STRONGLY CONVERGENT MATRIX MODELS FOR q-GAUSSIAN ALGEBRAS

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Abstract. We construct strongly convergent finite-dimensional random matrix models for finite q-Gaussian family in the range $|q| < \sqrt{2}-1$. The construction has two stages. First, we show that normalized sums of graph-product semicirculars over an Erdős–Rényi graph satisfy the q-Toeplitz relations up to an operator norm error converging to zero in probability. Using ultraproduct methods, this yields complete strong convergence, uniformly over all matrix coefficient dimensions, for noncommutative polynomials with bounded degree. Second, we use a quantitative tensor-GUE for graph-product semicirculars to convert these operator models into finite-dimensional random matrices. For every fixed polynomial degree, the convergence is uniform over coefficient dimensions that may be larger than the dimension of the random matrices. As applications, in the above range of q, the C^*-algebra generated by a finite q-Gaussian family is MF, and the Brown–Douglas–Fillmore extension semigroup of the nontrivial C^*-algebra generated by a finite q-Gaussian family is not a group.

1. Introduction

The q-Gaussian processes and q-semicircular variables, $-1 \leq q \leq 1$, have been introduced and developed by Bożejko and Speicher [BoSp91] and Bożejko, Kümmerer and Speicher [BKS97]. They interpolate between Bosonic classical fields of Gaussian variables ($q = 1$) and Fermionic CAR algebras ($q = -1$). The particular parameter $q = 0$ gives a free semicircular system and is of fundamental importance in Voiculescu's theory of free probability. The aim of this paper is to construct random matrix tuples such that norms of noncommutative polynomials in that tuple converge to norms of the polynomial evaluated at a q-semicircular system. The main feature of our construction is that convergence holds even when the coefficient dimension grows beyond the size of the random matrices.

Let us give an overview of the origin and motivation of strong convergence results. Firstly, distinction between weak convergence (convergence in moment) and strong convergence is essential. Voiculescu proved that the independent GUE ('Gaussian Unitary Ensemble') converges in joint $*$-moments to a free semicircular family [Voi91]. Haagerup and Thorbjørnsen subsequently proved convergence of the operator norm of every noncommutative polynomial [HaTh05]. Together with weak convergence, this is called *strong convergence*; it retains the limiting C^*-norm and rules out spectral outliers that are invisible to normalized traces. Subsequent developments include the real and symplectic Gaussian ensembles [Sch05], deterministic deformations of GUE matrices [CoMa14], and the Wigner-type ensembles [And13].

A substantially stronger question is whether strong convergence remains uniform under growing matrix coefficients. More precisely, assume an ensemble $X^{(N)}$ to converge to X. For matrices

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$A_w^{(N)} \in M_{D_N}(\mathbb{C})$ one considers

$$P_A(X^{(N)}) = \sum_{|w|\leq d} A_w^{(N)} \otimes X_w^{(N)},$$

where w is a word and $X_w^{(N)}$ is the monomial of the ensemble described by w. For every fixed $D_N = D$, this is a finite amplification of ordinary strong convergence. When $D_N \to \infty$, however, scalar strong convergence gives no uniform control over the expanding coefficient ball. Such uniformity probes the higher matrix norms of the bounded-degree polynomial space and hence its operator-space structure, rather than only its scalar C^*-norm. Equivalently, it is a finite-scale version of complete norm convergence and it is relevant to completely bounded maps and minimal tensor products; see [Pis03, Pis14].

Growing matrix coefficients have played an important role in several developments in random matrix theory and operator algebras. Pisier connected strong convergence with slowly growing coefficients to the theory of subexponential operator spaces [Pis14]. In the related tensor-product problem for independent GUE families, Pisier obtained strong convergence when the second matrix size is $o(N^{1/4})$; Collins–Guionnet–Parraud improved this to $o(N^{1/3})$ [CGP22]. Hayes [Hay22] showed that reaching the matrix model dimension regime is powerful enough to have major von Neumann algebraic consequences: the GUE strong convergence problem with equal coefficients sizes implies under certain assumptions the Peterson–Thom conjecture [PeTh11]; this problem was subsequently resolved in [BeCa22, BoCo23] based on Hayes' method. Thereafter, the polynomial method of [CGTvH26a, CGVvH26b] has provided significant improvement of quantitative strong convergence estimates with large matrix coefficients for several important random matrix models, and has established strong convergence for tensor-GUE, which was conjectured by [MaTh26]. In particular, C.-F. Chen–Garza-Vargas–van Handel proved strong convergence for the classical Gaussian ensembles with matrix coefficients of dimension $\exp(o(N))$ [CGVvH26b]. Further strong convergence results with matrix coefficients and explicit rate functions, preceding [CGVvH26b], are contained in [BaBoVH23, Par22, Par23, Par24, MaSa25], and for further background, we refer the reader to the surveys [vH26a, vH26b] and the references therein.

1.1. **Strongly convergent random matrix models for q-Gaussian families for $|q| < \sqrt{2}-1$.** The main result of the present paper is a strong convergence result with matrix coefficients growing faster than the model dimension for a finite tuple of q-semicircular variables (Theorem 6.11) in the range $|q| < \sqrt{2} - 1$.

In passing we note that for $|q| < q_0(r)$, free monotone transport [GuSh14] represents an r-tuple of q-Gaussians by analytic noncommutative functions of a free semicircular family. Combining this representation with strong GUE convergence gives an indirect strong approximation in a generator-dependent small-q regime.

To approach our main result here several finite-dimensional approximations of q-Gaussian families were previously known at the level of noncommutative distribution. Speicher's 'ϵ-free' central limit theorem [Spe92] produces the standard spin-matrix approximation in joint $*$-moments, while Śniady [Sni01] constructed Gaussian random-matrix models for $0 < q < 1$ interpolating between Gaussian and semicircular families. These results do not by themselves control the norms of arbitrary noncommutative polynomials.

In this paper we use yet another model that is rooted in the work of Młotkowski [Mlo04] and shows weak convergence of averages of large sums of semicirculars that randomly are free or commuting, where randomness is dictated by the Erdős-Rényi random graph with edge probability q, in case $0 \leq q < 1$. This model is still infinite dimensional but we approximate the semicirculars in this model by GUE's to obtain finite dimensional models.

To achieve this we need to overcome a difficulty that is of a different nature than prior results: besides the coefficient dimension, the tensor-interaction given by the Erdős-Rényi graph and the number of variables also grow. We establish the required uniformity for these varying tensor-GUE models and ultimately obtain q-Gaussian random matrix models with dimensions n_k whose admissible coefficient dimensions D_k satisfy

$$D_k > n_k.$$

Thus the approximation retains bounded-degree matrix norms beyond the matrix model size amplification level.

In fact, C.-F. Chen–Garza-Vargas–van Handel proved that tensor-GUE models associated with a fixed interaction graph converge strongly to the corresponding graph-product semicircular family [CGVvH26b, Theorem 9.8]. Their tensor-GUE strong convergence theorem concerns a fixed graph and, in its sated form, a fixed polynomial with fixed matrix-coefficient dimension. They also observe that a quantitative version can be obtained by bootstrapping their argument, based on their recent developed powerful polynomial methods [CGTvH26a, CGVvH26b]. However, the question for coefficient-uniform estimates remain open. Further, we note that their convergence rates worsen if the size of the graph increases causing a serious difficulty in the analysis.

Our use of tensor-GUE models differs in two respects. First, the interaction graph is an Erdős–Rényi graph whose number of vertices tends to infinity, and the normalized averages of the corresponding graph-product semicirculars converge to a q-Gaussian family. Second, we require the strong convergence of tensor-GUE are uniform over bounded-degree polynomials whose matrix coefficients and coefficient dimensions may vary with the random matrix size. Establishing this uniformity is a substantial part of the argument.

Building on their quantitative Hayes' model strong convergence results and universality argument, we establish a growing-coefficient refinement for tensor-GUE models, which allow the matrix coefficient dimension to exceed the dimension of the matrix models themselves.

The construction proceeds in two stages. In the first stage, we show that normalized sums of graph-product creation operators (when $0 \leq q < 1$) or Clifford-twisted graph-product creation operators (when $-1 \leq q < 0$) indexed by $\Gamma_k \sim \Gamma(rk, |q|)$, namely $L^{q,\Gamma_k}_{k,a}, a = 1, \cdots, r$, satisfy the multivariate approximate q-Toeplitz relations, i.e. q-Toeplitz relations up to an operator-norm error (see Theorem 3.12):

$$(L^{q,\Gamma_k}_{k,a})^* L^{q,\Gamma_k}_{k,b} = \delta_{ab} 1 + q L^{q,\Gamma_k}_{k,b} (L^{q,\Gamma_k}_{k,a})^* + E^{q,\Gamma_k}_{k,ab},$$

where, with probability tending to one,

$$\max_{a,b} \|E^{q,\Gamma_k}_{k,ab}\| = \mathcal{O}_{\mathbb{P},q,r}\left(\frac{\log k}{\sqrt{k}}\right).$$

The error estimate follows from the logarithmic size of the clique number of an Erdős–Rényi graph (see Lemma 3.6 or [BoEr76]), as well as the square-root size of the centered adjacency matrix of an Erdős–Rényi graph (see Theorem 3.7 from [BaYi88], [Tro12, Theorem 1.4]).

Conceptually, this relation implements a q-deformed non-backtracking word reduction: it separates immediate creation-annihilation contractions from reduced propagation, while the random-graph fluctuation are absorbed into the vanishing error $E_{k,ab}$; see Remark 4.2.

Then using the universal q-Toeplitz algebra and an ultraproduct argument, we identify the limiting representation with the q-Fock representation. Nuclearity and exactness then upgrade fixed matrix coefficient strong convergence to complete bounded-matrix coefficient strong convergence.

One of the key ingredient for the ultraproduct argument is applying the faithfulness of the Fock space representations of the universal q-Toeplitz algebras. Although it is expected to hold true for all $-1 < q < 1$, it was only proved for range $|q| < \sqrt{2} - 1$ ([PuWo89]). This is where our specific range of q for the later strong convergence of our random matrix models occur.

In the second stage, we develop a growing-coefficient refinement of the tensor-GUE argument of [CGVvH26b]. A dimension-free Fock-space comparison gives the quantitative estimate (see Lemma 6.2)

$$\sup_{\substack{D \geq 1 \\ \Lambda(A) \leq M}} \left| \|\mathcal{P}_A(s^{(T)})\| - \|\mathcal{P}_A(x^\Gamma)\| \right| \leq \frac{(2^L - 2)M}{\sqrt{T}}.$$

Combining this estimate with the quantitative Hayes's model strong convergence theorem and the universality comparison (developed in [BravH]) used in [CGVvH26b], we obtain that the norms of linear pencils of tensor-GUE models converge uniformly over coefficient dimensions $D \leq N^{2L}$. Then we apply exactness compression to obtain the corresponding uniform lower bound estimate, while a finite polynomial spectral detector and uniform linearization extend the result from linear pencils to bounded-degree noncommutative polynomials.

Finally, we combine these two stages by realizing every graph Γ_k through tensor-coordinate sets and then choosing the GUE dimension sufficiently large. This produces finite-dimensional random matrices $X_1^{(k)}, \dots, X_r^{(k)}$ such that, for every fixed degree d (see Theorem 6.11),

$$\frac{\|P_k(X_1^{(k)}, \dots, X_r^{(k)})\|}{\|P_k(s_1^{(q)}, \dots, s_r^{(q)})\|} \longrightarrow 1$$

in probability, uniformly for nonzero matrix-valued noncommutative polynomials P_k of degree at most d and coefficient dimensions up to D_k, where D_k may be chosen larger than the dimension of the random matrices themselves.

The construction of matrix models of this paper applies to the explicit generator-independent range $|q| < \sqrt{2} - 1$, and provides quantitative complete bounded-degree strong convergence at growing matrix-coefficient allowed larger than the dimension of the matrix model. To the best of our knowledge, this is the first finite-dimensional random-matrix models proved to converge strongly to nonfree q-Gaussian families throughout the explicit generator-independent range of non-zero q uniform over matrix-coefficient dimensions.

In this context we also mention that very recently Shlyakhtenko [Shl26] showed that 'CAR-UE' matrix models (see also [PiSh02]) do not strongly converge to semicircular variables. Heuristically this model replaces Gaussian random variables, appearing as matrix entries of the GUE's, by CAR algebras. On the other hand in [Shl26, Section 4] it is shown that if these Gaussian variables are replaced by q-Gaussians $-1 < q < 1$ strong convergence to the semicircular system holds. Note that in [Shl26] the limit model is semicircular ($q = 0$) whereas in the current paper we consider different, finite dimensional, models that converge to any q-Gaussian (at least if $|q| < \sqrt{2} - 1$).

1.2. **Sharp Erdős–Rényi Graph-Product Degree-One Khintchine Inequality.** As a first consequence of the approximate q-Toeplitz relation, we obtain an asymptotically sharp degree-one graph-product Khintchine inequality for Erdős–Rényi graph products of semicircular variables which for $-1 \le q < 1$ states that (see Theorem 3.11):

$$\left\| \frac{1}{\sqrt{k}} \sum_{v=1}^{k} x_v^{q,\Gamma_k} \right\| \le \frac{2}{\sqrt{1-q}} + \mathcal{O}_{\mathbb{P}}\left(\frac{\log k}{\sqrt{k}}\right), \qquad \left\| \frac{1}{\sqrt{k}} \sum_{v=1}^{k} x_v^{q,\Gamma_k} \right\| \longrightarrow \frac{2}{\sqrt{1-q}}. \tag{1.1}$$

in probability, where the semicircular variables x_v^{q,Γ_k} are labeled by $\Gamma(k,|q|)$; in case $-1 \le q < 0$ the variables are Clifford twisted versions of semicirculars. The limiting constant is optimal, since $2/\sqrt{1-q}$ is precisely the norm of a standard q-Gaussian variable. This complements the deterministic graph-product Khintchine inequalities of [CKL21] and the recent degree-one graph-product Khintchine inequalites of [CoMi26, STY25]. These works treat arbitrary operator coefficients for a fixed graph and bound the norm in terms of adjacency eigenvalues, clique numbers, or the associated trace monoid. These results are deterministic and thus hold for all graphs. For an Erdős–Rényi graph, applying their general clique-based graph-product Khintchine inequalities gives, where the first estimate holds everywhere,

$$\left\| \frac{1}{\sqrt{k}} \sum_{v=1}^{k} x_v^{\Gamma_k} \right\| \le 2\sqrt{\omega(\Gamma_k)} = \mathcal{O}_{\mathbb{P}}(\sqrt{\log k}),$$

whereas our probabilistic argument identifies the finite sharp limit $2/\sqrt{1-q}$ and provides a quantitative error estimate.

Note that the Khintchine inequality (1.1) is the simplest case of our strong convergence result where the polynomial is linear and scalar valued. We stress that this specific case is already new.

1.3. **MF Property of q-Gaussian C^*-algebras.** As a natural corollary from the strong convergent random matrix models for q-Gaussian family, applying a diagonal choice of the parameters of such random matrices gives strong convergence for all noncommutative polynomials and implies the MF property of the associated q-Gaussian C^*-algebra with $|q| < \sqrt{2} - 1$.

Then using the recent proof [AJW26] of the simplicity of the q-Gaussian C^*-algebras, we show that the q-Gaussian C^*-algebras $A_q(\mathbb{C}^r)$ with $r \ge 2$ are not quasidiagonal. Hence combining this with the MF property, by Brown's criterion [Bro04], we deduce that the Brown–Douglas–Fillmore extension semigroup $\mathrm{Ext}(A_q(\mathbb{C}^r))$ is not a group for $r \ge 2$ and $|q| < \sqrt{2} - 1$.

1.4. **Paper Overview.** The paper is organized as follows. In Section 2 we fix notation. In Section 3, we establish the Erdős–Rényi graph-product operators model approximation for q-Gaussians, as well as the sharp degree-one graph product Khintchine inequality for the Erdős–Rényi graph. In Section 4, we obtain a quantitative weak convergence for the operator model on the level of q-Toeplitz variables. In Section 5, we pass the single variable strong convergence of the operator model to complete strong convergence with growing matrix coefficients. In Section 6, we convert the operator models into finite random matrix models via the tensor-GUE models, and prove their strong convergence with growing matrix coefficients. The applications to the MF property and extension semigroup are proved in Section 7.

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2. Preliminaries

All C^*-algebras are complex and unital, all tensor product are minimal, and Hilbert-space innner products are linear in the first variable.

2.1. **Notation, polynmials and strong convergence.** We write $[m] = \{1, \dots, m\}$, $M_D = M_D(\mathbb{C})$, and $\mathrm{tr}_D(A) = D^{-1}\mathrm{Tr}(A)$. The notation $M_{D,N}$ stands for the $D \times N$ complex matrices, $(M_D)_{\mathrm{sa}}$ for the self-adjoint matrices, and id_D for the identity map on M_D. For a self-adjoint element a, its spectrum is $\mathrm{sp}(a)$. If $K, L \subset \mathbb{R}$ are nonempty compact sets, then

$$d_{\mathrm{H}}(K, L) = \max \left\{ \sup_{x \in K} \mathrm{dist}(x, L), \sup_{y \in L} \mathrm{dist}(y, K) \right\}$$

denotes their Hausdorff distance, and

$$K + [-\varepsilon, \varepsilon] = \{x + t : x \in K,\ |t| \le \varepsilon\}.$$

We write $Z_k = O_{\mathbb{P}}(b_k)$ if Z_k / b_k is bounded in probability, and $Z_k = o_{\mathbb{P}}(b_k)$ if $Z_k / b_k \to 0$ in probability. An event holds *with high probability* if its probability tends to one. For a family $(Z_{k,\theta})_{\theta \in I_k}$, uniform convergence in probability means

$$\sup_{\theta \in I_k} \mathbb{P}(|Z_{k,\theta}| > \varepsilon) \longrightarrow 0 \qquad (\forall \varepsilon > 0).$$

We use $\mathbb{C}\langle x_1, \dots, x_r \rangle$ for noncommutative polynomials in self-adjoint variables $x_1, \dots, x_r$. For a word $w = i_1 \cdots i_m$, put

$$|w| = m, \qquad x_w = x_{i_1} \cdots x_{i_m}, \qquad x_\varnothing = 1.$$

Every M_D-valued polynomial of degree at most d has a unique expression

$$P(x) = \sum_{|w| \le d} A_w \otimes x_w, \qquad \Lambda_d(P) = \sum_{|w| \le d} \|A_w\|.$$

Its involution is determined by $(A_w \otimes x_w)^* = A_w^* \otimes x_{w^{\mathrm{rev}}}$. For $A_0, \dots, A_V \in (M_D)_{\mathrm{sa}}$ and $x = (x_1, \cdots, x_V)$, we use the linear pencil

$$\mathcal{P}_A(x) = A_0 \otimes 1 + \sum_{v=1}^{V} A_v \otimes x_v, \qquad \Lambda(A) = \sum_{v=0}^{V} \|A_v\|.$$

If $E \subseteq A$ is an operator space, $M_D(E)$ carries the norm inherited from $M_D \otimes_{\min} A$. For a linear map $u : E \to F$,

$$\|u\|_{\mathrm{cb}} = \sup_{D \ge 1} \|\mathrm{id}_D \otimes u\|.$$

Let $X^{(N)}$ and x be self-adjoint r-tuples in tracial C^*-probability spaces (A_N, τ_N) and (A, τ), respectively. We say that $X^{(N)}$ converges *strongly in probability* to x if, for every scalar noncommutative polynomial P,

$$\tau_N(P(X^{(N)})) \longrightarrow \tau(P(x)), \qquad \|P(X^{(N)})\| \longrightarrow \|P(x)\|$$

in probability [Mal12].

