Inequality of spectral bound and growth bound for positive semigroups in rearrangement invariant Banach function spaces

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Extending a recent example of Arendt, we study the semigroup $T(t)f(s) = f(se^t)$ in rearrangement invariant Banach function spaces over $(1,\infty)$ and obtain various abstract conditions under which equality of spectral bound and growth bound fails.

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0. Introduction

It is well-known that the spectral bound $s(A)$ and the growth bound $\omega_0(\mathbf{T})$ of a C_0-semigroup $\mathbf{T} = \{T(t)\}_{t\geqslant 0}$ with generator A need not be equal, even if the underlying space is a Hilbert space [Za] or if the semigroup is a positive semigroup on a Banach lattice [GVW].

On the other hand, it is known that $s(A) = \omega_0(\mathbf{T})$ for positive C_0-semigroups on each of the spaces $L^1(\mu)$, $L^2(\mu)$ and $C_0(\Omega)$. Recently, L. Weis [We] announced a proof of the long-standing conjecture that the growth bound and the spectral bound of positive C_0-semigroups on $L^p(\mu)$, $1 \leqslant p < \infty$, coincide.

Since every rearrangement invariant Banach function space with order continuous norm is an exact interpolation space between L^1 and L^∞, this suggests that it might be possible to extend Weis's arguments to positive C_0-semigroups on certain rearrangement invariant Banach function spaces. However, somewhat earlier W. Arendt [Ar] had shown that for the positive C_0-semigroup $\mathbf{T} = \{T(t)\}_{t\geqslant 0}$ on the rearrangement invariant space $L^p(1,\infty) \cap L^q(1,\infty)$, $1 \leqslant p < q < \infty$, defined by

$$(T(t)f)(s) = f(se^t), \qquad t \geqslant 0, \quad s > 1, \quad f \in L^p(1,\infty) \cap L^q(1,\infty),$$

one has $s(A) \leqslant -\frac{1}{p} < -\frac{1}{q} \leqslant \omega_0(\mathbf{T})$.

In this paper, we study the semigroup $\mathbf{T}_E$ defined by $(T_E(t)f)(s) := f(se^t)$ in arbitrary rearrangement invariant Banach function spaces E over $(1,\infty)$ and show that in many of these spaces the equality $s(A_E) = \omega_0(\mathbf{T}_E)$ fails for $\mathbf{T}_E$. Since each operator $T_E(t)$ acts as the restriction to $(1,\infty)$ of a dilatation operator, we try to relate $s(A_E)$ and $\omega_0(\mathbf{T}_E)$ to the asymptotic behaviour of these operators. Recalling that the Boyd indices $\underline{\alpha}_E$ and $\overline{\alpha}_E$ are defined in terms of dilatation operators, for spaces over $(0,\infty)$ (rather than $(1,\infty)$) it is indeed fairly easy to show that $s(A_E) = \omega_0(\mathbf{T}_E) = -\underline{\alpha}_E$ (Theorem 2.6 below). For spaces over $(1,\infty)$, one again can prove that $\omega_0(\mathbf{T}_E) = -\underline{\alpha}_E$ but it is more difficult to obtain an estimate

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for $s(A_E)$. We obtain such an estimate, which in some special cases gives the correct value of $s(A_E)$ (Theorem 2.5). It is based on the simple observation that $\mathbf{T}_E$ acts nilpotently on functions of compact support.

In Sections 3 and 4, we apply our results to extend Arendt's example into two different directions. In Section 3, we give conditions on two spaces E and F ensuring that on their intersection we have $s(A_{E\cap F}) < \omega_0(\mathbf{T}_{E\cap F})$. These conditions are satisfied for $E = L^p$ and $F = L^q$, $1 \leqslant p < q < \infty$.

The intersection of two Orlicz spaces $E_\Phi \cap E_\Psi$ is an Orlicz space up to an equivalent norm, and its Young function is given by $\max\{\Phi, \Psi\}$ [dJ]. In particular, the space $L^p \cap L^q$ of Arendt's example is an Orlicz space. With this in mind, in Section 4 we give abstract sufficient properties for a Young function Φ in order that $s(A_{E_\Phi}) < \omega_0(\mathbf{T}_{E_\Phi})$ holds in the Orlicz space $E_\Phi(1,\infty)$. Since these conditions are trivially satisfied by the Young function $\Phi(t) = \max\{t^p, t^q\}$ of $L^p \cap L^q$, $1 \leqslant p < q < \infty$.

I would like to dedicate this paper to our boy Matthijs who was born during the preparation of this paper.

