LECTURE NOTES SUMMER SCHOOL POTCHEFSTROOM - SOUTH AFRICA

MARTIJN CASPERS

Abstract. These are the notes of lectures given at a spring school entitled 'An invitation to harmonic analysis' in Potchefstroom, South Africa, October 27-31, 2025. The lecture series was 6 full hours in total and these are the notes of the first 5 hours. It gives an introduction to transference results in harmonic analysis. The idea of these notes is to explain some basic principles in this area and to provide context and further references. The interested reader can use this as a starting point to go further in the literature. The proofs in the lecture notes have intentionally been given only partially, in which case we refer further to the literature for further reading.

Background and notation

Prerequisites. The notes are written for a reader with basic knowledge in functional analysis such as Banach and Hilbert spaces and bounded maps thereon. In principle the lectures should be accessible for students with only an introductory course in functional analysis or a more advanced audience coming from a different but related area. In particular for the larger part of the lecture series I will not assume perquisites on C*- or von Neumann algebras.

General notation.

- χ_A or $\chi_{>0}$ denotes an indicator function on a set A or on all elements larger than 0.
- All vector spaces are complex. Inner products are linear in the first coordinate and anti-linear in the second.
- M_n denotes the $n \times n$ matrices over the complex numbers.
- $C^1(\mathbb{R})$ denotes the continuously differentiable functions $\mathbb{R} \to \mathbb{C}$.

Lecture 1 - Schur multipliers

1.1. **Schur multipliers - an introduction.** In this lecture series we shall be dealing with *Schur multipliers* - in other words, entry wise matrix multiplication. At first glance such an operation may look a bit naive and one may wonder what its use is. But it turns out that Schur multipliers pop up naturally in harmonic analysis, differentiability properties of functional calculus, commutator estimates, perturbation theory, approximation properties of groups, et cetera. In this first section we start introducing Schur multipliers and give simple examples of where they appear. In the remaining chapters we focus on their analytical properties.

Definition 1.1. Let $\phi = (\phi_{ij})_{ij} \in M_n$ with indices $1 \leq i, j \leq n$ labeling the matrix units. We define the *Schur multiplier* with symbol ϕ as the linear map

$$M_\phi : M_n \to M_n : (x_{ij})_{ij} \mapsto (\phi_{ij} x_{ij})_{ij}.$$

Later on we will extend the notion from M_n to more general bounded operators on a Hilbert space or Schatten von Neumann classes.

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1.2. **Special examples of Schur multipliers.** At this point we introduce the divided difference function.

Definition 1.2. For $f \in C^1(\mathbb{R})$ we set the *divided difference function* $f^{[1]} : \mathbb{R}^2 \to \mathbb{C}$ as

$$f^{[1]}(\lambda, \mu) = \begin{cases} \frac{f(\lambda)-f(\mu)}{\lambda-\mu} & \lambda \neq \mu, \\ f'(\lambda) & \lambda = \mu. \end{cases}$$

Note in particular that $f^{[1]}$ is invariant under permutation of the variables, that is $f^{[1]}(\lambda, \mu) = f^{[1]}(\mu, \lambda)$.

Let us examine some important symbols of Schur multipliers:

(1) If $\phi_{ij} = \psi(i-j), 1 \leq i, j \leq n$ for some $\psi : \mathbb{Z} \cap [-n, n] \to \mathbb{C}$ then we say that ϕ is of *Toeplitz form.*
(2) If $\phi_{ij} = \psi(i+j), 1 \leq i, j \leq n$ for some $\psi : \mathbb{Z} \cap [-n, n] \to \mathbb{C}$ then we say that ϕ is of *Hänkel form.*
(3) Let $f \in C^1(\mathbb{R})$ and let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$. Then set the *(first order) divided difference function*,

$$\phi_{f,\lambda}(i,j) := f^{[1]}(\lambda_i, \lambda_j) := \begin{cases} \frac{f(\lambda_i)-f(\lambda_j)}{\lambda_i-\lambda_j} & \lambda_i \neq \lambda_j, \\ f'(\lambda_i) & \lambda_i = \lambda_j. \end{cases}$$

Later on when we change matrix algebras for bounded operators on a Hilbert space we may take ψ as in (1) and (2) to be a function $\mathbb{Z} \to \mathbb{C}$.

In the lecture series we will mostly focus on the relevance of symbols of Toeplitz form (1) and divided differences (3). We will immediately motivate that Schur multipliers of divided differences occur naturally in differentiability properties of functional calculus. The following proposition - being of fundamental relevance - is one of the many connections between noncommutative (Fréchet) differentiation and Schur multipliers.

Proposition 1.3. *Let $f(s) = s^k, k \in \mathbb{N}$. Let $A \in M_n$ be self-adjoint and let its diagonalization be given by*

$$A = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}, \qquad \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n.$$

Then for every $B \in M_n$ we have

$$\frac{d}{dt}|_{t=0} f(A+tB) = M_{\phi_{f,\lambda}}(B). \tag{1}$$

Proof. We have

$$\begin{aligned} \frac{d}{dt}|_{t=0} f(A+tB) &= \lim_{t \searrow 0} \frac{(A+tB)^k - A^k}{t} \\ &= \lim_{t \searrow 0} \frac{A^k + t\sum_{l=0}^{k-1} A^l B A^{k-l-1} + \mathcal{O}(t^2) - A^k}{t} \\ &= \sum_{l=0}^{k-1} A^l B A^{k-l-1} = T_{\phi_{f,\lambda}}(B), \end{aligned}$$

where the last equality follows as

$$(\phi_{f,\lambda})_{i,j} = \begin{cases} \frac{\lambda_i^k - \lambda_j^k}{\lambda_i - \lambda_j} = \sum_{l=0}^{k-1} \lambda_i^l \lambda_j^{k-l-1} & \text{if } \lambda_i \neq \lambda_j, \\ k\lambda_i^{k-1} = \sum_{l=0}^{k-1} \lambda_i^l \lambda_j^{k-l-1} & \text{if } \lambda_i = \lambda_j. \end{cases}$$

□

Remark 1.4. By linearity (1) holds for any polynomial f. In fact through continuous functional calculus (1) can be interpreted properly for any $f \in C^1(\mathbb{R})$ and equality still holds which can be proved by approximating f by polynomials.

1.3. **Literature.** Schur multipliers are closely related to the notion of double operator integrals. A tremendous amount of literature is available for double operator integrals of divided differences for which the book [SkTo19] is one of the most relevant sources and a good introduction in the topic treating both basic and advanced material.