For a sequence $D_N \ge 1$, convergence is *uniform up to coefficient dimension D_N at bounded degree* if, for every fixed d, M and $\varepsilon > 0$,

$$\sup_{\substack{1 \le D \le D_N,\ \deg P \le d \\ P \in M_D \otimes \mathbb{C}\langle x_1, \dots, x_r \rangle,\ \Lambda_d(P) \le M}} \mathbb{P}\Big(\big| \|P(X^{(N)})\| - \|P(x)\| \big| > \varepsilon \Big) \longrightarrow 0.$$

If the supremum is over all $D \ge 1$, we call this *complete bounded-degree convergence.*

2.2. **Graphs and Erdős–Rényi models.** A graph $\Gamma = (V\Gamma, E\Gamma)$ is finite, simple, and undirected. We write $u \sim_\Gamma v$ for adjacency, A_Γ for the zero-diagonal adjacency matrix. A clique is a subgraph whose vertices are pairwise adjacent, and $\omega(\Gamma)$ denotes the clique number, i.e. the maximal size of cliques in Γ.

We use reduced graph products in the sense of [CaFi17]. A *graph-product semicircular* family $(s_v)_{v\in\Gamma}$ is the canonical family in the reduced graph product of standard semicircular probability spaces. Thus adjacent vertex algebras commute and are tensor independent, while an edgeless graph gives free independence. The concrete graph-product Fock space is introduced in Section 3.

For $p \in [0,1]$, $\Gamma(k,p)$ denotes the *Erdős–Rényi graph* on $[k]$ whose edges occur independently with probability p. If we index the vertices of $\Gamma(\infty,p)$ with the natural numbers than we may naturally see $\Gamma(k,p)$ as a random subgraph of $\Gamma(\infty,p)$ by identifying it with the first k vertices. This way we can make sense of limits $k \to \infty$ that converge almost everywhere or that converge in probability by viewing $\Gamma(k,p)$ as subgraph of $\Gamma(\infty,p)$.

We write $\mathbb{P}_\Gamma$ and $\mathbb{E}_\Gamma$ for its law and expectation.

In the multivariable construction we sample one graph Γ_k on

$$V_k = [k] \times [r] = V_{k,1} \,\dot\cup \cdots \dot\cup\, V_{k,r}, \qquad V_{k,a} = [k] \times \{a\},$$

including all cross-block edges. Throughout the q-Toeplitz and q-Gaussian model, the edge probability is $p = |q|$.

For negative q we also use the Clifford C^*-algebra $\mathrm{Cliff}(V_k)$ generated by self-adjoint Clifford unitaries $(c_v)_{v\in V_k}$ satisfying

$$c_u c_v = -c_v c_u \quad (u \neq v), \qquad c_v^2 = 1,$$

equipped with its canonical trace τ_{Cl}, which is characterized by $\tau_{\mathrm{Cl}}(1) = 1$ and by vanishing on every nontrivial reduced word of Clifford unitaries.

2.3. **The q-Fock space and q-Toeplitz algebras.** Fix $-1 < q < 1$, let $H = \mathbb{C}^r$, and let $e_1, \dots, e_r$ be its standard basis. On

$$\mathcal{F}_{\mathrm{alg}}(H) = \mathbb{C}\Omega_q \oplus \bigoplus_{n\geq 1}^{\mathrm{alg}} H^{\otimes n},$$

where Ω_q is the vacuum vector, the q-inner product on $H^{\otimes n}$ is

$$\langle \xi, \eta\rangle_q = \langle P_q^{(n)}\xi, \eta\rangle, \qquad P_q^{(n)} = \sum_{\sigma\in S_n} q^{\mathrm{inv}(\sigma)} U_\sigma.$$

Here U_σ denotes the natural permutation of tensor factors. For $|q| < 1$ these forms are strictly positive, and their completion is the q-Fock space $\mathcal{F}_q(H)$ [BoSp91, DyNi93].

The left creation operator on $\mathcal{F}_q(H)$ is

$$\ell_q(h)\Omega_q = h, \qquad \ell_q(h)(\xi_1 \otimes \cdots \otimes \xi_n) = h \otimes \xi_1 \otimes \cdots \otimes \xi_n.$$

Writing $\ell_{q,a} = \ell_q(e_a)$, we has

$$\ell_{q,a}^* \ell_{q,b} = \delta_{ab} 1 + q \ell_{q,b} \ell_{q,a}^*.$$

Set

$$s_a^{(q)} = \ell_{q,a} + \ell_{q,a}^*, \qquad \mathcal{T}_{q,r}^{\mathrm{red}} = C^*(\ell_{q,1}, \dots, \ell_{q,r}),$$

the latter is called the *reduced q-Toeplitz algebra*. And write the *q-Gaussian C^*-algebra* and *q-Gaussian von Neumann algebra* respectively as

$$A_q(\mathbb{C}^r) = C^*(s_1^{(q)}, \dots, s_r^{(q)}), \qquad \Gamma_q(\mathbb{R}^r) = W^*(s_1^{(q)}, \dots, s_r^{(q)}).$$

The vacuum functional $\varphi_q(x) = \langle x\Omega_q, \Omega_q\rangle$, $x \in \mathcal{T}_{q,r}^{\mathrm{red}}$ is not tracial on $\mathcal{T}_{q,r}^{\mathrm{red}}$. Its restriction to $A_q(\mathbb{C}^r)$ is a faithful trace, denoted by τ_q [BKS97, Proposition 2.3]. We have

$$\|s_a^{(q)}\| = \frac{2}{\sqrt{1-q}} \qquad \text{and} \qquad \tau_q(s_{i_1}^{(q)} \cdots s_{i_m}^{(q)}) = \sum_{\pi\in\mathcal{P}_2(m)} q^{\mathrm{cr}(\pi)} \prod_{\{u,v\}\in\pi} \delta_{i_u,i_v}.$$

Here $\mathcal{P}_2(m)$ is the set of pair partitions of $[m]$, $\mathrm{cr}(\pi)$ is the crossing number of π, and the sum is zero for odd m [BKS97].

Let $\mathcal{T}_{q,r}$ be the unital $*$-algebra generated by $T_1, \dots, T_r$ with *the q-Toeplitz relations*

$$T_a^* T_b = \delta_{ab} 1 + q T_b T_a^*,$$

and let $\mathcal{T}_{q,r}^u$ be its universal C^*-completion. We call $\mathcal{T}_{q,r}^u$ *the universal q-Toeplitz algebra.* Then

$$\Phi_q : \mathcal{T}_{q,r}^u \to \mathcal{T}_{q,r}^{\mathrm{red}}$$

denotes the canononical quotient map.

2.4. **Ultaproduct and approximation techniques.** For nonprincipal ultrafilter $\mathcal{U}$ and C^*-algebras (A_k), their C^*-ultraproduct is

$$\prod_{k,\mathcal{U}} A_k = \ell^\infty((A_k)) \Big/ \{(a_k) : \lim_{k\to\mathcal{U}} \|a_k\| = 0\}, \qquad \|(a_k)_{\mathcal{U}}\| = \lim_{k\to\mathcal{U}} \|a_k\|.$$

We use the analogous Hilbert-space ultraproduct notation. We recall the Choi-Effros lifting theorem that we will use: a complete positive contraction from a separable C^*-algebra B to a quotient B/J has a complete positive contraction lift to B [ChEf76].

The following consequence of exactness is needed in Sections 5 and 6.

Lemma 2.1 (Fixed-family compression). *Let A be a exact C^*-algebra and let $E \subseteq A$ be finite dimensional subspace. For every $\delta > 0$ there is $m = m(E, \delta)$ such that, for every $D \geq 1$ and $X \in M_D(E)$, there are contractions $U, V \in M_{m,D}$ satisfying*

$$\|(U \otimes 1)X(V^* \otimes 1)\| \geq (1-\delta)\|X\|.$$

In particular, m is independent of D and X.

This is [MaSa25, Lemma 5.13]; it also follows from the local operator space characterization of exactness [Pis03, Corollary 17.5].

A separable C^*-algebra is *MF* if it embeds into

$$\prod_k M_{n_k} \Big/ \bigoplus_k M_{n_k}, \qquad \bigoplus_k M_{n_k} = \{(a_k) : a_k \in M_{n_k}, \|a_k\| \longrightarrow 0\}.$$

2.5. **Normalized GUE matrices.** A normalized $N \times N$ GUE matrix $H^{(N)}$ is a random self-adjoint matrix which has independent real Gaussian random variables on the diagonal entries of variance N^{-1} and independent real and imaginary parts of Gaussian random variables above the diagonal of variance $(2N)^{-1}$. Thus $\mathbb{E} \circ \mathrm{tr}_N\big((H^{(N)})^2\big) = 1$, and its limiting law is the standard semicircular distribution on $[-2, 2]$. All GUE matrices appearing together are independent and independent of the graph unless stated otherwise; their joint law is denoted by $\mathbb{P}_{\Gamma,\mathrm{GUE}}$.

3. Strong convergence of q-Gaussian models

Let Γ be a finite graph. Let $\mathcal{F}_\Gamma$ be the graph product Fock space of the graph product of semicircular variables with respect to the vacuum state.

$$\mathcal{F}_\Gamma = \bigoplus_{\mathcal{A}_\Gamma^+} \mathbb{C} = \ell^2(\mathcal{A}_\Gamma^+),$$

where $\mathcal{A}_\Gamma^+$ is the right-angled Artin monoid consisting of all words with letters in Γ; i.e. without formal inverses. For $w \in \mathcal{A}_\Gamma^+$ we let δ_w be the unit vector that is 1 in the summand indexed by w and which is 0 in all other summands. Denote $\Omega_\Gamma = \delta_e$ as the vacuum vector of $\mathcal{F}_\Gamma$.

3.1. **Graph Product Model.** For $v \in \Gamma$ we set the left creation operator

$$\ell_v : \mathcal{F}_\Gamma \to \mathcal{F}_\Gamma : \delta_w \mapsto \delta_{vw}.$$

Then ℓ_v^* is the left annihilation operator. We have the graph product creation relation

$$\ell_w^* \ell_v = \delta_{wv} I + (A_\Gamma)_{wv} \ell_v \ell_w^*.$$

Then $x_v^{\Gamma_k} = \ell_v + \ell_v^*$, $v \in \Gamma$ are the graph product semicirculars. We also set

$$L_\Gamma = \frac{1}{\sqrt{|\Gamma|}} \sum_{v\in\Gamma} \ell_v.$$

We define the following annihilation-gradient map:

$$D_\Gamma : \mathcal{F}_\Gamma \to \ell^2\Gamma \otimes \mathcal{F}_\Gamma \qquad \xi \mapsto \sum_{v\in\Gamma} \delta_v \otimes \ell_v^* \xi.$$

It is easy to see that $D_\Gamma^*(\delta_v \otimes \eta) = \ell_v \eta$, for $\eta \in \mathcal{F}_\Gamma$, and $D_\Gamma^* D_\Gamma = \sum_{v\in\Gamma} \ell_v \ell_v^*$.

3.2. **Clifford-twisted Graph Product Model.** Let $\{c_v\}_{v\in\Gamma}$ be self-adjoint Clifford unitaries satisfying

$$c_v^2 = 1, \qquad c_u c_v = -c_v c_u \quad (u \neq v).$$

Denote the (finite dimensional) von Neumann algebra geneated by them as $(\mathrm{Cliff}(\Gamma), \tau_{\mathrm{Cl}})$ where τ_{Cl} is the usual tracial state that is 0 on elements of nonzero degree. In this case define the Clifford-twisted left creation operator

$$\widetilde{\ell}_v := c_v \otimes \ell_v : L^2(\mathrm{Cliff}(\Gamma), \tau_{\mathrm{Cl}}) \otimes \mathcal{F}_\Gamma \to L^2(\mathrm{Cliff}(\Gamma), \tau_{\mathrm{Cl}}) \otimes \mathcal{F}_\Gamma.$$

Denote Ω_{Cl} as the vacuum vector of $L^2(\mathrm{Cliff}(\Gamma), \tau_{\mathrm{Cl}})$, then the vacuum vector of $L^2(\mathrm{Cliff}(\Gamma), \tau_{\mathrm{Cl}}) \otimes \mathcal{F}_\Gamma$ is $\Omega_{\mathrm{Cl}} \otimes \Omega_\Gamma$. Then $\widetilde{\ell}_v^*$ is the left annihilation operator. We have the Clifford-twisted graph product creation relation

$$\widetilde{\ell}_w^* \widetilde{\ell}_v = \delta_{wv} I - (A_\Gamma)_{wv} \widetilde{\ell}_v \widetilde{\ell}_w^*.$$

Then $x_v^{\Gamma,\mathrm{Cl}} = \widetilde{\ell}_v + \widetilde{\ell}_v^*$, $v \in \Gamma$ are the Clifford-twisted graph product semicirculars. Let

$$\widetilde{L}_\Gamma = \frac{1}{\sqrt{|\Gamma|}} \sum_{v\in\Gamma} \widetilde{\ell}_v.$$

We define the following annihilation-gradient map:

$$\widetilde{D}_\Gamma : L^2(\mathrm{Cliff}(\Gamma), \tau_{\mathrm{Cl}}) \otimes \mathcal{F}_\Gamma \to \ell^2\Gamma \otimes L^2(\mathrm{Cliff}(\Gamma), \tau_{\mathrm{Cl}}) \otimes \mathcal{F}_\Gamma \qquad \xi \mapsto \sum_{v\in\Gamma} \delta_v \otimes \widetilde{\ell}_v^* \xi.$$

Again, $\widetilde{D}_\Gamma^*(\delta_v \otimes \eta) = \widetilde{\ell}_v \eta$, for $\eta \in \mathcal{F}_\Gamma$, and $\widetilde{D}_\Gamma^* \widetilde{D}_\Gamma = \sum_{v\in\Gamma} \widetilde{\ell}_v \widetilde{\ell}_v^* = \sum_{v\in\Gamma} \ell_v \ell_v^*$.

3.3. **Approximate q-Toeplitz relation with the graph product and Clifford-twisted graph product model.** Consider the adjacency matrix as operator

$$A_\Gamma : \ell^2\Gamma \to \ell^2\Gamma : \delta_v \mapsto \sum_{w\in\Gamma} (A_\Gamma)_{wv}\delta_w.$$

For $-1 < q < 1$, put

$$\varepsilon_q = \begin{cases} 1, & q \geq 0, \\ -1, & q < 0. \end{cases}$$

For $\Gamma \sim \Gamma(k, |q|)$ define

$$A_\Gamma^{(q)} = \varepsilon_q A_\Gamma, \qquad B_\Gamma^{(q)} = A_\Gamma^{(q)} - q(J - I).$$

Here J is the all-ones matrix, and I is the identity matirx.

Remark 3.1. $B_\Gamma^{(q)}$ is symmetric matrix. The off-diagonal entries of $B_\Gamma^{(q)}$ are centered and have variance $|q|(1 - |q|)$. In particular, when $q \geq 0$, every off-diagonal entry of B_Γ^q is mean-zero Bernoulli random variable.

We now have the following.

Lemma 3.2. *For Γ a finite (deterministic) graph with k vertices and $-1 \leq q \leq 1$ we have*

$$L_\Gamma^* L_\Gamma = qL_\Gamma L_\Gamma^* + I - \frac{q}{k}\sum_{v\in\Gamma} \ell_v\ell_v^* + D_\Gamma^*(\frac{B_\Gamma}{k} \otimes I)D_\Gamma. \tag{3.1}$$

and

$$\widetilde{L}_\Gamma^* \widetilde{L}_\Gamma = q\widetilde{L}_\Gamma \widetilde{L}_\Gamma^* + I - \frac{q}{k}\sum_{v\in\Gamma} \widetilde{\ell}_v\widetilde{\ell}_v^* + \widetilde{D}_\Gamma^*(\frac{\widetilde{B}_\Gamma}{k} \otimes I)\widetilde{D}_\Gamma. \tag{3.2}$$

Proof. Proof of (3.1). The graph-product creation operators satisfy

$$\ell_u^*\ell_v = \delta_{uv} I + (A_\Gamma)_{uv}\ell_v\ell_u^*.$$

Therefore

$$L_\Gamma^* L_\Gamma = \frac{1}{k}\sum_{u,v\in\Gamma} \ell_u^*\ell_v = \frac{1}{k}\sum_{u,v\in\Gamma} (\delta_{uv} I + (A_\Gamma)_{uv}\ell_v\ell_u^*) = I + \frac{1}{k}\sum_{u,v\in\Gamma} (A_\Gamma)_{uv}\ell_v\ell_u^*.$$

Since A_Γ has zero diagonal, we can write this identity as

$$L_\Gamma^* L_\Gamma = I + \frac{1}{k}\sum_{\substack{u,v\in\Gamma \\ u\neq v}} (A_\Gamma)_{uv}\ell_v\ell_u^*.$$

Next, we have

$$L_\Gamma L_\Gamma^* = \frac{1}{k}\sum_{u,v\in\Gamma} \ell_v\ell_u^*.$$

Hence

$$qL_\Gamma L_\Gamma^* - \frac{q}{k}\sum_{v\in\Gamma} \ell_v\ell_v^* = \frac{q}{k}\sum_{u,v\in V\Gamma u\neq v} \ell_v\ell_u^*.$$

Since $B_\Gamma = A_\Gamma - q(J - I)$, for $u \neq v$,we have

$$(B_\Gamma)_{uv} = (A_\Gamma)_{uv} - q \quad \text{and} \quad (B_\Gamma)_{vv} = 0.$$

We now compute the D_Γ-term. By $D_\Gamma \xi = \sum_{u\in\Gamma} \delta_u \otimes \ell_u^* \xi$, and $D_\Gamma^*(\delta_v \otimes \eta) = \ell_v \eta$, we have

$$D_\Gamma^* \left(\frac{B_\Gamma}{k} \otimes I \right) D_\Gamma = D_\Gamma^* \left(\frac{1}{k} \sum_{u,v\in\Gamma} (B_\Gamma)_{vu}\, \delta_v \otimes \ell_u^* \right) = \frac{1}{k} \sum_{u,v\in\Gamma} (B_\Gamma)_{vu} \ell_v \ell_u^*.$$

Using $(B_\Gamma)_{vv} = 0$, we obtain

$$D_\Gamma^* \left(\frac{B_\Gamma}{k} \otimes I \right) D_\Gamma = \frac{1}{k} \sum_{\substack{u,v\in\Gamma \\ u\neq v}} \big((A_\Gamma)_{uv} - q\big) \ell_v \ell_u^*.$$

Consequently,

$$\begin{aligned} qL_\Gamma L_\Gamma^* - \frac{q}{k} \sum_{v\in\Gamma} \ell_v \ell_v^* + D_\Gamma^* \left(\frac{B_\Gamma}{k} \otimes I \right) D_\Gamma &= \frac{q}{k} \sum_{\substack{u,v\in\Gamma \\ u\neq v}} \ell_v \ell_u^* + \frac{1}{k} \sum_{\substack{u,v\in\Gamma \\ u\neq v}} \big((A_\Gamma)_{uv} - q\big) \ell_v \ell_u^* \\ &= \frac{1}{k} \sum_{\substack{u,v\in\Gamma \\ u\neq v}} (A_\Gamma)_{uv} \ell_v \ell_u^*. \end{aligned}$$

Adding the identity term I we get

$$L_\Gamma^* L_\Gamma = qL_\Gamma L_\Gamma^* + I - \frac{q}{k} \sum_{v\in\Gamma} \ell_v \ell_v^* + D_\Gamma^* \left(\frac{B_\Gamma}{k} \otimes I \right) D_\Gamma,$$

which gives the first identity.

