ON THE ACTION OF LIPSCHITZ FUNCTIONS ON VECTOR-VALUED RANDOM SUMS

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Abstract. Let X be a Banach space and let $(\xi_j)_{j\geqslant 1}$ be an i.i.d. sequence of symmetric random variables with finite moments of all orders. We prove that the following assertions are equivalent:

(1) There exists a constant K such that
$$\Big(\mathbb{E}\Big\|\sum_{j=1}^{n}\xi_j f(x_j)\Big\|^2\Big)^{\frac12} \leqslant K\|f\|_{\rm Lip}\Big(\mathbb{E}\Big\|\sum_{j=1}^{n}\xi_j x_j\Big\|^2\Big)^{\frac12}$$
for all Lipschitz functions $f : X \to X$ satisfying $f(0) = 0$ and all finite sequences $x_1, \dots, x_n$ in X.

(2) X is isomorphic to a Hilbert space.

For Banach spaces X and Y let $\mathrm{Lip}_0(X, Y)$ denote the Banach space of all Lipschitz continuous functions $f : X \to Y$ satisfying $f(0) = 0$ with norm $\|f\|_{\rm Lip} := L_f$, the Lipschitz constant of f. Our main result relates the action of functions $f \in \mathrm{Lip}_0(X, Y)$ on random sums in X with the cotype and type of X and Y, respectively. Since the best constants are obtained for Gaussian variables, we state the result for this case first.

Theorem 1. *Let X and Y be Banach spaces with $\dim X = \infty$ and $\dim Y \geqslant 1$, and let $(\gamma_j)_{j\geqslant 1}$ be a sequence of independent standard Gaussian random variables. The following assertions are equivalent:*

(i) *For all finite sequences $x_1, \dots, x_n \in X$, all scalars $a_1, \dots, a_n > 0$, and all $f_1, \dots, f_n \in \mathrm{Lip}_0(X, Y)$ we have*
$$\Big(\mathbb{E}\Big\|\sum_{j=1}^{n}\gamma_j a_j^{-1} f_j(a_j x_j)\Big\|^2\Big)^{\frac12} \leqslant K\big(\max_{1\leqslant j\leqslant n}\|f_j\|_{\rm Lip}\big)\Big(\mathbb{E}\Big\|\sum_{j=1}^{n}\gamma_j x_j\Big\|^2\Big)^{\frac12},$$
where K is a constant depending on X and Y only.

(ii) *For all finite sequences $x_1, \dots, x_n \in X$ there exist scalars $a_1, \dots, a_n > 0$ such that for all $f \in \mathrm{Lip}_0(X, Y)$ we have*
$$\Big(\mathbb{E}\Big\|\sum_{j=1}^{n}\gamma_j a_j^{-1} f(a_j x_j)\Big\|^2\Big)^{\frac12} \leqslant K\|f\|_{\rm Lip}\Big(\mathbb{E}\Big\|\sum_{j=1}^{n}\gamma_j x_j\Big\|^2\Big)^{\frac12},$$
where K is a constant depending on X and Y only.

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(iii) *X has cotype 2 and Y has type 2.*

If (i) *or* (ii) *holds with constant K, then the Gaussian cotype 2 constant of X and the Gaussian type 2 constant of Y satisfy $C_2^\gamma(X) \leqslant K$ and $T_2^\gamma(Y) \leqslant \sqrt{2}K$.*

Proof. The implication (i)⇒(ii) is trivial.

(ii)⇒(iii): First we prove that X has cotype 2 with $C_2^\gamma(X) \leqslant K$. Fix a norm one vector $y_0 \in Y$ and define $f \in \mathrm{Lip}_0(X,Y)$ by $f(x) := \|x\| y_0$. Since $\|f\|_{\mathrm{Lip}} = 1$ it follows that for $x_1, \dots, x_n \in X$ we have, with the $a_1, \dots, a_n > 0$ as in (ii),

$$\sum_{j=1}^n \|x_j\|^2 = \mathbb{E}\Big\|\sum_{j=1}^n \gamma_j f(x_j)\Big\|^2 = \mathbb{E}\Big\|\sum_{j=1}^n \gamma_j a_j^{-1} f(a_j x_j)\Big\|^2 \leqslant K^2 \mathbb{E}\Big\|\sum_{j=1}^n \gamma_j\, x_j\Big\|^2.$$