Lecture 2 - Fourier multipliers on the torus

In this section we will introduce the concept of a Fourier multiplier on the torus. For general theory of Fourier multipliers and harmonic analysis the standard works are those by Grafakos [Gra04] and the more specialized book by Stein [Ste70].

2.1. **Standard notation.** Let

$$\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$$

be the torus which we identify as the complex unit circle. We equip $\mathbb{T}$ with the *Haar measure* given by

$$\int_{\mathbb{T}} f(z)dz = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})d\theta.$$

We denote

$$\begin{aligned} C(\mathbb{T}) =& \{f : \mathbb{T} \to \mathbb{C} \mid f \text{ is continuous}\}, \\ L^p(\mathbb{T}) =& \{f : \mathbb{T} \to \mathbb{C} \mid f \text{ measurable and } \int_{\mathbb{T}} |f(z)|^p ds < \infty\} \text{ modulo equivalence}, \end{aligned}$$

where two functions are equivalent if the absolute value of their difference integrates to 0. $L^p(\mathbb{T})$ carries the norm

$$\|f\|_p = (\int_{\mathbb{T}} |f(z)|^p dz)^{\frac{1}{p}},$$

for which it is a Banach space. $L^\infty(\mathbb{T})$ is than the space of essentially bounded Borel functions $\mathbb{T} \to \mathbb{C}$ modulo equivalence equipped with the $\| \, \|_\infty$-norm (essential supremum). The space $L^2(\mathbb{T})$ is a Hilbert space with inner product

$$\langle f, g\rangle = \int_{\mathbb{T}} f(z)\overline{g(z)}dz.$$

Note that our convention is that inner products are anti-linear in the second variable.

With mild abuse of notation we use the symbol z both for an element of $\mathbb{T}$ and for the identitiy function

$$z : \mathbb{T} \to \mathbb{T} : \lambda \mapsto \lambda.$$

We have $z^k z^l = z^{k+l}$. For $k \in \mathbb{Z}, k \neq 0$ we get that

$$\int_{\mathbb{T}} z^k dz = 0.$$

Therefore $(z^k)_{k\in\mathbb{Z}}$ are orthonormal vectors in $L^2(\mathbb{T})$. In fact one checks through the Stone-Weierstrass theorem that the linear span of $z^k, k \in \mathbb{Z}$ is dense in $C(\mathbb{T})$. As $C(\mathbb{T})$ is dense in $L^2(\mathbb{T})$ it follows that the span of $z^k, k \in \mathbb{Z}$ is dense in $L^2(\mathbb{T})$ and thus $(z^k)_{k\in\mathbb{Z}}$ forms an orthonormal basis of $L^2(\mathbb{T})$.

As every vector ξ in a (separable) Hilbert space H with orthonormal basis $(e_k)_{k\in\mathbb{Z}}$ has an orthonormal basis expansion

$$\xi = \sum_{k\in\mathbb{Z}} \langle \xi, e_k \rangle e_k,$$

we have proved the following theorem.

Theorem 2.5. *For every $f \in \ell^2(\mathbb{Z})$ we have*

$$f(z) = \sum_{k\in\mathbb{Z}} \widehat{f}(k) z^k, \tag{2}$$

where

$$\widehat{f}(k) = \langle f, z^k \rangle = \int_{\mathbb{T}} f(z) z^{-k} dz.$$

The series (2) *converges in the norm of $L^2(\mathbb{T})$ and the* **Fourier coefficients** $(\widehat{f}(k))_{k\in\mathbb{Z}}$ *are a sequence in $\ell^2(\mathbb{Z})$.*

Definition 2.6. A function $m \in \ell^\infty(\mathbb{Z})$ defines a bounded map

$$T_m : L^2(\mathbb{T}) \to L^2(\mathbb{T}) : z^k \mapsto m(k) z^k,$$

with norm $\|m\|_\infty$. Let $p \in [1, \infty)$. If T_m maps $L^2(\mathbb{T}) \cap L^p(\mathbb{T})$ to $L^2(\mathbb{T}) \cap L^p(\mathbb{T})$ and extends to a bounded map $L^p(\mathbb{T}) \to L^p(\mathbb{T})$ then T_m is called a L^p-**Fourier multiplier** with **symbol** m.

Remark 2.7. By mild abuse of terminology we shall also refer to the symbol m as being the L^p-Fourier multiplier instead of T_m. If p is clear from the context we sometimes call m or T_m a Fourier multiplier without specifying p.

The following remark gives all basic properties of the set of Fourier multipliers.

Remark 2.8. Let $\mathcal{M}_p \subseteq \ell^\infty(\mathbb{Z})$ be the set of symbols of L^p-Fourier multipliers. Then,

- $\mathcal{M}_p$ is a vector space (by the triangle inequality for operators).
- $\mathcal{M}_2 = \ell^\infty(\mathbb{Z})$ (by definition).
- $\mathcal{M}_p = \mathcal{M}_q$ in case $\frac{1}{p} + \frac{1}{q} = 1$ (by duality).
- $\mathcal{M}_p \supseteq \mathcal{M}_r$ in case $2 \leq p < r$ or $r < p \leq 2$ (complex interpolation, Riesz Thorin Theorem 2.10 [BeLo76]).
- In the previous bullet the inclusion is strict (see [CoFo76]).
- $\mathcal{M}_p$ contains c_{00}, the sequences with finite support, and all constant functions.

Remark 2.9. In general it is completely out of reach to give a comprehensive description of $\mathcal{M}_p$ in case $p \neq 1, 2, \infty$. For $p = 1, \infty$ there are characterizations of the set of *completely bounded* (not discussed here) Fourier multiplier that go back to Grothendieck, see [Pis01]. It is however known that not every bounded multiplier is also completely bounded due to a result by Pisier [Pis98] that is based on transference techniques between Fourier and Schur multipliers.

The following theorem is a standard tool when working with maps on L^p-spaces; it can be found in [BeLo76].