Proof of (3.2). For Clifford-twisted graph product model we have

$$\widetilde{\ell}_u \widetilde{\ell}_v^* = \delta_{uv} I - (A_\Gamma)_{uv} \widetilde{\ell}_v \widetilde{\ell}_u^*.$$

Thus the same proof works with the Clifford-twisted adjacency matrix $-A_\Gamma$ in place of A_Γ. Equivalently, we just replace B_Γ by $\widetilde{B}_\Gamma$, then we get

$$\widetilde{L}_\Gamma^* \widetilde{L}_\Gamma = q\widetilde{L}_\Gamma \widetilde{L}_\Gamma^* + I - \frac{q}{k} \sum_{v\in\Gamma} \widetilde{\ell}_v \widetilde{\ell}_v^* + \widetilde{D}_\Gamma^* \left(\frac{\widetilde{B}_\Gamma}{k} \otimes I \right) \widetilde{D}_\Gamma,$$

which gives the second identity.

□

Next we turn to the probabilistic setting. We first make the following remark.

Remark 3.3. Assume $0 \le q < 1$. We interpret the terms in (3.1) as follows:

$$\overbrace{L_\Gamma^* L_\Gamma = qL_\Gamma L_\Gamma^* + I}^{q-\text{Toeplitz relation}} - \overbrace{\frac{q}{k} \sum_{v\in\Gamma} \ell_v \ell_v^*}^{\text{remainder 1}} + \overbrace{D_\Gamma^* (\frac{B_\Gamma}{k} \otimes 1) D_\Gamma}^{\text{remainder 2}}.$$

We will show in the remainder of this section that if $\Gamma = \Gamma(k,q)$ and we let $k \to \infty$ then the two remainder terms will go to 0 with high probability and the q-Toeplitz relation remains. This is essentially why our proof works. The analogous observation holds for (3.2) for $-1 < q < 0$.

We first consider the necessary estimates that control remainder 1 of Remark 3.3.

Remark 3.4 (Starting clique)**.** For $w \in \mathcal{A}_\Gamma^+$ there is a uniquely defined largest clique $\mathrm{Init}_\Gamma(w)$ such that $|\mathrm{Init}_\Gamma(w) w| = |w| - |\mathrm{Init}_\Gamma(w)|$, i.e. $\mathrm{Init}_\Gamma(w)$ is the largest clique at the start of w.

The following lemma is now straightforward and we omit the proof.

Lemma 3.5. *We have for $w \in \mathcal{A}_\Gamma^+$ that $\sum_{v\in\Gamma} \ell_v \ell_v^* \, \delta_w = |\mathrm{Init}_\Gamma(w)|\delta_w$. In particular,*

$$\Big\| \sum_{v\in\Gamma} \ell_v \ell_v^* \Big\| = \Big\| \sum_{v\in\Gamma} \widetilde{\ell}_v \widetilde{\ell}_v^* \Big\| = \omega(\Gamma).$$

In case of the random graph $\Gamma = \Gamma(k,q)$ we have good control over the maximal clique size $\omega(\Gamma(k,q))$. In fact, it is a classical result by Bollobás and Erdős [BoEr76, Corollary 1] that $\omega(\Gamma(k,q))$ is highly concentrated around only two neighbouring natural numbers that are of $\mathcal{O}(\log(k))$. Here we use a slightly different statement about the tail bound that can be found in [AlSp08, Section 10.2, p. 184]. Its proof is only stated for $q = \frac{1}{2}$ but holds verbatim for other values of q as is also mentioned in [AlSp08]. We give the proof for convenience of the reader.

Lemma 3.6. *Let $0 < q < 1$. For every constant $C_q > \frac{2}{\log(1/q)}$, there exists a constant $c_q > 0$ such that*

$$\mathbb{P}(\omega(\Gamma(k,q)) \geq C_q \log k) \leq \exp\big(-c_q \log^2 k\big)$$

for all sufficiently large k.

Proof. Let $m = \lceil C_q \log k \rceil$. If $\omega(\Gamma(k,q)) \geq m$, then the graph contains an m-clique. Hence, by the Markov's inequality,

$$\mathbb{P}(\omega(\Gamma(k,q)) \geq m) \leq \mathbb{E}[X_m],$$

where X_m denotes the number of m-cliques in $\Gamma(k,q)$.

Since every m-subset of vertices forms a clique with probability $q^{\binom{m}{2}}$, we have

$$\mathbb{E}[X_m] = \binom{k}{m} q^{\binom{m}{2}}.$$

Using the elementary estimate $\binom{k}{m} \leq \left(\frac{ek}{m}\right)^m$, we obtain

$$\mathbb{P}(\omega(\Gamma(k,q)) \geq m) \leq \left(\frac{ek}{m}\right)^m q^{m(m-1)/2}.$$

Taking logarithms gives

$$\log \mathbb{P}(\omega(\Gamma(k,q)) \geq m) \leq m \log\left(\frac{ek}{m}\right) - \frac{\log(1/q)}{2} m(m-1).$$

Since $m = C_q \log k + \mathcal{O}(1)$, we have

$$m \log\left(\frac{ek}{m}\right) = C_q \log^2 k + \mathcal{O}(\log k \log\log k),$$

while

$$\frac{\log(1/q)}{2} m(m-1) = \frac{C_q^2 \log(1/q)}{2} \log^2 k + \mathcal{O}(\log k).$$

Therefore

$$\log \mathbb{P}(\omega(\Gamma(k,q)) \geq m) \leq \left(C_q - \frac{C_q^2 \log^2(1/q)}{2} \right) \log^2 k + \mathcal{O}(\log k \, \log\log k).$$

Since $C_q > \frac{2}{\log(1/q)}$, the coefficient $C_q - \frac{C_q^2 \log(1/q)}{2}$ is strictly negative. Hence there exists $c_q > 0$ such that

$$\log \mathbb{P}(\omega(\Gamma(k,q)) \geq m) \leq -c_q \log^2 k$$

for all sufficiently large k. Exponentiating both sides yields

$$\mathbb{P}(\omega(\Gamma(k,q)) \geq C_q \log k) \leq \exp\bigl(-c_q \log^2 k\bigr) .$$

□

To control remainder 2 in Remark 3.3 we recall the following. Note that by Remark 3.1 we are indeed in the setting of the cited references.

Theorem 3.7 (Theorem A of [BaYi88] and Theorem 1.4 of [Tro12])**.** *For $0 \leq q \leq 1$, we have*

$$\lim_{k\to\infty} k^{-\frac{1}{2}} \|B_{\Gamma(k,q)}\| = 2\sqrt{q(1-q)}.$$

in probability. Further, there exists $b_q > 0$ and $T > 0$ such that for every $t > T$ and $k \in \mathbb{N}_{\geq 1}$ we have

$$\mathbb{P}\left(\|B_{\Gamma(k,q)}\| > t\right) \leq 2k \exp\left(-\frac{b_q t^2}{k+t}\right).$$

Proof. The convergence

$$k^{-\frac{1}{2}} ||B_{\Gamma(k,q)}|| \to 2\sqrt{q(1-q)}$$

follows from Bai–Yin's result [BaYi88] applied to the centered Bernoulli Wigner matrix $B_{\Gamma(k,q)}$. The non-asymptotic tail estimate

$$\mathbb{P}(\|B_{\Gamma(k,q)}\| > t) \leq 2k \exp\left(-\frac{b_q t^2}{k+t}\right)$$

follows from Tropp's matrix Bernstein inequality [Tro12]; alternatively see Oliveira's result [Oli10] for adjacency-matrix concentration in random graphs with independent edges. □

Corollary 3.8. *Let $\alpha \in (\frac{1}{2}, 1)$. For any $0 \leq q \leq 1$, there exists a constant $b'_q > 0$ such that for all $k \in \mathbb{N}_{\geq 1}$ we have,*

$$\mathbb{P}\left(\|B_{\Gamma(k,q)}\| > k^{\alpha}\right) \leq k \exp\left(-b'_q k^{2\alpha-1}\right).$$

Now,when $0 \leq q \leq 1$, we denote

$$L_k^{(q)} := L_{\Gamma(k,q)} \quad \text{and} \quad D_k^{(q)} := D_{\Gamma(k,q)};$$

when $-1 \leq q < 0$, we denote

$$L_k^{(q)} := \widetilde{L}_{\Gamma(k,-q)} \quad \text{and} \quad D_k^{(q)} := \widetilde{D}_{\Gamma(k,-q)}.$$

Combining Theorems 3.6 and 3.8, we are now able to estimate $L_{\Gamma(k,q)}$ with high probability.

Theorem 3.9. *For every $-1 \leq q < 1$ there exists $C_q > 0$ such that for every $\alpha \in (\frac{1}{2}, 1)$ and $k \in \mathbb{N}$ we have:*

(1) for nonnegative q:

$$\mathbb{P}\left(\|L_k^{(q)}\|^2 \leq \frac{1}{1-q} + C_q \frac{\log k + k^{\alpha}\log k}{k}\right) \geq 1 - \exp\bigl(-c_q \log^2 k\bigr) - k\exp\bigl(-b'_{|q|}k^{2\alpha-1}\bigr),$$

(2) for negative q:

$$\mathbb{P}\left(\|L_k^{(q)}\|^2 \leq 1 + C_q \frac{\log k + k^{\alpha}\log k}{k}\right) \geq 1 - \exp\bigl(-c_q \log^2 k\bigr) - k\exp\bigl(-b'_{|q|}k^{2\alpha-1}\bigr).$$

Proof. By Theorem 3.5, we have for every $-1 \le q \le 1$, $\|D_k^{(q)}\|^2 = \omega(\Gamma(k,|q|))$. Hence

$$\left\| (D_k^{(q)})^* \left(\frac{B_k^{(q)}}{k} \otimes 1 \right) D_k^{(q)} \right\| \le \frac{\omega(\Gamma(k,|q|))}{k} \|B_k^{(q)}\|.$$

We now estimate $\|L_k^{(q)}\|^2$.

If $0 \le q < 1$, then by Lemma 3.2,

$$\|L_k^{(q)}\|^2 = \|(L_k^{(q)})^* L_k^{(q)}\| \le 1 + q\|L_k^{(q)}\|^2 + \frac{q}{k}\omega(\Gamma(k,q)) + \frac{\omega(\Gamma(k,q))}{k}\|B_k^{(q)}\|.$$

Therefore

$$(1-q)\|L_k^{(q)}\|^2 \le 1 + \frac{q}{k}\omega(\Gamma(k,q)) + \frac{\omega(\Gamma(k,q))}{k}\|B_k^{(q)}\|.$$

So for $C_q = 3q/(1-q)$

$$\|L_k^{(q)}\|^2 \le \frac{1}{1-q} + \frac{C_q}{3} \left(\frac{\omega(\Gamma(k,|q|))}{k} + \frac{\omega(\Gamma(k,|q|))\|B_k^{(q)}\|}{k} \right). \tag{3.3}$$

If $-1 \le q < 0$, then the term $qL_k^{(q)}(L_k^{(q)})^*$ is negative, so in operator order it can be discarded for an upper bound. Therefore by Lemma 3.2,

$$\|L_k^{(q)}\|^2 \le 1 + \frac{|q|}{k}\omega(\Gamma(k,|q|)) + \frac{\omega(\Gamma(k,|q|))}{k}\|B_k^{(q)}\|. \tag{3.4}$$

We shall now derive (1) from (3.3); the derivation of (2) from (3.4) follows in exactly the same manner. Let

$$\mathcal{G}_k := \left\{ \omega(\Gamma(k,|q|)) \le 3\log_{\frac{1}{|q|}} k \right\} \cap \left\{ \|B_k^{(q)}\| \le k^\alpha \right\}.$$

On $\mathcal{G}_k$, by Eq. (3.3) we get

$$\|L_\Gamma\|^2 \le \frac{1}{1-q} + C_q \left(\frac{\log_{\frac{1}{|q|}} k}{k} + \frac{k^\alpha \log_{\frac{1}{|q|}} k}{k} \right).$$

We now bound the probability of $\mathcal{G}_k$. First, by Theorem 3.6, we have

$$\mathbb{P}\big(\omega(\Gamma(k,|q|)) > 3\log_{\frac{1}{|q|}} k\big) \le \exp\big(-c_q \log^2 k\big).$$

Second, by Theorem 3.8, we get for some constant $b'_{|q|} > 0$,

$$\mathbb{P}\big(\|B_\Gamma^{(q)}\| > k^\alpha\big) \le k\exp\big(-b'_{|q|}k^{2\alpha-1}\big).$$

By the union bound,

$$\mathbb{P}(\mathcal{G}_k) \ge 1 - \exp\big(-c_q \log^2 k\big) - k\exp\big(-b'_{|q|}k^{2\alpha-1}\big).$$

This proves the theorem. □

Theorem 3.11 below is one of the main theorems of this paper and shows that with high probability we have a Khintchine inequality for sums of semi-circular variables that randomly are either free or commutative. Moreover the inequality is sharp by Theorem 5.4 we prove below applied to the 1-variable case with a scalar valued linear polynomial and the fact that a q-Gaussian generator s_1^q has norm $2/\sqrt{1-q}$.

The deterministic analogue of this problem has been addressed in [CoMi26, STY25]; however a bandwidth between the upper and lower bound remained.

Lemma 3.10. *Let $-1 \le q < 1$ and $T \in \mathcal{B}(\mathcal{H})$. If $r = \|T^*T - I - qTT^*\|$, then we have*

$$\|T + T^*\| \le \frac{2}{\sqrt{1-q}}\sqrt{1+r}.$$

Proof. Denote $R = T^*T - I - qTT^*$, then $r = \|R\|$.

When $0 \le q < 1$, we have

$$\begin{aligned}\|T\|^2 = \|T^*T\| &\le \|I\| + q\|TT^*\| + \|T^*T - I - qTT^*\| \\ &= 1 + q\|T\|^2 + \|R\|.\end{aligned}$$

This give that

$$\|T\| \le \sqrt{\frac{1+r}{1-q}}.$$

Therefore

$$\|T + T^*\| \le 2\|T\| \le \frac{2}{\sqrt{1-q}}\sqrt{1+r}.$$

When $-1 \le q < 0$, let $p = -q$. Take a unit vector $\xi \in \mathcal{H}$, we have

$$\|T\xi\|^2 + p\|T^*\xi\|^2 = 1 + \langle R\xi, \xi\rangle.$$

But $|\langle T\xi, \xi\rangle| \le \|T\xi\|$ and $|\langle T\xi, \xi\rangle| = |\langle \xi, T^*\xi\rangle| \le \|T^*\xi\|$, we get

$$|\langle T\xi, \xi\rangle|^2 \le \min\{\|T\xi\|^2, \|T^*\xi\|^2\}.$$

On the other hand, we have for $a, b \ge 0$, $\min\{a, b\} \le \frac{a+pb}{1+p}$, therefore

$$|\langle T\xi, \xi\rangle|^2 \le \frac{\|T\xi\|^2 + p\|T^*\xi\|^2}{1+p} = \frac{1 + \langle R\xi, \xi\rangle}{1+p}.$$

But

$$|\langle (T+T^*)\xi, \xi\rangle| = 2|\Re\langle T\xi, \xi\rangle| \le 2|\langle T\xi, \xi\rangle| \le 2\sqrt{\frac{1+\langle R\xi, \xi\rangle}{1+p}},$$

we finally get

$$\|T + T^*\| = \sup_{\|\xi\|=1} |\langle (T+T^*)\xi, \xi\rangle| \le \frac{2}{\sqrt{1+p}}\sqrt{1+r}.$$

$\square$

Theorem 3.11 (Sharp Erdős–Rényi graph-product degree-one Khintchine inequality)**.** *Let $-1 < q < 1$. Then for every $\alpha \in (\frac{1}{2}, 1)$ there exist constants $C_q, c_q > 0$ such that for every $k \in \mathbb{N}$*

$$\mathbb{P}\left(\left\|\frac{1}{\sqrt{k}}\sum_{v\in\Gamma(k,|q|)} x_v^{q,\Gamma_k}\right\| \le \frac{2}{\sqrt{1-q}} + C_q k^{\alpha-1}\log k\right) \ge 1 - \exp\bigl(-c_q \log^2 k\bigr) - 2k\exp(-c_q k^{2\alpha-1}).$$

Consequently, for fixed $\varepsilon > 0$, there is constant $c_{q,\varepsilon} > 0$ such that for large k

$$\mathbb{P}\left(\left\|\frac{1}{\sqrt{k}}\sum_{v\in\Gamma(k,|q|)} x_v^{q,\Gamma_k}\right\| \le \frac{2}{\sqrt{1-q}} + \varepsilon\right) \ge 1 - \exp\left(-c_{q,\varepsilon}\log^2 k\right) - 2k\exp\left(-c_{q,\varepsilon}\frac{k}{(\log k)^2}\right),$$

where

$$x_v^{q,\Gamma_k} = \begin{cases} x_v^{\Gamma_k}, & q \ge 0, \\ x_v^{\Gamma_k,\mathrm{Cl}}, & q < 0. \end{cases}$$

More sharp, but without tail bound, we have in probability

$$\left\| \frac{1}{\sqrt{k}} \sum_{v \in \Gamma(k,|q|)} x_v^{q,\Gamma_k} \right\| \leq \frac{2}{\sqrt{1-q}} + \mathcal{O}(\frac{\log k}{\sqrt{k}}).$$

Proof. By Lemma 3.2

$$\left(L_k^{(q)}\right)^* L_k^{(q)} = I + qL_k^{(q)} \left(L_k^{(q)}\right)^* + E_k^{(q)},$$

where

$$E_k^{(q)} = -\frac{q}{k} \sum_v \ell_v^{(q)} \left(\ell_v^{(q)}\right)^* + (D_k^{(q)})^* \left(\frac{B_\Gamma^{(q)}}{k} \otimes I \right) D_k^{(q)}.$$

Since

$$(D_k^{(q)})^* D_k^{(q)} = \sum_v \ell_v^{(q)} \left(\ell_v^{(q)}\right)^*,$$

and

$$\|D_k^{(q)}\|^2 = \omega(\Gamma(k, |q|)),$$

we have

$$\|E_k^{(q)}\| \leq \frac{|q|}{k} \omega(\Gamma(k, |q|)) + \frac{\omega(\Gamma(k, |q|))}{k} \|B_k^{(q)}\|.$$

Again, define the good event

$$\mathcal{G}_k = \left\{ \omega(\Gamma(k, |q|)) \leq 3 \log_{\frac{1}{|q|}} k \right\} \cap \left\{ \|B_k^{(q)}\| \leq k^\alpha \right\}.$$

Then on $\mathcal{G}_k$,

$$\|E_k\| \leq C_q \frac{\log k + \log(k) k^\alpha}{k} = \mathcal{O}(k^{\alpha-1} \log k).$$

Applying Theorem 3.10 with $T_k = L_k^{(q)}$ gives

$$\left\| \frac{1}{\sqrt{k}} \sum_{v \in \Gamma(k,|q|)} x_v^{q,\Gamma_k} \right\| = \|L_k^{(q)} + (L_k^{(q)})^*\| \leq \frac{2}{\sqrt{1-q}} + \mathcal{O}(k^{\alpha-1} \log k)$$

on $\mathcal{G}_k$. Finally, Theorem 3.6 yields

$$\mathbb{P}\left(\omega(\Gamma(k, |q|)) > 3 \log_{\frac{1}{|q|}} k \right) \leq \exp(-c_q \log^2 k),$$

while Theorem 3.8 gives

$$\mathbb{P}\left(\|B_k^{(q)}\| > k^\alpha \right) \leq 2k \exp(-c_q k^{2\alpha-1}).$$

And then take the union bound to completes the proof. □

3.4. **Multivariable approximate q-Toeplitz relation.** Now we turn to the r-variable case. We recall the notation for graphs in multivariable case in Section 2.2. Now we sample the Erdős–Rényi graph on $V_k = [k] \times [r]$, and denote this graph as Γ_k. For $a \in [r]$, let $\Gamma_{k,a}$ on $V_{k,a} = [k] \times \{a\}$ be the subgraph of Γ_k inheriting all edges from Γ_k. Let $\mathcal{A}^+_{\Gamma_k}(\Gamma_{k,a})$ be the words in $\mathcal{A}^+_{\Gamma_k}$ that do not start with a letter in $\mathcal{A}^+_{\Gamma_{k,a}}$. For $w \in \mathcal{A}^+_{\Gamma_k}(\Gamma_{k,a})$ the map $\delta_v \mapsto \delta_{uv}$ identifies $\mathcal{F}_{\Gamma_{k,a}}$ isometrically as a subspace of $\mathcal{F}_{\Gamma_k}$. Moreover, this map intertwines the actions of $\ell_v, v \in \Gamma_{k,a}$ and $L_{\Gamma_{k,a}}$. Hence,

$$\mathcal{F}_{\Gamma_k} = \bigoplus_{w \in \mathcal{A}^+_{\Gamma_k}(\Gamma_{k,a})} \mathcal{F}_{\Gamma_{k,a} w} \simeq \bigoplus_{w \in \mathcal{A}^+_{\Gamma_k}(\Gamma_{k,a})} \mathcal{F}_{\Gamma_{k,a}}. \tag{3.5}$$

It follows in particular that 3.2 holds for $L_{\Gamma_{k,a}}$ acting on $\mathcal{F}_{\Gamma_k}$ through (3.5).

