{1}{|x-y|^{n-1}} dy \leq C_n m(E)^{\frac{1}{n}} = C. \tag{12}$$

Hence,

$$|I_1 f(x)| \leq C^{\frac{1}{q}} \left(\int_E \frac{|f(y)|^p}{|x-y|^{n-1}} dy \right)^{\frac{1}{p}} \tag{13}$$

a.e. on $\mathbb{R}^n$. Note that this inequality trivially holds for $p = 1$. Using (12) and (13) we find for all $1 \leq p < \infty$ that

$$\begin{aligned} \int_E |I_1 f(x)|^p \, dx &\leq C^{\frac{p}{q}} \int_E \left(\int_E \frac{|f(y)|^p}{|x-y|^{n-1}} dy \right) dx \\ &= C^{p-1} \int_E |f(y)|^p \left(\int_E \frac{1}{|x-y|^{n-1}} dx \right) dy \\ &\leq C^p \int_E |f(y)|^p \, dy \end{aligned}$$

with $C = C_n m(E)^{\frac{1}{n}}$, which completes the proof. ■

3 Sobolev spaces.

In this section we will discuss some of the relevant properties of the Sobolev spaces which we will need in the sequel.

3.1 Definitions.

By α we denote a multi-index, i.e., $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_k \in \mathbb{N}$. The length of α is defined by $|\alpha| = \alpha_1 + \cdots + \alpha_n$. For such a multi-index α we define the differential operator D^α by

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

Let Ω be an open subset of . If $f \in L^1_{loc}(\nleq)$ and α is a multi-index and if there exists $g \in L^1_{loc}(\nleq)$ such that

$$\int_\Omega f D^\alpha \varphi dx = (-1)^{|\alpha|} \int_\Omega g \varphi dx$$

for all $\varphi \in C_c^\infty(\nleq)$, then we denote $g = D^\alpha f$, and g is called the weak (partial) derivative of f corresponding to the multi-index α. If $D^\alpha f$ exists, it is uniquely determined up to sets of measure zero. If $m \in \mathbb{N}$ and $f \in C^m(\Omega)$, then all weak derivatives $D^\alpha f$ exist for $|\alpha| \leq m$ and coincide with the classical derivatives of f. In case $|\alpha| = 1$ we denote the weak derivatives $D^\alpha f$ also by $D_j f = \frac{\partial f}{\partial x_j}$.

Remark 3.1 1. *Suppose that $f \in L^1_{loc}(\nleq)$, where $\Omega \subseteq \mathbb{R}^n$ is open. Now consider the collection $\mathcal{U}$ of all open subsets $U \subseteq \Omega$ such that $f = 0$ a.e. on U. Let $O = \bigcup \{U : U \in \mathcal{U}\}$. Then $f = 0$ a.e. on O (since $f = 0$ a.e. on every compact subset of O). Hence O is the largest open subset of Ω on which $f = 0$ a.e. The set $\Omega \diagdown U$ is called the (essential) support of f and will be denoted by* $\operatorname{supp}(f)$. *If f has a weak derivative $D^\alpha f$, then* $\operatorname{supp}(D^\alpha f) \subseteq \operatorname{supp}(f)$. *Indeed, if $U \subseteq \Omega$ is open and $f = 0$ on U, then*

$$\int_\Omega (D^\alpha f)\, \varphi dx = (-1)^{|\alpha|} \int_\Omega f D^\alpha \varphi dx = 0$$

for all $\varphi \in C_c^\infty(U)$, which implies that $D^\alpha f = 0$ a.e. on U.

2. *Suppose that $f \in L^1_{loc}(\nleq)$, where $\Omega \subseteq \mathbb{R}^n$ is open with weak derivative $D^\alpha f$. Suppose that U is an open subset of Ω. Then $f_{|U}$ has weak derivative $D^\alpha \left(f_{|U}\right) = (D^\alpha f)_{|U}$.*

3. *If $f \in L^1_{loc}(\nleq)$ has weak derivative $\frac{\partial f}{\partial x_j}$ and $\varphi \in C_c^\infty(\Omega)$, then $f\varphi$ has weak derivative*

$$\frac{\partial}{\partial x_j}(f\varphi) = \frac{\partial f}{\partial x_j}\varphi + f\frac{\partial \varphi}{\partial x_j}.$$

Definition 3.2 *For* $1 \leq p \leq \infty$ *and* $k \in \mathbb{N}$ *the Sobolev spaces* $W^{k,p}(\Omega)$ *are defined by*

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \text{ for all } |\alpha| \leq k\}.$$

For $u \in W^{k,p}(\Omega)$ *the norm is defined by*

$$\|u\|_{k,p} = \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|_p^p \right)^{\frac{1}{p}}$$

if $1 \leq p < \infty$ *and*

$$\|u\|_{k,\infty} = \max_{|\alpha| \leq k} \|D^\alpha u\|_\infty .$$

For all the values of p and k the space $\left(W^{k,p}(\Omega), \|\cdot\|_{k,p}\right)$ is a Banach space, as is easily verified. From now on we will only consider $1 \leq p < \infty$. The Hilbert space $W^{k,2}(\Omega)$ is sometimes also denoted by $H^k(\Omega)$. It is clear that $C_c^\infty(\Omega)$ is a linear subspace of $W^{k,p}(\Omega)$.

Definition 3.3 *For* $1 \leq p < \infty$ *and* $k \in \mathbb{N}$ *the closure of* $C_c^\infty(\Omega)$ *in* $W^{k,p}(\Omega)$ *is denoted by* $W_0^{k,p}(\Omega)$.

Furthermore we will denote by $W_c^{k,p}(\Omega)$ the subspace of $W^{k,p}(\Omega)$ consisting of all $u \in W^{k,p}(\Omega)$ for which $\operatorname{supp}(u)$ is compact.

Remark 3.4 *In the spaces* $W^{k,p}(\Omega)$ *there are several other possible natural norms which are equivalent with the above introduced norm* $\|\cdot\|_{k,p}$. *For example, in the space* $W^{1,p}(\Omega)$ *such an equivalent norm is given by*

$$\|u\| = \left(\|u\|_p^p + \|\nabla u\|_p^p \right)^{\frac{1}{p}} .$$

Here we denote $\nabla u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n} \right)$ *and*

$$\|\nabla u\|_p = \left(\int_\Omega |\nabla u|^p \, dx \right)^{\frac{1}{p}} ,$$

where $|\nabla u|$ *is the euclidean length of the vector* ∇u *in* $\mathbb{C}^n$.

3.2 Approximation.

Now we discuss the approximation of functions in $W^{k,p}(\Omega)$ by smooth functions. For sake of simplicity we will restrict the discussion to the case $k = 1$. First we recall some facts concerning convolutions.

a. If $\varphi \in C_c^\infty(\mathbb{R}^n)$ and $u \in L_{loc}^1(\mathbb{R}^n)$, then $\varphi * u \in C^\infty(\mathbb{R}^n)$ and
$$D^\alpha(\varphi * u) = (D^\alpha \varphi) * u$$
for all multi-indices α.

b. If $\varphi \in C_c^\infty(\mathbb{R}^n)$ and $u \in L^p(\mathbb{R}^n)$, then $\varphi * u \in C^\infty(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for all $1 \leq p \leq \infty$.

c. Let $\psi_1 \in C_c^\infty(\mathbb{R}^n)$ be such that $\operatorname{supp}(\psi_1) \subseteq B(0,1)$, $\psi_1 \geq 0$ and $\int_{\mathbb{R}^n} \psi_1 dx = 1$. For $\varepsilon > 0$ define $\psi_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ by $\psi_\varepsilon(x) = \varepsilon^{-n}\psi_1\left(\frac{x}{\varepsilon}\right)$. Then $\{\psi_\varepsilon : \varepsilon > 0\}$ is called a regularizer (or Dirac-system). Observe that $\operatorname{supp}(\psi_\varepsilon) \subseteq B(0,\varepsilon)$ and $\int_{\mathbb{R}^n} \psi_\varepsilon dx = 1$ for all $\varepsilon > 0$.

d. Let $\{\psi_\varepsilon\}_{\varepsilon>0}$ be a regularizer. If $u \in C_c(\mathbb{R}^n)$ then $\psi_\varepsilon * u \in C_c^\infty(\mathbb{R}^n)$ for all $\varepsilon > 0$ and $\|\psi_\varepsilon * u - u\|_\infty \to 0$ as $\varepsilon \downarrow 0$.

e. Let $\{\psi_\varepsilon\}_{\varepsilon>0}$ be a regularizer. If $1 \leq p < \infty$ and $u \in L^p(\mathbb{R}^n)$, then $\|\psi_\varepsilon * u - u\|_p \to 0$ as $\varepsilon \downarrow 0$.

The next proposition gives a local approximation of functions in $W^{1,p}(\Omega)$ by functions in $C^\infty(\Omega)$.

Proposition 3.5 *Let $\Omega \subseteq \mathbb{R}^n$ be open and $1 \leq p < \infty$. Suppose that Ω_0 is an open subset of Ω such that $\overline{\Omega}_0 \subseteq \Omega$ and $\overline{\Omega}_0$ is compact. Let $\{\psi_\varepsilon\}_{\varepsilon>0}$ be a regularizer and put $u_\varepsilon = \psi_\varepsilon * u$ for all $u \in W^{1,p}(\Omega)$ and all $\varepsilon > 0$. Then*
$$\left\| (u_\varepsilon)_{|\Omega_0} - u_{|\Omega_0} \right\|_{1,p} \to 0 \tag{14}$$
as $\varepsilon \downarrow 0$.

Proof. It follows from **b**. above that $u_\varepsilon \in C^\infty(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for all $\varepsilon > 0$ and from **e**. we know that $\|u_\varepsilon - u\|_{L^p(\mathbb{R}^n)} \to 0$ as $\varepsilon \downarrow 0$. Since $\overline{\Omega}_0$ is compact, it is clear that $(u_\varepsilon)_{|\Omega_0} \in W^{1,p}(\Omega_0)$. Furthermore, it follows from **a**. that
$$\begin{aligned} \frac{\partial u_\varepsilon}{\partial x_j}(x) &= \left(\frac{\partial \psi_\varepsilon}{\partial x_j} * u \right)(x) = \int_\Omega \frac{\partial \psi_\varepsilon}{\partial x_j}(x-y)\, u(y)\, dy \\ &= -\int_\Omega \frac{\partial}{\partial y_j} \psi_\varepsilon(x-y)\, u(y)\, dy \end{aligned}$$

for all $x \in \mathbb{R}^n$. Since $\overline{\Omega}_0 \subseteq \Omega$ and $\overline{\Omega}_0$ is compact it follows that there exists $\varepsilon_0 > 0$ such that $\operatorname{dist}(x, \partial\Omega) > \varepsilon_0$ for all $x \in \Omega_0$. Hence, if $x \in \Omega_0$ and $0 < \varepsilon < \varepsilon_0$ then the function $y \longmapsto \psi_\varepsilon(x-y)$ belongs to $C_c^\infty(\Omega)$. Now it follows from the definition of the weak derivative that

$$-\int_\Omega \frac{\partial}{\partial y_j}\psi_\varepsilon(x-y)\,u(y)\,dy = \int_\Omega \psi_\varepsilon(x-y)\,\frac{\partial u}{\partial x_j}(y)\,dy$$

and so

$$\frac{\partial u_\varepsilon}{\partial x_j}(x) = \left(\psi_\varepsilon * \frac{\partial u}{\partial x_j}\right)(x)$$

for all $x \in \Omega_0$ and all $0 < \varepsilon < \varepsilon_0$. Hence

$$\begin{aligned}\left\|\frac{\partial u_\varepsilon}{\partial x_j} - \frac{\partial u}{\partial x_j}\right\|_{L^p(\Omega_0)} &= \left\|\psi_\varepsilon * \frac{\partial u}{\partial x_j} - \frac{\partial u}{\partial x_j}\right\|_{L^p(\Omega_0)} \\ &\le \left\|\psi_\varepsilon * \frac{\partial u}{\partial x_j} - \frac{\partial u}{\partial x_j}\right\|_{L^p(\mathbb{R}^n)} \to 0\end{aligned}$$

as $\varepsilon \downarrow 0$. Therefore we may conclude that (14) holds. ■

The following proposition is proved by a similar argument.