Theorem 2.10 (Riesz-Thorin theorem). *Let $1 \leq p_0 < p_1 \leq \infty$ and let $\theta \in [0,1]$. Let p_θ be such that $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Suppose that $T : L^{p_0}(\mathbb{T}) \cap L^{p_1}(\mathbb{T}) \to L^{p_0}(\mathbb{T}) \cap L^{p_1}(\mathbb{T})$ be such that $\|T : L^{p_i}(\mathbb{T}) \to L^{p_i}(\mathbb{T})\| < \infty$ for $i = 0,1$. Then*

$$\|T : L^{p_\theta}(\mathbb{T}) \to L^{p_\theta}(\mathbb{T})\| \leq \|T : L^{p_0}(\mathbb{T}) \to L^{p_0}(\mathbb{T})\|^{1-\theta} \|T : L^{p_1}(\mathbb{T}) \to L^{p_1}(\mathbb{T})\|^{\theta}.$$

Lemma 2.11. *The rank 1 map $f \mapsto \int_{\mathbb{Z}} f(z)dz \cdot 1$ extends to a bounded map $L^p(\mathbb{T}) \to L^p(\mathbb{T})$.*

Proof. For $p = 1$ and $p = \infty$ this is obvious. For $1 \leq p \leq \infty$ the proof follows by Theorem 2.10. Alternatively, by Hölder,

$$|\int_{\mathbb{T}} f(z)dz| \leq \int_{\mathbb{T}} |f(z)|dz \leq \int_{\mathbb{T}} |f(z)|^p dz^{\frac{1}{p}} \int_{\mathbb{T}} 1^q dz^{\frac{1}{q}} = \|f\|_p,$$

which also concludes the proof. □

We will now show that $\mathcal{M}_p$ contains some non-trivial functions as well. The following proof goes back to Cotlar [Cot55] but we have chosen to present the proof for the discrete case as was given by Mei-Ricard [MeRi17] and González-Pérez, Parcet and Xia [GPX]. The proof here is taken from [GPX]. We refer also to [LuXi] for extensions of these results to Coxeter groups. The multiplier $H = M_{\chi_{>0}}$ is also known as the *Hilbert transform* and plays a fundamental role in many parts of harmonic analysis.

Theorem 2.12. *Let $m(k) = \chi_{>0}(k)$ and let $1 < p < \infty$. Then $m \in \mathcal{M}_p$.*

Proof. Set $H = M_m$. For $g = \sum_{k\in\mathbb{Z}} \alpha_k z^k \in L^\infty(\mathbb{T})$ with finitely many $\alpha_k \in \mathbb{C}$ non-zero we set

$$g^\circ = g - \int_{\mathbb{T}} g(z)dz = \sum_{k\in\mathbb{Z}, k\neq 0} \alpha_k z^k.$$

The crucial step in the proof is to verify the following equality which is known as the *Cotlar identity*:

$$\underbrace{(H(f)H(\overline{f}))^\circ}_{\text{I}} = \underbrace{H(f\overline{H(f)})}_{\text{II}} + \underbrace{H(fH(\overline{f}))}_{\text{III}} - \underbrace{H(\overline{H(f\overline{f})})}_{\text{IV}}. \tag{3}$$

Proof of Cotlar identity: Consider again $f = \sum_{k\in\mathbb{Z}} \widehat{f}(k) z^k$ with only finitely many Fourier coefficients non-zero. Each of the expressions I, II, III, IV is then computable and we find

$$\begin{aligned}
\text{I} &= \sum_{k\neq 0} \sum_l \widehat{f}(k+l)\overline{\widehat{f}(l)} m(k+l)\overline{m(l)} z^k, \\
\text{II} &= \sum_{k\neq 0} \sum_l \widehat{f}(k+l)\overline{\widehat{f}(l)} m(k)\overline{m(l)} z^k, \\
\text{III} &= \sum_{k\neq 0} \sum_l \widehat{f}(k+l)\overline{\widehat{f}(l)} m(k+l)\overline{m(-k)} z^k, \\
\text{IV} &= \sum_{k\neq 0} \underbrace{\sum_l \widehat{f}(k+l)\overline{\widehat{f}(l)} m(k)\overline{m(-k)}}_{\text{Fourier coefficients}} z^k.
\end{aligned}$$

Now if we collect the Fourier coefficients of the combined term I − II − III + IV then we get

$$\sum_l \widehat{f}(k+l)\overline{\widehat{f}(l)}(m(k+l) - m(k))\overline{(m(l) - m(-k))}.$$

We claim that each of the summands of this expression is 0 for any l, k. Indeed, if $m(l)-m(-k) = 0$ then we are done. Otherwise if $m(l) - m(-k) \neq 0$ then as $m = \chi_{>0}$ this means that l and k have the same sign (or are 0). Therefore also $k+l$ and k have the same sign and $m(k+l) - m(k) = 0$. This concludes (3).

Proof of the theorem: Let $c_p = \| H : L^p(\mathbb{T}) \to L^p(\mathbb{T}) \|$. Note that $c_2 = \| \chi_{>0} \|_\infty = 1$. It now follows that

$$\begin{aligned} \|H(f)\|_{2p}^2 =& \|H(f)\overline{H(f)}\|_p \\ \leq& \| \int_{\mathbb{T}} H(f)\overline{H(f)} dz \|_p + \|(H(f)\overline{H(f)})^\circ\|_p. \end{aligned} \tag{4}$$

Now as $\int_{\mathbb{Z}} H(f)\overline{H(f)}dz = \langle H(f), H(f) \rangle$ is the scalar product we find using Lemma 2.11:

$$\begin{aligned} \| \int_{\mathbb{T}} H(f)\overline{H(f)} dz \|_p =& \| \int_{\mathbb{T}} H(f)\overline{H(f)} dz \|_2 \leq c_2^2 \| \int_{\mathbb{T}} f\overline{f} dz \|_2 \\ =& \| \int_{\mathbb{T}} f\overline{f} dz \|_p \leq \|f\overline{f}\|_p = \|f\|_{2p}^2. \end{aligned}$$

Recall $c_2 = 1$. We continue (4) and find using Cotlar's identity,

$$\begin{aligned} \|H(f)\|_{2p}^2 \leq& \|f\|_{2p}^2 + \|H(f\overline{H(f)})\|_p + \|H(fH(\overline{f}))\|_p - \|H(\overline{H(f\overline{f})})\|_p \\ \leq& (1 + 2c_p c_{2p} + c_p^2) \|f\|_{2p}^2. \end{aligned}$$

Hence

$$c_{2p}^2 \leq 1 + 2c_p c_{2p} + c_p^2. \tag{5}$$

We see this as a quadratic equation in the variable c_{2p} and inequality holds in case

$$c_p - \sqrt{2c_p^2 + 1} \leq c_{2p} \leq c_p + \sqrt{2c_p^2 + 1}.$$

As c_{2p} is positive the lower estimate holds trivially. Therefore (5) holds if and only if $c_{2p} \leq c_p + \sqrt{2c_p^2 + 1}$. Applying this inequality inductively we find that for all $k \in \mathbb{N}$ we have,

$$c_{2^k} \leq (1 + \sqrt{2})^k.$$

Hence $c_{2^k} < \infty$. By the Riesz-Thorin Theorem 2.10 we get that $c_p < \infty$ for any $2 \leq p < \infty$; for $1 < p \leq 2$ the same conclusion holds by duality. ☐

Remark 2.13. The Fourier multiplier $H = M_m$ in Theorem 2.12 is called the *Hilbert transform* and forms a fundamental tool in every branch of harmonic analysis.