Define

$$L_{k,a}^{q,\Gamma_k} = \frac{1}{\sqrt{k}} \sum_{v\in\Gamma_{k,a}} \ell_{(a,i)}^{q,\Gamma_k}, \qquad \text{and} \qquad S_{k,a}^{q,\Gamma_k} = L_{k,a}^{q,\Gamma_k} + (L_{k,a}^{q,\Gamma_k})^*.$$

We shall now also show that this q-Toeplitz type relation holds for different generators. Sometime we moit the superscripts Γ_k in the above notations.

For $a, b \in [r]$, let $B_{k,ab}^{(q)}$ be the block of $B_{\Gamma_k}^{(q)}$ with rows indexed by $\Gamma_{k,a}$ and columns indexed by $\Gamma_{k,b}$. Define

$$D_{k,a}^{(q)}\xi = \sum_{u\in\Gamma_{k,a}} \delta_u \otimes (\ell_u^{(q)})^*\xi.$$

Lemma 3.12 (Multivariable approximate q-Toeplitz relation)**.** *Let $-1 < q < 1$ and $a, b \in [r]$. Then*

$$(L_{k,a}^{(q)})^* L_{k,b}^{(q)} = \delta_{ab}1 + qL_{k,b}^{(q)}(L_{k,a}^{(q)})^* + E_{k,ab}^{(q)}. \tag{3.6}$$

Here,

$$E_{k,ab}^{(q)} = -\frac{q\delta_{ab}}{k} \sum_{u\in\Gamma_a} \ell_u^{(q)}(\ell_u^{(q)})^* + (D_{k,b}^{(q)})^* \left(\frac{B_{k,ba}^{(q)}}{k} \otimes 1 \right) D_{k,a}^{(q)}.$$

Consequently,

$$\max_{a,b\in[r]} \|E_{k,ab}^{(q)}\| \le \frac{|q|\,\omega(\Gamma_k)}{k} + \frac{\omega(\Gamma_k)}{k}\|B_{\Gamma_k}^{(q)}\|. \tag{3.7}$$

In particular, there are events $\mathcal{G}_k$ with $\mathbb{P}_\Gamma(\mathcal{G}_k) \longrightarrow 1$ and deterministic numbers $\delta_k \to 0$ such that

$$\max_{a,b} \|E_{k,ab}^{(q)}\| \le \delta_k \qquad (\Gamma \in \mathcal{G}_k).$$

One may take

$$\delta_k = \mathcal{O}_{q,r}\left(\frac{\log k}{\sqrt{k}}\right).$$

Proof. The graph-product and Clifford-twisted graph-product creation operators satisfy, respectively,

$$\ell_u^*\ell_v = \delta_{uv}1 + (A_{\Gamma_k})_{uv}\ell_v\ell_u^*$$

and

$$(\widetilde{\ell}_u)^*\ell_v = \delta_{uv}1 - (A_{\Gamma_k})_{uv}\widetilde{\ell}_v(\widetilde{\ell}_u)^*.$$

Thus, in both cases,

$$(\ell_u^{(q)})^*\ell_v^{(q)} = \delta_{uv}1 + (A_{\Gamma_k}^{(q)})_{uv}\ell_v^{(q)}(\ell_u^{(q)})^*.$$

Summing over $u \in \Gamma_{k,a}$ and $v \in \Gamma_{k,b}$ gives

$$(L_{k,a}^{(q)})^* L_{k,b}^{(q)} = \delta_{ab}1 + \frac{1}{k} \sum_{\substack{u\in\Gamma_{k,a}\\ v\in\Gamma_{k,b}}} (A_\Gamma^{(q)})_{uv}\ell_v^{(q)}(\ell_u^{(q)})^*.$$

Subtracting

$$qL_{k,b}^{(q)}(L_{k,a}^{(q)})^* = \frac{q}{k} \sum_{\substack{u\in\Gamma_{k,a}\\ v\in\Gamma_{k,b}}} \ell_v^{(q)}(\ell_u^{(q)})^*$$

yields (3.6). The diagonal contribution occurs only when $a = b$ and equals

$$-\frac{q}{k}\sum_{u\in\Gamma_{k,a}} \ell_u^{(q)}(\ell_u^{(q)})^*.$$

Moreover,

$$\|D_{k,a}^{(q)}\|^2 = \left\|\sum_{u\in\Gamma_{k,a}} \ell_u^{(q)}(\ell_u^{(q)})^*\right\| \leq \omega(\Gamma_k),$$

and

$$\|B_{k,ba}^{(q)}\| \leq \|B_{\Gamma_k}^{(q)}\|.$$

This proves (3.7). The final assertion follows from the clique number estimate and the random-matrix norm estimate for $B_{\Gamma_k}^{(q)}$ from Theorem 3.6 and Theorem 3.7 . □

4. Vacuum Expectations Comparison Bound

In this section we prove Theorem 4.1 which shows in particular that q-Toeplitz generators $\ell_{q,a}$ can be approximated in moments by the models operators $L_{k,a}$. This strengthens a theorem by Młotkowski [Mlo04] who showed already that $\ell_{q,a} + \ell_{q,a}^*$ can be approximated in moments by $L_{k,a} + L_{k,a}^*$. Our estimate is moreover quantitative.

We use the notation of Section 2 and let Γ be a graph with edge labels $[k] \times [r]$ and Γ_i the subgraph with labels $[k] \times \{a\}$. Let

$$L_{k,a} = \frac{1}{\sqrt{k}}\sum_{i=1}^{k} \ell_{k,a}, \qquad a = 1, \cdots, r,$$

be the averaged creation operators on the graph-product Fock space $\mathcal{F}_{\Gamma_k}$.

Let

$$\Omega_{\Gamma_k}^q = \begin{cases} \Omega_k, & q \geq 0, \\ \xi_{\mathrm{Cl},k} \otimes \Omega_{\Gamma_k}, & q < 0, \end{cases}$$

where $\xi_{\mathrm{Cl},k}$ is the canonical tracial GNS vector of the Clifford algebra. Define the vacuum states on two models:

$$\varphi_{q,k}(T) = \langle T\Omega_{\Gamma_k}^q, \Omega_{\Gamma_k}^q\rangle, \qquad \varphi_q(T) = \langle T\Omega_q, \Omega_q\rangle.$$

In Lemma 3.12, we proved that

$$L_{k,a}^* L_{k,b} = \delta_{ab} I + q L_{k,b} L_{k,a}^* + E_{k,ab}. \tag{4.1}$$

Let $\delta_k = \max_{1\leq a,b\leq r} ||E_{k,ab}||$. For $\Gamma_k \sim \Gamma(kr, q)$, we have

$$\delta_k = \mathcal{O}_{q,r}(\frac{\log k}{\sqrt{k}})$$

in probability. On the other hand, we have

$$\ell_{q,a}^* \ell_{q,b} = \delta_{ab} I + q\ell_{q,b}\ell_{q,a}^*.$$

Since $L_{k,a}^*\Omega_{\Gamma_k}^q = 0, \ell_{q,a}^*\Omega_q = 0$, we easily get

$$\varphi_{q,k}(L_{k,a_1}\cdots L_{k,a_s}) = 0, \qquad \varphi_q(\ell_{q,a_1}\cdots \ell_{q,a_s}) = 0,$$

for any nonempty pure creator word. The analogous statement holds for a word consisting purely of annihilators.

Since by combining Theorem 3.9 and Lemma 3.12, and [BoSp91, Lemma 4],

$$||L_{k,a}||^2 \le \frac{1}{\sqrt{\min\{1-q,1\}}} + \delta_k, \qquad ||\ell_{q,a}|| \le \frac{1}{\sqrt{\min\{1-q,1\}}},$$

there is a uniform bound K_q such that for all k,a we have $||L_{k,a}|| \le K_q$ with probability going to 1 as $k \to \infty$, and $||\ell_{q,a}|| \le K_q$.

Let w be a word of letters $\{1,\cdots,r,1^*,\cdots,r^*\}$, and letter a means a creator, a^* means an annihilator. Let $L_{k,w}$ be the corresponding word of the graph product averaged creators/annihilators, $\ell_{q,w}$ be the corresponding word of the q-Fock space creators/annihilators. For instance, if $w = a_1a_2a_3^*a_4^*$, then $L_{k,w} = L_{k,a_1}L_{k,a_2}L_{k,a_3}^*L_{k,a_4}^*$, $\ell_{q,w} = \ell_{q,a_1}\ell_{q,a_2}\ell_{q,a_3}^*\ell_{q,a_4}^*$. When $w = \varnothing$, we just put $L_{k,\varnothing} = I, \ell_{q,\varnothing} = I$. For these two kinds of words, we prove the following word comparison bound, which tell that the difference between $\varphi_{q,k}(L_{k,w})$ and $\varphi_q(\ell_{q,w})$ will go to 0 being of order δ_k.

Theorem 4.1. *There are events $\mathcal{G}_k$ with $\mathbb{P}_\Gamma(\mathcal{G}_k) \longrightarrow 1$ as $k \to \infty$, such that for any word w of $\{1,\cdots,r,1^*,\cdots,r^*\}$, we have*

$$|\varphi_{q,k}(L_{k,w}) - \varphi_q(\ell_{q,w})| \le |w| C_q^{|w|} \delta_k.$$

Here C_q is a constant depending only on q.

Proof. For $t \in \mathbb{N}$, we define

$$D(t) := \sup_{|w|\le t} |\varphi_{q,k}(L_{k,w}) - \varphi_q(\ell_{q,w})|.$$

Denote $Q_q := \frac{1}{1-|q|}$. Let K_q be as in the paragraph preceding this theorem and assume we are in the event that $\|L_{k,a}\| \le K_q$ which has probability converging to 1 (Lemma 3.12). Choose a large C_q such that

$$K_q \le C_q, \qquad Q_q \le C_q^2. \tag{4.2}$$

In what follows we prove by induction that $D(t) \le tC_q^t\delta_k$.

For $t = 0,1$, $D(t) = 0$.

For $|w| = t \ge 2$, we assume $D(t-2) \le (t-2)C_q^{t-2}\delta_k$ and run the following procedure to reduce the word length:

If w ends in an annihilator, then since $L_{k,a}^*\Omega_{\Gamma_k}^q = 0, \ell_{q,a}^*\Omega_q = 0$, both traces vanish. If w has no annihilator and is nonempty, both traces again vanish.

Therefore we can assume w has at least one annihilator and does not end in annihilator. Write w of the form $w = \tilde{w}a^*a_1a_2\cdots a_s$, where a^* is the rightmost annihilator of w, and $\tilde{w}$ is the subword of w before a^*.

Then $L_{k,w} = L_{k,\tilde{w}}L_{k,a}^*L_{k,a_1}\cdots L_{k,a_s}$. Now, we move $L_{k,a}^*$ to the right using the approximating q-Toeplitz relation Eq. (4.1) iterately, then we obtain

$$\begin{aligned} L_{k,a}^*L_{k,a_1}\cdots L_{k,a_s} =& \sum_{i=1}^s q^{i-1}\delta_{aa_i}L_{k,a_1}\cdots L_{k,a_{i-1}}L_{k,a_{i+1}}\cdots L_{k,a_s} \\ &+ \sum_{i=1}^s q^{i-1}L_{k,a_1}\cdots L_{k,a_{i-1}}E_{k,aa_i}L_{k,a_{i+1}}\cdots L_{k,a_s} + q^sL_{k,a_1}\cdots L_{k,a_s}L_{k,a}^*. \end{aligned}$$

But $\varphi_{q,k}(L_{k,\tilde{w}}L_{k,a_1}\cdots L_{k,a_s}L_{k,a}^*)=0$ since $L_{k,a}^*\Omega_{\Gamma_k}^q=0$, therefore

$$\varphi_{q,k}(L_{k,w})=\overbrace{\sum_{i=1}^{s}q^{i-1}\delta_{aa_i}\varphi_{q,k}(L_{k,\tilde{w}}L_{k,a_1}\cdots L_{k,a_{i-1}}L_{k,a_{i+1}}\cdots L_{k,a_s})}^{\text{Contraction Terms}}$$
$$+\overbrace{\sum_{i=1}^{s}q^{i-1}\varphi_{q,k}(L_{k,\tilde{w}}L_{k,a_1}\cdots L_{k,a_{i-1}}E_{k,aa_i}L_{k,a_{i+1}}\cdots L_{k,a_s})}^{\text{Error Terms}}.$$

On the other hand, by the analogous arugment for the q-creation operators

$$\varphi_q(\ell_{q,w})=\sum_{i=1}^{s}q^{i-1}\delta_{aa_i}\tau_q(\ell_{\tilde{w}}\ell_{q,a_1}\cdots\ell_{q,a_{i-1}}\ell_{q,a_{i+1}}\cdots\ell_{q,a_s}).$$

First, we bound the difference of the contraction terms of two models. For each i, the contraction removes two letters a_i and a from w, thus the resulting words have length $t-2$, therefore the difference is bounded by $D(t-2)$. Then the difference between two total contraction terms is bounded by

$$\sum_{i=1}^{s}|q|^{i-1}D(t-2)\le\frac{1}{1-|q|}D(t-2).$$

Second, we bound the error terms of the graph product model. In i-th error term, $L_{k,a}^*L_{k,a_i}$ is replaced by E_{k,aa_i}, so there are at most $t-2$ ordinary L and L^* factors. Thus

$$\begin{aligned}|\varphi_{q,k}(L_{k,\tilde{w}}L_{k,a_1}\cdots L_{k,a_{i-1}}E_{k,aa_i}L_{k,a_{i+1}}\cdots L_{k,a_i})|&\le||L_{k,\tilde{w}}L_{k,a_1}\cdots L_{a_{i-1}}E_{k,aa_i}L_{k,a_{i+1}}\cdots L_{k,a_s}||\\&\le K_q^{t-2}\delta_k.\end{aligned}$$

Therefore the total error is bounded by

$$\sum_{i=1}^{s}|q|^{i-1}K_q^{t-2}\delta_k\le\frac{K_q^{t-2}}{1-|q|}\delta_k.$$

Combining these two bounds, we get that

$$D(t)\le\frac{1}{1-|q|}D(t-2)+\frac{K_q^{t-2}}{1-|q|}\delta_k. \tag{4.3}$$

Now, we use the induction assumption and the inequality Eq. (4.3), we have, by (4.2),

$$D(t)\le Q_q(t-2)C_q^{t-2}\delta_k+Q_qK_q^{t-2}\delta_k\le(t-2)C_q^t\delta_k+C_q^t\delta_k\le tC_q^t\delta_k,$$

which concludes the proof. □

Remark 4.2 (A non-backtracking viewpoint)**.** The reduction in the proof of Theorem 4.1 admits a useful non-backtracking interpretation. On the graph-product Fock space $\mathcal{F}_\Gamma=\ell^2(A_\Gamma^+)$, a creation operator follows a labelled edge of the Cayley graph of the right-angled Artin monoid, whereas the corresponding annihilation operator traverses such an edge backwards. The relation

$$(\ell_u^{(q)})^*\ell_v^{(q)}=\delta_{uv}1+(A_\Gamma^{(q)})_{uv}\ell_v^{(q)}(\ell_u^{(q)})^*$$

either contracts an immediate creation–annihilation reversal or moves the backward step past a creation operator whenever there is an edge in the graph. The Clifford twist supplies the appropriate sign when $q<0$.

After averaging, this relation becomes

$$L_{k,a}^* L_{k,b} = \delta_{ab} 1 + q L_{k,b} L_{k,a}^* + E_{k,ab}.$$

Thus, moving the rightmost annihilator through the creators to its right either contracts a matched pair, reducing the word length by two, or moves the annihilator one step further with weight q. The terminal uncontracted term has zero vacuum expectation. Iterating the procedure produces the usual q-weighted Wick contractions [EfPo03], while $E_{k,ab}$ collects the centered-adjacency and finite-graph corrections. In this sense, the estimate $\max_{a,b} \|E_{k,ab}\| \to 0$ is a Fock-space analogue of controlling the remainder in a non-backtracking expansion.

This viewpoint is related to the reduced-word framework underlying graph-product Khintchine inequalities [CKL21] and to the matrix-valued and operator-valued non-backtracking methods used in strong-convergence problems [BC19, BC24, FP23]. We emphasize, however, that this is an operator-theoretic analogy: our proof does not introduce a Hashimoto operator or use an Ihara–Bass identity.

5. Strong convergence for multivariate polynomials

In this section we prove that the reduced q-Toeplitz algebra admits strongly convergent models. We let $-1 < q < 1$ and consider the q-Toeplitz algebra$\mathcal{T}_{q,r}$ with r canonical generators $T_1, \ldots, T_r$ which subject to the q-Toeplitz relation

$$T_a^* T_b = q T_b T_a^* + \delta_{ab}.$$

Let again $\ell_{q,1}, \ldots, \ell_{q,r} \in \mathcal{B}(\mathcal{F}_q(\mathbb{C}^r))$ denote the q-creation operators and let $\mathcal{T}_{q,r}^{\mathrm{red}}$ be the reduced q-Toeplitz C*-algebra generated by them. The assignment

$$T_a \mapsto \ell_{q,a} \tag{5.1}$$

is a faithful representation $\mathcal{T}_{q,r} \to \mathcal{T}_{q,r}^{\mathrm{red}}$. For the universal q-Toeplitz C*-algebra $\mathcal{T}_{q,r}^u$ generated by $\mathcal{T}_{q,r}$, let

$$\Phi_q : \mathcal{T}_{q,r}^u \to \mathcal{T}_{q,r}^{\mathrm{red}},$$

be the map extending (5.1) by the universal property of $\mathcal{T}_{q,r}^u$.

Remark 5.1. In [PuWo89] it was proved that Φ_q is an isomorphism for $|q| < \sqrt{2} - 1$. Outside this range this is unknown and we refer to [DyNi93, Kuz23] for related results and further discussion on this.