Proposition 3.6 *Let $\Omega \subseteq \mathbb{R}^n$ be open. If $u \in W^{1,p}(\Omega)$ has compact support, then there exists a sequence $\{v_k\}_{k=1}^\infty$ in $C_c^\infty(\Omega)$ such that $\|u - v_k\|_{1,p} \to 0$ as $k \to \infty$. Equivalently, $W_c^{1,p}(\Omega) \subseteq W_0^{1,p}(\Omega)$.*

Proof. Take $u \in W_c^{1,p}(\Omega)$ and put $K = \operatorname{supp}(u)$. Since K is compact, there exists $\delta > 0$ such that the compact set

$$K_{2\delta} = \{x \in \mathbb{R}^n : \operatorname{dist}(x, K) \le 2\delta\}$$

is contained in Ω. Let $\{\psi_\varepsilon\}_{\varepsilon>0}$ be a regularizer and define $u_\varepsilon = \psi_\varepsilon * u$. We take $0 < \varepsilon < \delta$. We claim that $u_\varepsilon \in C_c^\infty(\Omega)$. Indeed, $x \notin K_\delta$ then

$$u_\varepsilon(x) = \int_{\mathbb{R}^n} \psi_\varepsilon(x-y)\,u(y)\,dy = \int_K \psi_\varepsilon(x-y)\,u(y)\,dy = 0$$

since the support of the function $y \longmapsto \psi_\varepsilon(x-y)$ is contained in $B(x,\varepsilon)$ and $B(x,\varepsilon) \cap K = \emptyset$. Hence $\operatorname{supp}(u_\varepsilon) \subseteq K_\delta$, which implies the claim.

Next we will show that

$$\frac{\partial u_\varepsilon}{\partial x_j} = \psi_\varepsilon * \frac{\partial u}{\partial x_j}. \tag{15}$$

Since $\operatorname{supp}(u_\varepsilon) \subseteq K_\delta$ it follows that also $\frac{\partial u_\varepsilon}{\partial x_j}$ has its support in K_δ, and so we only have to show that (15) holds on K_δ. Take $x \in K_\delta$. As in the proof of the previous proposition we have

$$\frac{\partial u_\varepsilon}{\partial x_j}(x) = -\int_\Omega \frac{\partial}{\partial y_j}\psi_\varepsilon(x-y)\,u(y)\,dy.$$

The function $y \longmapsto \psi_\varepsilon(x-y)$ is supported in $B(x,\varepsilon) \subseteq K_{2\delta} \subseteq \Omega$, so it follows from the definition of the weak derivative that

$$\frac{\partial u_\varepsilon}{\partial x_j}(x) = \int_\Omega \psi_\varepsilon(x-y)\frac{\partial u}{\partial x_j}(y)\,dy = \left(\psi_\varepsilon * \frac{\partial u}{\partial x_j}\right)(x)$$

and the proof of (15) if finished.

Since $\|u-\psi_\varepsilon * u\|_p \to 0$ and $\left\|\frac{\partial u}{\partial x_j} - \psi_\varepsilon * \frac{\partial u}{\partial x_j}\right\|_p \to 0$ for all $j = 1,\ldots,n$ as $\varepsilon \downarrow 0$ and hence $\|u-u_\epsilon\|_{1,p} \to 0$ as $\varepsilon \downarrow 0$. We are done. ■

To obtain smooth approximations on the whole open domain Ω we need the following standard construction of a smooth partition of unity.

Lemma 3.7 *Let Ω be an open subset of $\mathbb{R}^n$ and suppose that $\mathcal{U}$ is an open covering of Ω. Then there exists an open covering $\{V_i\}_{i=1}^\infty$ of Ω and a sequence $\{\varphi_i\}_{i=1}^\infty$ in $C_c^\infty(\Omega)$ such that $0 \le \varphi_i \le 1$ for all $i = 1,2,\ldots$ and:*

1. *for every V_i there exists a $U \in \mathcal{U}$ such that $V_i \subseteq U$;*
2. *the open covering $\{V_i\}_{i=1}^\infty$ is locally finite, i.e., for every compact subset $K \subseteq \Omega$ we have $K \cap V_i \neq \emptyset$ for only finitely many values of i;*
3. $\operatorname{supp}(\varphi_i) \subseteq V_i$ *for all $i = 1,2,\ldots$;*
4. $\sum_{i=1}^\infty \varphi_i(x) = 1$ *for all $x \in \Omega$;*
5. *for every compact $K \subseteq \Omega$ there exists an open set W such that $K \subseteq W \subseteq \Omega$ and there exists $m \in \mathbb{N}$ such that $\varphi_1(x) + \cdots \varphi_m(x) = 1$ for all $x \in W$.*

Proof. Let $\{q_k\}_{k=1}^\infty$ be a sequence which is dense in Ω and let $\{r_l\}_{l=1}^\infty$ be an enumeration of the positive rational numbers. Now consider the collection of all open balls $B(q_k,r_l)$ for which there exists a $U \in \mathcal{U}$ such that $B(q_k,r_l) \subseteq U$. We enumerate this collection of balls as $\{B_i\}_{i=1}^\infty$. For each i we denote by O_i the open ball with the same center and half the radius as B_i. It is clear that $\Omega = \bigcup_{i=1}^\infty O_i$.

For every i there exists a function $\eta_i \in C_c^\infty(\Omega)$ such that $0 \leq \eta_i \leq 1$, $\eta_i(x) = 1$ for all $x \in O_i$ and $\operatorname{supp}(\eta_i) \subseteq B_i$. Note that this implies that for every φ_i there exists a $U \in \mathcal{U}$ such that $\operatorname{supp}(\eta_i) \subseteq U$. Now define the sequence $\{\varphi_i\}_{i=1}^\infty$ in $C_c^\infty(\Omega)$ by $\varphi_1 = \eta_1$ and

$$\varphi_{i+1} = (1-\eta_1)\cdots(1-\eta_i)\,\eta_{i+1}$$

for $i \geq 1$. A simple induction argument shows that

$$\varphi_1 + \cdots + \varphi_m = 1 - (1-\eta_1)\cdots(1-\eta_m)$$

for all $m \geq 1$. This implies that $\varphi_1(x) + \cdots + \varphi_m(x) = 1$ whenever $x \in O_i$ and $1 \leq i \leq m$. Now it is clear that **4**. holds. If $K \subseteq \Omega$ is compact, then there exists $m \in \mathbb{N}$ such that $K \subseteq \bigcup_{i=1}^m O_i$ from which **5**. now follows.

Define the open sets $V_i = \{x \in \Omega : \varphi_i > 0\}$. Since **4**. hold for the functions $\{\varphi_i\}_{i=1}^\infty$ it is clear that $\{V_i\}_{i=1}^\infty$ is an open covering of Ω. Since $V_i \subseteq B_i$ and B_i is contained in some $U \in \mathcal{U}$, it follows that **1**. holds. Moreover, it follows from **5**. that $\{V_i\}_{i=1}^\infty$ is locally finite. Now we repeat the above construction with the covering $\mathcal{U}$ replaced by $\{V_i\}_{i=1}^\infty$. This produces a new sequence of functions in $C_c^\infty(\Omega)$, which we call $\{\varphi_i\}_{i=1}^\infty$ again. So **4**. and **5**. are satisfied. For each i there exists a V_{k_i} such that $\operatorname{supp}(\varphi_i) \subseteq V_{k_i}$. Finally replace the collection $\{V_i\}_{i=1}^\infty$ by $\{V_{k_i}\}_{i=1}^\infty$. Then **3**. is satisfied as well and the other properties are preserved. By this the construction is finished. ■

In the situation of the above proposition we say that $\{\varphi_i\}_{i=1}^\infty$ is a (locally finite) smooth partition of unity subordinate to the open covering $\mathcal{U}$ of Ω.

Theorem 3.8 *Let Ω be an open subset of $\mathbb{R}^n$. Then $C^\infty(\Omega) \cap W^{1,p}(\Omega)$ is a dense subspace of $W^{1,p}(\Omega)$.*

Proof. We apply the above lemma to $\mathcal{U} = \{\Omega\}$. So let $\{\varphi_i\}_{i=1}^\infty$ be any locally finite smooth partition of unity with corresponding open covering $\{V_i\}_{i=1}^\infty$ of Ω. Let $u \in W^{1,p}(\Omega)$ and $\varepsilon > 0$ be given. Then $u\varphi_i \in W^{1,p}(\Omega)$ and $\operatorname{supp}(u\varphi_i) \subseteq \operatorname{supp}(\varphi_i) \subseteq V_i$, so $u\varphi_i \in W_c^{1,p}(V_i)$. It follows from Proposition 3.6 that exists $g_i \in C_c^\infty(V_i)$ such that

$$\|u\varphi_i - g_i\|_{1,p} < 2^{-i}\varepsilon.$$

Define

$$g(x) = \sum_{i=1}^\infty g_i(x) \tag{16}$$

for all $x \in \Omega$. Since $\{V_i\}_{i=1}^\infty$ is locally finite, any closed ball $\bar{B}(y,r) \subseteq \Omega$ has a non-empty intersection with only a finite number of the V_i's and so (16) is a finite sum on $\bar{B}(y,r)$. This implies that $g \in C^\infty(\Omega)$.

Now let $\Omega_0 \subseteq \Omega$ be open such that $\overline{\Omega}_0$ is compact and $\overline{\Omega}_0 \subseteq \Omega$. Since $\{V_i\}_{i=1}^{\infty}$ is locally finite, there exists $N \in \mathbb{N}$ such that $V_i \cap \overline{\Omega}_0 = \emptyset$ for all $i > N$. Hence for all $x \in \Omega_0$ we have $\sum_{i=1}^{N} \varphi_i(x) = 1$ and

$$g(x) = \sum_{i=1}^{N} g_i(x).$$

Moreover, $u(x) = \sum_{i=1}^{N} u(x)\varphi_i(x)$ for all $x \in \Omega_0$. Consequently

$$\begin{aligned}
\left\|u_{|\Omega_0} - g_{|\Omega_0}\right\|_{1,p} &\leq \sum_{i=1}^{N} \left\|(u\varphi_i)_{|\Omega_0} - (g_i)_{|\Omega_0}\right\|_{1,p} \\
&\leq \sum_{i=1}^{N} \|u\varphi_i - g_i\|_{1,p} < \sum_{i=1}^{N} 2^{-i}\varepsilon < \varepsilon. \qquad (17)
\end{aligned}$$

Now let $\{\Omega_k\}_{k=1}^{\infty}$ be an increasing sequence of open subsets of Ω with $\overline{\Omega}_k$ compact and $\overline{\Omega}_k \subseteq \Omega$ for all k. Then (17) applies to each of the Ω_k, so it follows from the monotone convergence theorem that $\|u - g\|_{1,p} \leq \varepsilon$. Since this also implies that $g \in W^{1,p}(\Omega)$, the proof is complete. ■

3.3 Some estimates.

The following inequality will be useful.