1. Preliminaries

In this section we recall some facts about rearrangement invariant Banach function spaces and C_0-semigroups. For proofs and more details we refer to [KPS, Chapter II] (function spaces) and [P], [Na] (semigroups).

The terminology concerning rearrangement invariant Banach function spaces varies among different authors. For the sake of definiteness, throughout this paper a *rearrangement invariant Banach function space*, or briefly a *function space*, over $(0,\infty)$ is a Banach function space E over $(0,\infty)$ in the sense of [KPS] with the property that $f \in E$, $g : (0,\infty) \to \mathbb{C}$ measurable, and $g^* \leqslant f^*$ imply that $g \in E$ and $\|g\|_E \leqslant \|f\|_E$; here f^* and g^* denote the decreasing arrangements of $|f|$ and $|g|$, i.e. the unique non-negative, non-increasing, left continuous functions on $(0,\infty)$ equimeasurable with $|f|$ and $|g|$, respectively.

If a (f_n) converges to f in E, then some subsequence of (f_n) converges to f pointwise a.e.

The *fundamental function* of E is the function φ_E defined by $\varphi_E(t) := \|\chi_{H_t}\|_E$, where $H_t \subset (0,\infty)$ is any measurable subset of measure t and χ_{H_t} is its characteristic function. By the rearrangement invariance, this function is well-defined.

For $s > 0$ let $D_s : E \to E$ be the *dilatation operator* defined by $(D_s f)(t) = f(st)$, $t > 0$. This operator is bounded on E. With $h_E(s) := \|D_{\frac{1}{s}}\|_E$, we define

$$\underline{\alpha}_E := \sup_{0<s<1} \frac{\ln h_E(s)}{\ln s} = \lim_{s\downarrow 0} \frac{\ln h_E(s)}{\ln s}; \qquad \overline{\alpha}_E := \inf_{s>1} \frac{\ln h_E(s)}{\ln s} = \lim_{s\to\infty} \frac{\ln h_E(s)}{\ln s}.$$

The identities in these definitions are consequences of the theory of submultiplicative functions. The numbers $\underline{\alpha}_E$ and $\overline{\alpha}_E$ are called the *lower*- and *upper Boyd index* of E, respectively, and satisfy $0 \leqslant \underline{\alpha}_E \leqslant \overline{\alpha}_E \leqslant 1$.

Let E and F be two function spaces over $(0,\infty)$. We define $E \cap F$ as the set of all measurable functions on $(0,\infty)$ that belong to both E and F. This space is a function space over $(0,\infty)$ under the norm

$$\|f\|_{E\cap F} = \max\{\|f\|_E, \|f\|_F\}.$$

Similarly, the space $E+F$ is defined as the set of all measurable functions on $(0,\infty)$ that can be represented as a sum $f=g+h$ with $g\in E$ and $h\in F$. This space is a function space over $(0,\infty)$ under the norm

$$\|f\|_{E+F} := \inf\{\|g\|_E + \|h\|_F : g+h=f,\, g\in E,\, h\in F\}.$$

For every function space E over $(0,\infty)$, we have continuous inclusions $L^1\cap L^\infty \subset E \subset L^1+L^\infty$. In particular, bounded functions of compact support belong to E.

We recall the classical inequality of Hardy and Littlewood for decreasing rearrangements: for all $f\in E$ and $g\in E$ we have

$$\int_0^\infty |f(t)g(t)|\,dt \leqslant \int_0^\infty f^*(t)g^*(t)\,dt.$$

The *growth bound* of a semigroup $\mathbf{T}=\{T(t)\}_{t\geqslant 0}$ of bounded linear operators on a Banach space X is the quantity

$$\begin{aligned}\omega_0(\mathbf{T}) &:= \inf\{\omega\in\mathbb{R} : \exists M\geqslant 1 \text{ such that } \|T(t)\|\leqslant Me^{\omega t} \text{ for all } t\geqslant 0\}\\ &= \lim_{t\to\infty}\frac{\ln\|T(t)\|}{t}.\end{aligned}$$

The *spectral bound* of a C_0-semigroup is the quantity $s(A):=\sup\{\operatorname{Re}\lambda:\ \lambda\in\sigma(A)\}$. Here, A is the generator of $\mathbf{T}$ and $\sigma(A)$ is its spectrum. It is easy to see that $s(A)\leqslant\omega_0(\mathbf{T})$. We will need the following proposition, which summarizes [Na, Theorems A-IV.1.4, C-III.1.1, C-IV.1.3]. As usual, $D(A)$ denotes the domain of A.