Next we prove that Y has type 2 with $T_2^\gamma(Y) \leqslant K\sqrt{2}$. By an observation in [5] we have

$$T_2^\gamma(Y) = \sup\Big\{\mathbb{E}\Big(\Big\|\sum_{j=1}^n \gamma_j\, y_j\Big\|^2\Big)^{\frac12} :\ n \geqslant 1,\ \|y_1\| = \dots = \|y_n\| = n^{-\frac12}\Big\}. \tag{1}$$

Fix an integer $n \geqslant 1$ and vectors $y_1, \dots, y_n \in Y$ of norm 1. Let $(e_j)_{j=1}^n$ be the standard unit basis of l_n^2 and let $\varepsilon > 0$ be arbitrary and fixed. Since $\dim X = \infty$, by Dvoretzky's theorem [4] we can find an isomorphism T from l_n^2 onto an n-dimensional subspace X_0 of X such that $\|T\| \leqslant 1+\varepsilon$ and $\|T^{-1}\| = 1$. Let

$$x_j := Te_j, \quad j = 1, \dots, n. \tag{2}$$

Clearly, $1 \leqslant \|x_j\| \leqslant 1+\varepsilon$ and for all $1 \leqslant j \neq k \leqslant n$ and $a, b \in \mathbb{R}$ we have

$$\|ax_j - bx_k\| \geqslant \|T^{-1}\|^{-1}\|ae_j - be_k\| = \sqrt{a^2+b^2}. \tag{3}$$

Define $\varphi_j : X \to \mathbb{R}$ by

$$\varphi_j(x) := \max\big\{0, 1 - \sqrt{2}\|x - x_j\|\big\}.$$

Then φ_j is Lipschitz continuous with Lipschitz constant $\|\varphi_j\|_{\mathrm{Lip}} \leqslant \sqrt{2}$, we have $\varphi_j(x_j) = 1$, and $\varphi_j \equiv 0$ outside the open 'sector'

$$V_j := \Big\{x \in X : \exists t > 0 \text{ such that } \|tx - x_j\| < \tfrac12\sqrt{2}\Big\}.$$

Note that $0 \notin V_j$. We claim that the sectors V_j are disjoint. Indeed, given $x \in V_j$ we choose $t > 0$ such that $\|tx - x_j\| < \frac12\sqrt{2}$. Then for $j \neq k$ and all $s > 0$,

$$\|sx - x_k\| \geqslant \|t^{-1}sx_j - x_k\| - \|t^{-1}sx_j - sx\| \overset{(*)}{>} \sqrt{t^{-2}s^2+1} - \tfrac12 t^{-1}s\sqrt{2} \overset{(**)}{\geqslant} \tfrac12\sqrt{2}.$$

In $(*)$ we used (3) and the choice of t, while $(**)$ follows from the inequality $\sqrt{c^2+1} - \frac12 c\sqrt{2} \geqslant \frac12\sqrt{2}$.

Define $\psi_j : X \to \mathbb{R}$ by

$$\psi_j(x) := a_j \varphi_j\big(a_j^{-1}x\big),$$

where the $a_1, \dots, a_n > 0$ are chosen as in (ii). Then ψ_j is Lipschitz continuous with Lipschitz constant $\|\psi_j\|_{\mathrm{Lip}} \leqslant \sqrt{2}$, we have $\psi_j(a_j x_j) = a_j$, and $\psi_j \equiv 0$ outside V_j. Define $f : X \to Y$ by

$$f(x) := \sum_{j=1}^n \psi_j(x) y_j.$$

It is clear that $f(0) = 0$ and $f(a_j x_j) = a_j y_j$. We claim that $f \in \mathrm{Lip}_0(X, Y)$ with $\|f\|_{\mathrm{Lip}} \leqslant \sqrt{2}$. If $x, x' \in V_j$ for some j, then by the disjointness of V_j with the other V_k's and the fact that $\|y_j\| = 1$ we obtain

$$\|f(x) - f(x')\| = \|y_j\|\,|\psi_j(x) - \psi_j(x')| \leqslant \sqrt{2}\,\|x - x'\|.$$