Remark 2.14. The proof we presented here does not give the best possible bound for c_p; however using a mild variation it is possible to obtain a sharper upper bound as is explained in the language of the current proof in [GPX]; see also [Gra04] for the classical argument.

Lecture 3 - Connection between Fourier and Schur multipliers: algebraic level

This sections shows that Fourier multipliers are algebraically the same as Schur multipliers of Toeplitz form. Note however that on the analytic level we cannot yet compare them as we have not defined norms on matrix algebras yet.

Let $m \in \ell^\infty(\mathbb{Z})$ and recall that

$$T_m : \sum_k \alpha_k z^k \mapsto \sum_k \alpha_k z^k.$$

View z^k as multiplication operator

$$z^k : L^2(\mathbb{T}) \to L^2(\mathbb{T}) : f \mapsto z^k f.$$

Note that for $f = z^l$ we have

$$z^k : z^l \mapsto z^{k+l}.$$

Now as $(z^l)_{l\in\mathbb{Z}}$ is an orthonormal basis we see that

$$\langle z^k z^j, z^i \rangle = \delta_{k+j,i}.$$

In other words, in an informal infinite dimensional matrix representation we get

$$z^k \approx \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & & & \\ \dots & 0 & 1 & 0 & \dots & & \\ & \dots & 0 & 1 & 0 & \dots & \\ & & \dots & 0 & 1 & 0 & \dots \\ & & & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where the 1-s are located at the off-diagonal entries (i,j) with $i - j = k$. Further,

$$\begin{aligned} T_m : & \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & & & \\ \dots & c_{-1} & c_0 & c_1 & \dots & & \\ & \dots & c_{-1} & c_0 & c_1 & \dots & \\ & & \dots & c_{-1} & c_0 & c_1 & \dots \\ & & & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \\ \mapsto & \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & & & \\ \dots & m(-1)c_{-1} & m(0)c_0 & m(1)c_1 & \dots & & \\ & \dots & m(-1)c_{-1} & m(0)c_0 & m(1)c_1 & \dots & \\ & & \dots & m(-1)c_{-1} & m(0)c_0 & m(1)c_1 & \dots \\ & & & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \end{aligned} \tag{6}$$

And so we see that T_m acts as the Schur multiplier M_ϕ with symbol $\phi(i,j) = m(i-j)$ on the space of (infinite) Toeplitz matrices.

Lecture 4 - Schatten-von Neumann noncommutative L^p-spaces

Let H be a Hilbert space and let $B(H)$ be the bounded linear operators on H. For simplicity assume that H is separable, i.e. has a *countable* orthonormal basis. The material in this section can partly be found in [Mur90] and more extensively in [GoLa25] (general noncommutative L^p-spaces) or [PiXu03].

Definition 2.15. Let $(e_i)_i$ be an orthonormal basis of H. Let $\mathrm{Tr} : B(H)^+ \to [0,\infty]$ be the map given by

$$\mathrm{Tr}(x) = \sum_{i\in I} \langle x e_i, e_i \rangle.$$

Remark 2.16. The trace is independent of the choice of an orthonormal basis.

The following theorem introduced continuous functional calculus in an ad hoc way. Recall that we have chosen to set up these notes without assuming prerequisites on C*-algebras and therefore we have chosen to present the following theorem in a somewhat unusual way. The most common method of proving this theorem is to first prove the *Gelfand-Naimark theorem* which classifies commutative unital C*-algebras as continuous functions on a compact Hausdorff space. Then in case such a C*-algebra is generated by a single self-adjoint element $x \in B(H)$ one can show that this compact Hausdorff space can be identified with the spectrum $\sigma(x)$, which leads to the following theorem. The limit occuring in the theorem is understood as a limit in the operator norm; it us part of the theorem that this limit exists and does not depend on the choice the approximating sequence.

Theorem 2.17. *Let $x \in B(H)$ be self-adjoint with spectrum $\sigma(x) \subseteq \mathbb{R}$. Let $f : \sigma(x) \to \mathbb{C}$ be continuous. There exists a unique operator $f(x) \in B(H)$ such that:*

- *$f(x) = \sum_{k=0}^{n} \alpha_k x^k$ in case $f(s) = \sum_{k=0}^{n} \alpha_k s^k$ is a polynomial.*
- *$f(x) = \lim_n p_n(x)$ in case $p_n, n \in \mathbb{N}$ are polynomials that converge to f uniformly on $\sigma(x)$.*

Recall that an operator $x \in B(H)$ is positive if $x = x^*$ and $\sigma(x) \subseteq \mathbb{R}$.

Proposition 2.18. *Let $x \in B(H)$ be positive. Then there exists a unique positive operator $y \in B(H)$ such that $y^2 = x$.*

Proof sketch. We now get that $y = \sqrt{x}$ will work. □

Definition 2.19. For any $x \in B(H)$ we will set the absolute value

$$|x| = \sqrt{x^* x}.$$

We can now introduce noncommutative L^p-spaces.

Definition 2.20. Let $p \in [1, \infty)$. We set the *Schatten noncommutative L^p-space* as

$$S_p = \{x \in B(H) \mid \mathrm{Tr}(|x|^p) < \infty\},$$

and then define

$$\|x\|_p := \mathrm{Tr}(|x|^p)^{\frac{1}{p}}.$$

For $p = \infty$ we set $S_\infty = B(H)$ with $\| \cdot \|_\infty = \| \cdot \|$ the operator norm of $B(H)$.

Remark 2.21. It is also very common to define S_∞ as the space of compact operators in $B(H)$ with the operator norm but here we have chosen to work with the definition as above. We refer to S_p also as *Schatten classes* or *Schatten von Neumann classes*, *Schatten S_p*-spaces or any other variation. The name 'Schatten', the name of a mathematician, refers to the fact that these are noncommutative L^p-spaces of $B(H)$ and not of a more general von Neumann algebra.

We now summarize the basic properties of Schatten classes that resemble those of classical, commutative L^p-spaces.