Proposition 3.9 *Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded and convex set with diameter d. Suppose that $\varphi \in L^{\infty}(\Omega)$ such that $\int_\Omega \varphi dx = 1$. For $u \in W^{1,1}(\Omega)$ define*

$$\bar{u}_\varphi = \int_\Omega u(y)\varphi(y)\,dy.$$

Then for all $u \in W^{1,1}(\Omega)$ we have

$$|u(x) - \bar{u}_\varphi| \leq C \int_\Omega \frac{|\nabla u(y)|}{|x-y|^{n-1}} dy, \qquad (18)$$

for almost all $x \in \Omega$, where $C > 0$ is a constant depending only on n, d and $\|\varphi\|_\infty$ (actually we can take $C = n^{-1} d^n \|\varphi\|_\infty$).

Proof. We denote by S the unit sphere in $\mathbb{R}^n$, i.e.,

$$S = \{z \in \mathbb{R}^n : |z| = 1\},$$

and let σ be the normalized surface measure of S. Let $x \in \Omega$ be fixed. For any $z \in S$ we define

$$r(z) = \sup\{r > 0 : x + rz \in \Omega\}.$$

It is easy to see that r is a measurable function and it is clear that $r(z) \leq d$. Since Ω is convex it follows that

$$\Omega = \{x + rz : z \in S, 0 \leq r < r(z)\}.$$

Consequently, for every positive (or integrable) function $f : \Omega \to \mathbb{R}$ we have

$$\int_\Omega f(y)\, dy = \int_S \int_0^{r(z)} f(x + rz)\, r^{n-1} dr d\sigma(z).$$

First we assume that $u \in W^{1,1}(\Omega) \cap C^\infty(\Omega)$. Take $y \in \Omega$ and write $y = x + tz$ with $z \in S$ and $0 \leq t \leq r(z)$. Then

$$u(y) - u(x) = \int_0^t \langle \nabla u(x + rz), z\rangle\, dr$$

and so

$$|u(x) - u(y)| \leq \int_0^t |\nabla u(x + rz)|\, dr \leq \int_0^{r(z)} |\nabla u(x + rz)|\, dr.$$

Now it follows that

$$\begin{aligned}
|u(x) - \bar{u}_\varphi| &= \left|\int_\Omega [u(x) - u(y)]\, \varphi(y)\, dy\right| \\
&\leq \|\varphi\|_\infty \int_\Omega |u(x) - u(y)|\, dy \\
&= \|\varphi\|_\infty \int_S \int_0^{r(z)} |u(x) - u(x + rz)|\, r^{n-1} dr d\sigma(z) \\
&\leq \|\varphi\|_\infty \int_S \int_0^{r(z)} \int_0^{r(z)} |\nabla u(x + sz)|\, r^{n-1} ds dr d\sigma(z) \\
&\leq \|\varphi\|_\infty \int_S \int_0^{r(z)} \left(\int_0^d r^{n-1} dr\right) |\nabla u(x + sz)|\, ds d\sigma(z) \\
&= \frac{\|\varphi\|_\infty d^n}{n} \int_S \int_0^{r(z)} |\nabla u(x + sz)|\, ds d\sigma(z) \\
&= C \int_S \int_0^{r(z)} \frac{|\nabla u(x + sz)|}{s^{n-1}} s^{n-1} ds d\sigma(z) \\
&= C \int_\Omega \frac{|\nabla u(y)|}{|x - y|^{n-1}} dy.
\end{aligned}$$

This proves (18) in the case that $u \in W^{1,1}(\Omega) \cap C^{\infty}(\Omega)$ and for all $x \in \Omega$. Note that (18) can also be written as

$$|u(x) - \bar{u}_{\varphi}| \leq CI_1(|\nabla u|)(x)$$

where $I_1(|\nabla u|)$ denotes the Riesz potential corresponding to $\alpha = 1$.

Now let $u \in W^{1,1}(\Omega)$ be given. It follows from Theorem 3.8 that there exists a sequence $\{u_k\}_{k=1}^{\infty}$ in $W^{1,1}(\Omega) \cap C^{\infty}(\Omega)$ such that $\|u_k - u\|_{1,1} \to 0$ as $k \to \infty$. From the first part of the proof we know that

$$\left| u_k(x) - (\overline{u_k})_{\varphi} \right| \leq CI_1(|\nabla u_k|)(x)$$

for all $x \in \Omega$ and all k. Since $\|u_k - u\|_1 \to 0$, it follows that $(\overline{u_k})_{\varphi} \to \bar{u}_{\varphi}$ as $k \to \infty$. Furthermore, since $\||\nabla u_k| - |\nabla u|\|_1 \to 0$ as $k \to \infty$, Proposition 2.6 implies that

$$\|I_1(|\nabla u_k|) - I_1(|\nabla u|)\|_1 \to 0$$

as $k \to \infty$. Passing to a subsequence we may assume that

$$I_1(|\nabla u_k|)(x) \to I_1(|\nabla u|)(x)$$

as well as $u_k(x) \to u(x)$ a.e. on Ω as $k \to \infty$, and from this the result follows. ■

Given a bounded open subset Ω of $\mathbb{R}^n$ and $u \in L^1(\Omega)$ we will denote

$$\bar{u}_{\Omega} = \frac{1}{m(\Omega)} \int_{\Omega} u dx.$$

The result of the following theorem is sometimes referred to as Poincaré's inequality.

Theorem 3.10 *Let $\Omega \subseteq \mathbb{R}^n$ be open, convex and bounded. Let $d = \operatorname{diam}(\Omega)$. Suppose that $1 \leq p < \infty$. Then*

$$\int_{\Omega} |u(x) - \bar{u}_{\Omega}|^p dx \leq C^p d^p \int_{\Omega} |\nabla u(x)|^p dx$$

for all $u \in W^{1,p}(\Omega)$, where $C > 0$ is a constant depending only on n and the ratio $d^n/m(\Omega)$.

Proof. First note that

$$\bar{u}_{\Omega} = \int_{\Omega} u \varphi dx,$$

where $\varphi = m(\Omega)^{-1} \chi_\Omega$. Since $W^{1,p}(\Omega) \subseteq W^{1,1}(\Omega)$, it follows from Proposition 3.9 that

$$|u(x) - \bar{u}_\Omega| \leq C_0 I_1 (|\nabla u|)(x)$$

a.e. on Ω, where $C_0 = n^{-1} d^n m(\Omega)^{-1}$. Using Proposition 2.6 we find that

$$\int_\Omega |u - \bar{u}_\Omega|^p \, dx \leq C_0^p \int_\Omega I_1 (|\nabla u|)^p \, dx \leq C_0^p C_1^p \int_\Omega |\nabla u|^p \, dx,$$

where $C_1 = C_n m(\Omega)^{\frac{1}{n}}$. Finally observe that

$$C_0 C_1 = \left(n^{-1} C_n\right) d^n m(\Omega)^{-1} m(\Omega)^{\frac{1}{n}} = \left(n^{-1} C_n\right) \left[\frac{d^n}{m(\Omega)}\right]^{1-\frac{1}{n}} d.$$

■

Recall that

$$\bar{u}_{x,r} = \frac{1}{m(B_r)} \int_{B(x,r)} u(y) \, dy$$

for $u \in L^1(B(x,r))$.

Corollary 3.11 *For every n there exists a constant $C > 0$ such that for every ball $B(x,r) \subseteq \mathbb{R}^n$ and all $u \in W^{1,p}(B(x,r))$, $1 \leq p < \infty$, we have*

$$\int_{B(x,r)} |u(y) - \bar{u}_{x,r}|^p \, dy \leq C^p r^p \int_{B(x,r)} |\nabla u(y)|^p \, dy.$$

Proof. Since $d = \operatorname{diam}(B(x,r)) = 2r$ and $m(B(x,r)) = n^{-1} \omega_n r^n$, this is an immediate consequence of the above theorem. ■

Remark 3.12 *In the proof of the next theorem we will make use of the following extension of the Hölder inequality. Suppose that $(\Omega, \mathcal{A}, \mu)$ is any measure space and let $1 \leq p_1, \ldots, p_k < \infty$ be such that*

$$\frac{1}{p_1} + \cdots + \frac{1}{p_k} = 1.$$

Then

$$\int_\Omega |f_1 \cdots f_k| \, d\mu \leq \left(\int_\Omega |f_1|^{p_1}\right)^{\frac{1}{p_1}} \cdots \left(\int_\Omega |f_1|^{p_k}\right)^{\frac{1}{p_k}}$$

for all measurable functions $f_1, \ldots, f_k$ on Ω. This follows via a simple induction argument from the case $k = 2$. Note that this implies in particular that

$$\int_\Omega \prod_{j=1}^k |f_j|^{\frac{1}{k}} \, d\mu \leq \prod_{j=1}^k \left(\int_\Omega |f_j| \, d\mu\right)^{\frac{1}{k}}. \tag{19}$$

If $1 \leq p < n$ we define p^* by

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}.$$

The following inequality is sometimes called Sobolev's inequality.

Theorem 3.13 *Let $\Omega \subseteq \mathbb{R}^n$ be open and $1 \leq p < n$. Then*

$$\|u\|_{p^*} \leq \frac{(n-1)\,p}{n-p} \|\nabla u\|_p$$

for all $u \in W_0^{1,p}(\Omega)$.

Proof. First we will consider the case that $u \in C_c^1(\Omega) \subseteq C_c^1(\mathbb{R}^n)$ and $p = 1$. Note that $p^* = \frac{n}{n-1}$ in this case. For every $x = (x_1, \ldots, x_n)$ and all $j = 1, \ldots, n$ we have

$$u(x) = \int_{-\infty}^{x_j} \frac{\partial u}{\partial x_j}(x_1, \ldots, t_j, \ldots, x_n)\, dt_j,$$

and so

$$\begin{aligned} |u(x)| &\leq \int_{-\infty}^{\infty} \left| \frac{\partial u}{\partial x_j}(x_1, \ldots, t_j, \ldots, x_n) \right| dt_j \\ &\leq \int_{-\infty}^{\infty} |\nabla u(x_1, \ldots, t_j, \ldots, x_n)|\, dt_j. \end{aligned} \tag{20}$$

For the sake of notation we will write

$$\int_{-\infty}^{\infty} |\nabla u(x_1, \ldots, t_j, \ldots, x_n)|\, dt_j = \int_{-\infty}^{\infty} |\nabla u|\, dt_j.$$

It follows from (20) that

$$|u(x)|^{\frac{n}{n-1}} \leq \left(\int_{-\infty}^{\infty} |\nabla u|\, dt_1 \right)^{\frac{1}{n-1}} \prod_{j=2}^{n} \left(\int_{-\infty}^{\infty} |\nabla u|\, dt_j \right)^{\frac{1}{n-1}}$$

and so, using (19) we find that

$$\begin{aligned} \int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}} dx_1 &\leq \left(\int_{-\infty}^{\infty} |\nabla u|\, dt_1 \right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{j=2}^{n} \left(\int_{-\infty}^{\infty} |\nabla u|\, dt_j \right)^{\frac{1}{n-1}} dx_1 \\ &\leq \left(\int_{-\infty}^{\infty} |\nabla u|\, dt_1 \right)^{\frac{1}{n-1}} \prod_{j=2}^{n} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\nabla u|\, dt_j dx_1 \right)^{\frac{1}{n-1}} \\ &= \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\nabla u|\, dt_2 dx_1 \right)^{\frac{1}{n-1}} \left(\int_{-\infty}^{\infty} |\nabla u|\, dx_1 \right)^{\frac{1}{n-1}} \cdot \\ &\quad \prod_{j=3}^{n} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\nabla u|\, dt_j dx_1 \right)^{\frac{1}{n-1}}. \end{aligned}$$

Similarly, this implies that

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}}\, dx_1 dx_2 \\
\leq\ & \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\nabla u|\, dt_2 dx_1\right)^{\frac{1}{n-1}} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\nabla u|\, dx_1 dx_2\right)^{\frac{1}{n-1}} \cdot \\
& \prod_{j=3}^{n} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\nabla u|\, dt_j dx_1 dx_2\right)^{\frac{1}{n-1}} \\
=\ & \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\nabla u|\, dt_3 dx_1 dx_2\right)^{\frac{1}{n-1}} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\nabla u|\, dx_1 dx_2\right)^{\frac{2}{n-1}} \cdot \\
& \prod_{j=4}^{n} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\nabla u|\, dt_j dx_1 dx_2\right)^{\frac{1}{n-1}}.
\end{aligned}$$

Repeating this argument we finally get

$$\int_{\mathbb{R}^n} |u(x)|^{\frac{n}{n-1}}\, dx \leq \prod_{j=1}^{n} \left(\int_{\mathbb{R}^n} |\nabla u(x)|\, dx\right)^{\frac{1}{n-1}} = \left(\int_{\mathbb{R}^n} |\nabla u(x)|\, dx\right)^{\frac{n}{n-1}}$$

and so

$$\|u\|_{\frac{n}{n-1}} \leq \|\nabla u\|_1 . \tag{21}$$

This completes the proof in case $p = 1$ and $u \in C_c^1(\Omega)$.