Let again $L_{k,a} = \frac{1}{\sqrt{k}} \sum_{v=1}^k \ell_{k,a} \in \mathcal{B}(\mathcal{F}_\Gamma)$, see Section 4. For $d \geq 0$, put

$$\mathcal{E}_d^{(q)} = \operatorname{span}\left\{1,\ s_{i_1}^{(q)} \cdots s_{i_m}^{(q)} : 1 \leq m \leq d,\ i_1, \ldots, i_m \in [r]\right\} \subseteq \mathcal{T}_{q,r},$$

where $s_a^{(q)} = \ell_{q,a} + \ell_{q,a}^*$. For a graph Γ, set $S_{k,a}^{(q,\Gamma)} = L_{k,a}^{(q,\Gamma)} + (L_{k,a}^{(q,\Gamma)})^*$.

Remark 5.2. The monomials occurring in the definition of $\mathcal{E}_d^{(q)}$ are linearly independent. This is proved in (the proof of) [CIW21, Theorem 4.1] but we give the short argument for convenience of the reader. Indeed, if P has degree m, then the projection of $P(s^{(q)})\Omega_q$ onto the m-particle space is the coefficient tensor of the homogeneous degree-m part of P. The q-inner product is strictly positive for $|q| < 1$, so this tensor vanishes only when all degree-m coefficients vanish. Descending induction to the degree of P proves the assertion.

As a consequence of Remark 5.2, there is a well-defined linear map $\Theta^{(d)}_{k,\Gamma} : \mathcal{E}^{(q)}_d \longrightarrow \mathcal{B}(\mathcal{F}_\Gamma)$ given by

$$\Theta^{(d)}_{k,\Gamma}\left(s^{(q)}_{i_1}\cdots s^{(q)}_{i_m}\right) = S^{(q,\Gamma)}_{k,i_1}\cdots S^{(q,\Gamma)}_{k,i_m}.$$

We include the following standard lemma for completeness.

Lemma 5.3. *Let $\mathcal{E}$ be a finite dimensional operator space. There exists $c_{\mathcal{E}} > 0$ such that for every $R : \mathcal{E} \to \mathcal{B}(\mathcal{H})$ we have $\|R\|_{\mathrm{cb}} \leq c_{\mathcal{E}}\|R\|$.*

Proof. Let $e_1, \ldots, e_m$ be a basis of $\mathcal{E}$ and let $f^*_1, \ldots, f^*_m \in \mathcal{E}^*$ be the dual basis, i.e. $f^*_a(e_b) = \delta_{ab}$. We have $\|f^*_a\| = \|f^*_a\|_{\mathrm{cb}}$. Now let $X \in M_D(\mathbb{C})\otimes\mathcal{E}$ and write $X = \sum_{a=1}^m X_a \otimes e_a$. As $X_a = (\mathrm{id}\otimes f^*_a)(X)$ we find that

$$\|(\mathrm{id}\otimes R)(X)\| = \|\sum_{a=1}^m X_a \otimes R(e_a)\| \leq \sum_{a=1}^m \|X_a\|\|R\|\|e_a\| \leq \sum_{a=1}^m \|f^*_a\|\|X\|\|R\|\|e_a\|,$$

so that $c_{\mathcal{E}} = \sum_{a=1}^m \|f^*_a\|\|e_a\|$ does the job. □

For the upper bound in the following theorem we we were inspired by [Shl26, Theorem 6]. We show that in the nuclear case the argument can be changed to yield a completely bounded strong version of the strong convergence result.

Theorem 5.4 (Complete bounded-degree strong convergence)**.** *Let $-1 < q < 1$ satisfy $|q| < \sqrt{2}-1$, and fix $r, d \geq 1$. There are events $\mathcal{G}_k$ such that $\mathbb{P}_\Gamma(\mathcal{G}_k) \to 1$ and*

$$\sup_{\Gamma\in\mathcal{G}_k}\sup_{D\geq 1}\sup_{0\neq X\in M_D(\mathcal{E}^{(q)}_d)}\left|\frac{\|(id_{M_D}\otimes\Theta^{(d)}_{k,\Gamma})(X)\|}{\|X\|} - 1\right| \to 0.$$

Equivalently, for every sequence $D_k \geq 1$ and every sequence of nonzero matrix-valued noncommutative polynomials $P_k \in M_{D_k}(\mathbb{C})\otimes\mathbb{C}\langle x_1, \ldots, x_r\rangle$ of degree at most d,

$$\frac{\|P_k(S^{(q,\Gamma_k)}_{k,1}, \ldots, S^{(q,\Gamma_k)}_{k,r})\|}{\|P_k(s^{(q)}_1, \ldots, s^{(q)}_r)\|} \to 1$$

in probability.

Proof. Let $\mathcal{G}_k$ be the events from Lemma 3.12, enlarged if necessary so that

$$\max_{1\leq a\leq r}\|L^{(q,\Gamma)}_{k,a}\| \leq K_q \qquad (\Gamma\in\mathcal{G}_k)$$

for a constant K_q independent of k and $\Gamma \in \mathcal{G}_k$, and we have $\mathbb{P}_\Gamma(\mathcal{G}_k) \longrightarrow 1$.

We first establish fixed-matrix-level convergence. Let $\Gamma_k \in \mathcal{G}_k$ be an arbitrary deterministic sequence and let $\mathcal{U}$ be a nonprincipal ultrafilter. Put

$$\mathcal{A}_k = C^*(L^{(q,\Gamma_k)}_{k,1}, \ldots, L^{(q,\Gamma_k)}_{k,r}),$$

as C*-subalgebras of $\ell^\infty(\mathcal{G}_k; B(\mathcal{H}_\Gamma))$. By (4.1) the elements

$$\mathcal{L}_a = (L^{(q,\Gamma_k)}_{k,a})_{k,\mathcal{U}} \in \prod_{k,\mathcal{U}}\mathcal{A}_k$$

satisfy the exact q-Toeplitz relations

$$\mathcal{L}^*_a\mathcal{L}_b = \delta_{ab}1 + q\mathcal{L}_b\mathcal{L}^*_a.$$

Therefore the universal property gives a representation of the universal q-Toeplitz algebra. Since $\Phi_q : \mathcal{T}^u_{q,r} \longrightarrow \mathcal{T}^{\mathrm{red}}_{q,r}$ is an isomorphism for $|q| < \sqrt{2}-1$ [PuWo89], this representation induces a $*$-homomorphism

$$\psi_{\mathcal{U}} : \mathcal{T}^{\mathrm{red}}_{q,r} \longrightarrow \prod_{k,\mathcal{U}} \mathcal{A}_k, \qquad \psi_{\mathcal{U}}(\ell_{q,a}) = \mathcal{L}_a. \tag{5.2}$$

Let, for $0 \le q < 1$, $\mathcal{F}_{\mathcal{U}} = \prod_{k,\mathcal{U}} \mathcal{F}_{\Gamma(k,q)}$ be the corresponding Hilbert-space ultraproduct and let $\Omega_{\mathcal{U}}$ be the ultraproduct vacuum vector. In the Clifford-twisted case, $-1 \le q < 0$, use the canonical tracial GNS vector in the Clifford factor to define $\mathcal{F}_{\mathcal{U}}$. Theorem 4.1 gives, for every $*$-polynomial Q,

$$\langle Q(\mathcal{L}_1, \dots, \mathcal{L}_r, \mathcal{L}_1^*, \dots, \mathcal{L}_r^*)\Omega_{\mathcal{U}}, \Omega_{\mathcal{U}}\rangle = \left\langle Q(\ell_{q,1}, \dots, \ell_{q,r}, \ell_{q,1}^*, \dots, \ell_{q,r}^*)\Omega_q, \Omega_q\right\rangle.$$

Hence the cyclic representation generated by $\Omega_{\mathcal{U}}$ is unitarily equivalent to the q-Fock representation. The latter is faithful on the reduced algebra $\mathcal{T}^{\mathrm{red}}_{q,r}$; this easily follows as Ω_q is cyclic. Therefore $\psi_{\mathcal{U}}$ is injective, and hence completely isometric.

It follows that, for every fixed m,

$$\sup_{\Gamma \in \mathcal{G}_k} \sup_{\substack{X \in M_m(\mathcal{E}_d^{(q)}) \\ \|X\| \le 1}} \Big| \|(\mathrm{id}_{M_m} \otimes \Theta^{(d)}_{k,\Gamma})(X)\| - \|X\| \Big| \to 0. \tag{5.3}$$

Indeed, otherwise one could choose a violating sequence Γ_k and, by compactness of the unit ball of $M_m(\mathcal{E}_d^{(q)})$, obtain a contradiction to the complete isometry of (5.4).

We next prove the upper estimate at arbitrary matrix level. Suppose it fails. Then, after passing to a subsequence, there are

$$\Gamma_k \in \mathcal{G}_k, \qquad D_k \ge 1, \qquad X_k \in M_{D_k}(\mathcal{E}_d^{(q)})$$

such that $\|X_k\| = 1$ and

$$\|(\mathrm{id}_{M_{D_k}} \otimes \Theta^{(d)}_{k,\Gamma_k})(X_k)\| \ge 1 + \varepsilon \tag{5.4}$$

for some $\varepsilon > 0$.

The reduced q-Toeplitz algebra $\mathcal{T}^{\mathrm{red}}_{q,r}$ is nuclear: by [Kuz23, Theorem 1.2], it is isomorphic to the Cuntz–Toeplitz algebra. Recall that $\prod_{k,\mathcal{U}} \mathcal{A}_k$ is by definition the quotient of $\prod_k \mathcal{A}_k$ by the c_0 sequences with respect to $\mathcal{U}$. By the Choi–Effros lifting theorem, the homomorphism (5.2) has a completely positive contractive lift

$$\widetilde{\psi} = (\phi_k)_k : \mathcal{T}^{\mathrm{red}}_{q,r} \longrightarrow \prod_k \mathcal{A}_k. \tag{5.5}$$

Set

$$R_k = \Theta^{(d)}_{k,\Gamma_k} - \phi_k|_{\mathcal{E}_d^{(q)}}.$$

Since (5.5) lifts (5.2) and $\mathcal{E}_d^{(q)}$ is finite dimensional we have $\|R_k\| \to 0$ and by Lemma 5.3 even $\|R_k\|_{\mathrm{cb}} \to 0$. Consequently,

$$\begin{aligned}\|(\mathrm{id}_{M_{D_k}} \otimes \Theta^{(d)}_{k,\Gamma_k})(X_k)\| &\le \|(\mathrm{id}_{M_{D_k}} \otimes \phi_k)(X_k)\| + \|R_k\|_{\mathrm{cb}}\|X_k\| \\ &\le 1 + \mathcal{O}_{\mathcal{U}}(1),\end{aligned}$$

contradicting (5.4).

For the lower estimate, suppose that there are $\varepsilon > 0$ and sequences as above such that

$$\|X_k\| = 1, \qquad \|(\mathrm{id}_{M_{D_k}} \otimes \Theta^{(d)}_{k,\Gamma_k})(X_k)\| \le 1 - \varepsilon.$$

Fix $\delta \in (0, \varepsilon/3)$. Since $\mathcal{T}_{q,r}^{\mathrm{red}}$ is exact [KeNi11], in fact it is even nuclear [Kuz23], the fixed-family compression lemma [MaSa25, Lemma 5.13], applied to $E = \mathcal{E}_d^{(q)}$, gives an integer $m_0 = m_0(d, r, q, \delta)$ and contractions

$$U_k, V_k \in M_{m_0, D_k}(\mathbb{C})$$

such that

$$Y_k = (U_k \otimes 1) X_k (V_k^* \otimes 1) \in M_{m_0}(\mathcal{E}_d^{(q)})$$

satisfies

$$\|Y_k\| \geq 1 - \delta.$$

On the model side,

$$\|(\mathrm{id}_{M_{m_0}} \otimes \Theta_{k,\Gamma_k}^{(d)})(Y_k)\| \leq \|(\mathrm{id}_{M_{D_k}} \otimes \Theta_{k,\Gamma_k}^{(d)})(X_k)\| \leq 1 - \varepsilon. \tag{5.6}$$

But (5.3), applied at the fixed m_0, gives

$$\|(\mathrm{id}_{M_{m_0}} \otimes \Theta_{k,\Gamma_k}^{(d)})(Y_k)\| = \|Y_k\| + o_{\mathcal{U}}(1) \geq 1 - \delta + o_{\mathcal{U}}(1),$$

contradicting (5.6). □

Corollary 5.5. *For every fixed matrix-valued polynomial P,*

$$\|P(S_{k,1}^{(q,\Gamma)}, \dots, S_{k,r}^{(q,\Gamma)})\| \to \|P(s_1^{(q)}, \dots, s_r^{(q)})\|$$

in probability.

Proof. Apply Theorem 5.1 with $d = \deg P$ and the fixed coefficient dimension of P. □

Remark 5.6. Theorem 5.4 holds true for any $-1 < q < 1$ as long as Φ_q is an isomorphism.

6. Strongly convergent matrix models

In Section 5 we have shown that q-Toeplitz algebras admit strongly convergent models of random graph products of creation and annihilation operators with arbitrary matrix amplifications. The latter models are still infinite dimensional. In this section we focus on the subalgebra of q-semicircular elements and show that they can be approximated with GUE's yielding finite dimensional models that converge strongly. This is also the place where control of the growth rate of the matrix amplification is essential.

6.1. **Tensor-GUE models.** Tensor-GUE models are random operators on the multipartite space $(\mathbb{C}^N)^{\otimes L}$ in which each independent GUE matrix acts on a prescribed collection of tensor coordinates and as the identity on the complementary coordinates. The overlap pattern of these supports determines the limiting mixed independence: disjoint supports give commuting, classically independent variables, while the full joint limit is the graph-product semicircular family. This form of mixed classical and free independence is discussed in [Mlo04, SpWy16]. Weak convergence of the corresponding tensor-matrix models was proved in [ChCo21, Theorem 4]. Strong convergence was obtained for particular tensor configurations in [BeCa22, CY26], and for every fixed finite interaction pattern in [CGVvH26b, Section 9.4, Theorem 9.8].

We recall the construction and comparison scheme of [CGVvH26b, Section 9.4]. The fixed-parameter strong-convergence theorem is not by itself sufficient for our application, because the matrix-coefficient dimension below grows with N. Section 6.2 establishes the required uniform refinement.

The results of this section can be found in [CGVvH26b].

Fix integers $L, V \geq 1$ and nonempty subsets $K_1, \dots, K_V \subseteq [L] := \{1, \dots, L\}$. For $N \geq 1$, let

$$G_v^{(N)} = H_v^{(N)} \otimes I_{N^{L-|K_v|}} \in M_{N^L}(\mathbb{C}), \qquad 1 \leq v \leq V,$$

where $\big(H_v^{(N)}\big)_{v=1}^V$ are independent normalized GUE matrices of dimension $N^{|K_v|}$. Set $G^{(N)} = \big(G_1^{(N)}, \dots, G_V^{(N)}\big)$. Its limiting model is the Γ-independent semicircular family $x^\Gamma = (x_1^\Gamma, \dots, x_V^\Gamma)$, where Γ is the graph on $[V]$ defined by

$$v \sim_\Gamma w \quad \Longleftrightarrow \quad K_v \cap K_w = \varnothing.$$

For self-adjoint coefficient matrices $A_0, \dots, A_V \in M_D(\mathbb{C})_{\mathrm{sa}}$, define the self-adjoint linear pencil

$$\mathcal{P}_A(x) = A_0 \otimes 1 + \sum_{v=1}^V A_v \otimes x_v.$$

And we denote For $T \geq 1$, define

$$G_v^{(N,T)} = \frac{1}{\sqrt{T}} \sum_{t=1}^T \left(\bigotimes_{j \in K_v} G_{v,t,j}^{(N)} \right) \otimes I_{N^{L-|K_v|}},$$

where all matrices $G_{v,t,j}^{(N)}$ are independent normalized $N \times N$ GUE matrices. Write

$$G^{(N,T)} = \big(G_1^{(N,T)}, \dots, G_V^{(N,T)}\big).$$

Correspondingly, let

$$s_v^{(T)} = \frac{1}{\sqrt{T}} \sum_{t=1}^T \left(\bigotimes_{j \in K_v} s_{v,t,j} \right) \otimes 1,$$

where the $s_{v,t,j}$'s form the Hayes' limiting model, and set $s^{(T)} = \big(s_1^{(T)}, \dots, s_V^{(T)}\big)$.

By [CGVvH26b, Lemma 9.9], $G^{(N,T)}$ and $G^{(N)}$ have the same entrywise mean and covariance (for the real and imaginary parts of their entries). Moreover, [CGVvH26b, Lemma 9.10] gives, for every fixed coefficient dimension and every fixed linear pencil P_A,

$$\|P_A(s^{(T)})\| \longrightarrow \|P_A(x^\Gamma)\|, \qquad T \to \infty.$$

Below, the quantitative Hayes-model estimate [CGVvH26b, Theorem 9.7] and the universality principle [BravH, Theorem 2.8] are combined to make this comparison uniform over growing matrix-coefficient dimensions.

6.2. **Uniform large coefficient dimensions strong convergence of tensor-GUE.**

Definition 6.1 (Dimension-free limiting CLT modulus)**.** For $M \geq 1$, define

$$\Delta_{L,V,M}(T) := \sup_{\substack{D \geq 1,\ A_0, \dots, A_V \in M_D(\mathbb{C})_{\mathrm{sa}} \\ \Lambda(A) \leq M}} \Big| \|\mathcal{P}_A(s^{(T)})\| - \|\mathcal{P}_A(x^\Gamma)\| \Big|,$$

The following lemma tells why $\Delta_{L,V,M}(T)$ is called dimension-free: it is precisely what removes the matrix-coefficient dimension from the limiting $T \to \infty$ step.

Lemma 6.2. *For fixed L, V, M and every $T \geq 1$, we have*

$$\Delta_{L,V,M}(T) \leq \frac{(2^L - 2)M}{\sqrt{T}}.$$

Consequently, $\Delta_{L,V,M}(T) \to 0$ as $T \to \infty$.