If $x \in V_j$ and $x' \in V_k$ for $j \neq k$, we choose convex combinations ξ and ξ' of x and x', say $\xi = (1-s)x + sx'$ and $\xi' = (1-t)x + tx'$ with $0 \leqslant s \leqslant t \leqslant 1$, such that $\xi \in \partial V_j$ and $\xi' \in \partial V_k$. Clearly, $f(\xi) = f(\xi') = 0$. It follows from the previous case that

$$\begin{aligned}\|f(x) - f(x')\| &\leqslant \|f(x) - f(\xi)\| + \|f(\xi') - f(x')\| \\ &\leqslant \sqrt{2}\|x - \xi\| + \sqrt{2}\|\xi' - x'\| \\ &= \sqrt{2}(s + (1-t))\|x - x'\| \leqslant \sqrt{2}\|x - x'\|.\end{aligned}$$

The case where $x \in V_j$ and $x' \notin \bigcup_k V_k$ is handled similarly. Finally if $x, x' \notin \bigcup_k V_k$, then $f(x) = f(x') = 0$. This concludes the proof of the claim.

Recalling that $f(0) = 0$, $\|f\|_{\mathrm{Lip}} \leqslant \sqrt{2}$, $\|T\| \leqslant 1 + \varepsilon$, $\|y_j\| = 1$, we obtain

$$\begin{aligned}\mathbb{E}\Big\|\sum_{j=1}^n \gamma_j\, y_j\Big\|^2 &= \mathbb{E}\Big\|\sum_{j=1}^n \gamma_j\, a_j^{-1} f(a_j x_j)\Big\|^2 \leqslant 2K^2\mathbb{E}\Big\|\sum_{j=1}^n \gamma_j\, x_j\Big\|^2 \\ &\leqslant 2K^2(1+\varepsilon)^2\mathbb{E}\Big\|\sum_{j=1}^n \gamma_j\, e_j\Big\|^2 = 2K^2(1+\varepsilon)^2\sum_{j=1}^n \|y_j\|^2.\end{aligned}$$

By (1) this proves that Y has type 2 with $T_2^\gamma(Y) \leqslant K\sqrt{2}(1+\varepsilon)$. Since $\varepsilon > 0$ was arbitrary, the proof is complete.

(iii)$\Rightarrow$(i): Assume that X has cotype 2 and Y has type 2. For all $x_1, \dots, x_n \in X$, $a_1, \dots, a_n > 0$, and $f_1, \dots, f_n \in \mathrm{Lip}_0(X, Y)$ we have

$$\begin{aligned}\mathbb{E}\Big\|\sum_{j=1}^n \gamma_j a_j^{-1} f_j(a_j x_j)\Big\|^2 &\leqslant T_2^\gamma(Y)^2\big(\max_{1\leqslant j\leqslant n}\|f_j\|_{\mathrm{Lip}}\big)^2\sum_{j=1}^n a_j^{-2}\|a_j x_j\|^2 \\ &\leqslant T_2^\gamma(Y)^2\big(\max_{1\leqslant j\leqslant n}\|f_j\|_{\mathrm{Lip}}\big)^2 C_2^\gamma(X)^2\,\mathbb{E}\Big\|\sum_{j=1}^n \gamma_j x_j\Big\|^2.\end{aligned}$$

□

By a celebrated theorem of Kwapień [8], a Banach space X has type 2 and cotype 2 if and only if X is isomorphic to a Hilbert space. Thus if we take $X = Y$ in the theorem, then assertion (iii) may be replaced by:

(iii)′ X is isomorphic to a Hilbert space.