Now we consider the case in which $1 < p < n$ and $u \in C_c^1(\Omega)$. Let $q = \frac{(n-1)p}{n-p}$. Note that $q > 1$ and that $q\frac{n}{n-1} = p^*$. Denote by p' the conjugate exponent of p and observe that $(q-1)\,p' = p^*$. Define $v = |u|^q$. Since $q > 1$ it follows that $v \in C_c^1(\Omega)$ and $\nabla v = q\,\mathrm{sgn}(u)\,|u|^{q-1}\,\nabla u$. Applying (21) to the function v we find that

$$\begin{aligned}
\|u\|_{p^*}^q &= \left(\int_{\mathbb{R}^n} |u|^{p^*}\, dx\right)^{\frac{q}{p^*}} = \left(\int_{\mathbb{R}^n} v^{\frac{n}{n-1}} dx\right)^{\frac{n-1}{n}} \\
&= \|v\|_{\frac{n}{n-1}} \leq \|\nabla v\|_1 = q\int_{\mathbb{R}^n} |u|^{q-1}\,|\nabla u|\, dx \\
&\leq q\left(\int_{\mathbb{R}^n} |u|^{p^*}\, dx\right)^{\frac{1}{p'}} \left(\int_{\mathbb{R}^n} |\nabla u|^p\, dx\right)^{\frac{1}{p}} \\
&= q\,\|u\|_{p^*}^{\frac{p^*}{p'}}\,\|\nabla u\|_p = q\,\|u\|_{p^*}^{q-1}\,\|\nabla u\|_p ,
\end{aligned}$$

and so $\|u\|_{p^*} \leq q\,\|\nabla u\|_p$. By this we have proved the theorem for $u \in C_c^1(\Omega)$ and all $1 \leq p < n$.

Now take $u \in W_0^{1,p}(\Omega)$, $1 \leq p < n$. By definition there exists a sequence $\{u_k\}_{k=1}^{\infty}$ in $C_c^{\infty}(\Omega)$ such that $\|u - u_k\|_{1,p} \to 0$, i.e., $\|u - u_k\|_p \to 0$ and $\|\nabla(u - u_k)\|_p \to 0$ as $k \to \infty$. From the first part of the present proof, applied to the functions $u_k - u_l$, we know that

$$\|u_k - u_l\|_{p^*} \leq q \|\nabla(u_k - u_l)\|_p$$

for all $k, l \geq 1$, so $\{u_k\}_{k=1}^{\infty}$ is a Cauchy sequence in $L^{p^*}(\Omega)$. Now $\|u - u_k\|_p \to 0$ implies that $\|u - u_k\|_{p^*} \to 0$ as $k \to \infty$. Finally, since $\|u_k\|_{p^*} \leq q \|\nabla u_k\|_p$ for all k, $u_k \to u$ in $L^{p^*}(\Omega)$ and $|\nabla u_k| \to |\nabla u|$ in $L^p(\Omega)$ we may conclude that $\|u\|_{p^*} \leq q \|\nabla u\|_p$, which completes the proof of the theorem. ■

For sake of completeness we mention the following corollary.

Corollary 3.14 *Let $\Omega \subseteq \mathbb{R}^n$ be open with $m(\Omega) < \infty$. Suppose that $p, r \geq 1$ are such that*

$$\frac{1}{r} - \frac{1}{p} + \frac{1}{n} = \alpha \geq 0.$$

Then

$$\|u\|_r \leq C m(\Omega)^{\alpha} \|\nabla u\|_p$$

for all $u \in W_0^{1,p}(\Omega)$, where $C > 0$ is a constant only depending on n and r.

In particular, for $p \geq 1$ there exists a constant $C > 0$ only depending on n and p such that $\|u\|_p \leq C m(\Omega)^{\frac{1}{n}} \|\nabla u\|_p$ for all $u \in W_0^{1,p}(\Omega)$.

Proof. First observe that the case $n = 1$ is easy. Indeed, it is sufficient to consider $u \in C_c^{\infty}(\Omega) \subseteq C_c^{\infty}(\mathbb{R})$. Then $u(x) = \int_{-\infty}^{x} u'(t)\,dt$, so

$$|u(x)| \leq \int_{-\infty}^{\infty} |u'(t)|\,dt \leq m(\Omega)^{\frac{1}{p'}} \|u'\|_p,$$

for all x, hence $\|u\|_{\infty} \leq m(\Omega)^{\frac{1}{p'}} \|u'\|_p$. This implies for all $r \geq 1$ that

$$\|u\|_r \leq m(\Omega)^{\frac{1}{r}} \|u\|_{\infty} \leq m(\Omega)^{\frac{1}{r} + \frac{1}{p'}} \|u'\|_p = m(\Omega)^{\alpha} \|u'\|_p,$$

without any additional restrictions on p and r.

Now we assume that $n \geq 2$. Define $s = \frac{nr}{n+r}$. It is easy to check that $1 \leq s < n$, $s \leq p$ (as $\alpha \geq 0$) and $r = s^*$. In particular $W_0^{1,p}(\Omega) \subseteq W_0^{1,s}(\Omega)$. Now, with $C = \frac{(n-1)s}{n-s}$, it follows from the above theorem that

$$\begin{aligned} \|u\|_r &= \|u\|_{s^*} \leq C \|\nabla u\|_s \\ &\leq C m(\Omega)^{\frac{1}{s} - \frac{1}{p}} \|\nabla u\|_p = C m(\Omega)^{\alpha} \|\nabla u\|_p \end{aligned}$$

for all $u \in W_0^{1,p}(\Omega)$. ■

4 Hölder continuous functions.

In this section we discuss some characterizations of Hölder continuous functions in terms of average values.

4.1 Definitions.

First we recall some of the relevant definitions. We assume that Ω is an open subset of $\mathbb{R}^n$ and that $0 < \alpha \leq 1$. A function $u : \Omega \to \mathbb{C}$ is called (uniformly) Hölder continuous with exponent α if there exists a constant $K > 0$ such that

$$|u(x) - u(y)| \leq K |x - y|^{\alpha} \tag{22}$$

for all $x, y \in \Omega$. In the case that $\alpha = 1$ the function u is called Lipschitz continuous. It is clear that any Hölder continuous function u is uniformly continuous on Ω and hence has a unique continuous extension (which we will denote by u as well) to the closure $\overline{\Omega}$. It is clear that this extension satisfies (22) for all $x, y \in \overline{\Omega}$. If a function $u : \Omega \to \mathbb{C}$ is bounded and if there exists $\varepsilon_0 > 0$ such that (22) holds for all $x, y \in \Omega$ satisfying $|x - y| < \varepsilon_0$, then u is uniformly Hölder continuous with exponent α.

The space of all bounded Hölder continuous functions with exponent α on the open set $\Omega \subseteq \mathbb{R}^n$ is denoted by $C^{0,\alpha}\left(\overline{\Omega}\right)$. It is easy to see that this space is a Banach space equipped with the norm given by

$$\|u\|_{0,\alpha} = \|u\|_{\infty} + \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x) - u(y)|}{|x - y|^{\alpha}}$$

for all $u \in C^{0,\alpha}\left(\overline{\Omega}\right)$.

A function $u : \Omega \to \mathbb{C}$ is called locally Hölder continuous with exponent $0 < \alpha \leq 1$ if for every open $\Omega_0 \subseteq \Omega$ with $\overline{\Omega}_0 \subseteq \Omega$ and $\overline{\Omega}_0$ compact we have $u_{|\Omega_0} \in C^{0,\alpha}\left(\overline{\Omega}_0\right)$. The space of all locally Hölder continuous functions with exponent α on Ω is denoted by $C^{0,\alpha}(\Omega)$. Although $C^{0,\alpha}(\Omega)$ is not a Banach space, it has the obvious structure of a Fréchet space.

4.2 The oscillation of a function.

Let $\Omega \subseteq \mathbb{R}^n$ be open. For any measurable function $u : \Omega \to \mathbb{C}$ we define the oscillation of u over Ω by

$$\operatorname{osc}(u; \Omega) = \inf \left\{\operatorname{diam}(D) : D \subseteq \mathbb{C} \text{ such that } u(x) \in D \text{ a.e. on } \Omega\right\}.$$

It is easy to see that the infimum is actually a minimum. In particular, if $\varepsilon > 0$, then $\operatorname{osc}(u;\Omega) \leq \varepsilon$ if and only if $|u(x) - u(y)| \leq \varepsilon$ for almost all $x, y \in \Omega$. Obviously, if Ω_0 is an open subset of Ω, then $\operatorname{osc}(u;\Omega_0) \leq \operatorname{osc}(u;\Omega)$.

If $u : \Omega \to \mathbb{C}$ is measurable then for all $x \in \Omega$ and all $r > 0$ we define

$$\omega_u(x, r) = \operatorname{osc}(u; B(x, r) \cap \Omega).$$

The following observation will be useful.

Lemma 4.1 *Let $\Omega \subseteq \mathbb{R}^n$ be open and $u : \Omega \to \mathbb{C}$ measurable. Then there exists a continuous function $\bar{u} : \Omega \to \mathbb{C}$ such that $u = \bar{u}$ a.e. on Ω if and only if $\lim_{r\downarrow 0} \omega_u(x, r) = 0$ for all $x \in \Omega$.*

Proof. If such a continuous function $\bar{u}$ exists, then it is clear that $\lim_{r\downarrow 0} \omega_u(x, r) = \lim_{r\downarrow 0} \omega_{\bar{u}}(x, r) = 0$ for all $x \in \Omega$.