Proof. We use the full Fock-space realization of the free semicircular families appearing in the Hayes' limiting model. For each $j \in [L]$, let
$$\mathcal{F}_j = \mathcal{F}\big(\ell_2([V] \times [T])\big),$$
and denote by $a^{(j)}_{v,t}$, $v \in [V]$, $t \in [T]$ the corresponding left creation operators. Thus $s_{v,t,j} = a^{(j)}_{v,t} + a^{(j)*}_{v,t}$. We work on $\mathcal{H}_T = \bigotimes_{j=1}^L \mathcal{F}_j$. For $v \in [V]$, put
$$W^{(T)}_{v,t} = \bigotimes_{j=1}^{L} b^{(v,t)}_j, \qquad b^{(v,t)}_j = \begin{cases} a^{(j)}_{v,t}, & j \in K_v, \\ 1, & j \notin K_v, \end{cases}$$
and define
$$\lambda^{(T)}_v = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} W^{(T)}_{v,t}.$$
We first identify the joint operator model of $(\lambda^{(T)}_v)_{v=1}^V$. Since K_v is nonempty, $W^{(T)*}_{v,t} W^{(T)}_{v,u} = \delta_{tu} 1$, and hence $\lambda^{(T)*}_v \lambda^{(T)}_v = 1$. Moreover, if $u \neq v$ and $K_u \cap K_v \neq \varnothing$, then a tensor coordinate in $K_u \cap K_v$ contains the factor $a^{(j)*}_{u,t} a^{(j)}_{v,r} = 0$, so that $\lambda^{(T)*}_u \lambda^{(T)}_v = 0$. If $K_u \cap K_v = \varnothing$, the two operators act on disjoint tensor coordinates and therefore doubly commute:
$$\lambda^{(T)}_u \lambda^{(T)}_v = \lambda^{(T)}_v \lambda^{(T)}_u, \qquad \lambda^{(T)*}_u \lambda^{(T)}_v = \lambda^{(T)}_v \lambda^{(T)*}_u.$$
Thus the family $(\lambda^{(T)}_v)_{v=1}^V$ satisfies precisely the graph-product creation relations associated with
$$u \sim_\Gamma v \quad \Longleftrightarrow \quad K_u \cap K_v = \varnothing.$$
For completeness, we verify that this representation is an amplication of the graph Fock representation. Let $P_v = \lambda^{(T)}_v \lambda^{(T)*}_v$. The preceding relations imply that the projections P_v commute: if $u \not\sim_\Gamma v$, then $P_u P_v = 0$, while if $u \sim_\Gamma v$, this follows from double commutation. Set
$$Q_T = \prod_{v=1}^{V} (1 - P_v), \qquad \mathcal{K}_T = Q_T \mathcal{H}_T.$$
For $w \in \mathcal{A}_\Gamma^+$ denote by $\lambda^{(T)}_w$ the corresponding product of the $\lambda^{(T)}_v$. The graph-product relations imply
$$Q_T \lambda^{(T)*}_w \lambda^{(T)}_{w'} Q_T = \delta_{w,w'} Q_T.$$
Indeed, using the normal form in the right-angled Artin monoid, one first cancels the maximal common left divisor of w and w'. If residual letters remain, either a noncommuting pair gives zero, or the graph relations move a residual creation or annihilation operator next to Q_T. The latter term vanishes because
$$Q_T \lambda^{(T)}_v = 0, \qquad \lambda^{(T)*}_v Q_T = 0.$$
Therefore the map
$$U_T : \ell_2(\mathcal{A}_\Gamma^+) \otimes \mathcal{K}_T \longrightarrow \mathcal{H}_T, \qquad U_T(\delta_w \otimes \xi) = c^{(T)}_w \xi,$$
is an isometry. It is also onto. Indeed, grade $\mathcal{H}_T$ by total Fock degree. Each $c^{(T)}_v$ raises the total degree by $|K_v| \geq 1$, whereas P_v and Q_T preserve the grading. If $\mathcal{H}^{(n)}_T$ denotes the homogeneous subspace of total degree n, then
$$(1 - Q_T)\mathcal{H}^{(n)}_T \subseteq \sum_{v=1}^{V} P_v \mathcal{H}^{(n)}_T \subseteq \sum_{v=1}^{V} c^{(T)}_v \mathcal{H}^{(n-|K_v|)}_T.$$

Induction on n shows that every homogeneous subspace is contained in the range of U_T. Hence U_T is unitary.

Since we have $\lambda_v^{(T)} = U_T(\ell_v \otimes 1_{\mathcal{K}_T})U_T^*$. Consequently, for every coefficient dimension D,

$$\left\| A_0 \otimes 1 + \sum_{v=1}^{V} A_v \otimes \left(\lambda_v^{(T)} + \lambda_v^{(T)*}\right) \right\| = \left\| A_0 \otimes 1 + \sum_{v=1}^{V} A_v \otimes (\ell_v + \ell_v^*) \right\| = \|\mathcal{P}_A(x^\Gamma)\|. \tag{6.1}$$

We now compare $s_v^{(T)}$ with $\lambda_v^{(T)} + \lambda_v^{(T)*}$. Expanding

$$\bigotimes_{j \in K_v} \left(a_{v,t}^{(j)} + a_{v,t}^{(j)*}\right),$$

in the expression

$$s_v^{(T)} := T^{-\frac{1}{2}} \sum_{t=1}^{T} \bigotimes_{j \in K_v} \left(a_{v,t}^{(j)} + a_{v,t}^{(j)*}\right),$$

the all-creation and all-annihilation terms give $\lambda_v^{(T)}$ and $\lambda_v^{(T)*}$, respectively. Thus

$$s_v^{(T)} = \lambda_v^{(T)} + \lambda_v^{(T)*} + r_v^{(T)},$$

where

$$r_v^{(T)} = \sum_{\varnothing \neq S \subsetneq K_v} r_{v,S}^{(T)}$$

and

$$r_{v,S}^{(T)} = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \bigotimes_{j=1}^{L} d_j^{(v,t,S)},$$

with

$$d_j^{(v,t,S)} = \begin{cases} a_{v,t}^{(j)}, & j \in S, \\ a_{v,t}^{(j)*}, & j \in K_v \setminus S, \\ 1, & j \notin K_v. \end{cases}$$

Fix a nonempty proper subset $S \subsetneq K_v$. Because $S \neq \varnothing$, the cross terms with distinct t vanish in $r_{v,S}^{(T)*} r_{v,S}^{(T)}$. Hence

$$r_{v,S}^{(T)*} r_{v,S}^{(T)} = \frac{1}{T} \sum_{t=1}^{T} \left(\bigotimes_{j \in S} 1 \right) \otimes \left(\bigotimes_{j \in K_v \setminus S} a_{v,t}^{(j)} a_{v,t}^{(j)*} \right) \otimes 1.$$

As $K_v \setminus S \neq \varnothing$, the summands in the last display are pairwise orthogonal projections. It follows that $\|r_{v,S}^{(T)}\| = \frac{1}{\sqrt{T}}$. There are $2^{|K_v|} - 2$ nonempty proper subsets of K_v, so

$$\|r_v^{(T)}\| \leq \frac{2^{|K_v|} - 2}{\sqrt{T}} \leq \frac{2^L - 2}{\sqrt{T}}. \tag{6.2}$$

Combining (6.1) and (6.2), we obtain

$$\begin{aligned} \left| \|\mathcal{P}_A(s^{(T)})\| - \|\mathcal{P}_A(x^\Gamma)\| \right| &\leq \left\| \sum_{v=1}^{V} A_v \otimes r_v^{(T)} \right\| \leq \sum_{v=1}^{V} \|A_v\| \, \|r_v^{(T)}\| \\ &\leq \frac{2^L - 2}{\sqrt{T}} \sum_{v=1}^{V} \|A_v\| \leq \frac{(2^L - 2)M}{\sqrt{T}}. \end{aligned}$$

The estimate is independent of D, and taking the supremum proves the statement of the lemma. $\square$

Remark 6.3. We record a stronger consequence of the preceding construction in Lemma 6.2. There is a faithful representation

$$\pi_T : C^*(x_1^\Gamma, \dots, x_V^\Gamma) \longrightarrow B(\mathcal{H}_T)$$

such that

$$\max_{1\le v\le V} \|s_v^{(T)} - \pi_T(x_v^\Gamma)\| \le \frac{2^L-2}{\sqrt{T}}.$$

Consequently, for every self-adjoint linear pencil with $\Lambda(A) \le M$,

$$\Big\|\mathcal{P}_A(s^{(T)}) - (\mathrm{id}_{M_D} \otimes \pi_T)(\mathcal{P}_A(x^\Gamma))\Big\| \le \frac{(2^L-2)M}{\sqrt{T}}.$$

As $\mathrm{id}_{M_D} \otimes \pi_T$ is faithful, spectral stability under self-adjoint perturbations gives

$$d_{\mathrm{H}}\Big(\mathrm{sp}(\mathcal{P}_A(s^{(T)})), \mathrm{sp}(\mathcal{P}_A(x^\Gamma))\Big) \le \frac{(2^L-2)M}{\sqrt{T}}.$$

Let C_0 be the universal constant in [CGVvH26b, Theorem 9.7], and define

$$c_H(L,V,T) := L^{-2}\big(C_0 LVT\big)^{-2L}. \tag{6.3}$$

The exponent L occurs because a linear pencil in the tensor variables becomes a polynomial of degree at most L in the Hayes' variables, while the number of Hayes' variables is VT. More precisely, [CGVvH26b, Theorem 9.7] states that the quantitative constant satisfies $c^{-1} = q_0^2(CLr)^{2q_0}$, where $q_0 \le L$ and $r = VT$.

Theorem 6.4 (Quantitative tensor-GUE transfer)**.** *Fix L, V, M. There exist constants*

$$C_1 = C_1(L,V,M), \qquad C_2 = C_2(L,V), \qquad c_2 = c_2(L,V) > 0,$$

such that the following holds:

Let $D, N, T \ge 1$, let $t > 0$, and let $\rho \in (0,1]$. Suppose

$$T \le e^N$$

and further, with c_H is defined in (6.3), assume

$$D \le \exp\big(c_H(L,V,T)N\rho^2\big). \tag{6.4}$$

Then for every self-adjoint linear pencil $\mathcal{P}_A$ satisfying $\Lambda(A) \le M$, we have

$$\begin{aligned}&\mathbb{P}\Big[\|\mathcal{P}_A(G^{(N)})\| > \|\mathcal{P}_A(x^\Gamma)\| + \Delta_{L,V,M}(T) + C_2 M\rho + C_1\big(N^{-1/2}t^{1/2} + T^{-1/12}t^{2/3} + T^{-1/4}t\big)\Big]\\ &\le 2DN^L e^{-t} + C_1 VT e^{-c_2 N} + \frac{N^L}{c_H(L,V,T)\rho} e^{-c_H(L,V,T)N\rho^2}.\end{aligned} \tag{6.5}$$

Proof. Write

$$X_{A,N,T} = \mathcal{P}_A(G^{(N,T)}) = A_0 \otimes 1 + \frac{1}{\sqrt{T}} \sum_{v=1}^{V} \sum_{t=1}^{T} A_v \otimes Z_{v,t},$$

where each $Z_{v,t}$ is a tensor product of at most L independent normalized $N \times N$ GUE matrices, amplified by identities on the unused tensor coordinates. Its associated Gaussian matrix with the same mean and covariance has the same law as $\mathcal{P}_A(G^{(N)})$.

We estimate the parameters in the universality theorem uniformly over all coefficient dimensions D and all coefficient tuples satisfying $\Lambda(A) \le M$. Repeating the estimates in the proof of [CGVvH26b, Theorem 9.8], every occurrence of the coefficient matrices is bounded by $\sum_{v=1}^{V} \|A_v\| \le M$. Consequently, there exist a constant $C_{L,V} > 0$ such that (in the notation from [BravH])

$$\sigma_*(X_{A,N,T}) \le C_{L,V} M N^{-1/2}, \qquad \sigma(X_{A,N,T}) \le C_{L,V} M,$$

and

$$\overline{R}(X_{A,N,T}) \le C_{L,V} M T^{-1/2}.$$

Moreover, there exist constants $c_{L,V} > 0$ such that for $T \le e^N$,

$$\mathbb{P}\left[\max_{v,t} \|Z_{v,t}\| > C_{L,V}\right] \le C_{L,V} V T e^{-c_{L,V} N}.$$

Set $R_0 = C_{L,V} M T^{-1/4}$, where the constant is enlarged if necessary. Since

$$\overline{R}(X_{A,N,T}) \le C_{L,V} M T^{-1/2} \quad \text{and} \quad \sigma(X_{A,N,T}) \le C_{L,V} M,$$

we have

$$R_0 \ge \overline{R}(X_{A,N,T})^{1/2} \sigma(X_{A,N,T})^{1/2} + \sqrt{2}\, \overline{R}(X_{A,N,T}).$$

Moreover, on the event $\max_{v,t} \|Z_{v,t}\| \le C_{L,V}$, every independent summand satisfies

$$\left\| \frac{1}{\sqrt{T}} A_v \otimes Z_{v,t} \right\| \le C_{L,V} M T^{-1/2} \le R_0.$$

Consequently, the error term in the unbounded spectrum universality theorem satisfies for $t > 0$

$$\begin{aligned} &\sigma_*(X_{A,N,T}) t^{1/2} + R_0^{1/3} \sigma(X_{A,N,T})^{2/3} t^{2/3} + R_0 t \\ &\le C_1 \left(N^{-1/2} t^{1/2} + T^{-1/12} t^{2/3} + T^{-1/4} t \right), \end{aligned}$$

where $C_1 = C_1(L, V, M)$. Therefore the Brailovskaya–van Handel universality comparison ([BravH]), applied as in the proof of [CGVvH26b, Theorem 9.8], gives

$$\mathrm{sp}(\mathcal{P}_A(G^{(N)})) \subseteq \mathrm{sp}(\mathcal{P}_A(G^{(N,T)})) + [-u, u]$$

except on an event of probability at most

$$2 D N^L e^{-t} + C_1 V T e^{-c_2 N},$$

where

$$u = C_1 \left(N^{-1/2} t^{1/2} + T^{-1/12} t^{2/3} + T^{-1/4} t \right).$$

Here the ambient matrix dimension in the universality theorem is DN^L, which accounts for the prefactor DN^L.

Next, $\mathcal{P}_A(G^{(N,T)})$ is a polynomial of degree at most L in a Hayes' model with VT GUE variables. Hence [CGVvH26b, Theorem 9.7] yields

$$\|\mathcal{P}_A(G^{(N,T)})\| \le (1 + \rho) \|\mathcal{P}_A(s^{(T)})\|$$

except on an event of probability at most

$$\frac{N^L}{c_H(L, V, T)\rho} e^{-c_H(L,V,T) N \rho^2},$$

provided

$$D \le \exp\big(c_H(L, V, T) N \rho^2\big).$$

Indeed, the constant c in [CGVvH26b, Theorem 9.7] satisfies $c \geq c_H(L,V,T)$, and therefore

$$c^{-1} \leq c_H(L,V,T)^{-1}, \qquad e^{-cN\rho^2} \leq e^{-c_H(L,V,T)N\rho^2}.$$

The limiting approximants satisfy the uniform norm estimate

$$\sup_{T\geq 1} \max_{1\leq v\leq V} \|s_v^{(T)}\| \leq C_2,$$

by the uniform Haagerup inequality used in Appendix B of [CGVvH26b]. Therefore,

$$\|\mathcal{P}_A(s^{(T)})\| \leq C_2 M.$$

Moreover, by the definition of $\Delta_{L,V,M}(T)$,

$$\|\mathcal{P}_A(s^{(T)})\| \leq \|\mathcal{P}_A(x^\Gamma)\| + \Delta_{L,V,M}(T).$$

Combining these estimates gives

$$\begin{aligned}\|\mathcal{P}_A(G^{(N)})\| \leq \|\mathcal{P}_A(G^{(N,T)})\| + u &\leq (1+\rho)\|\mathcal{P}_A(s^{(T)})\| + u \\ &\leq \|\mathcal{P}_A(x^\Gamma)\| + \Delta_{L,V,M}(T) + C_2 M\rho + u.\end{aligned}$$

Taking the union bound over the two exceptional events proves (6.5). □

Denote the full tensor-GUE matrix dimension as $n_N = N^L$. Now choose

$$D_N = N^{2L} = n_N^2. \tag{6.6}$$

Then $D_N > n_N \geq N$ for $N \geq 2$.

Fix $\varepsilon, \eta \in (0,1)$, and put

$$\rho = \min\left\{1, \frac{\varepsilon}{6C_2 M}\right\}. \tag{6.7}$$

Define

$$t_N = \log\left(\frac{12 D_N N^L}{\eta}\right) = \log\left(\frac{12}{\eta}\right) + 3L\log N$$

and

$$T_N = \left\lceil (1+t_N)^9 \right\rceil.$$

Corollary 6.5. *For fixed $L, V, M, \varepsilon, \eta$, there exists N_0 such that, for every $N \geq N_0$, every $D \leq N^{2L}$, and every self-adjoint linear pencil $\mathcal{P}_A$ satisfying $\Lambda(A) \leq M$, we has*

$$\mathbb{P}\left[\|\mathcal{P}_A(G^{(N)})\| > \|\mathcal{P}_A(x^\Gamma)\| + \varepsilon\right] \leq \eta.$$

Proof. As $N \to \infty$,

$$t_N = \mathcal{O}(\log N), \qquad T_N = \mathcal{O}\big((\log N)^9\big).$$

In particular, $T_N \leq e^N$ for every sufficiently large N, so condition (6.2) is satisfied. Therefore, $N^{-1/2}t_N^{1/2} \to 0$, while $T_N^{-1/12}t_N^{2/3} = \mathcal{O}(t_N^{-1/12}) \to 0$ and $T_N^{-1/4}t_N = \mathcal{O}(t_N^{-5/4}) \to 0$. Moreover, $\Delta_{L,V,M}(T_N) \to 0$ by Lemma 6.2. Recall that $c_H(L,V,T_N) = L^{-2}(C_0 LVT_N)^{-2L}$. Since $T_N = \mathcal{O}((\log N)^9)$, we have

$$c_H(L,V,T_N)N\rho^2 \asymp_{L,V,\rho} \frac{N}{(\log N)^{18L}}.$$

Consequently,

$$\frac{\log D_N}{c_H(L,V,T_N)N\rho^2} \asymp_{L,V,\rho} \frac{(\log N)^{18L+1}}{N} \longrightarrow 0.$$

Thus, for all sufficiently large N, we get $D_N \le \exp\bigl(c_H(L,V,T_N)N\rho^2\bigr)$, so that condition (6.4) holds for every $D \le D_N$. By the definition of t_N, we have $2D_N N^L e^{-t_N} = \frac{\eta}{6}$. Moreover, $C_1 V T_N e^{-c_2 N} \to 0$. Also,

$$c_H(L,V,T_N)^{-1} = \mathcal{O}_{L,V}\bigl((\log N)^{18L}\bigr),$$

and hence

$$\log\left(\frac{N^L}{c_H(L,V,T_N)\rho}\right) = L\log N + \mathcal{O}_{L,V,\rho}(\log\log N).$$

On the other hand,

$$c_H(L,V,T_N)N\rho^2 \asymp_{L,V,\rho} \frac{N}{(\log N)^{18L}}.$$

Consequently,

$$\frac{N^L}{c_H(L,V,T_N)\rho}e^{-c_H(L,V,T_N)N\rho^2} \asymp_{L,V,\rho} e^{L\log(N)-\frac{N}{(\log N)^{18L}}+\mathcal{O}_{L,V,\rho}(\log\log N)} \longrightarrow 0.$$

Finally, by (6.7), we have $C_2 M\rho \le \frac{\varepsilon}{6}$, and hence, for sufficiently large N,

$$\Delta_{L,V,M}(T_N) + C_2 M\rho + u(N,T_N,t_N) \le \varepsilon.$$

Applying Theorem 6.4 completes the proof. □

Remark 6.6. Thus, the quantitative tensor-GUE method allows $D_N = N^{2L} > N^L$. More generally, every fixed polynomial growth regime $D_N = N^{aL}$, $a > 1$ is allowed.

We have the quantitative upper bound. In the following we prove that the lower bound is uniform in the growing coefficient dimension by a finite-dimensional operator-space argument.