In Theorem 1 we may replace the Gaussian sequence $(\gamma_j)_{j\geqslant 1}$ by a Rademacher sequence $(r_j)_{j\geqslant 1}$, in which case we obtain the estimates

$$C_2^r(X) \leqslant K \quad\text{and}\quad T_2^r(Y) \leqslant \frac{2}{\sqrt{\pi}}K.$$

Here $C_2^r(X)$ and $T_2^r(Y)$ denote the Rademacher cotype 2 constant of X and the Rademacher type 2 constant of Y, respectively. For the second estimate we recall from [9, Lemma 4.5] that $T_2^r(X) \leqslant \frac{1}{m_1^\gamma} T_2^\gamma(X)$, where $m_1^\gamma := \mathbb{E}\,|\gamma_j| = \sqrt{2/\pi}$ and that by an observation in [5] we have

$$T_2^\gamma(Y) = \sup\Big\{\mathbb{E}\Big(\Big\|\sum_{j=1}^n r_j\, y_j\Big\|^2\Big)^{\frac12} :\ n \geqslant 1,\ \|y_1\| = \dots = \|y_n\| = n^{-\frac12}\Big\}. \tag{4}$$

The proof of (ii)⇒(iii) may now be repeated verbatim.

Next let $(\xi_j)_{j\geqslant 1}$ be an arbitrary sequence of i.i.d. symmetric random variables with $\mathbb{E}\,|\xi_j|^2 = 1$. We denote by $T_2^\xi(X)$ and $C_2^\xi(X)$ the ξ-type 2 and ξ-cotype 2 constant of a Banach space, respectively. By a standard randomization argument, every Banach space X with (co)type 2 has ξ-(co)type 2 with constants $T_2^\xi(X) \leqslant T_2^r(X)$ and $C_2^\xi(X) \leqslant C_2^r(X)$. Conversely, if X has ξ-type 2, then again by [9, Lemma 4.5],

$$\Big(\mathbb{E}\Big\|\sum_{j=1}^n r_j x_j\Big\|^2\Big)^{\frac12} \leqslant \frac{1}{m_1^\xi}\Big(\mathbb{E}\Big\|\sum_{j=1}^n \xi_j x_j\Big\|^2\Big)^{\frac12} \leqslant \frac{1}{m_1^\xi} T_2^\xi(X)\Big(\sum_{j=1}^n \|x_j\|^2\Big)^{\frac12},$$

where $m_1^\xi := \mathbb{E}\,|\xi_j|$. It follows that X has type 2 with $T_2^r(X) \leqslant \frac{1}{m_1^\xi} T_2^\xi(X)$. If X has ξ-cotype 2 and all moments of ξ_j are finite, then X has finite cotype (we are grateful to Tuomas Hytönen for pointing this out to us). In fact, by means of elementary estimates it can be shown that c_0 does not have finite ξ-cotype. The Rademacher cotype 2 of X then follows from the Maurey-Pisier theorem; cf. [9, Section 9.2].

At the expense of slightly worse estimate for the type 2 constant it is possible to generalize Theorem 1 to sequences of random variables $(\xi_j)_{j\geqslant 1}$ as above. This is achieved by a slightly modified argument which does not require normalizations as in (1) and (4) and which has the additional virtue that for each n the scalars $a_1, \dots, a_n$ are allowed to depend not only on the vectors $x_1, \dots, x_n$ but also on the function f.

Theorem 2. *Let X and Y be Banach spaces with $\dim X = \infty$ and $\dim Y \geqslant 1$, and let $\xi = (\xi_j)_{j\geqslant 1}$ be a sequence of i.i.d. random variables with $\mathbb{E}\,|\xi_j|^2 = 1$. The following assertions are equivalent:*

(i) *For all $f_1, \dots, f_n \in \mathrm{Lip}_0(X, Y)$, all finite sequences $x_1, \dots, x_n \in X$, and all scalars $a_1, \dots, a_n > 0$ we have*

$$\Big(\mathbb{E}\Big\|\sum_{j=1}^n \xi_j a_j^{-1} f_j(a_j x_j)\Big\|^2\Big)^{\frac12} \leqslant K\big(\max_{1\leqslant j\leqslant n} \|f_j\|_{\mathrm{Lip}}\big)\Big(\mathbb{E}\Big\|\sum_{j=1}^n \xi_j x_j\Big\|^2\Big)^{\frac12},$$

where K is a constant depending on X and Y only.