Proposition 1.1. *Let $\mathbf{T}$ be a positive C_0-semigroup with generator A on a Banach lattice E.*

(i) If $\sigma(A)\neq\varnothing$, then $s(A)\in\sigma(A)$.

(ii) For all $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda > s(A)$, for the resolvent $R(\lambda,A):=(\lambda-A)^{-1}$ we have

$$R(\lambda,A)x = \lim_{\tau\to\infty}\int_0^\tau e^{-\lambda t}T(t)x\,dt, \qquad x\in E,$$

the convergence being in the norm of E. In fact, $s(A)$ is the infimum of all $s\in\mathbb{R}$ such that (1.1) holds for all $\lambda > s$ (and hence for all $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda > s$).

(iii) $s(A)$ is the infimum of all $\lambda\in\mathbb{R}$ such that $\int_0^\infty e^{-\lambda t}\|T(t)f\|\,dt<\infty$ for all $f\in D(A)$.

2. The main results

It will be necessary to consider function spaces over $(0,\infty)$ and $(1,\infty)$ simultaneously. For this purpose, it is useful to introduce the following notation. If $E(0,\infty)$ is a function space over $(0,\infty)$, we define $E(1,\infty)$ to be the set of all functions f on $(1,\infty)$ such that the function $\tilde f$ defined by $\tilde f(s):=f(s+1)$, $s>0$, belongs to $E(0,\infty)$. The space $E(1,\infty)$ is a function space over $(1,\infty)$ with respect to the norm $\|f\|_{E(1,\infty)}:=\|\tilde f\|_{E(0,\infty)}$. If $E(1,\infty)$ is a function space over $(1,\infty)$, one can define a function space over $(0,\infty)$ in the analogous way. If $E(0,\infty)$ and $E(1,\infty)$ are related in this way, one has $\varphi_{E(0,\infty)}(t)=\varphi_{E(1,\infty)}(t)$ for all $t\geqslant 0$. Therefore, we will simply write $\varphi_E(t)$ to denote both numbers. We define the Boyd indices of a function space $E(1,\infty)$ over $(1,\infty)$ to be those of the translated space $E(0,\infty)$ and denote both the indices of $E(1,\infty)$ and $E(0,\infty)$ by $\underline{\alpha}_E$ and $\overline{\alpha}_E$.

Let $E = E(1,\infty)$ be a function space over $(1,\infty)$. We define the semigroup $\mathbf{T}_E$ on E by

$$(T_E(t)f)(s) = f(se^t), \qquad t \geqslant 0, \quad s > 1, \quad f \in E(1,\infty). \tag{2.1}$$

By considering the deceasing rearrangements of f and $T_E(t)f$ it is easy to see that $\|T_E(t)\| \leqslant 1$ for all $t \geqslant 0$. Our first result is an expression for the growth bound of $\mathbf{T}_E$.

Theorem 2.1. $\omega_0(\mathbf{T}_E) = -\underline{\alpha}_E$.

Proof: For $f \in E(1,\infty)$ and $t \geqslant -1$ let $f_{[t]} \in E(0,\infty)$ be defined by

$$f_{[t]}(s) = \begin{cases} f(s-t), & s > t+1; \\ 0, & 0 < s < t+1. \end{cases}$$

For $t \geqslant 0$, $f_{[t]}$ can be identified with an element in $E(1,\infty)$ which will be denoted by $f_{\{t\}}$. Also, for $f \in E(0,\infty)$ and $t \geqslant 0$ we define $f_{(t)} \in E(0,\infty)$ by

$$f_{(t)}(s) = \begin{cases} f(s-t), & s > t; \\ 0, & 0 < s < t. \end{cases}$$

Let $f \in E(1,\infty)$ of norm one be fixed. Since $\|f_{[0]}\|_{E(0,\infty)} = 1$, for all $t \geqslant 0$ we have

$$\|T_E(t)f\|_{E(1,\infty)} = \|f(\cdot\, e^t)\|_{E(1,\infty)} \leqslant \|D_{e^t} f_{[0]}\|_{E(0,\infty)} \leqslant \|D_{e^t}\|_{E(0,\infty)}.$$

Therefore, for all $t \geqslant 0$, $\|T_E(t)\|_{E(1,\infty)} \leqslant \|D_{e^t}\|_{E(0,\infty)}$.

