MSc thesis projects – Martijn Caspers – June, 2022

Below you can find a list of possible MSc topics. All topics have a literature component and a (modest but genuinely interesting) research component. In the literature component you learn about a certain more specialized subject. Collecting and exposing the literature about a certain subject in an original way can surely contribute to a Master thesis or sometimes it can even constitute a whole Master thesis. In the research component you will also try to solve a small problem. Solving such a problem should typically use variations of existing methods (at least I believe so), but behind the horizon there are also many open problems (that would go beyond the scope of a thesis).

Prerequisites. There is a very diverse set of possible projects in the list below. They have in common that (almost all of them) have to do with the theory of bounded operators on a Hilbert space. In particular some of them deal with C*-algebras (closed $*$-subalgebras of $B(H)$) or von Neumann algebras (strong operator topology-closed $*$-subalgebras of $B(H)$).

For these projects it is *necessary* that you have taken the course on Applied Functional Analysis in Delft plus for most projects at least a follow-up course that treats some aspect of C*-algebras. This could be Spectral Theory of Linear Operators (at Delft) or Functional Analysis (Mastermath) or Operator Algebras (Mastermath). Note that Spectral Theory of Linear Operators and Mastermath Functional Analysis have a large overlap in the material it treats, so I advise not to follow both.

Other topics. There are several other topics available on request taking into account the set of courses you have followed. Feel free to inform!

Almost invariant neighbourhoods of the identity. The first topic on the list is very geometric in nature but has applications in operator theory. Let G be a group and assume it is locally compact or just think of G as the invertible $n \times n$ matrices. Let $F \subseteq G$ be a finite set. Let $\mathcal{U} = (U_i)_i$ be a shrinking sequence of open neighbourhoods of the identity $1 \in G$ such that $\cap_i U_i = \{1\}$. Set $\delta_F(\mathcal{U}) = \liminf_i \frac{|\cap_{s \in F} s U_i s^{-1}|}{|U_i|}$. Let δ_F be the $\sup_{\mathcal{U}}$ of $\delta_F(\mathcal{U})$ over all possible $\mathcal{U}$ as before. Then δ_F somehow quantifies how much neighbourhoods of the identity of G change under conjugation with elements of F. The aim of this project is to determine δ_F for specific groups G and specific sets F. This is has very important applications as was shown in Theorem A of [CJKM]. In fact for certain Lie groups and certain sets F we have computed δ_F: namely for nilpotent Lie groups and real reductive Lie groups. There are several questions now. One could be to find more explicit neighbourhoods for nilpotent Lie groups; another project could be to look at more general Lie groups and their Levi decomposition. Learning what these Lie groups are can be part of the project; it would however be desirable if you have followed the course on differential geometry (Bachelor) or Lie theory (Mastermath). This project would then be supervised jointly with Bas Janssens.

Literature: [CJKM].

Uniqueness of C*-norms on $*$-algebras. This project deals with the following question. Given a $*$-algebra A, how many norms can one put on A such that its completion is a C*-algebra? The question has in particular been studied for the case that A is the group algebra $\mathbb{C}[\Gamma]$ of a discrete group Γ. Here the group algebra $\mathbb{C}[\Gamma]$ is defined as follows. As a (complex) vector space it has basis $e_\gamma, \gamma \in \Gamma$ and the multiplication is given by $e_\gamma e_\mu = e_{\gamma\mu}$ which can be extended linearly. By now there are examples where $\mathbb{C}[\Gamma]$ has a unique C*-norm and there are whole classes of groups for which it does not have a unique C*-norm. The idea is to look for C*-uniqueness for other natural classes of algebras. Most natural examples come from crossed products, but also other constructions can be taken into account.

Literature: [KeAl19], [CaSk19], [Sca20].