(ii) *For all $f \in \mathrm{Lip}_0(X, Y)$ and all finite sequences $x_1, \dots, x_n \in X$ there exist scalars $a_1, \dots, a_n > 0$ such that*

$$\Big(\mathbb{E}\Big\|\sum_{j=1}^n \xi_j a_j^{-1} f(a_j x_j)\Big\|^2\Big)^{\frac12} \leqslant K\|f\|_{\mathrm{Lip}}\Big(\mathbb{E}\Big\|\sum_{j=1}^n \xi_j x_j\Big\|^2\Big)^{\frac12},$$

where K is a constant depending on X and Y only.

(iii) *X has ξ-cotype 2 and Y has ξ-type 2.*

If (ii) *holds, then $C_2^\xi(X) \leqslant K$ and $T_2^\xi(Y) \leqslant (1 + 2\sqrt{2})K$. If the ξ_j have finite moments of all orders, then* (iii) *is equivalent to*

(iv) *X has cotype 2 and Y has type 2.*

Proof. Only the proof that Y has ξ-type 2 in the implication (ii)⇒(iii) needs to be adapted. Fix arbitrary nonzero vectors $y_1, \dots, y_n \in Y$. Following the arguments in the proof of (ii)⇒(iii) in Theorem 1, we replace (2) by

$$x_j := \|y_j\| T e_j, \quad j = 1, \dots, n,$$

and define $\varphi_j : X \to \mathbb{R}$ by $\varphi(0) = 0$ and

$$\varphi_j(x) := \max\Big\{0, 1 - \sqrt{2}(1+\varepsilon) d_j(x)\Big\} \|x\|,$$

where $d_j : X \setminus \{0\} \to \mathbb{R}$ is the function

$$d_j(x) := \Big\| \frac{x}{\|x\|} - \frac{x_j}{\|x_j\|} \Big\|.$$

Then φ_j is Lipschitz continuous with $\|\varphi_j\|_{\mathrm{Lip}} \leqslant L_\varepsilon := 2\sqrt{2}(1+\varepsilon) + 1$, we have $\varphi_j(a x_j) = a\|x_j\|$ for all $a > 0$, and $\varphi_j \equiv 0$ outside the sector

$$V_j := \Big\{ x \in X \setminus \{0\} : \ d_j(tx) < \tfrac{1}{2}\sqrt{2}(1+\varepsilon)^{-1} \Big\}.$$

As before, V_j and V_k are disjoint for $j \neq k$. Indeed if $x \in V_j$, then for $j \neq k$ we have

$$\begin{aligned} \Big\| \frac{x}{\|x\|} - \frac{x_k}{\|x_k\|} \Big\| &\geqslant \Big\| \frac{x_j}{\|x_j\|} - \frac{x_k}{\|x_k\|} \Big\| - \Big\| \frac{x_j}{\|x_j\|} - \frac{x}{\|x\|} \Big\| \\ &> \sqrt{\|Te_j\|^{-2} + \|Te_k\|^{-2}} - \tfrac{1}{2}\sqrt{2}(1+\varepsilon)^{-1} \\ &\geqslant \sqrt{2}(1+\varepsilon)^{-1} - \tfrac{1}{2}\sqrt{2}(1+\varepsilon)^{-1} = \tfrac{1}{2}\sqrt{2}(1+\varepsilon)^{-1}, \end{aligned}$$

which shows that $x \notin S_k$. Define $f : X \to Y$ by

$$f(x) = \sum_{j=1}^n \varphi_j(x) \frac{y_j}{\|x_j\|}.$$

Then $f(0) = 0$, $f(a x_j) = a y_j = a f(x_j)$ for $a > 0$, and f is Lipschitz continuous with $\|f\|_{\mathrm{Lip}} \leqslant L_\varepsilon$. With the $a_1, \dots, a_n > 0$ as in (ii), estimating as before we obtain

$$\mathbb{E}\Big\| \sum_{j=1}^n \xi_j\, y_j \Big\|^2 = \mathbb{E}\Big\| \sum_{j=1}^n \xi_j\, a_j^{-1} f(a_j x_j) \Big\|^2 \leqslant \|f\|_{\mathrm{Lip}}^2 K^2 (1+\varepsilon)^2 \sum_{j=1}^n \|y_j\|^2.$$

This proves that Y has ξ-type 2 with

$$T_2^\xi(Y) \leqslant K \|f\|_{\mathrm{Lip}} (1+\varepsilon) \leqslant K\big(1 + 2\sqrt{2}(1+\varepsilon)\big)(1+\varepsilon).$$

Since $\varepsilon > 0$ was arbitrary, the proof is complete. □

If the ξ_j have finite moments of all orders, for $X = Y$ we obtain an isomorphic characterization of Hilbert spaces as before.