Next, let $f \in E(1,\infty)$ of norm one be arbitrary and fix $t \geqslant 0$. Then, for all $s > 0$ we have $(T_E(t)f_{\{e^t\}})_{[0]}(s) = (D_{e^t} f_{[-1]})_{(1+e^{-t})}(s)$. Since $\|f_{\{e^t\}}\|_{E(1,\infty)} = 1$, it follows that

$$\begin{aligned} \|T_E(t)\|_{E(1,\infty)} &\geqslant \|T_E(t)f_{\{e^t\}}\|_{E(1,\infty)} = \|(T_E(t)f_{\{e^t\}})_{[0]}\|_{E(0,\infty)} \\ &= \|(D_{e^t} f_{[-1]})_{(1+e^{-t})}\|_{E(0,\infty)} = \|D_{e^t} f_{[-1]}\|_{E(0,\infty)}. \end{aligned}$$

If f ranges over all norm one functions in $E(1,\infty)$, then $f_{[-1]}$ ranges over all norm one functions in $E(0,\infty)$ and therefore it follows that $\|T_E(t)\|_{E(1,\infty)} \geqslant \|D_{e^t}\|_{E(0,\infty)}$.

We have proved that $\|T_E(t)\|_{E(1,\infty)} = \|D_{e^t}\|_{E(0,\infty)}$. Therefore,

$$\frac{1}{t}\ln\|T_E(t)\|_{E(1,\infty)} = \frac{1}{t}\ln\|D_{e^t}\|_{E(0,\infty)} = \frac{1}{t}\ln h_{E(0,\infty)}(e^{-t}) = \frac{\ln h_{E(0,\infty)}(e^{-t})}{-\ln e^{-t}}.$$

Letting $t \to \infty$, by (1.2) we obtain $\omega_0(\mathbf{T}_E) = -\underline{\alpha}_E$. ////

We are now going to investigate the spectral bound of the generator A_E of $\mathbf{T}_E$. In order to be able to define the generator, we first need to make sure that $\mathbf{T}_E$ is strongly continuous. The following lemma gives a sufficient condition for this. A *simple function* is a finite linear combination of characteristic functions of sets of finite measure.

Lemma 2.2. *Let $E = E(1,\infty)$ be a function space over $(1,\infty)$ and assume that $\varphi_E(0+) = 0$. Then the restriction $\mathbf{T}_{E_0}$ of $\mathbf{T}_E$ to the closure E_0 of the simple functions in E is strongly continuous. Let A_{E_0} denote its generator. For $\lambda > s(A_{E_0})$, the resolvent is given by*

$$(R(\lambda, A_{E_0})f)(s) = s^{\lambda} \int_s^{\infty} t^{-\lambda-1} f(t)\,dt, \qquad f \in E_0, \quad \text{a.a. } s > 1. \tag{2.2}$$

Proof: Since $\varphi_E(0+) = 0$, it is not difficult to see that $\mathbf{T}$ acts in a strongly continuous way on each characteristic function of a set of finite measure. Hence the restriction of $\mathbf{T}_E$ to E_0 is strongly continuous. The formula (2.2) follows by a straightforward calculation from Proposition 1.1 and (2.1), using the fact that norm convergent sequences have pointwise a.e. convergent subsequences. ////

In particular, $\mathbf{T}_E$ is strongly continuous on E if $\varphi_E(0+) = 0$ and the simple functions are dense in E. This is the case whenever E has order continuous norm. For this reason, from now on all function spaces will be assumed to have order continuous norm. This is merely a matter of convenience; all results to follow are valid without this assumption provided $\varphi_E(0+) = 0$ and E is replaced by E_0.

The next result gives a lower bound for $s(A_E)$.

Theorem 2.3. *Let $E = E(1,\infty)$ be a function space over $(1,\infty)$ with order continuous norm. If there exist constants $0 \leqslant \beta \leqslant 1$ and $C > 0$ such that $\varphi_E(t) \leqslant Ct^{\beta}$ for all $t \geqslant 1$, then $s(A_E) \geqslant -\beta$.*

Proof: Let $\alpha > \beta$ be arbitrary. We claim that the function f_α defined by $f_\alpha(s) = s^{-\alpha}$, $s > 1$, defines an element of E. Indeed, by estimating f_α on the interval $(2^n, 2^{n+1})$ by $2^{-n\alpha}$, for all $M > N \in \mathbb{N}$ we have

$$\|f_\alpha|_{(1,2^M)} - f_\alpha|_{(1,2^N)}\|_E \leqslant \sum_{n=N}^{M-1} 2^{-n\alpha} \cdot \varphi_E(2^{n+1} - 2^n) \leqslant \sum_{n=N}^{\infty} 2^{-n\alpha} \cdot C 2^{n\beta} = C \frac{2^{-N(\alpha-\beta)}}{1 - 2^{\beta-\alpha}}.$$