Remark 2.22. The following properties hold true:

- The space $(S_p, \| \|_p), p \in [1, \infty]$ is a Banach space.
- $S_p \subseteq S_r$ in case $1 \leq p \leq r \leq \infty$.
- We have a Hölder inequality meaning that for $1 \leq p, r, s, \leq \infty$ with $\frac{1}{r} + \frac{1}{s} = \frac{1}{p}$ we have for $x \in S_p, y \in S_q$ that $xy \in S_r$ and

$$\|xy\|_r \leq \|x\|_r \|y\|_s.$$

- The space S_1 is spanned by $S_1^+ = \{x \in B(H)^+ \mid \mathrm{Tr}(x) < \infty\}$; moreover every element $x \in S_1$ can be written as $x = x_1 - x_2 + ix_3 - ix_4$ with $x_j \in S_1^+$ and $\|x_j\|_1 \leq \|x\|_1$.
- The trace Tr extends linearly to a well-defined map $\mathrm{Tr} : S_1 \to \mathbb{C}$.
- Let $1 \leq p \leq q \leq \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. The pairing
$$\langle x, y\rangle_{p,q} = \mathrm{Tr}(xy), \qquad x \in S_p, y \in S_q.$$
isometrically identifies S_q as the dual Banach space of S_p.

A much deeper fact is that S_p is known not to be isometrically isomorphic to a classical, commutative L^p-space. So we have found new Banach spaces that at least carry a number of the standard properties of a classical L^p-space, but are genuinely different.

Example 2.23. In this example let all indices k run over some countable set. Let $(e_k)_k$ and $(f_k)_k$ be orthonormal bases and let $(\alpha_k)_k$ be a sequence in $\mathbb{C}$ that converges to 0. Then consider the operator

$$x = \sum_k \alpha_k \left| e_k \right\rangle \left\langle f_k \right| \tag{7}$$

where the sum converges in norm and $\left| e_k \right\rangle \left\langle f_k \right| \xi = \langle \xi, f_k\rangle e_k$. We have

$$x^* x = \sum_k |\alpha_k|^2 \left| f_k \right\rangle \left\langle f_k \right|, \qquad |x| = \sqrt{x^* x} = \sum_k |\alpha_k| \left| f_k \right\rangle \left\langle f_k \right|.$$

We find that $x \in S_p$ if and only if $(\alpha_k)_k \in \ell^p$. It can be verified, using the spectral theorem, that any operator in S_p is compact and thus of the form (7). We thus conclude that S_p consists of all compact operators whose singular value sequence $(\alpha_k)_k$ belongs to ℓ^p.

Remark 2.24. S_2 is a Hilbert space with inner product

$$\langle x, y\rangle = \mathrm{Tr}(y^* x), \qquad x, y \in S_2.$$

Note that

$$\|x\|_2^2 = \langle x, x\rangle = \mathrm{Tr}(x^* x) = \sum_i \langle x^* x e_i, e_i\rangle = \sum_i \|x e_i\|^2 = \sum_{i,j} |\langle x e_i, e_j\rangle|^2.$$

In other words the S_2-norm of x is the ℓ^2-norm of its matrix entries with respect to the orthonormal basis $(e_i)_i$ (or any other orthonormal basis).

Completely bounded maps. We may identify the tensor product of vector spaces $S_p \odot S_p$ with a subspace of $S_p(H \otimes H)$ from which it inherits its norm. The closure of $S_p \odot S_p$ with respect to this norm will be denoted by $S_p \otimes S_p$ and it can be shown that with the previous identificaiton in fact $S_p \otimes S_p = S_p(H \otimes H)$.

Similarly consider $L^p(\mathbb{T}; S_p)$ the space of Bochner measurable functions $f : \mathbb{T} \to S_p$ equipped with norm

$$\|f\|_p = \left(\int_{\mathbb{T}} \|f(z)\|_p^p dz\right)^{\frac{1}{p}}.$$

We may linearly identify an element $x \otimes g \in S_p \odot L^p(\mathbb{T})$ with the function $g \cdot x \in L^p(\mathbb{T}; S_p)$ given by $(g \cdot x)(z) = g(z)x$. This way $S_p \odot L^p(\mathbb{T})$ inherits a norm from $L^p(\mathbb{T}; S_p)$ and its completion is denoted by $S_p \otimes L^p(\mathbb{T})$.

We will now introduce the notion of a completely bounded map. In fact S_p has a so-called operator space structure and this leads to the notion of a completely bounded map; we omit details but they can be found in Pisier's book [Pis03] or [Pis98]. It is a theorem of Pisier [Pis98] that this notion of completely bounded map is equivalent to the one given in the following theorem, which we will take as a definition.

Theorem 2.25. *A map $T : S_p \to S_p$ is* **completely bounded** *if*

$$\|T : S_p \to S_p\|_{cb} := \|\mathrm{id} \otimes T : S_p \otimes S_p \to S_p \otimes S_p\| < \infty.$$

A map $M : L^p(\mathbb{T}) \to L^p(\mathbb{T})$ is **completely bounded** *if*

$$\|M : L^p(\mathbb{T}) \to L^p(\mathbb{T})\|_{cb} := \|\mathrm{id} \otimes T : S_p \otimes L^p(\mathbb{T}) \to S_p \otimes L^p(\mathbb{T})\| < \infty.$$

Lecture 5: Transference of Fourier and Schur multipliers

In this section we consider $H = \ell^2(\mathbb{Z})$ and we let $S_p = S_p(\ell^2(\mathbb{Z})$ denote the Schatten class associated with the Hilbert space $\ell^2(\mathbb{Z})$. Note that as all separable Hilbert spaces can be identified unitarily the space S_p does not depend on the choice of the Hilbert space. However, to introduce Schur multipliers we shall require to pick some natural basis which is why it is more convenient to work with $H = \ell^2(\mathbb{Z})$. We extend our notion of Schur multipliers.

Definition 2.26. Let $\phi : \mathbb{Z}^2 \to \mathbb{C}$. We define the *Schur multiplier* with symbol ϕ as the linear map

$$M_\phi : S_2 \to S_2 : x \mapsto M_\phi(x),$$

where $M_\phi(x) \in S_2$ is determined by $\langle M_\phi(x)\delta_j, \delta_i\rangle = \phi(i,j)\delta_{i,j}$.

Remark 2.27. The finite rank operators $|\delta_i\rangle\langle\delta_j|, i,j \in \mathbb{Z}$ form an orthonormal basis of S_2, see Remark 2.24. Each orthonormal basis vector $|\delta_i\rangle\langle\delta_j|, i,j \in \mathbb{Z}$ is an eigenvector of M_ϕ with eigenvalue $\phi(i,j)$. Therefore, in particular

$$\|M_\phi : S_2 \to S_2\| = \|\phi\|_\infty.$$

The following theorem was proved by Neuwirth-Ricard [NeRi11]. For general amenable locally compact groups it was proved in [CaSa15].