Lemma 6.7 (Uniform tensor-GUE lower bound)**.** *Under the hypotheses of Theorem 6.5, for every $\varepsilon > 0$,*

$$\lim_{N\to\infty} \sup_{\substack{D\le N^{2L}\\ A_0,\dots,A_V\in M_D(\mathbb{C})_{\mathrm{sa}}\\ \Lambda(A)\le M}} \mathbb{P}\Bigl[\|\mathcal{P}_A(G^{(N)})\| < \|\mathcal{P}_A(x^\Gamma)\| - \varepsilon\Bigr] = 0.$$

Proof. Put $\mathcal{A} = C^*(x_1^\Gamma,\dots,x_V^\Gamma)$. Each vertex algebra $C^*(x_v^\Gamma) \simeq C([-2,2])$ is nuclear, hence exact. Since reduced graph products preserve exactness [CaFi17, Corollary 2.17], the reduced graph-product algebra $\mathcal{A} = C^*(x_1^\Gamma,\dots,x_V^\Gamma)$ is exact. Fix $\delta \in (0,1)$. Applying the fixed-family exactness compression lemma Theorem 2.1 to $1, x_1^\Gamma,\dots,x_V^\Gamma \in \mathcal{A}$ gives an integer $m = m(L,V,\delta)$ with the following property. For every $D \ge 1$ and every $A_0,\dots,A_V \in M_D(\mathbb{C})$, there are contractions $U, V \in M_{m,D}(\mathbb{C})$ such that, on putting

$$B_v = UA_vV^* \in M_m(\mathbb{C}), \qquad 0 \le v \le V,$$

we have

$$(1-\delta)\|\mathcal{P}_A(x^\Gamma)\| \le \left\| B_0\otimes 1 + \sum_{v=1}^V B_v\otimes s_v\right\|. \tag{6.8}$$

Here we have used the canonical isometric flip between the two minimal tensor factors. Notice also that

$$\Lambda(B) = \sum_{v=0}^V \|UA_vV^*\| \le \Lambda(A) \le M.$$

For $B = (B_0, \dots, B_V)$, write

$$\mathcal{Q}_B(x) = B_0 \otimes 1 + \sum_{v=1}^{V} B_v \otimes x_v,$$

and define

$$R_N := \sup_{\substack{B_0,\dots,B_V \in M_m(\mathbb{C}) \\ \Lambda(B) \le M}} \Big| \|\mathcal{Q}_B(G^{(N)})\| - \|\mathcal{Q}_B(x^\Gamma)\| \Big|.$$

We claim that

$$R_N \longrightarrow 0 \quad \text{in probability.} \tag{6.9}$$

Indeed, the coefficient set

$$\mathcal{K}_m(M) = \big\{ (B_0, \dots, B_V) \in M_m(\mathbb{C})^{V+1} : \Lambda(B) \le M \big\}$$

is compact. For each fixed $B \in \mathcal{K}_m(M)$, [CGVvH26b, Theorem 9.8] gives

$$\|\mathcal{Q}_B(G^{(N)})\| - \|\mathcal{Q}_B(x^\Gamma)\| \longrightarrow 0$$

in probability, in fact almost surely.

Moreover, for $B, C \in \mathcal{K}_m(M)$,

$$\Big| \|\mathcal{Q}_B(G^{(N)})\| - \|\mathcal{Q}_C(G^{(N)})\| \Big| \le \|B_0 - C_0\| + \sum_{v=1}^{V} \|B_v - C_v\| \, \|G_v^{(N)}\|,$$

$$\Big| \|\mathcal{Q}_B(x^\Gamma)\| - \|\mathcal{Q}_C(x^\Gamma)\| \Big| \le \|B_0 - C_0\| + \sum_{v=1}^{V} \|B_v - C_v\| \, \|x_v^\Gamma\|.$$

By [CGVvH26b, Theorem 9.8], applied to the coordinate polynomials, the family $\max_{1 \le v \le V} \|G_v^{(N)}\|$ is bounded in probability. A finite-net argument on $\mathcal{K}_m(M)$ therefore upgrades the pointwise convergence to the uniform convergence (6.9).

On the random-matrix side, $\mathcal{Q}_B(G^{(N)}) = (U \otimes 1)\mathcal{P}_A(G^{(N)})(V^* \otimes 1)$, and consequently $\|\mathcal{Q}_B(G^{(N)})\| \le \|\mathcal{P}_A(G^{(N)})\|$. Combining this with (6.8) gives, uniformly over all admissible D and A,

$$\|\mathcal{P}_A(G^{(N)})\| \ge \|\mathcal{Q}_B(G^{(N)})\| \ge \|\mathcal{Q}_B(x^\Gamma)\| - R_N \ge (1-\delta)\|\mathcal{P}_A(s)\| - R_N. \tag{6.10}$$

Set $C_s = \max\{1, \|x_1^\Gamma\|, \dots, \|x_V^\Gamma\|\}$. Then $\|\mathcal{P}_A(x^\Gamma)\| \le C_s \Lambda(A) \le C_s M$. Given $\varepsilon > 0$, choose $0 < \delta < \frac{\varepsilon}{2 C_s M}$. It follows from (6.9) and (6.10) that

$$\sup_{\substack{D \le N^{2L} \\ \Lambda(A) \le M}} \mathbb{P}\Big[\|\mathcal{P}_A(G^{(N)})\| < \|\mathcal{P}_A(x^\Gamma)\| - \varepsilon \Big] \le \mathbb{P}\Big[R_N > \frac{\varepsilon}{2} \Big] \longrightarrow 0.$$

This proves the lemma. $\square$

Lemma 6.8 (Finite polynomial spectral detector)**.** *For every $K, \varepsilon > 0$, there are finitely many real polynomials*

$$h_1, \dots, h_J \in \mathbb{R}[t]$$

such that the following holds. If X, Y are nonempty compact subsets of $[-K, K]$ and $X \not\subseteq Y + [-\varepsilon, \varepsilon]$, then, for some j,

$$\sup_{x \in X} |h_j(x)| > 2 \sup_{y \in Y} |h_j(y)|.$$

The polynomials may moreover be chosen strictly positive on $[-K, K]$.

Proof. Choose a finite $\varepsilon/8$-net $z_1, \dots, z_J$ in $[-K, K]$. For each j, choose a continuous function $f_j : [-K, K] \to [0, 1]$ such that

$$f_j(t) = 1 \quad \text{if } |t - z_j| \le \varepsilon/8, \qquad f_j(t) = 0 \quad \text{if } |t - z_j| \ge \varepsilon/2.$$

By the Weierstrass approximation theorem, choose $h_j \in \mathbb{R}[t]$ satisfying

$$\left\| h_j - \left(\frac{1}{8} + f_j \right) \right\|_{C([-K,K])} < \frac{1}{64}.$$

Then $h_j \ge 7/64$ on $[-K, K]$.

If $x \in X$ satisfies $\mathrm{dist}(x, Y) > \varepsilon$, choose j with $|x - z_j| \le \varepsilon/8$. We then have $h_j(x) \ge \frac{71}{64}$. For every $y \in Y$,

$$|y - z_j| \ge |y - x| - |x - z_j| > \frac{7\varepsilon}{8},$$

so $f_j(y) = 0$ and $|h_j(y)| \le \frac{9}{64}$. This proves (6.15). $\square$

Combining Corollary 6.5 and Lemma 6.7 gives the full norm convergence. Note that (6.13) must be interpreted as a spectral gap condition.

Theorem 6.9 (Growing-coefficient tensor-GUE convergence)**.** *Fix L, V, M. Then the following statements hold uniformly over all $D \le N^{2L}$ and all self-adjoint pencils $\mathcal{P}_A$ satisfying $\Lambda(A) \le M$.*

(1) For every $\varepsilon > 0$,

$$\mathbb{P}\left[\Big| \|\mathcal{P}_A(G^{(N)})\| - \|\mathcal{P}_A(x^\Gamma)\| \Big| > \varepsilon \right] \longrightarrow 0. \tag{6.11}$$

(2) For every $\varepsilon > 0$,

$$\mathbb{P}\left[\mathrm{sp}(\mathcal{P}_A(G^{(N)})) \not\subseteq \mathrm{sp}(\mathcal{P}_A(x^\Gamma)) + [-\varepsilon, \varepsilon] \right] \longrightarrow 0. \tag{6.12}$$

Consequently, for every $\gamma > 0$,

$$\sup_{\substack{D \le N^{2L},\, \Lambda(A) \le M \\ \mathrm{dist}(0, \mathrm{sp}(\mathcal{P}_A(x^\Gamma))) \ge \gamma}} \mathbb{P}\left[\mathcal{P}_A(G^{(N)}) \textit{ is not invertible, or } \|\mathcal{P}_A(G^{(N)})^{-1}\| > \frac{2}{\gamma} \right] \longrightarrow 0. \tag{6.13}$$

Proof. Statement (6.11) follows from Corollary 6.5 and Lemma 6.7. It remains to prove (6.12).

Put

$$a_L = 2^L - 2, \qquad C_s = \max\{1, \|x_1^\Gamma\|, \dots, \|x_V^\Gamma\|\}, \qquad K_0 = (C_s + a_L)M.$$

By (6.2”),

$$\|\mathcal{P}_A(s^{(T)})\| \le K_0 \qquad (T \ge 1). \tag{6.19}$$

Fix $\varepsilon > 0$. Apply Lemma 6.8 with

$$K = 2K_0, \qquad \varepsilon \text{ replaced by } \frac{\varepsilon}{3},$$

and let $\mathcal{C}$ denote the resulting finite family of polynomials. Adjoin to $\mathcal{C}$ the polynomial $g(t) = 1 + \frac{t^2}{K_0^2}$. Let $q = \max_{h \in \mathcal{C} \cup \{g\}} \deg h$. For $N \ge 2$, set

$$t_N = (3L + 3) \log N, \qquad T_N = \left\lceil (1 + t_N)^9 \right\rceil,$$

and define

$$c_{H,q}(L, V, T) = (Lq)^{-2} (C_0 L V T)^{-2Lq}. \tag{6.20}$$

If $h \in \mathcal{C} \cup \{g\}$, then $h(\mathcal{P}_A(G^{(N,T)}))$ is a polynomial of degree at most Lq in the Hayes variables. Therefore [CGVvH26b, Theorem 9.7], with its parameter $\rho = 1$, gives

$$\mathbb{P}\left[\|h(\mathcal{P}_A(G^{(N,T_N)}))\| \geq 2\|h(\mathcal{P}_A(s^{(T_N)}))\|\right] \leq \frac{N^L}{c_{H,q}(L,V,T_N)} e^{-c_{H,q}(L,V,T_N)N}, \tag{6.21}$$

provided $D \leq \exp\big(c_{H,q}(L,V,T_N)N\big)$. This condition holds for every $D \leq N^{2L}$ and all sufficiently large N, since

$$c_{H,q}(L,V,T_N)^{-1} = \mathcal{O}\big((\log N)^{18Lq}\big).$$

Taking a union bound over the finite set $\mathcal{C} \cup \{g\}$, the total probability in (6.21) tends to zero. On the complementary event, (6.19) gives $\|\mathcal{P}_A(s^{(T_N)}))\| \leq K_0$, and hence $\|\mathcal{P}_A(G^{(N,T_N)}))\| < 2K_0$. Since $\mathcal{P}_A(G^{(N,T_N)})$ is self-adjoint,

$$1 + \frac{\|\mathcal{P}_A(G^{(N,T_N)})\|^2}{K_0^2} < 5,$$

so

$$\mathrm{sp}(\mathcal{P}_A(G^{(N,T_N)})) \subseteq [-2K_0, 2K_0]. \tag{6.22}$$

Suppose now that

$$\mathrm{sp}(\mathcal{P}_A(G^{(N,T_N)})) \not\subseteq \mathrm{sp}(\mathcal{P}_A(s^{(T_N)})) + \left[-\frac{\varepsilon}{3}, \frac{\varepsilon}{3}\right].$$

By (6.19), (6.22), and the spectral detector Lemma 6.8, there is $h \in \mathcal{H}$ such that

$$\|h(\mathcal{P}_A(G^{(N,T_N)}))\| > 2\|h(\mathcal{P}_A(s^{(T_N)}))\|,$$

contradicting the complementary event in (6.21). Therefore

$$\mathrm{sp}(\mathcal{P}_A(G^{(N,T_N)})) \subseteq \mathrm{sp}(\mathcal{P}_A(s^{(T_N)})) + \left[-\frac{\varepsilon}{3}, \frac{\varepsilon}{3}\right] \tag{6.23}$$

outside an event whose probability tends to zero uniformly.

The universality comparison gives

$$\mathrm{sp}(\mathcal{P}_A(G^{(N)})) \subseteq \mathrm{sp}(\mathcal{P}_A(G^{(N,T_N)})) + [-u_N, u_N] \tag{6.24}$$

outside an event of probability at most $2DN^L e^{-t_N} + C_1 V T_N e^{-c_2 N}$, where

$$u_N = C_1\left(N^{-1/2} t_N^{1/2} + T_N^{-1/12} t_N^{2/3} + T_N^{-1/4} t_N\right) \longrightarrow 0.$$

Because $D \leq N^{2L}$, $2DN^L e^{-t_N} \leq 2N^{-3}$, so this exceptional probability also tends to zero.

Finally, (6.2") gives

$$\mathrm{sp}(\mathcal{P}_A(s^{(T_N)})) \subseteq \mathrm{sp}(\mathcal{P}_A(x^\Gamma)) + \left[-\frac{a_L M}{\sqrt{T_N}}, \frac{a_L M}{\sqrt{T_N}}\right]. \tag{6.25}$$

Combining (6.23)–(6.25), and using $u_N + \frac{a_L M}{\sqrt{T_N}} < \frac{2\varepsilon}{3}$ for all sufficiently large N, proves (6.17).

If $\mathrm{dist}\big(0, \mathrm{sp}(\mathcal{P}_A(x^\Gamma))\big) > \gamma$, apply (6.12) with $\varepsilon = \gamma/2$. Then, with probability tending uniformly to one, $\mathrm{dist}\big(0, \mathrm{sp}(P_A(G^{(N)}))\big) \geq \frac{\gamma}{2}$. This proves (6.13). If $\mathrm{dist}\big(0, \mathrm{sp}(\mathcal{P}_A(x^\Gamma))\big) = 0$ then the statement follows from (6.12) as well. $\square$

Corollary 6.10 (Bounded-degree polynomial tensor-GUE convergence)**.** *Fix L, V, d, M. There is an integer $c_{V,d} \geq 1$ such that, for every $\varepsilon > 0$,*

$$\lim_{N\to\infty} \sup_{\substack{D\geq 1,\, c_{V,d}D\leq N^{2L}\\ P=P^*,\, \deg P\leq d\\ \Lambda_d(P)\leq M}} \mathbb{P}\left[\left|\|P(G^{(N)})\| - \|P(x^\Gamma)\|\right| > \varepsilon\right] = 0. \tag{6.26}$$

Moreover,

$$\lim_{N\to\infty} \sup_{\substack{D\geq 1,\, c_{V,d}D\leq N^{2L}\\ P=P^*,\, \deg P\leq d\\ \Lambda_d(P)\leq M}} \mathbb{P}\left[\mathrm{sp}(P(G^{(N)})) \not\subseteq \mathrm{sp}(P(x^\Gamma)) + [-\varepsilon, \varepsilon]\right] = 0. \tag{6.27}$$

For a non-self-adjoint polynomial, the norm-convergence statement (6.26) remains valid after replacing $c_{V,d}$ by $2c_{V,d}$ and applying the preceding argument to the Hermitian dilation

$$\mathcal{D}(P) = \begin{pmatrix} 0 & P \\ P^* & 0 \end{pmatrix}, \qquad \|\mathcal{D}(P)\| = \|P\|.$$

We do not claim any spectral-inclusion statement for the generally nonnormal operator $P(G^{(N)})$.

Proof. We use the self-adjoint linearization trick; see [HaTh05, Sections 2 and 7], [And13, Section 2.6], and [Mal12, Section 3, Step 1].

For fixed V and d, the standard self-adjoint Schur-complement linearization gives constants $c_{V,d}$ and $C_{V,d}$ with the following properties.

For every self-adjoint polynomial P of degree at most d and every real λ, there is a self-adjoint linear pencil

$$L_{P,\lambda}(x) = B_0(\lambda) \otimes 1 + \sum_{v=1}^{V} B_v \otimes x_v$$

such that:

(1) the coefficient dimension of $L_{P,\lambda}$ is at most $c_{V,d}D$;
(2)

$$\lambda - P(x) \text{ is invertible} \quad \Longleftrightarrow \quad L_{P,\lambda}(x) \text{ is invertible;} \tag{6.14}$$

(3) for λ in a fixed bounded interval,

$$\sum_{v=0}^{V} \|B_v\| \leq C_{V,d}(1 + M + |\lambda|); \tag{6.15}$$

(4) for every $R, \varepsilon > 0$ there is

$$C = C(V, d, M, R, \varepsilon)$$

such that, whenever $\max_v |x_v| \leq R$ and

$$\mathrm{dist}\big(\lambda, \mathrm{sp}(P(x))\big) \geq \varepsilon,$$

one has

$$\|L_{P,\lambda}(x)^{-1}\| \leq C. \tag{6.16}$$

Moreover, $L_{P,\lambda}(x) - L_{P,\mu}(x) = (\lambda - \mu)E$, where $\|E\| = 1$. The last assertion follows directly from the block inverse formula for the Schur-complement linearization: the auxiliary block is unitriangular, while the only resolvent term is $(\lambda - P(x))^{-1}$.

Choose $R > \max_{1\le v\le V} \|s_v\| + 1$. By Theorem 6.9 applied to the coordinate linear pencils,

$$\mathbb{P}\left[\max_v \|G_v^{(N)}\| > R\right] \longrightarrow 0.$$

On the complementary event,

$$\|P(G^{(N)})\|, \|P(x^\Gamma)\| \le K := M \max\{1, R\}^d.$$

Thus all relevant spectra are contained in the fixed interval $I = [-K, K]$. Moreover, for every $\lambda \in I$, (6.15) gives

$$\sum_{v=0}^{V} \|B_v(\lambda)\| \le M' := C_{V,d}(1 + M + K).$$

The constants K and M' are independent of D, P, and λ.

Fix $\varepsilon > 0$. By (6.16), there exists $C = C(V, d, M, R, \varepsilon/2)$ such that $\|L_{P,\lambda}(x^\Gamma)^{-1}\| \le C$ whenever $\mathrm{dist}(\lambda, \mathrm{sp}(P(x^\Gamma))) \ge \frac{\varepsilon}{2}$.

Set $\gamma = C^{-1}$. Choose $0 < \delta < \min\left\{\frac{\varepsilon}{2}, \frac{\gamma}{4}\right\}$, and let $\lambda_1, \dots, \lambda_m$ be a finite δ-net of the bounded interval containing all relevant spectra. For every net point satisfying $\mathrm{dist}(\lambda_j, \mathrm{sp}(P(x^\Gamma))) \ge \frac{\varepsilon}{2}$, we have $\mathrm{dist}\left(0, \mathrm{sp}(L_{P,\lambda_j}(x^\Gamma))\right) \ge \gamma$.

Applying (6.18) to the finitely many linear pencils L_{P,λ_j}, and taking a union bound, we obtain that, with probability tending uniformly to one,

$$\|L_{P,\lambda_j}(G^{(N)})^{-1}\| \le \frac{2}{\gamma}$$

for every such net point.

Now let λ satisfy $\mathrm{dist}(\lambda, \mathrm{sp}(P(x^\Gamma))) \ge \varepsilon$, and choose λ_j with $|\lambda - \lambda_j| < \delta$. Then $\mathrm{dist}(\lambda_j, \mathrm{sp}(P(x^\Gamma))) \ge \frac{\varepsilon}{2}$, and $L_{P,\lambda}(G^{(N)}) = L_{P,\lambda_j}(G^{(N)}) + (\lambda - \lambda_j)E$, where $\|E\| = 1$. Therefore

$$\left\|L_{P,\lambda_j}(G^{(N)})^{-1}\right\| \left\|\left(L_{P,\lambda}(G^{(N)}) - L_{P,\lambda_j}(G^{(N)})\right)\right\| < \frac{1}{2},$$

because $\delta < \frac{\gamma}{4}$. Hence a standard von Neumann series argument (see [Mur90, Theorem 1.2.3]) shows that $L_{P,\lambda}(G^{(N)})$ is invertible.

Using the linearization equivalence (6.28), we conclude that

$$\mathrm{sp}(P(G^{(N)})) \subseteq \mathrm{sp}(P(x^\Gamma)) + [-\varepsilon, \varepsilon]$$

with probability tending uniformly to one. This proves (6.27), and therefore the upper estimate in (6.26).