Therefore, the sequence $(f_\alpha|_{(1,2^N)})_{N\in\mathbb{N}}$ is Cauchy in E. Since norm convergent sequences have pointwise a.e. convergent subsequences, the limit must be the function f_α. Therefore, $f_\alpha \in E$. By (2.2), for $\lambda > 0$ we have

$$(R(\lambda, A)f_\alpha)(s) = s^{\lambda} \int_s^{\infty} t^{-\lambda-\alpha-1}\,dt = \frac{1}{\lambda+\alpha} f_\alpha(s), \qquad \text{a.a. } s > 1.$$

It follows that $f_\alpha \in D(A_E)$ and $A_E f_\alpha = -\alpha f_\alpha$, so $-\alpha$ is an eigenvalue of A_E. Hence, $s(A_E) \geqslant -\alpha$. Since $\alpha > \beta$ is arbitrary, it follows that $s(A_E) \geqslant -\beta$. ////

In particular, it follows that $s(A_E) \geqslant -\overline{\alpha}_E$. Indeed, it is an easy consequence of the definition of $\overline{\alpha}_E$ that for all $\alpha > \overline{\alpha}_E$ there exists a constant C such that $\varphi_E(t) \leqslant Ct^{\alpha}$ for all $t \geqslant 1$. Combining this with Theorem 2.1, we see that

$$-\overline{\alpha}_E \leqslant s(A_E) \leqslant \omega_0(\mathbf{T}_E) = -\underline{\alpha}_E.$$

The Boyd indices of $L^p \cap L^q$, $1 \leqslant p \leqslant q \leqslant \infty$, are given by $\underline{\alpha}_{L^p\cap L^q} = \frac{1}{q}$ and $\overline{\alpha}_{L^p\cap L^q} = \frac{1}{p}$. Theorem 2.1 and the fact that $s(A_{L^p\cap L^q}) \leqslant -\frac{1}{p} \leqslant -\frac{1}{q} \leqslant \omega_0(\mathbf{T}_{L^p\cap L^q})$ might suggest that in general one has $s(A_E) = -\overline{\alpha}_E$. This need not be the case, as is shown by the following example.

Example 2.4. Let $E = L^p(1,\infty) + L^q(1,\infty)$, where $1 \leqslant p \leqslant q < \infty$. It is well-known that $\underline{\alpha}_{L^p+L^q} = \frac{1}{q}$ and $\overline{\alpha}_{L^p+L^q} = \frac{1}{p}$. On the other hand, $\varphi_{L^p+L^q}(t) = t^{\frac{1}{q}}$ for all $t \geqslant 1$. From Theorems 2.1 and 2.3 it follows that

$$-\frac{1}{q} \leqslant s(A_{L^p+L^q}) \leqslant \omega_0(\mathbf{T}_{L^p+L^q}) = -\underline{\alpha}_{L^p+L^q} = \frac{1}{q},$$

and therefore $s(A_{L^p+L^q}) = \omega_0(\mathbf{T}_{L^p+L^q}) = -\frac{1}{q}$.

Thus, the problem of giving an estimate from above for $s(A_E)$ is rather subtle.

In order to motivate the next result, let us make the following observation. Proposition 1.1 shows that in order to find an estimate for $s(A_E)$, we have to find exponential growth bounds for the maps $t \mapsto T_E(t)f$ with $f \in D(A_E)$. Now if f is compactly supported, then $T_E(t)f = 0$ for all sufficiently large t. Therefore, we are free to redefine an arbitrary $f \in D(A_E)$ on a compact interval. We will see that by the above lemma we may also assume that f is non-increasing. So, without loss of generality, we may assume that f is non-increasing and constant on the interval $(1,2)$. Then, f is bounded and its decreasing rearrangement satisfies $f^* \geqslant \|f\|_\infty \chi_{(0,1)}$.

Theorem 2.5. *Let $E = E(1,\infty)$ be a function space over $(1,\infty)$ with order continuous norm. Assume there exist constants $0 \leqslant \beta \leqslant 1$ and $C > 0$ such that for all $f \in E(0,\infty) \cap L^\infty(0,\infty)$ satisfying $f^* \geqslant \|f\|_\infty \chi_{(0,1)}$ we have*

$$\|D_{\frac{1}{s}} f\|_{E(0,\infty)} \geqslant C s^\beta \|f\|_{E(0,\infty)}, \qquad \forall s \geqslant 1. \tag{2.3}$$

Then $s(A_E) \leqslant -\beta$.