Theorem 2.28. *Let $1 \le p \le \infty$. Let $m \in \ell^\infty(\mathbb{Z})$ and set $\phi(i,j) = m(i-j)$. Then,*

$$\|T_m : L^p(\mathbb{T}) \to L^p(\mathbb{T})\|_{cb} = \|M_m : S_p \to S_p\|_{cb}.$$

Remark 2.29. In fact we prove that

$$\begin{aligned} \|T_m : L^p(\mathbb{T}) \to L^p(\mathbb{T})\| &\le \|M_m : S_p \to S_p\|, \\ \|T_m : L^p(\mathbb{T}) \to L^p(\mathbb{T})\|_{cb} &\ge \|M_m : S_p \to S_p\|_{cb}. \end{aligned} \tag{8}$$

The first of these inequality also holds in the completely bounded setting by tensoring with S_p everywhere in the argument and considering completely bounded norms.

Proof of the inequalities (8). We first prove $\ge$. Let $p_k : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be the orthogonal projection onto $\mathbb{C}\delta_k, k \in \mathbb{Z}$. Let

$$u = \sum_{k\in\mathbb{Z}} p_k \otimes z^k \in B(\ell^2(\mathbb{Z}) \otimes L^2(\mathbb{T})).$$

The sum converges in the strong operator topology and u is unitary. For any $x \in S_p = S_p(\ell^2(\mathbb{Z}))$ we have

$$\|x\|_{S_p} = \|x \otimes 1\|_{S_p \otimes L^p(\mathbb{T})} = \|u(x \otimes 1)u^*\|_{S_p \otimes L^p(\mathbb{T})}.$$

From this point we just write $\|\cdot\|_p$ for any of the p-norms in the latter equality and suppress the space in our notation.

We have for $x \in S_p$ that

$$\begin{aligned}(\mathrm{id} \otimes T_m)(u(x \otimes 1)u^*) =& T_m(\sum_{i,j \in \mathbb{Z}} p_i x p_j \otimes z^{i-j}) = \sum_{i,j \in \mathbb{Z}} p_i x p_j \otimes m(i-j) z^{i-j} \\ =& \sum_{i,j \in \mathbb{Z}} \phi(i,j) p_i x p_j \otimes z^{i-j} = (M_m \otimes \mathrm{id})(u(x \otimes 1)u^*).\end{aligned}$$

Combining this we obtain that for $x \in S_p$ we have

$$\begin{aligned}\|M_\phi(x)\|_p =& \|(M_m \otimes \mathrm{id})(u(x \otimes 1)u^*)\|_p = \|(\mathrm{id} \otimes T_m)(u(x \otimes 1)u^*)\|_p \\ =& \|(\mathrm{id} \otimes T_m)(u(x \otimes 1)u^*)\|_p \leq \|T_m\|_{cb} \|x\|_p.\end{aligned}$$

This concludes the estimate $\geq$.

Next we prove the estimate $\leq$. Recall that $(z^k)_{k \in \mathbb{Z}}$ forms an orthonormal basis of $L^2(\mathbb{T})$. For $F \subseteq \mathbb{Z}$ we let $P_F : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ be the orthonormal projection onto the linear span of $z^k, k \in F$. Assume that F is finite; we shall later choose F more specifically. Consider the maps,

$$\varphi_F^p : L^p(\mathbb{T}) \to S_p(x) : x \mapsto |F|^{-\frac{1}{p}} P_F x P_F.$$

We now prove three claims.

Claim 1: φ_F^p is a contraction.

Proof of claim 1: For $p = \infty$ this states that $x \mapsto P_F x P_F$ is a contraction from $L^\infty(\mathbb{T})$ to $B(L^2(\mathbb{T}))$ which is obviously true.

For $p = 1$ we take $x \in L^1(\mathbb{T})$. Write $x = ab$ with $a, b \in L^2(\mathbb{T})$ and $\|x\|_1 = \|a\|_2 \|b\|_2$; note that we may take $b = |x|^{\frac{1}{2}}$ and $a = x|x|^{-\frac{1}{2}}$ on the support of x. We have Fourier expansions $a = \sum_{k \in \mathbb{Z}} a_k z^k$ and $b = \sum_{k \in \mathbb{Z}} b_k z^k$. Now we have

$$|F|^{-1} \|P_F x P_F\|_1 \leq |F|^{-1} \|P_F a\|_2 \|b P_F\|_2$$

Now $\|P_F a\|_2^2$ is the sum of the squares of the absolute values of the matrix entries of $P_F a$ where we recall that with respect to the basis $(z^k)_{k \in \mathbb{Z}}$ the matrix entries are:

$$P_F a = \text{ indexed by } F \left\{ \overbrace{\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & & & \\ \dots & a_{-1} & a_0 & a_1 & \dots & & \\ & \dots & a_{-1} & a_0 & a_1 & \dots & \\ & & \dots & a_{-1} & a_0 & a_1 & \dots \\ & & & \vdots & \vdots & \vdots & \ddots \end{pmatrix}}^{\text{indexed by } \mathbb{Z}} \right. .$$

On a fixed row we find the sequence $(a_k)_{k \in \mathbb{Z}}$, possibly shifted, and there are precisely $|F|$ rows. This means that we have

$$\|P_F a\|_2 = |F|^{\frac{1}{2}} \left(\sum_{k \in \mathbb{Z}} |a_k|^2 \right)^{\frac{1}{2}} = |F|^{\frac{1}{2}} \|a\|_2,$$

and similarly

$$\|b P_F\|_2 = |F|^{\frac{1}{2}} \left(\sum_{k \in \mathbb{Z}} |b_k|^2 \right)^{\frac{1}{2}} = |F|^{\frac{1}{2}} \|b\|_2.$$

Therefore

$$\|\varphi^1_F(x)\|_1 = |F|^{-1}\|P_F x P_F\|_1 \leq |F|^{-1}\|P_F a\|_2 \|b P_F\|_2 \leq |F|^{-1}\|a\|_2\|b\|_2 = \|x\|_1.$$

This concludes the claim for $p = 1$.

For arbitrary $1 \leq p \leq \infty$ we conclude by a version of complex interpolation [BeLo76]. Note that the maps depend on p so we cannot use the most primitive version of complex interpolation; nevertheless it is possible to interpolate analytic families of operators.

Claim 2: $\varphi^p_F \circ T_m = M_\phi \circ \varphi^p_F$.

Proof of claim 2: This follows from the description of Fourier multipliers given in (6).

Claim 3: For $x \in L^p(\mathbb{T})$ and $y \in L^q(\mathbb{T})$ we have

$$\lim_{j\to\infty} \langle \varphi^p_{F_j}(x), \varphi^q_{F_j}(y)\rangle_{p,q} = \langle x, y\rangle_{p,q},$$

where $F_j = [-j, j]$. In particular the limit exists.