Now assume that $\lim_{r\downarrow 0} \omega_u(x, r) = 0$ for all $x \in \Omega$. Fix $x_0 \in \Omega$ and $R_0 > 0$ such that $\bar{B}(x_0, R_0) \subseteq \Omega$. We will show that there exists a null set A such that u is uniformly continuous on $\bar{B}(x_0, R_0) \setminus A$. To this end take $0 < k \in \mathbb{N}$ fixed for the moment. For every $x \in \bar{B}(x_0, R_0)$ there exists $r_x > 0$ such that $\omega_u(x, r_x) < \frac{1}{k}$ and $B(x_i, r_i) \subseteq \Omega$. Using that $\bar{B}(x_0, R_0)$ is compact it follows that there exist $x_1, \ldots, x_N \in \bar{B}(x_0, R_0)$ and $r_1, \ldots, r_N > 0$ such that

$$\bar{B}(x_0, R_0) \subseteq \bigcup_{i=1}^{N} B\left(x_i, \frac{r_i}{2}\right)$$

and $\omega_u(x_i, r_i) < \frac{1}{k}$ for all $i = 1, \ldots, N$. For every i there exists a null set F_i such that $|u(x) - u(y)| < \frac{1}{k}$ for all $x, y \in B(x_i, r_i) \setminus F_i$. Define $A_k = F_1 \cup \cdots \cup F_N$ and $\delta_k = \frac{1}{2} \min\{r_1, \ldots, r_N\}$. We claim that $|u(x) - u(y)| < \frac{1}{k}$ for all $x, y \in \bar{B}(x_0, R_0) \setminus A_k$ with $|x - y| < \delta_k$. Indeed, there exists an i such that $x \in B\left(x_i, \frac{r_i}{2}\right)$ and from the definition of δ_k it follows that $B(x, \delta_k) \subseteq B(x_i, r_i)$, so in particular $y \in B(x_i, r_i)$. Consequently $x, y \in B(x_i, r_i) \setminus F_i$ and so $|u(x) - u(y)| < \frac{1}{k}$, which proves that claim. Define $A = \bigcup_{k=1}^{\infty} A_k$. We claim that u is uniformly continuous on $\bar{B}(x_0, R_0) \setminus A$. Indeed, let $\varepsilon > 0$ be given. Take $0 < k \in \mathbb{N}$ such that $\frac{1}{k} < \varepsilon$. If $x, y \in \bar{B}(x_0, R_0) \setminus A$ are such that $|x - y| < \delta_k$, then in particular $x, y \in \bar{B}(x_0, R_0) \setminus A_k$ and so $|u(x) - u(y)| < \frac{1}{k} < \varepsilon$, by which that claim is proved.

Since $\bar{B}(x_0, R_0) \setminus A$ is dense in $\bar{B}(x_0, R_0)$ it follows that there exists a unique continuous function $\bar{u}_0 : \bar{B}(x_0, R_0) \to \mathbb{C}$ such that $u = \bar{u}_0$ a.e. on $\bar{B}(x_0, R_0)$. If $\bar{B}(x_1, R_1)$ and $\bar{B}(x_2, R_2)$ are closed balls contained in Ω and $\bar{u}_1$ and $\bar{u}_2$ are continuous functions on $\bar{B}(x_1, R_1)$ and $\bar{B}(x_2, R_2)$ respectively such that $u = \bar{u}_1$ and $u = \bar{u}_2$ a.e., then $\bar{u}_1 = \bar{u}_2$ on $B(x_1, R_1) \cap B(x_1, R_1)$. From this observation it follows now immediately that there exists a continuous function $\bar{u} : \Omega \to \mathbb{C}$ such that $u = \bar{u}$ a.e. on Ω. ∎

Remark 4.2 *Let $u : \Omega \to \mathbb{C}$ be a measurable function. If $x \in \Omega$ and if there exists a measurable function $\bar{u}_x : \Omega \to \mathbb{C}$ which is continuous at x such that $u = \bar{u}_x$ a.e. on Ω, then it follows that $\lim_{r\downarrow 0} \omega_u(x,r) = 0$. Therefore, if for every $x \in \Omega$ there exists a measurable function $\bar{u}_x$ on Ω which is continuous at x such that $u = \bar{u}_x$ a.e. on Ω, then $\lim_{r\downarrow 0} \omega_u(x,r) = 0$ for all $x \in \Omega$ and so by the above lemma there exists a continuous function $\bar{u} : \Omega \to \mathbb{C}$ such that $u = \bar{u}$ a.e. on Ω.*

For later reference we include the following simple observation.

Lemma 4.3 *For any open set $\Omega \subseteq \mathbb{R}^n$ and measurable function $u : \Omega \to \mathbb{C}$ the following two statements are equivalent:*

1. *$u \in C^{0,\alpha}\left(\overline{\Omega}\right)$;*

2. *u is bounded and there exists a constant $K > 0$ and $r_0 > 0$ such that $\omega_u(x,r) \leq Kr^\alpha$ for all $0 < r < r_0$ and all $x \in \Omega$.*

Proof. It is clear that 1. implies 2. Now assume that u satisfies 2. First note that 2. implies that $\lim_{r\downarrow 0} \omega_u(x,r) = 0$ and so, by Lemma 4.1 we may assume that u is continuous. Suppose that $x, y \in \Omega$ are such that $|x-y| < r_0$ and take r such that $|x-y| < r < r_0$. Then $y \in B(x,r)$ and so, since u is continuous it follows that $|u(x) - u(y)| \leq \omega_u(x,r) \leq Kr^\alpha$. Letting $r \downarrow |x-y|$ gives $|u(x) - u(y)| \leq K|x-y|^\alpha$. As observed already in Section 4.1, this suffices to show that $u \in C^{0,\alpha}\left(\overline{\Omega}\right)$. ■

The following lemma gives a sufficient condition for a function u to belong to $C^{0,\alpha}(\Omega)$.

Lemma 4.4 *Suppose that $\Omega \subseteq \mathbb{R}^n$ is open and $u : \Omega \to \mathbb{C}$ is measurable such that there exists a constant $K > 0$ such that $\omega_u(x,r) \leq Kr^\alpha$ for all $x \in \Omega$ and all $r > 0$ such that $\bar{B}(x,2r) \subseteq \Omega$. Then $u \in C^{0,\alpha}(\Omega)$.*

Proof. The condition on u clearly imply that $\lim_{r\downarrow 0} \omega_u(x,r) = 0$ for all $x \in \Omega$. Hence, by Lemma 4.1 we may assume that u is continuous. Let $\Omega_0 \subseteq \Omega$ be open such that $\overline{\Omega}_0$ is compact and $\overline{\Omega}_0 \subseteq \Omega$. Then u is bounded on Ω_0. Since $\overline{\Omega}_0$ is compact there exist $r_0 > 0$ such that $\bar{B}(x,2r) \subseteq \Omega$ for all $x \in \Omega_0$ and all $0 < r < r_0$. Hence $\omega_u(x,r) \leq Kr^\alpha$ for all $x \in \Omega_0$ and all $0 < r < r_0$. This clearly implies that u satisfies condition 2. of the above lemma on Ω_0, and so $u \in C^{0,\alpha}\left(\overline{\Omega}_0\right)$. ■

4.3 Characterizations of Hölder continuity.

Suppose that Ω is an open subset of $\mathbb{R}^n$. Recall that if $u \in L^1_{loc}(\Omega)$ and $\bar{B}(x,r) \subseteq \Omega$, then we denote

$$\bar{u}_{x,r} = \frac{1}{m(B_r)} \int_{B(x,r)} u(y)\, dy.$$

The following result is due to S. Campanato (1963).

Theorem 4.5 *Assume that $u \in L^1_{loc}(\Omega)$ and $0 < \alpha \leq 1$. If there exists a constant $M > 0$ such that*

$$\frac{1}{m(B_r)} \int_{B(x,r)} |u(y) - \bar{u}_{x,r}|\, dy \leq M r^\alpha$$

for all balls with $\bar{B}(x,r) \subseteq \Omega$, then

$$\omega_u(x,r) \leq C M r^\alpha$$

for all $x \in \Omega$ and all $r > 0$ for which $\bar{B}(x,2r) \subseteq \Omega$, where $C > 0$ is a constant only depending on n and α. In particular, $u \in C^{0,\alpha}(\Omega)$.

Proof. We start the proof with the following observation. Suppose that $z \in \Omega$ and $r > 0$ are such that $\bar{B}(z,r) \subseteq \Omega$. Now take $x \in \Omega$ such that $|x - z| < \frac{1}{2}r$. We claim that

$$\left|\bar{u}_{x,\frac{r}{2}} - \bar{u}_{z,r}\right| \leq 2^n M r^\alpha. \tag{23}$$

Indeed, using that $B\left(x, \frac{r}{2}\right) \subseteq B(z,r)$, we find that

$$\begin{aligned}
\left|\bar{u}_{x,\frac{r}{2}} - \bar{u}_{z,r}\right| &= \left|\frac{1}{m\left(B_{\frac{r}{2}}\right)} \int_{B\left(x,\frac{r}{2}\right)} \{u(y) - \bar{u}_{z,r}\}\, dy\right| \\
&\leq \frac{1}{m\left(B_{\frac{r}{2}}\right)} \int_{B\left(x,\frac{r}{2}\right)} |u(y) - \bar{u}_{z,r}|\, dy \\
&\leq \frac{2^n}{m(B_r)} \int_{B(z,r)} |u(y) - \bar{u}_{z,r}|\, dy \leq 2^n M r^\alpha,
\end{aligned}$$

which proves that claim.

As before we denote by $\mathcal{L}_u$ the Lebesgue points of the function u. Fix $x \in \mathcal{L}_u$ and $r > 0$ such that $\bar{B}(x,r) \subseteq \Omega$. It follows in particular from (23) that

$$\left|\bar{u}_{x,2^{-j}r} - \bar{u}_{x,2^{-j+1}r}\right| \leq 2^n M 2^{(-j+1)\alpha} r^\alpha$$

for all $j = 1, 2, \ldots$ and so

$$\begin{aligned}\left|\bar{u}_{x,2^{-k}} - \bar{u}_{x,r}\right| &\leq \sum_{j=1}^{k} \left|\bar{u}_{x,2^{-j}r} - \bar{u}_{x,2^{-j+1}r}\right| \leq 2^n M r^\alpha \sum_{j=1}^{k} 2^{(-j+1)\alpha} \\ &= 2^n \frac{1-2^{-k\alpha}}{1-2^{-\alpha}} M r^\alpha \leq 2^n \frac{1}{1-2^{-\alpha}} M r^\alpha = C_1 M r^\alpha\end{aligned}$$

for all $k = 1, 2, \ldots$. Since $x \in \mathcal{L}_u$ we have

$$\lim_{r \downarrow 0} \bar{u}_{x,2^{-k}} = u(x),$$

hence

$$|u(x) - \bar{u}_{x,r}| \leq C_1 M r^\alpha.$$

Now take $x \in \Omega$ and $r > 0$ such that $\bar{B}(x, 2r) \subseteq \Omega$. For any $y \in B(x,r) \cap \mathcal{L}_u$ we find that

$$\begin{aligned}|u(y) - \bar{u}_{x,2r}| &\leq |u(y) - \bar{u}_{y,r}| + |\bar{u}_{y,r} - \bar{u}_{x,2r}| \\ &\leq C_1 M r^\alpha + 2^n M (2r)^\alpha = [C_1 + 2^n 2^\alpha] M r^\alpha.\end{aligned}$$

This implies that

$$|u(y) - u(z)| \leq C M r^\alpha$$

for all $y, z \in B(x,r) \cap \mathcal{L}_u$ and hence $\omega_u(x,r) \leq C M r^\alpha$. The last statement of the theorem now follows from Lemma 4.4. ■

The following theorem goes back to C.B. Morrey.

Theorem 4.6 *Let Ω be an open subset of $\mathbb{R}^n$ and let $u \in W^{1,1}(\Omega)$. Suppose that $0 < \alpha \leq 1$ and that there exists a constant $M > 0$ such that*

$$\frac{1}{m(B_r)} \int_{B(x,r)} |\nabla u| \, dx \leq M r^{\alpha - 1} \tag{24}$$

for all open balls $B(x,r) \subseteq \Omega$. Then

$$\omega_u(x,r) \leq C M r^\alpha$$

for all $x \in \Omega$ and all $r > 0$ for which $B(x, 2r) \subseteq \Omega$, where $C > 0$ is a constant only depending on n and α. In particular, $u \in C^{0,\alpha}(\Omega)$.