Proof: Fix $f \in D(A_E)$ and $\lambda > s(A_E)$ arbitrary and let $g = (\lambda - A_E)f$. Let $\tilde{g}$ be the function on $(1,\infty)$ defined by $\tilde{g}(s) = g^*(s-1)$, $s > 1$, and let $\tilde{f} = R(\lambda, A_E)\tilde{g}$. By the inequality of Hardy and Littlewood and (2.2),

$$0 \leqslant |f| \leqslant R(\lambda, A_E)|g| \leqslant R(\lambda, A_E)\tilde{g} \leqslant \tilde{f}. \tag{2.4}$$

Moreover, $\tilde{f}$ is non-increasing on $(1,\infty)$ by (2.2) and the identity

$$s^\lambda \int_s^\infty t^{-\lambda-1} \tilde{f}(t)\, dt = \int_1^\infty t^{-\lambda-1} \tilde{f}(st)\, dt.$$

Fix $t > 0$ and put $\varepsilon_t = \operatorname{essinf}_{1<s<e^t} \tilde{f}(s)$. Then, $(\tilde{f} - \tilde{f} \wedge \varepsilon_t)(s) = 0$ for almost all $s > e^t$ and therefore

$$T_E(t)(\tilde{f} - \tilde{f} \wedge \varepsilon_t) = 0. \tag{2.5}$$

Also, we have $\tilde{f} \wedge \varepsilon_t \geqslant \varepsilon_t \chi_{(1,e^t)} = \|\tilde{f} \wedge \varepsilon_t\|_\infty \chi_{(1,e^t)}$. Therefore,

$$(\tilde{f} \wedge \varepsilon_t)^* \geqslant \|\tilde{f} \wedge \varepsilon_t\|_\infty \chi_{(0,e^t-1)}.$$

Let $h_t := D_{e^t-1}((\tilde{f} \wedge \varepsilon_t)^*)$. Then, $h_t \in E(0,\infty) \cap L^\infty(0,\infty)$ and $(h_t)^* = h_t \geqslant \|h_t\|_\infty \chi_{(0,1)}$. If $t \geqslant \ln 2$, then $e^t - 1 \geqslant 1$ and (2.3) implies

$$\begin{aligned}
\|(\tilde{f} \wedge \varepsilon_t)^*\|_{E(0,\infty)} &= \|D_{(e^t-1)^{-1}} h_t\| \\
&\geqslant C(e^t-1)^\beta \|h_t\|_{E(0,\infty)} \\
&= C(e^t-1)^\beta \|D_{e^t-1}((\tilde{f} \wedge \varepsilon_t)^*)\|_{E(0,\infty)}.
\end{aligned}$$

Hence, if $t \geqslant \ln 2$, using (2.4), (2.5), and the notation of Theorem 2.1, we have

$$\begin{aligned} \|T_E(t)f\|_E &\leqslant \|T_E(t)\tilde{f}\|_E = \|T_E(t)(\tilde{f} \wedge \varepsilon_t)\|_E \\ &\leqslant \|D_{e^t}((\tilde{f} \wedge \varepsilon_t)_{[0]})\|_{E(0,\infty)} = \|D_{e^t}((\tilde{f} \wedge \varepsilon_t)^*)\|_{E(0,\infty)} \\ &\leqslant \|D_{e^t-1}((\tilde{f} \wedge \varepsilon_t)^*)\|_{E(0,\infty)} \leqslant C^{-1}(e^t-1)^{-\beta}\|(\tilde{f} \wedge \varepsilon_t)^*\|_{E(0,\infty)} \\ &= C^{-1}(e^t-1)^{-\beta}\|\tilde{f} \wedge \varepsilon_t\|_E \leqslant C^{-1}(e^t-1)^{-\beta}\|\tilde{f}\|_E. \end{aligned} \tag{2.6}$$

So far, $t \geqslant \ln 2$ was fixed. Since the constant C in (2.6) does not depend on t, it follows that for all $\lambda > -\beta$,

$$\int_0^\infty e^{-\lambda t}\|T_E(t)f\|_E\,dt < \infty.$$

Since $f \in D(A_E)$ is arbitrary, by Proposition 1.1 we obtain $s(A_E) \leqslant -\beta$. ////

The final result of this section shows the relevance of working in spaces over $(1,\infty)$:

Theorem 2.6. *Let $E = E(0,\infty)$ be a function space over $(0,\infty)$ with order continuous norm. Then the semigroup $\mathbf{U}_E$ defined by $(U_E(t)f)(s) = f(se^t)$, $s > 0$, is a strongly continuous semigroup on E. Let B_E denote its generator. Then, $s(B_E) = \omega_0(\mathbf{U}_E) = -\underline{\alpha}_E$.*

Proof: Strong continuity is proved as in Lemma 2.2. Since $U_E(t) = D_{e^t}$ for all $t \geqslant 0$, we have $\|U_E(t)\| = \|D_{e^t}\| = h_E(e^{-t})$ and hence $\omega_0(\mathbf{U}_E) = \lim_{t\to\infty} t^{-1}\ln h_E(e^{-t}) = -\underline{\alpha}_E$. Next, as in Lemma 2.2 we have $(R(\lambda, B_E)f)(s) = s^\lambda \int_s^\infty t^{-\lambda-1}f(t)\,dt$ for all $f \in E$ and almost all $s > 0$. Since order continuous norms have the Fatou property, the proof of [BS, Thm. 3.5.15] shows that for real λ the right hand side defines a bounded operator on E if and only if $\lambda > -\underline{\alpha}_E$. Therefore, $s(B_E) = -\underline{\alpha}_E$ by Proposition 1.1. ////

3. Intersection of function spaces

In this section, we will consider two function spaces E and F and give abstract conditions implying that spectral bound and growth bound of $\mathbf{T}_{E\cap F}$ are different.

Theorem 3.1. *Let $E = E(1,\infty)$ and $F = F(1,\infty)$ be function spaces over $(1,\infty)$ with order continuous norms and assume there is an inclusion $E \cap L^\infty \subset F$. Then $-\overline{\alpha}_E \leqslant s(A_{E\cap F}) \leqslant -\underline{\alpha}_E$.*

Proof: Note that $E \cap F$ has order continuous norm since E and F have.

Let $f \in E(0,\infty) \cap F(0,\infty) \cap L^\infty(0,\infty)$ be such that $f^* \geqslant \|f\|_\infty \chi_{(0,1)}$. Then $\|f\|_{E(0,\infty)} \geqslant \varphi_E(1)\|f\|_\infty$ and hence

$$\|f\|_{F(0,\infty)} \leqslant K\max\{\|f\|_{E(0,\infty)}, \|f\|_\infty\} \leqslant K\max\{1, (\varphi_E(1))^{-1}\}\|f\|_{E(0,\infty)}, \tag{3.1}$$

where K is the norm of the inclusion map $E(0,\infty) \cap L^\infty(0,\infty) \to F(0,\infty)$. Applying this to $f = \chi_{(0,t)}$ with $t \geqslant 1$ shows that $\varphi_{E\cap F}(t) \leqslant C\varphi_E(t)$ for some C and all $t \geqslant 1$. Therefore, by Theorem 2.3 and the remark following it, $s(A_{E\cap F}) \geqslant -\overline{\alpha}_E$.

Next, we claim that for each $\alpha < \underline{\alpha}_E$ there is a constant $C > 0$ such that for all $g \in E(0,\infty)$,

$$\|D_{\frac{1}{s}}g\|_{E(0,\infty)} \geqslant Cs^\alpha\|g\|_{E(0,\infty)}, \qquad \forall s \geqslant 1.$$

The definition of $\underline{\alpha}_E$ implies that for all $t > 0$ small enough, $\|D_{\frac{1}{t}}\|_{E(0,\infty)} \leqslant t^{\alpha}$. Hence, $\|D_{\frac{1}{t}}\|_{E(0,\infty)} \leqslant ct^{\alpha}$ for some constant c and all $0 < t \leqslant 1$. With $s = t^{-1}$, it follows that for all $g \in E(0,\infty)$ and all $s \geqslant 1$,

$$\|g\|_{E(0,\infty)} = \|D_s(D_{\frac{1}{s}}g)\|_{E(0,\infty)} \leqslant \frac{c}{s^{\alpha}}\|D_{\frac{1}{s}}g\|_{E(0,\infty)}.$$

This proves the claim.