Proof of claim 3: Let $x = \sum_{k\in\mathbb{Z}} x_k z^k$ and $y = \sum_{k\in\mathbb{Z}} y_k z^k$ be the Fourier expansions of x and y. Note that for any $F \subseteq \mathbb{Z}$ finite,

$$P_F x P_F = \text{ indexed by } F \overbrace{\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & & & \\ \dots & x_{-1} & x_0 & x_1 & \dots & & \\ & \dots & x_{-1} & x_0 & x_1 & \dots & \\ & & \dots & x_{-1} & x_0 & x_1 & \dots \\ & & & \vdots & \vdots & \vdots & \ddots \end{pmatrix}}^{\text{indexed by } F}.$$

And so

$$\langle \varphi^p_F(x), \varphi^q_F(y)\rangle_{p,q} = \mathrm{Tr}(\varphi^p_F(x)\varphi^q_{F_j}(y)) = |F|^{-1}\mathrm{Tr}(P_F x P_F y P_F),$$

is an expression that is computable. A short matrix computation shows that

$$|F|^{-1}\mathrm{Tr}(P_F x P_F y P_F) = \sum_{k\in\mathbb{Z}} x_k y_{-k} \frac{|F \cap F + k|}{|F|}.$$

Now take $F = F_j = [-j, j]$ and let $j \to \infty$. The previous two equalities yield

$$\lim_{j\to\infty} \langle \varphi^p_{F_j}(x), \varphi^q_{F_j}(y)\rangle_{p,q} = \lim_{j\to\infty} \sum_{k\in\mathbb{Z}} x_k y_{-k} \frac{|F_j \cap F_j + k|}{|F_j|} = \sum_{k\in\mathbb{Z}} x_k y_{-k} = \langle x, y\rangle_{p,q}.$$

Remainder of the proof. Now take $x \in L^p(\mathbb{T})$ and $y \in L^q(\mathbb{T})$. Then by using Claim 3, Claim 2, Remark 2.22 (last bullet), and then Claim 1, we find:

$$\begin{aligned} |\langle T_m(x), y\rangle_{p,q}| =& \lim_j |\langle \varphi^p_{F_j}(T_m(x)), \varphi^q_{F_j}(y)\rangle_{p,q}| = \lim_j |\langle M_\phi(\varphi^p_{F_j}(x)), \varphi^q_{F_j}(y)\rangle_{p,q}| \\ \leq& \|M_\phi : S_p \to S_p\| \|\varphi^p_{F_j}(x)\|_p \|\varphi^q_{F_j}(x)\|_q \leq \|M_\phi : S_p \to S_p\| \|x\|_p \|y\|_q. \end{aligned}$$

Taking the supremum over x and y in the unit ball yields

$$\|T_m : L^p(\mathbb{T}) \to L^p(\mathbb{T})\| \leq \|M_\phi : S_p \to S_p\|,$$

and thus concludes the proof.

□

Remark 2.30. Theorem 2.28 can be generalized after the introduction of locally compact groups and their group von Neumann algebras, see [NeRi11, CaSa15]. In the analogous statement the inequality $\geq$ then holds for any locally compact group and the inequality $\leq$ holds for any so-called *amenable* locally compact group. The question whether there exists a counter example in case the group is not amenable remains open. In [CaSa15] it is shown that the proof technique does not extend beyond amenable groups.

Remark 2.31. The proof Theorem 2.28 is based on the principle of *transference*; and let us say that both directions of the inequality are transference proof. They have in common that one L^p-space is isometrically embedded in another L^p-space in such a way that the Fourier and Schur multipliers are intertwined by the embedding. There are many variations of this transference principle available in the literature and we list a number of them that were relevant in noncommutative analysis - the list shall surely be incomplete:

- In [Lee65] Karel de Leeuw proved a transference result for Fourier multipliers on $\mathbb{R}^d$ to Fourier multipliers on $\mathbb{Z}^d$. This was generalized to locally compact groups, with conditions, in [CPPR15] using again a new transference proof.
- In [Pis98] Pisier constructed a Fourier multiplier that is bounded but not completely bounded. Part of the proof used transference from Fourier to Schur multipliers exactly as in Theorem 2.28; note that Pisier's result dates back further.
- In [**?**] Fourier multipliers on $SL(n, \mathbb{R})$ were constructed. The proof uses a 'local' version of transference, see that paper. In fact the paper uses several applications between Fourier and Schur multipliers back and forth.
- In [CGPT22] sharp Hörmander-Mikhlin conditions were found for when the symbol of a Schur multiplier acts boundedly on S_p. The proof uses transference of twisted Fourier multipliers.
- Theorem 2.28 was generalized for actions of groups in [Gon18].
- In [PoSu11] boundedness of Schur multipliers of divided differences was shown using a transference proof that we shall outline in the next section. The proof also follows as a special case of [CGPT22].
- There are several multilinear transference results. A multilinear version of Theorem 2.28 can be found in [CKV22]. A multilinear De Leeuw theorem was proved in [CJKM24] for locally compact groups with conditions. Multilinear transference was applied in [CaRe25] to tackle problems on Schur multipliers of higher order divided differences.
- Several multiplier theorems on quantum tori have been obtained, see e.g. [CXY13, Ric16] (this list may be far from complete).

Lecture 6: Transference and Schur multipliers of divided differences

In this lecture we give another applicaton of a transference result: the main result of [PoSu11]. To do this we need to develop our theory a little bit further.

We shall need to do harmonic analysis on $\mathbb{T}^d$ and in particular for $d = 2$. All the theory about Fourier and Schur multipliers makes perfect sense in case we change $\mathbb{T}$ for $\mathbb{T}^d$ and $\mathbb{Z}$ by $\mathbb{Z}^d$. To keep the notation simple let us do this for $d = 2$ so that $\mathbb{T}^2 = \mathbb{T} \times \mathbb{T}$. In this case, we have the coordinate functions

$$z_1 : (z_1, z_2) \mapsto z_1, \qquad z_2 : (z_1, z_2) \mapsto z_2.$$

We have used the same abuse of notation as before and denote by z_i both the coordinate and the coordinate function.

Definition 2.32. A function $m \in \ell^\infty(\mathbb{Z}^2)$ defines a bounded map

$$T_m : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2) : z_1^{k_1} z_2^{k_2} \mapsto m(k_1, k_2) z_1^{k_1} z_2^{k_2},$$

with norm $\|m\|_\infty$. Let $p \in [1, \infty)$. If T_m maps $L^2(\mathbb{T}^2) \cap L^p(\mathbb{T}^2)$ to $L^2(\mathbb{T}^2) \cap L^p(\mathbb{T}^2)$ and extends to a bounded map $L^p(\mathbb{T}^2) \to L^p(\mathbb{T}^2)$ then T_m is called a L^p-**Fourier multiplier** with **symbol** m.