For the lower norm bound, apply the fixed-target exactness compression argument of Theorem 6.7 to the finite family $\{w(x^\Gamma) : |w| \le d\}$. The graph-product algebra $C^*(x_1^\Gamma, \dots, x_V^\Gamma)$ is exact. Consequently, arbitrary coefficient matrices may be compressed to a fixed matrix dimension while preserving $\|P(x^\Gamma)\|$ from below. At that fixed dimension, the polynomial tensor-GUE strong convergence theorem [CGVvH26b, Theorem 9.8], together with compactness of the coefficient ball, gives uniform convergence. This proves the lower bound and completes (6.26).

For a non-self-adjoint polynomial use the Hermitian dilation

$$\mathcal{D}(P) = \begin{pmatrix} 0 & P \\ P^* & 0 \end{pmatrix}, \qquad \|\mathcal{D}(P)\| = \|P\|.$$

$\square$

6.3. **Growing coefficient strong convergence of the Erdős–Rényi q-Gaussian matrix model.** We now combine the graph-product approximation of Section 5 with the growing-coefficient tensor-GUE approximation established above. For a fixed finite graph, the underlying tensor-GUE strong convergence is [CGVvH26b, Theorem 9.8]. In the present setting, however, the interaction graph has rk vertices and varies with k, while the allowable coefficient dimension grows with, and is eventually chosen to exceed, the dimension of the matrix models.

We therefore fix k, condition on a realization of the interaction graph, realize that graph as the disjointness graph of finitely many tensor supports, and apply Corollary 6.10 at a suitably chosen local dimension N_k. Since there are only finitely many graphs on V_k, the choice of N_k can be made uniformly over all graph realizations. A diagonal choice of N_k then combines this matrix approximation with Theorem 5.4.

Fix $r \geq 1$ and $-1 < q < 1$, and assume that the canonical map $\Phi_q : \mathcal{T}^u_{q,r} \longrightarrow \mathcal{T}^{\mathrm{red}}_{q,r}$ is an isomorphism. In particular, the conclusions below hold whenever $|q| < \sqrt{2} - 1$ [PuWo89].

Let

$$V_k = [k] \times [r], \qquad m_k = |V_k| = rk.$$

We introduce a tensor-coordinate set consisting of one private coordinate for each vertex and one pair coordinate for each unordered pair of distinct vertices:

$$I_k = \{p_v : v \in V_k\} \sqcup \{p_{\{u,v\}} : \{u, v\} \in \binom{V_k}{2}\};$$

so the p_v and $p_{\{u,v\}}$ are by definition the labels of the tensor coordinates. And we denote

$$L_k := |I_k| = m_k + \binom{m_k}{2}.$$

For a graph Γ with vertices V_k and $v \in V_k$, set

$$K^\Gamma_v = \{p_v\} \cup \big\{\{u, v\} : u \neq v,\ u \not\sim_\Gamma v\big\}.$$

Then

$$K^\Gamma_u \cap K^\Gamma_v = \varnothing \quad \iff \quad u \sim_\Gamma v.$$

For $0 \leq q < 1$, let $d_k = 1$. For $-1 < q < 0$, let $\rho_k : \mathrm{Cliff}(V_k) \longrightarrow M_{d_k}(\mathbb{C})$ be a faithful finite-dimensional representation that preserves the canonical Clifford trace, for example, the GNS representation of the canonical trace. We put $c^{(k)}_v = \rho_k(c_v)$.

For $N \geq 1$, let $G^{(N,\Gamma)}_v$ be the tensor-GUE variable associated with $K^\Gamma_v \subseteq I_k$, and put

$$\widehat{G}^{(N,\Gamma)}_v = \begin{cases} G^{(N,\Gamma)}_v, & 0 \leq q < 1, \\ c^{(k)}_v \otimes G^{(N,\Gamma)}_v, & -1 < q < 0. \end{cases}$$

Finally, define

$$X^{(N,\Gamma)}_{k,a} = \frac{1}{\sqrt{k}} \sum_{i=1}^{k} \widehat{G}^{(N,\Gamma)}_{(i,a)}, \qquad a \in [r]. \tag{6.17}$$

The dimension of matrices in (6.17) is

$$n_{k,N} = d_k N^{L_k}.$$

For $D, d \geq 1$, write

$$\mathscr{P}_d(D) = \{P \in M_D(\mathbb{C}) \otimes \mathbb{C}\langle x_1, \ldots, x_r\rangle : \deg P \leq d,\ \Lambda_d(P) \leq 1\}.$$

Theorem 6.11 (Strongly convergent q-Gaussian matrix model)**.** *Fix $r, d \geq 1$ and assume that Φ_q is an isomorphism. In particular, this holds for $|q| < \sqrt{2} - 1$. There exist integers $N_k \to \infty$ and $D_k \to \infty$ satisfying $D_k > n_k$, such that, with*

$$X^{(k)} = \big(X_{k,1}^{(N_k,\Gamma_k)}, \dots, X_{k,r}^{(N_k,\Gamma_k)}\big), \qquad n_k = d_k N_k^{L_k},$$

the following holds:

(1) *For every $\varepsilon > 0$,*

$$\sup_{1 \leq D \leq D_k} \sup_{P \in \mathscr{P}_d(D)} \mathbb{P}_{\Gamma,\mathrm{GUE}} \left(\Big| \|P(X^{(k)})\| - \|P(s^{(q)})\| \Big| > \varepsilon \right) \longrightarrow 0. \tag{6.18}$$

(2) *For every $\varepsilon > 0$,*

$$\sup_{1 \leq D \leq D_k} \sup_{\substack{0 \neq P \in M_D \otimes \mathbb{C}\langle x_1, \dots, x_r\rangle \\ \deg P \leq d}} \mathbb{P}_{\Gamma,\mathrm{GUE}} \left(\left| \frac{\|P(X^{(k)})\|}{\|P(s^{(q)})\|} - 1 \right| > \varepsilon \right) \longrightarrow 0. \tag{6.19}$$

(3) *The weak converge: for every scalar noncommutative polynomial Q,*

$$\mathrm{tr}_{n_k}\big(Q(X^{(k)})\big) \longrightarrow \tau_q\big(Q(s^{(q)})\big) \tag{6.20}$$

in probability.

Proof. Let $S_k^{(q,\Gamma)} = \big(S_{k,1}^{(q,\Gamma)}, \dots, S_{k,r}^{(q,\Gamma)}\big)$ be the graph-product or Clifford-twisted graph-product tuple from Section 5. By Theorem 5.4 and the bound

$$\|P(s^{(q)})\| \leq \Lambda_d(P) \max\left\{1, \frac{2}{\sqrt{1-q}}\right\}^d,$$

there are positive numbers $\varepsilon_k \to 0$ and $\beta_k \to 0$ such that

$$\sup_{D \geq 1} \sup_{P \in \mathscr{P}_d(D)} \mathbb{P}_{\Gamma} \left(\Big| \|P(S_k^{(q,\Gamma_k)})\| - \|P(s^{(q)})\| \Big| > \varepsilon_k \right) \leq \beta_k. \tag{6.21}$$

For a graph Γ on V_k and $P \in \mathscr{P}_d(D)$, substitute the averages in (6.17) and denote the resulting polynomial in the m_k tensor-GUE variables by $\widetilde{P}_\Gamma$. More precisely, put

$$\widetilde{P}_\Gamma(z_{(i,a)}) = P\left(\frac{1}{\sqrt{k}} \sum_{i=1}^{k} \widehat{z}_{(i,1)}, \ \dots, \ \frac{1}{\sqrt{k}} \sum_{i=1}^{k} \widehat{z}_{(i,r)} \right),$$

where

$$\widehat{z}_{(i,a)} = \begin{cases} z_{(i,a)}, & 0 \leq q < 1, \\ c_{(i,a)}^{(k)} \otimes z_{(i,a)}, & -1 < q < 0. \end{cases}$$

Then

$$P\big(X_k^{(N,\Gamma)}\big) = \widetilde{P}_\Gamma\big(G^{(N,\Gamma)}\big), \qquad P\big(S_k^{(q,\Gamma)}\big) = \widetilde{P}_\Gamma(x^\Gamma),$$

where, for $q < 0$, the Clifford matrices are absorbed into the matrix coefficients of $\widetilde{P}_\Gamma$. Then $\deg(\widetilde{P}_\Gamma) \leq d$, the coefficient dimension is $d_k D$, and $\Lambda_d(\widetilde{P}_\Gamma) \leq k^{d/2}$. Indeed, every monomial of degree m produces at most k^m summands, each carrying the coefficient $k^{-m/2}$, while every Clifford product has norm one. Therefore Theorem 6.10 applies directly to $\widetilde{P}_\Gamma$ with $L = L_k, V = m_k = rk, M = k^{d/2}$.

Let $c_k := c_{m_k,d}$ denote the corresponding linearization constant (see Theorem 6.10). It is independent of Γ, D, and the particular polynomial in $\mathscr{P}_d(D)$. Choose N_k sufficiently large that

$$N_k^{L_k} > 2c_k d_k^2 + 2. \tag{6.22}$$

and

$$\sup_{\Gamma} \sup_{\substack{D\geq 1 \\ c_k d_k D \leq N_k^{2L_k}}} \sup_{P\in\mathscr{P}_d(D)} \mathbb{P}_{\mathrm{GUE}}\left(\left|\|P(X_k^{(N_k,\Gamma)})\| - \|P(S_k^{(q,\Gamma)})\|\right| > \varepsilon_k\right) \leq k^{-1}. \tag{6.23}$$

We enlarge N_k, if necessary, so that the moment-convergence part of the tensor-GUE theorem also gives, for every graph Γ on V_k and every word w in r letters of length at most k,

$$\mathbb{P}_{\mathrm{GUE}}\left(\left|\mathrm{tr}_{n_k,N_k}\big(w(X_k^{(N_k,\Gamma)})\big) - \varphi_{q,k}\big(w(S_k^{(q,\Gamma)})\big)\right| > k^{-1}\right) \leq k^{-1}. \tag{6.24}$$

This simultaneous choice is possible, because for fixed k, only finitely many graphs and finitely many words occur. In the case $q < 0$, trace-preserving ρ_k identifies the limiting trace with the Clifford-twisted graph-product vacuum state.

Now define

$$D_k = \left\lfloor \frac{N_k^{2L_k}}{c_k d_k} \right\rfloor.$$

Writing $x_k = N_k^{L_k}$, (6.22) gives

$$D_k \geq \frac{x_k^2}{c_k d_k} - 1 > 2d_k x_k - 1.$$

Since $n_k = d_k x_k > 0$, it follows that $D_k > n_k$ as in the requirements of the theorem.

By the triangle inequality, conditioning on Γ_k, (6.21) and (6.23) yield,

$$\sup_{1\leq D\leq D_k} \sup_{P\in\mathscr{P}_d(D)} \mathbb{P}_{\Gamma,\mathrm{GUE}}\left(\left|\|P(X^{(k)})\| - \|P(s^{(q)})\|\right| > 2\varepsilon_k\right) \leq k^{-1} + \beta_k \longrightarrow 0. \tag{6.25}$$

This proves (6.18).

Finally, since the monomials of degree at most d in the q-Gaussians are linearly independent (see Remark 5.2), their coordinate functionals are completely bounded. Hence there is a constant $C_{q,r,d}$ such that, at every matrix coefficient dimension,

$$\Lambda_d(P) \leq C_{q,r,d}\|P(s^{(q)})\|.$$

Scaling (6.25) therefore proves (6.19).

Finally, fix a word w. For all sufficiently large k, $|w| \leq k$. By (6.24),

$$\mathrm{tr}_{n_k}\big(w(X^{(k)})\big) - \varphi_{q,k}\big(w(S_k^{(q,\Gamma_k)})\big) \longrightarrow 0$$

in probability. On the other hand, Theorem 4.1 gives

$$\varphi_{q,k}\big(w(S_k^{(q,\Gamma_k)})\big) \longrightarrow \tau_q\big(w(s^{(q)})\big)$$

in probability. Hence (6.20) holds for words, and therefore for all scalar polynomials by linearity. $\square$

7. Applications

Theorem 7.1 (MF property)**.** *Let $r \geq 1$ and let $|q| < \sqrt{2}-1$. Then $A_q(\mathbb{C}^r)$ is an MF algebra; equivalently, there exist positive integers n_m and a unital injective $*$-homomorphism*

$$A_q(\mathbb{C}^r) \longrightarrow \prod_{m=1}^{\infty} M_{n_m}(\mathbb{C}) \Big/ \bigoplus_{m=1}^{\infty} M_{n_m}(\mathbb{C}),$$

where

$$\bigoplus_{m=1}^{\infty} M_{n_m}(\mathbb{C}) = \Big\{ (a_m)_m : \lim_{m\to\infty} \|a_m\| = 0 \Big\}.$$

Proof. Let $\mathcal{P}_0 = \mathbb{Q}(i)\langle x_1, \ldots, x_r\rangle$ be the countable $*$-algebra of noncommutative polynomials with coefficients in $\mathbb{Q}(i)$, where the variables are seen to be self-adjoint.

Enumerate $\mathcal{P}_0 = \{P_1, P_2, \ldots\}$. By the strongly convergent matrix model for q-Gaussian with $|q| < \sqrt{2}-1$, for any fixed noncommutative polynomial there are random self-adjoint matrix tuples $X^{(k)} = (X_1^{(k)}, \cdots, X_r^{(k)}) \in M_{N_k}(\mathbb{C})^r_{sa}$ such that

$$\|P(X^{(k)})\| \longrightarrow \|P(s_1^{(q)}, \cdots, s_r^{(q)})\|$$

in probability.

After passing to an increasing subsequence of matrix model parameters, for every m we may choose a tuple of deterministic matrices

$$Y^{(m)} = (Y_1^{(m)}, \cdots, Y_r^{(m)}) \in M_{n_m}(\mathbb{C})^r_{sa}$$

such that

$$\max_{1\leq j\leq m} \Big| \|P_j(Y^{(m)})\| - \|P_j(s_1^{(q)}, \cdots, s_r^{(q)})\| \Big| \leq \frac{1}{m}.$$

Indeed, convergence in probability and the union bound show that the corresponding event has positive probability once the parameter of the matrix model is sufficiently large.

Consequently, for every $P \in \mathcal{P}_0$,

$$\lim_{m\to\infty} \|P_j(Y_1^{(m)}, \cdots, Y_r^{(m)})\| \longrightarrow \|P_j(s_1^{(q)}, \cdots, s_r^{(q)})\|. \tag{7.1}$$

Put

$$\mathcal{Q} := \prod_{m=1}^{\infty} M_{n_m}(\mathbb{C}) \Big/ \bigoplus_{m=1}^{\infty} M_{n_m}(\mathbb{C})$$

and denote the quotient map by

$$\pi : \prod_{m=1}^{\infty} M_{n_m}(\mathbb{C}) \longrightarrow \mathcal{Q}$$

Let $\mathcal{P}_0(s^{(q)})$ be the polynomial algebra generated by the q-Gaussian system, and define $\Psi_0 : \mathcal{P}_0(s^{(q)}) \to \mathcal{Q}$ by

$$\Psi_0(P(s_1^{(q)}, \cdots, s_r^{(q)})) = \pi\big((P(Y_1^{(m)}, \cdots, Y_r^{(m)}))_{m=1}^{\infty}\big).$$

This map is well-defined. Indeed, if $P(s_1^{(q)}, \cdots, s_r^{(q)}) = 0$, then (7.1) gives that

$$\lim_{m\to\infty} \|P_j(Y_1^{(m)}, \cdots, Y_r^{(m)})\| \to 0,$$

so the corresponding sequence belongs to $\bigoplus_{m=1}^{\infty} M_{n_m}(\mathbb{C})$; in fact $P = 0$ by Remark 5.2.

Since the norm in $\mathcal{Q}$ is a quotient norm, we have

$$\|\Psi_0(P(s^{(q)}))\| = \limsup_{m\to\infty} \|P(Y^{(m)})\| = \|P(s^{(q)})\|.$$

Thus Ψ_0 is a isomerty. Since $\mathcal{P}_0(s^{(q)})$ is norm dense in $A_q(\mathbb{C}^r)$, Ψ_0 extends uniquely to a unital injective $*$-homomorphism $\Psi : A_q(\mathbb{C}^r) \to \mathcal{Q}$. Hence $A_q(\mathbb{C}^r)$ is MF algebra. ☐

Theorem 7.2 (The extension semigroup)**.** *Let $r \geq 2$ and let $|q| < \sqrt{2} - 1$. Then the Brown–Douglas–Fillmore extension semigroup* $\mathrm{Ext}\big(A_q(\mathbb{C}^r)\big)$ *is not a group.*

Proof. The Brown criterion states that if a unital separable C^*-algebra is MF but not quasidiagonal, then its Brown-Douglas-Fillmore extension semigroup is not a group; see [Bro04]. Therefore we only need to show that $A_q(\mathbb{C}^r)$ is not quasidiagonal.

Suppose, for contradiction, that $A_q(\mathbb{C}^r)$ is quasidiagonal. Then by the caraterization of quasidiagonal, there exist unital completely positive maps $\phi_n : A_q(\mathbb{C}^r) \to M_{k_n}(\mathbb{C})$ such that, for $a, b \in A_q(\mathbb{C}^r)$,

$$\begin{aligned} &\|\phi_n(ab) - \phi_n(a)\phi_n(b)\| \longrightarrow 0; \\ &\|\phi_n(a)\| \longrightarrow \|a\|. \end{aligned} \tag{7.2}$$

Now, put

$$\tau_n := \mathrm{tr}_{k_n} \circ \phi_n : A_q(\mathbb{C}^r) \to \mathbb{C},$$

where tr_{k_n} is the normalized trace on $M_{k_n}(\mathbb{C})$. After passing to a subsequence, the state τ_n converge in the weak-* topology to a state τ on $A_q(\mathbb{C}^r)$.

Then the state τ is tracial. Indeed, for $a, b \in A_q(\mathbb{C}^r)$, because $\mathrm{tr}_{k_n}(\phi_n(a)\phi_n(b)) = \mathrm{tr}_{k_n}(\phi_n(b)\phi_n(a))$, we have

$$|\tau_n(ab) - \tau_n(ba)| \leq \|\phi_n(ab) - \phi_n(a)\phi_n(b)\| + \|\phi_n(ba) - \phi_n(b)\phi_n(a)\|.$$

Therefore $\tau(ab) = \tau(ba)$.

$A_q(\mathbb{C}^r)$ has a unique tracial state, namely the canonical vacuum trace τ_q, which was proved recently in [AJW26, Theorem C]. Hence $\tau = \tau_q$. The map ϕ_n, together with (7.2), therefore show that τ_q is a quasidiagonal trace. In particular, it is an amenable trace.

On the other hand, since $A_q(\mathbb{C}^r)$ is exact [Kuz23, KeNi11], the von Nuemann algebra generated by the GNS representation w.r.t. an amenable trace is injective; see, for example, [BroOza]. Therefore, $\pi_q(A_q(\mathbb{C}^r))'' = \Gamma_q(\mathbb{R}^r)$ is injective. But by Nou's result ([Nou04, Theorem 2]), $\Gamma_q(\mathbb{R}^r)$ is noninjective whenever $r \geq 2$, this leads to a contraction. Thus $A_q(\mathbb{C}^r), r \geq 2$ is not quasidiagonal, which completes the proof of this theorem. ☐

Remark 7.3. The assumption $r \geq 2$ in Theorem 7.2 is necessary. When $r = 1$, the algebra is generated by one self-adjoint operator, and hence $A_q(\mathbb{C}) = C^*(s_1^{(q)}) \cong C(\mathrm{sp}(s_1^{(q)}))$. is a separable nuclear C^*-algebra. The Choi-Effros' theorem on liftings of ucp maps on nuclear C^*-algebras ([ChEf76]) implies that $\mathrm{Ext}(A_q(\mathbb{C}))$ is a group.

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