Proof. It follows from Corollary 3.11 that there exists a constant $C_1 > 0$, only depending on n, such that

$$\int_{B(x,r)} |u - \bar{u}_{x,r}| \, dy \leq C_1 r \int_{B(x,r)} |\nabla u| \, dy$$

for all $x \in \Omega$ and $r > 0$ such that $B(x,r) \subseteq \Omega$. Hence it follows from (24) that

$$\frac{1}{m(B_r)} \int_{B(x,r)} |u - \bar{u}_{x,r}|\, dy \leq C_1 M r^{\alpha}$$

for all balls $B(x,r) \subseteq \Omega$. This shows that the conditions of Theorem 4.5 are fulfilled, by which the result follows. ■

Corollary 4.7 *Let Ω be an open subset of $\mathbb{R}^n$ and let $u \in W^{1,p}(\Omega)$, $1 \leq p < \infty$. Suppose that $0 < \alpha \leq 1$ and that there exists a constant $M > 0$ such that*

$$\frac{1}{m(B_r)} \int_{B(x,r)} |\nabla u|^p\, dx \leq M^p r^{p\alpha - p}$$

for all open balls $B(x,r) \subseteq \Omega$. Then

$$\omega_u(x,r) \leq CMr^{\alpha}$$

for all $x \in \Omega$ and all $r > 0$ for which $B(x,2r) \subseteq \Omega$, where $C > 0$ is a constant only depending on n and α. In particular, $u \in C^{0,\alpha}(\Omega)$.

Proof. For all $B(x,r) \subseteq \Omega$ we have

$$\frac{1}{m(B_r)} \int_{B(x,r)} |\nabla u|\, dy \leq \left(\frac{1}{m(B_r)} \int_{B(x,r)} |\nabla u|^p\, dy \right)^{\frac{1}{p}} \leq M r^{\alpha-1}.$$

This shows that the assumptions of the previous theorem are satisfied and the result follows. ■

5 Bounded mean oscillation.

In this section we discuss some of the basic properties of functions of bounded mean oscillation. In particular we will present a proof of the John-Nirenberg theorem.

5.1 Definitions.

Let Ω be an open subset of $\mathbb{R}^n$. We will assume that Ω is connected. It will be more convenient in this section to use closed cubes in $\mathbb{R}^n$ instead of balls. The collection of all closed cubes in $\mathbb{R}^n$ will be denoted by $\mathcal{Q}$. If $f \in L^1_{loc}(\Omega)$ and if Q is a closed cube such that $Q \subseteq \Omega$ then we will denote the mean value of f on Q by f_Q, i.e.,

$$f_Q = \frac{1}{m(Q)} \int_Q f(x)\, dx.$$

Definition 5.1 *We say that a function $f \in L^1_{loc}(\Omega)$ is of bounded mean oscillation on Ω if*

$$\|f\|_{BMO} = \sup_{\substack{Q \in \mathcal{Q} \\ Q \subseteq \Omega}} \frac{1}{m(Q)} \int_Q |f(x) - f_Q| \, dx < \infty.$$

The collection of all such functions will be denoted by $BMO(\Omega)$.

It is clear that $BMO(\Omega)$ is a vector space and that $\|\cdot\|_{BMO}$ is a seminorm. If $f \in L^1_{loc}(\Omega)$, then $\|f\|_{BMO} = 0$ if and only if f is a.e. constant on Ω. Identifying functions which differ by a constant, i.e., replacing $BMO(\Omega)$ by the quotient space $BMO(\Omega) \diagup \mathbb{C}\mathbf{1}$, we obtain the normed space $(BMO(\Omega) \diagup \mathbb{C}\mathbf{1}, \|\cdot\|_{BMO})$. In the sequel however we will consider $BMO(\Omega)$ as a space of functions and not of such equivalence classes.

It is easy to see that $L^\infty(\Omega) \subseteq BMO(\Omega)$ and that $\|f\|_{BMO} \leq 2\|f\|_\infty$ for all $f \in L^\infty(\Omega)$ (and actually $\|f\|_{BMO} \leq \|f\|_\infty$ for all $f \in L^\infty(\Omega)$).

Lemma 5.2

1. *If $f \in L^1_{loc}(\Omega)$ and if there exists a constant $A \geq 0$ such that for every closed cube $Q \subseteq \Omega$ there is an $\alpha_Q \in \mathbb{C}$ with the property that*
$$\frac{1}{m(Q)} \int_Q |f(x) - \alpha_Q| \, dx \leq A, \tag{25}$$
then $f \in BMO(\Omega)$ and $\|f\|_{BMO} \leq 2A$.

2. *If $f \in BMO(\Omega)$ then $|f| \in BMO(\Omega)$ and $\||f|\|_{BMO} \leq 2\|f\|_{BMO}$. In particular $BMO(\Omega)$ is a lattice.*

3. *In the case that $\Omega = \mathbb{R}^n$, the space $BMO(\mathbb{R}^n)$ is invariant under translations and dilations. To be more precise, if $f \in BMO(\mathbb{R}^n)$ and if we define for $a \in \mathbb{R}^n$ and $\delta > 0$ the functions $T_a f$ and $D_\delta f$ by $(T_a f)(x) = f(x+a)$ and $(D_\delta f)(x) = f(\delta x)$ respectively for all $x \in \mathbb{R}^n$, then $T_a f, D_\delta f \in BMO(\mathbb{R}^n)$ and $\|T_a f\|_{BMO} = \|D_\delta f\|_{BMO} = \|f\|_{BMO}$.*

Proof.

1. First note that (25) implies that

$$\begin{aligned} |f_Q - \alpha_Q| &= \left| \frac{1}{m(Q)} \int_Q (f(x) - \alpha_Q) \, dx \right| \\ &\leq \frac{1}{m(Q)} \int_Q |f(x) - \alpha_Q| \, dx \leq A. \end{aligned}$$

Hence

$$\frac{1}{m(Q)}\int_Q |f(x)-f_Q|\,dx \tag{26}$$
$$\leq \frac{1}{m(Q)}\int_Q |f(x)-\alpha_Q|\,dx + |f_Q-\alpha_Q| \leq 2A.$$

2. For any closed cube $Q \subseteq \Omega$ we have

$$\frac{1}{m(Q)}\int_Q \left||f(x)|-|f_Q|\right|dx \leq \frac{1}{m(Q)}\int_Q |f(x)-f_Q|\,dx \leq \|f\|_{BMO}$$

and so the result follows immediately from 1. (applied with $\alpha_Q = |f_Q|$).

3. For $T_a f$ the statement follows immediately from the translation invariance of the Lebesgue measure in $\mathbb{R}^n$. We indicate the proof for $D_\delta f$. Let Q be any closed cube in $\mathbb{R}^n$. We denote $\delta Q = \{\delta x : x \in Q\}$. Then

$$\begin{aligned}(D_\delta f)_Q &= \frac{1}{m(Q)}\int_Q f(\delta x)\,dx = \frac{1}{\delta^n m(Q)}\int_{\delta Q} f(y)\,dy \\ &= \frac{1}{m(\delta Q)}\int_{\delta Q} f(y)\,dy = f_{\delta Q}.\end{aligned}$$

Hence

$$\frac{1}{m(Q)}\int_Q \left|D_\delta f(x)-(D_\delta f)_Q\right|dx = \frac{1}{m(\delta Q)}\int_{\delta Q} |f(y)-f_{\delta Q}|\,dy, \tag{27}$$

from which the claim follows.

■

Remark 5.3 *We consider the case $\Omega = \mathbb{R}^n$. It is easy to see that a function $f \in L^1_{loc}(\mathbb{R}^n)$ belongs to $BMO(\mathbb{R}^n)$ if and only if there exists a constant $A > 0$ such that*

$$\frac{1}{m(B)}\int_B |f(x)-f_B|\,dx \leq A$$

for all open (or closed) balls $B \subseteq \mathbb{R}^n$. It is clear that (27) and (26) also hold if we replace the cube Q by a ball B.

Example 5.4 *Take $\Omega = \mathbb{R}^n$ and define $f(x) = \log|x|$. Then $f \in BMO(\mathbb{R}^n)$. To verify this statement first observe that it follows from (27) that for any open ball $B \subseteq \mathbb{R}^n$ and all $\delta > 0$ we have*

$$\begin{aligned}\frac{1}{m(B)}\int_B |f(x)-f_B|\,dx &= \frac{1}{m\left(\delta^{-1}B\right)}\int_{\delta^{-1}B} |D_\delta f(x)-(D_\delta f)_{\delta^{-1}B}|\,dx \\ &= \frac{1}{m\left(\delta^{-1}B\right)}\int_{\delta^{-1}B} |f(x)-f_{\delta^{-1}B}|\,dx,\end{aligned}$$

as $D_\delta f(x) = \log(|\delta x|) = \log|x| + \log(\delta)$. *Therefore it is sufficient to consider balls* B *with radius equal to* 1 *only. Now we consider two cases:*

(i). $B = B(x_0, 1)$ *with* $|x_0| \leq 2$. *Then* $B \subseteq B(0,3)$ *and so*

$$\frac{1}{m(B)} \int_B |f(x)|\, dx \leq \frac{1}{m(B)} \int_{B(0,3)} |\log|x||\, dx < \infty.$$

(ii). $B = B(x_0, 1)$ *with* $|x_0| > 2$. *In this case we have*

$$\frac{1}{m(B)} \int_B |f(x) - f(x_0)|\, dx = \frac{1}{m(B)} \int_B \left| \log \frac{|x|}{|x_0|} \right| dx \leq \log 2,$$

since

$$\frac{1}{2} \leq \frac{|x|}{|x_0|} \leq 2$$

for all $x \in B$.

Via (26) we may now conclude that $f \in BMO(\mathbb{R}^n)$. *It should be observed that (in case* $n = 1$*) the function* $f(x) = \log|x|\, \chi_{(0,\infty)}(x)$ *does not belong to* $BMO(\mathbb{R})$ *(although it belongs to* $BMO(0,\infty)$*).*

Proposition 5.5 $(BMO(\Omega) \diagup \mathbb{C}\mathbf{1}, \|\cdot\|_{BMO})$ *is a Banach space.*

Proof. We start with the following observation. Suppose that $Q_1, Q_2 \in \mathcal{Q}$ such that $Q_1 \subseteq Q_2 \subseteq \Omega$. For any $f \in BMO(\Omega)$ we then have

$$\begin{aligned} |f_{Q_1} - f_{Q_2}| &= \left| \frac{1}{m(Q_1)} \int_{Q_1} (f(x) - f_{Q_2})\, dx \right| \\ &\leq \frac{1}{m(Q_1)} \int_{Q_1} |f(x) - f_{Q_2}|\, dx \\ &\leq \left(\frac{m(Q_2)}{m(Q_1)} \right) \frac{1}{m(Q_2)} \int_{Q_2} |f(x) - f_{Q_2}|\, dx \qquad (28) \\ &\leq \left(\frac{m(Q_2)}{m(Q_1)} \right) \|f\|_{BMO}. \end{aligned}$$

¿From this observation it follows that for any two closed cubes $Q_1, Q_2 \subseteq \Omega$ such that $\operatorname{int}(Q_1 \cap Q_2) \neq \emptyset$ there exists a constant $c(Q_1, Q_2) > 0$ such that

$$|f_{Q_1} - f_{Q_2}| \leq c(Q_1, Q_2) \|f\|_{BMO} \qquad (29)$$

for all $f \in BMO(\Omega)$. Indeed, take a closed cube $Q_3 \subseteq Q_1 \cap Q_2$. Then it follows from (28) that

$$\begin{aligned} |f_{Q_1} - f_{Q_2}| &\leq |f_{Q_1} - f_{Q_3}| + |f_{Q_3} - f_{Q_2}| \\ &\leq \left(\frac{m(Q_1)}{m(Q_3)} + \frac{m(Q_2)}{m(Q_3)} \right) \|f\|_{BMO} \end{aligned}$$

for all $f \in BMO(\Omega)$.