Theorems 1 and 2 bear a striking resemblance to [1, Proposition 1.13] which states that X has type 2 and Y has cotype 2 if and only if every uniformly bounded family $\mathscr{T}$ in $\mathscr{L}(X, Y)$ is R-bounded. Recall that $\mathscr{T}$ is called *R-bounded* if there exists a constant K such that for all choices $x_1, \dots, x_n \in X$ we have

$$\Big(\mathbb{E}\Big\| \sum_{j=1}^n r_j\, T_j x_j \Big\|^2 \Big)^{\frac{1}{2}} \leqslant K \Big(\mathbb{E}\Big\| \sum_{j=1}^n r_j\, x_j \Big\|^2 \Big)^{\frac{1}{2}}.$$

This result is elementary (it suffices to consider suitably chosen families of rank one operators) and the role of the Rademacher variables can be replaced by any i.i.d. sequence of mean zero random variables with finite second moment. The precise relationship between [1, Proposition 1.13] and our results remains unclear, since we see no obvious way to relate finitely many linear operators in $\mathscr{L}(X,Y)$ to a single nonlinear function in $\mathrm{Lip}_0(X,Y)$. In this connection it is worthwhile to point out that it appears to be an unsolved open problem whether for every pair of Banach spaces X and Y there exists a constant $c(X,Y)$ such that, given any distinct elements $x_1,\dots,x_n \in X$ and elements $y_1,\dots,y_n \in Y$, there exists a Lipschitz function $f : X \to Y$ satisfying $f(x_j) = y_j$ for all $j = 1,\dots,n$ and

$$\|f\|_{\mathrm{Lip}} \leqslant c(X,Y) \max_{\substack{1\leqslant j,k\leqslant n\\ j\neq k}} \frac{\|y_j - y_k\|}{\|x_j - x_k\|}. \tag{5}$$

The important point here is that $c(X,Y)$ should be independent of n. Indeed, it was shown in [6] that for fixed n, (5) can be achieved with a constant $c(n,X,Y)$ of order $\log n$.

As an application of Theorem 1 we will prove next that $\mathrm{Lip}_0(X)$ acts in the operator ideal $\gamma(l^2,X)$ of γ-radonifying operators from l^2 to X if and only if X is isomorphic to a Hilbert space.

Let H be a Hilbert space. We denote by $\gamma(H,X)$ the completion of the vector space of all finite rank operators $u : H \to X$ with respect to the norm

$$\|u\|_{\gamma(H,X)} := \sup \Big(\mathbb{E}\Big\| \sum_j \gamma_j\, uh_j\Big\|^2\Big)^{\frac12}. \tag{6}$$

The supremum is taken over all finite orthonormal systems (h_j) in H. As is well known, $\gamma(H,X)$ is an operator ideal in the sense that for all bounded linear operators $v : \tilde H \to H$ and $w : X \to \tilde X$ we have $wuv \in \gamma(\tilde H, \tilde X)$ and

$$\|wuv\|_{\gamma(\tilde H,\tilde X)} \leqslant \|w\|\,\|u\|_{\gamma(H,X)}\,\|v\|.$$

For more information we refer to [3, Chapter 12].