We shall use the following theorem as a black box.

Theorem 2.33. *There exists a function $m : \mathbb{Z}^2 \to \mathbb{C}$ such that for $(i, j) \in \mathbb{Z}^2, (i, j) \neq (0, 0)$ with $|i| \leq |j|$ we have*

$$m(i, j) = \frac{i}{j},$$

and such that m is the symbol of an L^p-Fourier multiplier for any $1 < p < \infty$.

Proof sketch. We shall give some references to the literature from which one can reconstruct the proof. The original way to prove this theorem is through [PoSu11, Lemma 6] and then applying a completely bounded version of the Marcinkiewicz multiplier theorem which is due to Bourgain [Bou86] and relies on the fact that the Hilbert transform from Section 1.3 is in fact completely bounded (from this fact the total proof is only a page or 3). Combining this with De Leeuw's discretization theorem [Lee65], which is again a transference result, gives the result. □

The following theorem was proved by Potapov and Sukochev [PoSu11]. We state it a different - in fact less elegant - form to avoid some small technicalities. We let S_p^n be the Schatten von Neumann class associated with $\ell^2([1, n] \cap \mathbb{Z})$.

Theorem 2.34. *Let $1 < p < \infty$. There exists a constant $C_p > 0$ such that for every $f \in C^1(\mathbb{R})$ with $\|f'\|_\infty < \infty$, for every $n \in \mathbb{N}$, and for every $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ we have*

$$\|M_{\phi_{f,\lambda}} : S_p^n \to S_p^n\| \leq C_p \|f'\|_\infty.$$

Proof. We will give the proof except that we shall accept that the following reductions can be carried out, the second one being trivial:

- Without loss of generality we can assume that there exists $N \in \mathbb{N}$ such that $\lambda_i \in \frac{1}{N}\mathbb{Z}$ and $f(\lambda_i) \in \frac{1}{N}\mathbb{Z}$. This assumption can be made by approximating arbitrary $\lambda_i \in \mathbb{R}$ with elements from $\frac{1}{N}\mathbb{Z}$. A similar approximation is also carried out in [PoSu11].
- Without loss of generality, we may assume that $\|f'\|_\infty \leq 1$ as we may replace f by $\|f'\|_\infty^{-1} f$ if f is not the constant function. If f is constant both f' and $\phi_{f,\lambda}$ are 0 and so the statement is trivial.

Now let $N \in \mathbb{N}$ be as above and assume that $\lambda_1, \ldots, \lambda_n \in \frac{1}{N}\mathbb{Z}$. Define the unitary

$$u = \sum_{j=1}^{n} p_j \otimes z^{Nj} \otimes z^{N f(\lambda_j)} \in S_p^n \otimes L^p(\mathbb{T}) \otimes L^p(\mathbb{T}) = S_p^n \otimes L^p(\mathbb{T}^2).$$

Again we have for any $x \in S_p^n$ that

$$\|u(x \otimes 1 \otimes 1)u^*\|_p = \|x\|_p.$$

Let m be the multiplier from Theorem 2.33. For $x \in S_p^n$ we have the intertwining property.

$$\begin{aligned}(\mathrm{id} \otimes T_m)(u(x \otimes 1 \otimes 1)u^*) =&(\mathrm{id} \otimes T_m)(\sum_{i,j=1}^n p_i x p_j \otimes z^{N(\lambda_i - \lambda_j)} \otimes z^{N(f(i)-f(j))}) \\ =&\sum_{i,j=1}^n \frac{N(f(\lambda_i) - f(\lambda_j))}{N(\lambda_i - \lambda_j)} p_i x p_j \otimes z^{N(i-j)} \otimes z^{N(f(i)-f(j))} \\ =&\sum_{i,j=1}^n \frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j} p_i x p_j \otimes z^{N(i-j)} \otimes z^{N(f(i)-f(j))} \\ =&(M_{\phi_f,\lambda} \otimes \mathrm{id})(\sum_{i,j=1}^n p_i x p_j \otimes z^{N(\lambda_i - \lambda_j)} \otimes z^{N(f(i)-f(j))}) \\ =&u(M_{\phi_f,\lambda}(x) \otimes 1)u^*.\end{aligned}$$

Now combining the latter two properties we see that for $x \in S_p^n$ we have

$$\begin{aligned}\|M_{\phi_{f,\lambda}}(x)\|_p =&\|u(M_{\phi_{f,\lambda}}(x) \otimes 1)u^*\|_p = \|(\mathrm{id} \otimes T_m)(u(x \otimes 1)u^*)\|_p \\ \leq&\|T_m\|_{cb}\|u(x \otimes 1)u^*\|_p \leq \|T_m\|_{cb}\|x\|_p.\end{aligned}$$

□

Theorem 2.34 may at first glance look like a technical result but it has a number of beautiful consequences of which we mention at least the following.

Theorem 2.35. *Let $1 < p < \infty$. There exists a constant $C_p > 0$ such that for every $n \in \mathbb{N}$, every $A, B, Cx \in M_n$ with A, B, C self-adjoint and every $f \in C^1(\mathbb{R})$ with $\|f'\|_\infty < \infty$ we have the following commutator estimate:*

$$\|[f(A), x]\|_p \leq C_p \|f'\|_\infty \|[A, x]\|_\infty,$$

and the following Lipschitz estimate

$$\|f(B) - f(C)\|_p \leq C_p \|B - C\|_p.$$

Proof. We first prove the commutator estimate. Since A is self-adjoint we may assume it is diagonal with eigenvalues $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n_{\geq 0}$. Now we have

$$[f(A), x] = ((f(\lambda_i) - f(\lambda_j))x_{ij})_{ij} = M_{\phi_f,\lambda}(((\lambda_i - \lambda_j)x_{ij})_{ij}) = M_{\phi_f,\lambda}([A, x]).$$

Therefore by Theorem 2.34,

$$\|[f(A), x]\|_p \leq 2C_p \|[A, x]\|_p,$$

and we conclude the commutator estimate.

The Lipschitz estimate follows as we may take

$$A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}, \qquad x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then for the commutators we get:

$$[A, x] = \begin{pmatrix} 0 & B - C \\ C - B & 0 \end{pmatrix}, \qquad [f(A), x] = \begin{pmatrix} 0 & B - C \\ C - B & 0 \end{pmatrix}.$$

Taking norms concludes the theorem. □

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TU Delft, EWI/DIAM, P.O.Box 5031, 2600 GA Delft, The Netherlands
Email address: `M.P.T.Caspers@tudelft.nl`