Now assume that $\{f_k\}_{k=1}^{\infty}$ in $BMO(\Omega)$ is such that $\|f_k - f_l\|_{BMO} \to 0$ as $k, l \to \infty$. We have to prove that there exists $f \in BMO(\Omega)$ such that $\|f_k - f\|_{BMO} \to 0$ as $k \to \infty$. Take a fixed closed cube $Q_0 \subseteq \Omega$. Replacing f_k by $f_k - (f_k)_{Q_0}$ we may assume that $(f_k)_{Q_0} = 0$ for all k. Now let $\mathcal{Q}_c$ be the collection of all closed cubes $Q \subseteq \Omega$ with the property that $\left\{ (f_k)_Q \right\}_{k=1}^{\infty}$ is convergent in $\mathbb{C}$ and put $\Omega_c = \bigcup \{\text{int}(Q) : Q \in \mathcal{Q}_c\}$. We claim that $\Omega \diagdown \Omega_c$ is open. Indeed, let Q_1 be any closed cube such that $Q_1 \subseteq \Omega$ and $Q_1 \cap \Omega_c \neq \emptyset$. Then there exists $Q_2 \in \mathcal{Q}_c$ such that $\text{int}(Q_1 \cap Q_2) \neq \emptyset$ and so it follows from (29) that

$$\begin{aligned} & \left| \left\{ (f_k)_{Q_1} - (f_l)_{Q_1} \right\} - \left\{ (f_k)_{Q_2} - (f_l)_{Q_2} \right\} \right| \\ = & \left| (f_k - f_l)_{Q_1} - (f_k - f_l)_{Q_2} \right| \leq c(Q_1, Q_2) \|f_k - f_l\|_{BMO}. \end{aligned}$$

Since $\left\{ (f_k)_{Q_2} \right\}_{k=1}^{\infty}$ is convergent, this implies that $\left\{ (f_k)_{Q_1} \right\}_{k=1}^{\infty}$ is convergent and hence $Q_1 \in \mathcal{Q}_c$. Hence $Q_1 \subseteq \Omega_c$, from which the claim follows. Since Ω is assumed to be connected this shows that $\Omega = \Omega_c$. We thus have shown that for any closed cube $Q \subseteq \Omega$ the sequence $\left\{ (f_k)_Q \right\}_{k=1}^{\infty}$ is convergent. ¿From this it follows that

$$\begin{aligned} & \frac{1}{m(Q)} \int_Q |f_k - f_l| \, dx \\ \leq & \frac{1}{m(Q)} \int_Q \left| (f_k - f_l) - (f_k - f_l)_Q \right| dx + \left| (f_k)_Q - (f_l)_Q \right| \\ \leq & \|f_k - f_l\|_{BMO} + \left| (f_k)_Q - (f_l)_Q \right|. \end{aligned}$$

Therefore, the restrictions of $\{f_k\}$ to Q are a Cauchy sequence in $L^1(Q)$ and so there exists $f^Q \in L^1(Q)$ such that $\left\| f_k - f^Q \right\|_{L^1(Q)} \to 0$ as $k \to \infty$. If Q_1 and Q_2 are two such closed cubes, then it is clear that $f^{Q_1} = f^{Q_2}$ a.e. on $Q_1 \cap Q_2$. Hence there exists $f \in L^1_{loc}(\Omega)$ such that $f_{|Q} = f^Q$ for every closed cube $Q \subseteq \Omega$. It remains to show that $f \in BMO(\Omega)$ and that

$\|f_k - f\|_{BMO} \to 0$ as $k \to \infty$. To this end let $\varepsilon > 0$ be given. Then there exists $N \in \mathbb{N}$ such that $\|f_k - f_l\| \leq \varepsilon$ for all $k, l \geq N$. Take a closed cube $Q \subseteq \Omega$. Then

$$\frac{1}{m(Q)} \int_Q \left| f_k - f_l - (f_k)_Q - (f_l)_Q \right| dx \leq \varepsilon$$

for all $k, l \geq N$. Since $(f_l)_{|Q} \to f_{|Q}$ in $L^1(Q)$ implies that $(f_l)_Q \to f_Q$, it follows via Fatou's lemma that

$$\frac{1}{m(Q)} \int_Q \left| (f_k - f) - (f_k - f)_Q \right| dx \leq \varepsilon$$

for all $k \geq N$. This shows that

$$\sup_{\substack{Q \in \mathcal{Q} \\ Q \subseteq \Omega}} \frac{1}{m(Q)} \int_Q \left| (f_k - f) - (f_k - f)_Q \right| dx \leq \varepsilon$$

for all $k \geq N$. Consequently $f_k - f \in BMO(\Omega)$, so $f \in BMO(\Omega)$, and

$$\|f_k - f\|_{BMO} \leq \varepsilon$$

for all $k \geq N$, which completes the proof of the proposition. ■

5.2 The John-Nirenberg theorem.

For the proof of this theorem we will use the following form of the so-called Calderón-Zygmund decomposition. For sake of convenience we will say that two closed cubes in $\mathbb{R}^n$ are disjoint if their interiors are disjoint.

Proposition 5.6 *Let Q_0 be a closed cube in $\mathbb{R}^n$ and suppose that $u \in L^1(Q_0)$. Let α be a constant such that*

$$\frac{1}{m(Q_0)} \int_{Q_0} |u| \, dx < \alpha.$$

Then there exists an at most countable collection $\{Q_j\}$ of mutually disjoint subcubes of Q_0 such that

1. *$|u| \leq \alpha$ a.e. on $Q_0 \diagdown \left(\bigcup_j Q_j \right)$;*

2. *for all j we have*

$$\alpha \leq \frac{1}{m(Q_j)} \int_{Q_j} |u| \, dx < 2^n \alpha;$$

3. $\sum_j m(Q_j) \le \alpha^{-1} \int_{Q_0} |u|\, dx$.

Proof. Without loss of generality we may assume that $Q_0 = [0,1]^n$. The following terminology will be convenient. If $Q \subseteq Q_0$ is a subcube, then we will say that:

- Q is a case I cube if $m(Q)^{-1} \int_Q |u|\, dx < \alpha$;
- Q is a case II cube if $m(Q)^{-1} \int_Q |u|\, dx \ge \alpha$.

By hypothesis, Q_0 is a case I cube. Now we partition Q_0 in 2^n equal disjoint cubes. If one of these subcubes is a case II cube, then we put this cube in the collection $\{Q_j\}$. On the remaining case I cubes we repeat the above procedure.

We claim that the in this way constructed collection $\{Q_j\}$ has the desired properties. Indeed, it is clear that the collection $\{Q_j\}$ is pairwise disjoint. Now take $x \in Q_0 \diagdown \left(\bigcup_j Q_j\right)$. Then any dyadic cube $\Delta \subseteq Q_0$ with $x \in \Delta$ must be case I and so

$$\frac{1}{m(\Delta)} \int_\Delta |u|\, dx < \alpha.$$

Now assume in addition that x is a Lebesgue point of $|u|$ and take a sequence $\{\Delta_k\}_{k=1}^\infty$ of dyadic cubes in Q_0 such that $x \in \Delta_k$ for all k and $m(\Delta_k) \to 0$ as $k \to \infty$. Then

$$\frac{1}{m(\Delta_k)} \int_{\Delta_k} |u|\, dx \to |u(x)| \quad \text{as } k \to \infty$$

and so $|u(x)| \le \alpha$. This shows that $\{Q_j\}$ satisfies (i).

To prove property (ii), take any of the cubes Q_j. Let Q_j^* be the dyadic ancestor of Q_j (i.e., Q_j was obtained by subdivision of Q_j^*). Since Q_j^* was not selected, it must be a case I cube. Hence,

$$\alpha > \frac{1}{m\left(Q_j^*\right)} \int_{Q_j^*} |u|\, dx \ge \frac{1}{2^n m(Q_j)} \int_{Q_j} |u|\, dx \ge 2^{-n}\alpha$$

and this is (ii).

Finally, since the cubes $\{Q_j\}$ are mutually disjoint, it follows immediately from (ii) that

$$\sum_j m(Q_j) \le \sum_j \frac{1}{\alpha} \int_{Q_j} |u|\, dx \le \frac{1}{\alpha} \int_{Q_0} |u|\, dx,$$

which is (iii). ■

Theorem 5.7 (John-Nirenberg) *Let Q_0 be a closed cube in $\mathbb{R}^n$ and suppose that $f \in BMO(Q_0)$. Then for every cube $Q \subseteq Q_0$ and all $\lambda > 0$ we have*

$$m(\{x \in Q : |f(x) - f_Q| > \lambda\}) \leq Cm(Q) \exp\left(\frac{-c\lambda}{\|f\|_{BMO}}\right),$$

where $C, c > 0$ are constants only depending on n.

Proof. We may assume that $\|f\|_{BMO} \leq 1$, i.e., that

$$\frac{1}{m(Q)} \int_Q |f(x) - f_Q| \, dx \leq 1$$

for all cubes $Q \subseteq Q_0$.

Now let Q be a fixed cube in Q_0. We apply Proposition 5.6 to the function $u = |f - f_Q|$ with $\alpha = \frac{3}{2}$. This yields a collection $\left\{Q_j^1\right\}_{j=1}^{\infty}$ of disjoint subcubes of Q such that

(i). $|f - f_Q| \leq \frac{3}{2}$ a.e. on $Q \setminus \left(\bigcup_j Q_j^1\right)$;

(ii). for all j we have

$$\left|f_{Q_j^1} - f_Q\right| \leq \frac{1}{m\left(Q_j^1\right)} \int_{Q_j^1} |f(x) - f_Q| \, dx < 3.2^{n-1};$$

(iii).

$$\sum_j m\left(Q_j^1\right) \leq \frac{2}{3} \int_Q |f - f_Q| \, dx \leq \frac{2}{3} m(Q).$$

Note that the first inequality in (ii) follows from

$$\left|f_{Q_j^1} - f_Q\right| = \left|\frac{1}{m\left(Q_j^1\right)} \int_{Q_j^1} (f(x) - f_Q) \, dx\right| \leq \frac{1}{m\left(Q_j^1\right)} \int_{Q_j^1} |f(x) - f_Q| \, dx.$$

Now apply Proposition 5.6 to each cube Q_j^1 and the function $\left|f - f_{Q_j^1}\right|$, again with $\alpha = \frac{3}{2}$. This gives a collection $\left\{Q_j^2\right\}$ of disjoint cubes (each Q_j^2 is contained in some Q_j^1). For almost all $x \in \left(\bigcup Q_j^1\right) \setminus \left(\bigcup Q_j^2\right)$ we have

$$|f(x) - f_Q| \leq \left|f(x) - f_{Q_j^1}\right| + \left|f_{Q_j^1} - f_Q\right| < \frac{3}{2} + 3.2^{n-1} < 2.3.2^{n-1}.$$