We will be interested in the particular case where H equals $L^2 := L^2(S,\Sigma,\mu)$ for some σ-finite measure space (S,Σ,μ) and $u_\phi : L^2 \to X$ is an integral operator of the form

$$u_\phi h = \int_S h(s)\phi(s)\,d\mu(s), \quad h \in L^2,$$

for suitable functions $\phi : S \to X$. Operators in $\gamma(L^2,X)$ arising in this way have been investigated recently in [7]. If ϕ is a simple function, i.e., a function of the form $\sum_{j=1}^n \mathbf{1}_{S_j} \otimes x_j$ with vectors x_j taken from X and disjoint sets $S_j \in \Sigma$ satisfying $0 < \mu(S_j) < \infty$, it is easily checked that $u_\phi \in \gamma(L^2,X)$ and by considering the orthonormal functions $h_j := \mu(S_j)^{-\frac12}\mathbf{1}_{S_j}$, the γ-norm of ϕ is computed as

$$\|u_\phi\|^2_{\gamma(L^2,X)} = \mathbb{E}\Big\|\sum_{j=1}^n \gamma_j u_\phi h_j\Big\|^2 = \mathbb{E}\Big\|\sum_{j=1}^n \gamma_j\,\mu(S_j)^{\frac12} x_j\Big\|^2. \tag{7}$$

The subspace of all $u \in \gamma(L^2,X)$ of the form $u = u_\phi$ for some simple function $\phi : S \to X$ will be denoted by $\gamma_{\mathrm{simple}}(L^2,X)$. An easy approximation argument shows that this is a dense subspace of $\gamma(L^2,X)$.

If X has type 2, the mapping $\phi \mapsto u_\phi$ defined for simple functions ϕ as above, extends to a continuous embedding from $L^2(X) := L^2(S,\Sigma,\mu;X)$ into $\gamma(L^2,X)$.

Indeed, for a simple function $\phi = \sum_{j=1}^n \mathbf{1}_{S_j} \otimes x_j$ we have, using (7),

$$\begin{aligned} \|u_\phi\|_{\gamma(L^2,X)}^2 &= \mathbb{E}\Big\| \sum_{j=1}^n \gamma_j\, \mu(S_j)^{\frac12} x_j \Big\|^2 \\ &\leqslant T_2^\gamma(X)^2 \sum_{j=1}^n \mu(S_j)\|x_j\|^2 = T_2^\gamma(X)^2 \|\phi\|_{L^2(X)}^2, \end{aligned} \tag{8}$$

and the claim follows by a density argument. Similarly, if X has cotype 2, then $u_\phi \mapsto \phi$ extends to a continuous embedding from $\gamma(L^2, X)$ into $L^2(X)$.

If $\phi = \sum_{j=1}^n \mathbf{1}_{S_j} \otimes x_j$ is a simple X-valued function, then for each $f \in \mathrm{Lip}_0(X, Y)$,

$$f(\phi) = \sum_{j=1}^n \mathbf{1}_{S_j} \otimes f(x_j)$$

is a simple Y-valued function. In this way we obtain a mapping $\tilde f : \gamma_{\rm simple}(L^2, X) \to \gamma_{\rm simple}(L^2, Y)$ by putting

$$\tilde f(u_\phi) := u_{f(\phi)}.$$

We are interested in conditions ensuring that $\tilde f$ extends to a Lipschitz continuous mapping from $\gamma(L^2, X)$ to $\gamma(L^2, Y)$. From $f(0) = 0$ we see that a necessary condition is that there should exist a constant K such that

$$\|u_{f(\phi)}\|_{\gamma(L^2,Y)} \leqslant K\|f\|_{\rm Lip}\|u_\phi\|_{\gamma(L^2,X)}$$

for all simple functions $\phi : S \to X$. The next result gives a converse and relates both conditions to the geometry of the spaces X and Y.

Theorem 3. *Let X and Y be Banach spaces, let $L^2 := L^2(S, \Sigma, \mu)$ as before, and assume that $\dim X = \infty$, $\dim Y \geqslant 1$, and $\dim L^2 = \infty$. Let $(\gamma_j)_{j\geqslant1}$ be a sequence of independent standard Gaussian random variables. The following assertions are equivalent:*

(i) *For all $f \in \mathrm{Lip}_0(X, Y)$ and all simple functions $\phi : S \to X$ we have*

$$\|u_{f(\phi)}\|_{\gamma(L^2,Y)} \leqslant K\|f\|_{\rm Lip}\|u_\phi\|_{\gamma(L^2,X)},$$

where K is a constant depending on X and Y only.