Multi-linear harmonic analysis. This topic concerns a multi-linear versions of Fourier multipliers. In the most classical version, an L_p-Fourier multiplier with $1 \leq p < \infty$, is a function $m : \mathbb{R} \to \mathbb{R}$ such that $\mathcal{F}_2 \circ m \circ \mathcal{F}_2^{-1} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ extends to a bounded map $L_p(\mathbb{R}) \to L_p(\mathbb{R})$. Morally such a map is of the form

$$\left(x \mapsto e^{irx}\right) \mapsto \left(x \mapsto m(r)e^{irx}\right).$$

This is only morally, since $x \mapsto e^{irx}$ is not an L_2-function, but if you decompose an L_2-function as an integral of trigonometric functions, then this is what happens to the integrants. The theory of L_p-multipliers is already vast and very powerful, see [Gra08]. The idea of this project is to analyse some theorems here in the multilinear case, where one takes a function $m : \mathbb{R}^2 \to \mathbb{R}$ and and looks at maps $L_{p_1}(\mathbb{R}) \times L_{p_2}(\mathbb{R}) \to L_q(\mathbb{R})$ with $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q}$ that are formally of the form

$$\left(x \mapsto e^{irx}\right) \times \left(x \mapsto e^{isx}\right) \mapsto \left(x \mapsto m(r,s)e^{i(r+s)x}\right).$$

There are several options to study problems here. On of them would be to construct multipliers on Heisenberg groups of dimension $2n+1$ or nilpotent (homogeneous) Lie groups.

Literature: [Gra08] and more depending on the direction.

Non-commutative differentiation and estimates on operator integrals. In [PoSu11] the following result was proved. Let $p \in (1, \infty)$. There exists a constant $C_p > 0$ such that for any $A, B \in B(H)$ self-adjoint and any Lipschitz function $f : \mathbb{R} \to \mathbb{C}$ with Lipschitz constant ≤ 1 we have

$$\|f(A) - f(B)\|_p \leq \|A - B\|_p.$$

Here the norm is defined as $\|X\|_p = \mathrm{Tr}(|X|^p)^{\frac{1}{p}}$ (if you want it is the ℓ_p-norm of the eigenvalues (or singular values) of X). The crucial part of the proof relies on the fact that the *entry-wise* matrix multiplication $M_n(\mathbb{C}) \to M_n(\mathbb{C})$ with a matrix of the form

$$\left(\frac{f(i) - f(j)}{i - j}\right)_{i,j \in F}, \qquad F \subseteq \mathbb{R}, |F| = n.$$

is bounded if $M_n(\mathbb{C})$ is equipped with the $\| \|_p$-norm. The aim would be to study this proof. Ideally you would also look at the higher order analogue of the problem studied in [PSS13], [CSZ20] and try to sharpen current estimates (I have concrete suggestions how to do that which can be part of the project).

Literature: [PoSu11], [PSS13], [CSZ20].

Haagerup property and the relative Haagerup property. Let Γ be a countable discrete group. We say that Γ has the Haagerup property if the following statement holds:

(1) There exists a sequence of positive definite functions $\varphi_n : \Gamma \to \mathbb{C}$ with $\varphi_n(e) = 1$ such that for every $\gamma \in \Gamma$ we have $\varphi_n(\gamma) \to 1$ as $n \to \infty$. That φ_n is positive definite means that for every $\gamma_1, \ldots, \gamma_r \in \Gamma$ we have that the matrix $(\varphi_n(\gamma_i^{-1}\gamma_j))_{1 \leq i,j \leq r}$ is a positive matrix.

Groups having the Haagerup property are free groups, amenable groups, Coxeter groups, $SL(2,\mathbb{Z})$, and many more. Groups without the Haagerup property are typically $SL(n,\mathbb{Z}), n \geq 3$. The Haagerup property has *massive* applications in (geometric) group theory, the theory of C*-algebras and von Neumann algebras.

The aim of this project is to study several equivalent notions of the Haagerup property and meanwhile also look at certain generalizations of the Haagerup property. For instance, the relative Haagerup property for an inclusion of two countable discrete groups $\Lambda \subseteq \Gamma$. This property is best defined as (1) but with the extra condition that φ_n restricted to Λ should be the constant function 1.

Literature: [KeAl19], [CaSk19], [Sca20].

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