This inequality certainly holds for $x \in Q \diagdown \left(\bigcup Q_j^1\right)$. Hence

$$|f(x) - f_Q| < 2.3.2^{n-1}$$

for all $x \in Q \diagdown \left(\bigcup Q_j^2\right)$. Furthermore,

$$\left|f_{Q_j^2} - f_{Q_j^1}\right| \leq \frac{1}{m\left(Q_j^2\right)} \int_{Q_j^2} \left|f(x) - f_{Q_j^1}\right| dx < 3.2^{n-1}$$

whenever $Q_j^2 \subseteq Q_j^1$ and so

$$\left|f_{Q_j^2} - f_Q\right| \leq \left|f_{Q_j^2} - f_{Q_j^1}\right| + \left|f_{Q_j^1} - f_Q\right| < 2.3.2^{n-1}.$$

Moreover,

$$\sum_j m\left(Q_j^2\right) \leq \frac{2}{3} \sum_j m\left(Q_j^1\right) \leq \left(\frac{2}{3}\right)^2 m(Q).$$

Continuing this process we obtain at stage k a collection $\left\{Q_j^k\right\}$ of mutually disjoint cubes such that

$$\begin{cases} |f(x) - f_Q| < 3k2^{n-1} \text{ a.e. on } Q \diagdown \left(\bigcup_j Q_j^k\right) \\ \sum_j m\left(Q_j^k\right) \leq \left(\frac{2}{3}\right)^k m(Q) \end{cases}.$$

Now take $\lambda > 0$ and suppose that $3k2^{n-1} < \lambda \leq 3(k+1)2^{n-1}$ for some $k \geq 1$. Then

$$\{x \in Q : |f(x) - f_Q| > \lambda\} \subseteq \bigcup_j Q_j^k$$

and so

$$\begin{aligned} m(\{x \in Q : |f(x) - f_Q| > \lambda\}) &\leq \sum_j m\left(Q_j^k\right) \leq \left(\frac{2}{3}\right)^k m(Q) \\ &\leq e^{-c\lambda} m(Q) \end{aligned}$$

with e.g. $c = 3^{-1}2^{-n} \log\left(\frac{3}{2}\right)$. If $0 < \lambda \leq 3.2^{n-1}$, then

$$m(\{x \in Q : |f(x) - f_Q| > \lambda\}) \leq m(Q) \leq e^{3.2^{n-1}c} e^{-c\lambda} m(Q)$$

and so we can take $C = \exp(3.2^{n-1}c)$. ■

5.3 Some consequences.

Next we will discuss some consequences of the John-Nirenberg theorem. First we recall some useful formulas from general integration theory.

Suppose that $(\Omega, \mathcal{A}, \mu)$ is a measure space. Let $f : \Omega \to \mathbb{C}$ be a measurable function. The distribution function $d_{|f|} : [0, \infty) \to [0, \mu(\Omega)]$ of $|f|$ is then defined by

$$d_{|f|}(\lambda) = \mu\{x \in \Omega : |f(x)| > \lambda\}.$$

Note that $d_{|f|}$ is decreasing, left continuous and if there exists λ_0 such that $d_{|f|}(\lambda_0) < \infty$, then $d_{|f|}(\lambda) \to 0$ as $\lambda \to \infty$.

Lemma 5.8 *Let $\varphi : [0, \infty) \to [0, \infty)$ be measurable and define $\Phi(t) = \int_0^t \varphi(s)\, ds$ for all $t \geq 0$. For every measurable function $f : \Omega \to \mathbb{C}$ we have*

$$\int_\Omega \Phi(|f|)\, d\mu = \int_0^\infty \varphi(s)\, d_{|f|}(s)\, ds.$$

Proof. Using Fubini's theorem it follows that

$$\begin{aligned}
\int_\Omega \Phi(|f|)\, d\mu(x) &= \int_\Omega \int_0^{|f(x)|} \varphi(s)\, ds d\mu(x) \\
&= \int_0^\infty \int_\Omega \chi_{\{|f|>s\}} \varphi(s)\, d\mu(x)\, ds \\
&= \int_0^\infty \varphi(s)\, \mu\{x \in \Omega : |f(x)| > s\}\, ds \\
&= \int_0^\infty \varphi(s)\, d_{|f|}(s)\, ds.
\end{aligned}$$

■

Applying the above lemma to the functions $\varphi(s) = ps^{p-1}$ and $\varphi(s) = ke^{ks}$ respectively we immediately get the following corollary.

Corollary 5.9 *For every measurable function $f : \Omega \to \mathbb{C}$ we have:*

1. $$\int_\Omega |f|^p\, d\mu = p \int_0^\infty s^{p-1} d_{|f|}(s)\, ds \tag{30}$$

   for all $1 \leq p < \infty$;

2. $$\int_\Omega \left(e^{k|f|} - 1\right) d\mu = k \int_0^\infty e^{ks} d_{|f|}(s)\, ds \tag{31}$$

   for all $k \in \mathbb{R}$.

Now we return to BMO-functions on $\mathbb{R}^n$. Suppose that Q is a cube in $\mathbb{R}^n$ and that $f \in BMO(Q)$. For notational convenience we denote the distribution function of $|f - f_Q|$ on Q simply by d. By the John-Nirenberg theorem, this distribution function d satisfies

$$d(\lambda) \leq Cm(Q) \exp\left(\frac{-c\lambda}{\|f\|_{BMO}}\right) \tag{32}$$

for all $\lambda > 0$, where $c, C > 0$ are two constants depending only on n.

Proposition 5.10 *There exist constants $c_1, c_2 > 0$, only depending on n, such that for all $k < c_2$ and all $f \in BMO(Q)$ we have*

$$\frac{1}{m(Q)} \int_Q \exp\left(\frac{k}{\|f\|_{BMO}} |f - f_Q|\right) dx \leq c_1.$$

Proof. Let $c > 0$ denote the same constant as in (32) and take $c_2 = \frac{c}{2}$. Fix a function $f \in BMO(Q)$ and denote the distribution function of $|f - f_Q|$ by d. Take $k < c_2$. Using (31) and the estimate (32) we find that

$$\begin{aligned}
& \int_Q \left[\exp\left(\frac{k}{\|f\|_{BMO}} |f - f_Q|\right) - 1\right] dx \\
= \quad & \frac{k}{\|f\|_{BMO}} \int_0^\infty \exp\left(\frac{k}{\|f\|_{BMO}} s\right) d(s)\, ds \\
\leq \quad & \frac{k}{\|f\|_{BMO}} Cm(Q) \int_0^\infty \exp\left(\frac{k}{\|f\|_{BMO}} s\right) \exp\left(\frac{-c}{\|f\|_{BMO}} s\right) ds \\
\leq \quad & \frac{k}{\|f\|_{BMO}} Cm(Q) \int_0^\infty \exp\left(\frac{-c}{2\|f\|_{BMO}} s\right) ds \\
= \quad & \frac{k}{\|f\|_{BMO}} Cm(Q) \frac{2\|f\|_{BMO}}{c} \leq Cm(Q).
\end{aligned}$$

This implies that

$$\int_Q \exp\left(\frac{k}{\|f\|_{BMO}} |f - f_Q|\right) dx \leq (C+1)\, m(Q)$$

and so we can take $c_1 = C + 1$. ■

Another consequence of the John-Nirenberg theorem is the following proposition.

Proposition 5.11 *For every $1 \leq p < \infty$ there exists a constant C_p, only depending on n and p, such that for any closed cube $Q \subseteq \mathbb{R}^n$ and all $f \in BMO(Q)$ we have*

$$\left(\frac{1}{m(Q)} \int_Q |f - f_Q|^p \, dx \right)^{\frac{1}{p}} \leq C_p \|f\|_{BMO}.$$

Proof. Take $f \in BMO(Q)$ and let d be the distribution function of $|f - f_Q|$ on Q. Then it follows from (30) and (32) that

$$\begin{aligned} \int_Q |f - f_Q|^p \, dx &= p \int_0^\infty s^{p-1} d(s) \, ds \\ &\leq pCm(Q) \int_0^\infty s^{p-1} \exp\left(\frac{-cs}{\|f\|_{BMO}} \right) ds \\ &= pCm(Q) \frac{\|f\|_{BMO}^p}{c^p} \int_0^\infty s^{p-1} e^{-s} ds \\ &= C_p^p m(Q) \|f\|_{BMO}^p, \end{aligned}$$

which implies the result of the proposition. ■

Corollary 5.12 *Let $\Omega \subseteq \mathbb{R}^n$ be open and $f \in L^1_{loc}(\Omega)$. The following statements are equivalent:*

1. *$f \in BMO(\Omega)$;*
2. *for all (some) $1 \leq p < \infty$ we have*

$$\sup_{\substack{Q \in \mathcal{Q} \\ Q \subseteq \Omega}} \left(\frac{1}{m(Q)} \int_Q |f - f_Q|^p \, dx \right)^{\frac{1}{p}} < \infty \tag{33}$$

Moreover, (33) defines an equivalent norm on $BMO(\Omega)$.

Proof. First assume that $f \in BMO(\Omega)$ and let $1 \leq p < \infty$ be given. Then it is clear from the definition that $f \in BMO(Q)$ for any closed cube $Q \subseteq \Omega$ with $\|f\|_{BMO(Q)} \leq \|f\|_{BMO(\Omega)}$. Now it follows from Proposition 5.11 that

$$\left(\frac{1}{m(Q)} \int_Q |f - f_Q|^p \, dx \right)^{\frac{1}{p}} \leq C_p \|f\|_{BMO(Q)} \leq C_p \|f\|_{BMO(\Omega)},$$

which implies 2.

Now assume that $f \in L^1_{loc}(\Omega)$ is such that (33) is satisfied for some $1 \leq p < \infty$. Since

$$\frac{1}{m(Q)} \int_Q |f - f_Q| \, dx \leq \left(\frac{1}{m(Q)} \int_Q |f - f_Q|^p \, dx \right)^{\frac{1}{p}}$$

for any closed cube $Q \subseteq \Omega$, it follows immediately that $f \in BMO(\Omega)$. The final statement of the corollary is now also clear form the proof. ■

Before formulating the next result we recall the following definition. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and suppose that $f : \Omega \to \mathbb{C}$ is a measurable function such that $d_{|f|}(\lambda) < \infty$ for some $\lambda > 0$. The decreasing rearrangement $f^* : (0, \infty) \to [0, \infty)$ of $|f|$ is then defined by

$$f^*(t) = \inf \left\{ \lambda > 0 : d_{|f|}(\lambda) < t \right\}$$

for all $t > 0$. The function f^* is decreasing, left-continuous and equimeasurable with $|f|$ (i.e., the functions f^* and $|f|$ have the same distribution function).

Proposition 5.13 *Let $Q \subseteq \mathbb{R}^n$ be a closed cube and $f \in BMO(Q)$. Then*

$$(f - f_Q)^*(t) \leq \frac{\|f\|_{BMO}}{c} \log^+ \left(\frac{Cm(Q)}{t} \right)$$

for all $t > 0$, where $c, C > 0$ are constants depending only on n.

Proof. Take $f \in BMO(Q)$ and denote the distribution function of $|f - f_Q|$ by $d(\lambda)$. Using (32) it follows that

$$\begin{aligned}
(f - f_Q)^*(t) &= \inf \{ \lambda > 0 : d(\lambda) < t \} \\
&\leq \inf \left\{ \lambda > 0 : Cm(Q) \exp\left(\frac{-c\lambda}{\|f\|_{BMO}} \right) < t \right\} \\
&= \inf \left\{ \lambda > 0 : -\frac{c\lambda}{\|f\|_{BMO}} < \log\left(\frac{t}{Cm(Q)} \right) \right\} \\
&= \inf \left\{ \lambda > 0 : \lambda > \frac{\|f\|_{BMO}}{c} \log\left(\frac{Cm(Q)}{t} \right) \right\} \\
&= \frac{\|f\|_{BMO}}{c} \log^+ \left(\frac{Cm(Q)}{t} \right),
\end{aligned}$$

and we are done. ■