(ii) *X has cotype 2 and Y has type 2.*

If (i) *holds, then $C_2^\gamma(X) \leqslant K$ and $T_2^\gamma(Y) \leqslant \sqrt2 K$, and for all $f \in \mathrm{Lip}_0(X, Y)$ the mapping $\tilde f$ uniquely extends to an element of $\mathrm{Lip}_0(\gamma(L^2, X), \gamma(L^2, Y))$ satisfying*

$$\|\tilde f\|_{\rm Lip} \leqslant C_2^\gamma(X) T_2^\gamma(Y)\|f\|_{\rm Lip}.$$

Proof. (i)⇒(ii): Let $x_1, \dots, x_n \in X$ be arbitrary. By the σ-finiteness of (S, Σ, μ) and the assumption that $\dim L^2 = \infty$ there exist disjoint sets $S_1, \dots, S_n \in \Sigma$ satisfying $0 < \mu(S_j) < \infty$ for $j = 1, \dots, n$ and define $\phi : S \to X$ by $\phi := \sum_{j=1}^n h_j \otimes x_j$, where $h_j = \mu(S_j)^{-1/2}\mathbf{1}_{S_j}$ for all j. It follows from (7) that for all $f \in \mathrm{Lip}_0(X, Y)$,

$$\begin{aligned} \Big(\mathbb{E}\Big\| \sum_{j=1}^n \gamma_j \mu(S_j)^{\frac12} f\big(\mu(S_j)^{-\frac12} x_j\big)\Big\|^2\Big)^{\frac12} &= \|u_{f(\phi)}\|_{\gamma(L^2,Y)} \\ &\leqslant K\|f\|_{\rm Lip}\|u_\phi\|_{\gamma(L^2,X)} = K\|f\|_{\rm Lip}\Big(\mathbb{E}\Big\|\sum_{j=1}^n \gamma_j x_j\Big\|^2\Big)^{\frac12}. \end{aligned}$$

By an application of Theorem 1 with $a_j = \mu(S_j)^{-\frac{1}{2}}$ we obtain (ii).

(ii)⇒(i): Assume that X has cotype 2 and Y has type 2 and fix $f \in \mathrm{Lip}_0(X, Y)$. For simple functions $\phi, \psi : S \to X$ we have, by (8) and its cotype 2 analogue,

$$\begin{aligned}
&\|\tilde f(u_\phi) - \tilde f(u_\psi)\|_{\gamma(L^2,Y)} \\
&\quad = \|u_{f(\phi)} - u_{f(\psi)}\|_{\gamma(L^2,Y)} \leqslant T_2^\gamma(Y)\|f(\phi) - f(\psi)\|_{L^2(Y)} \\
&\quad \leqslant T_2^\gamma(Y)\|f\|_{\mathrm{Lip}}\|\phi - \psi\|_{L^2(X)} \leqslant C_2^\gamma(X)T_2^\gamma(Y)\|f\|_{\mathrm{Lip}}\|u_\phi - u_\psi\|_{\gamma(L^2,X)}.
\end{aligned}$$

Since $\gamma_{\mathrm{simple}}(L^2, X)$ is dense in $\gamma(L^2, X)$ it follows that $\tilde f$ has a unique Lipschitz continuous extension from $\gamma(L^2, X)$ to $\gamma(L^2, Y)$ with $\|\tilde f\|_{\mathrm{Lip}} \leqslant C_2^\gamma(X)T_2^\gamma(Y)\|f\|_{\mathrm{Lip}}$. This proves the final assertion, and (i) follows by taking $\psi = 0$. □

Theorem 3 is motivated by the result from [10, 11] that a function $\phi : (0, T) \to X$ is stochastically integrable with respect to a Brownian motion if and only if the operator u_ϕ is well defined and belongs to $\gamma(L^2(0, T), X)$. The question whether $\tilde f$ extends continuously to $\gamma(L^2(0, T), X)$ for all $f \in \mathrm{Lip}_0(X, X)$ thus amounts to asking whether $f(\phi)$ is stochastically integrable whenever ϕ has this property. This question arises naturally in the study of stochastic differential equations in X driven by multiplicative noise satisfying Lipschitz conditions; cf. [2] for the Hilbert space case. Theorem 3 applied to $X = Y$ shows that in general the answer is negative unless X is isomorphic to a Hilbert space.

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