Let $\varepsilon > 0$ be arbitrary. By (3.1) and the claim, for all $s \geqslant 1$ we have

$$\begin{aligned}\|D_{\frac{1}{s}}f\|_{E(0,\infty)\cap F(0,\infty)} &= \max\{\|D_{\frac{1}{s}}f\|_{E(0,\infty)}, \|D_{\frac{1}{s}}f\|_{F(0,\infty)}\}\\ &\geqslant Cs^{\underline{\alpha}_E-\varepsilon}\|f\|_{E(0,\infty)}\\ &\geqslant C\min\{1, K^{-1}, K^{-1}\varphi_E(1)\}s^{\underline{\alpha}_E-\varepsilon}\|f\|_{E(0,\infty)\cap F(0,\infty)},\end{aligned}$$

where C is a constant only depending on ε and E. Therefore, by Theorem 2.5, $s(A_{E\cap F}) \leqslant -\underline{\alpha}_E + \varepsilon$. Since ε is arbitrary, the proof is complete. ////

Since $L^p \cap L^{\infty} \subset L^q$ if $1 \leqslant p \leqslant q < \infty$ with continuous inclusion, Theorem 3.1 implies $s(A_{L^p\cap L^q}) \leqslant -\frac{1}{p}$. Thus, Arendt's example is a special case of Theorems 2.1 and 3.1. In fact, we have $s(A_{L^p\cap L^q}) = -\frac{1}{p}$ by Theorem 2.3.

Corollary 3.2. *Let $E = E(1,\infty)$ and $F = F(1,\infty)$ be function spaces over $(1,\infty)$ with order continuous norms and assume that there is a continuous inclusion $E \cap L^{\infty} \subset F$. If $\overline{\alpha}_F \leqslant \underline{\alpha}_E$, then $s(A_{E\cap F}) \leqslant -\underline{\alpha}_E \leqslant -\overline{\alpha}_F \leqslant \omega_0(\mathbf{T}_{E\cap F})$.*

Proof: In view of Theorems 2.1 and 3.1, all that remains to be shown is that $\overline{\alpha}_F \geqslant \underline{\alpha}_{E\cap F}$.

By definition of $\overline{\alpha}_F$, for each $\alpha > \overline{\alpha}_F$ there is a constant $C > 0$ such that $\|D_{\frac{1}{t}}\|_{F(0,\infty)} \leqslant Ct^{\alpha}$ for all $t \geqslant 1$. Therefore, for all $f \in E(0,\infty) \cap F(0,\infty)$ and $t \geqslant 1$,

$$\|D_t f\|_{E(0,\infty)\cap F(0,\infty)} \geqslant \|D_t f\|_{F(0,\infty)} \geqslant C^{-1}t^{-\alpha}\|D_{\frac{1}{t}}(D_t f)\|_{F(0,\infty)} = C^{-1}t^{-\alpha}\|f\|_{F(0,\infty)}.$$

In particular, applying this to $f = \chi_{(0,1)} \in E(0,\infty) \cap F(0,\infty)$ shows that

$$\|D_t\|_{E(0,\infty)\cap F(0,\infty)} \geqslant \frac{\varphi_F(1)}{C\max\{\varphi_E(1), \varphi_F(1)\}}t^{-\alpha}$$

for all $t \geqslant 1$. The definition of $\underline{\alpha}_{E\cap F}$ implies that $\underline{\alpha}_{E\cap F} \leqslant \alpha$. ////

In particular, if $\overline{\alpha}_F < \underline{\alpha}_E$, then $s(A_{E\cap F}) < \omega_0(\mathbf{T}_{E\cap F})$.

4. Application to Orlicz spaces

In this section, we apply our abstract results to construct Orlicz space $E_{\Phi} = E_{\Phi}(1,\infty)$ over $(1,\infty)$ in which we have $s(A_{E_{\Phi}}) < \omega_0(\mathbf{T}_{E_{\Phi}})$. The idea is motivated by the example in $L^p \cap L^q$: the Young function Φ should reflect the essential properties of the fundamental function of $L^p \cap L^q$.

For the general theory of Orlicz spaces we refer to the book [Z]. Let $\Phi : (0,\infty) \to [0,\infty]$ be a Young function, i.e. $\Phi(t) = \int_0^t \phi(s)\,ds$ for some non-decreasing function $\phi : (0,\infty) \to [0,\infty]$ not identically 0 or ∞. We denote by E_Φ be the Orlicz space over $(0,\infty)$ corresponding to Φ. Explicitly, E_Φ is the space of all measurable functions f on $(0,\infty)$ such that

$$\|f\| := \|f\|_{E_\Phi} := \inf\left\{k \geqslant 0 : \int_0^\infty \Phi(k^{-1}f^*(t))\,dt \leqslant 1\right\} < \infty. \tag{4.1}$$

With the norm $\|\cdot\|$, the space E_Φ is a rearrangement invariant Banach function space over $(0,\infty)$. If $0 < \phi(t) < \infty$ for all $t > 0$, then Φ is strictly increasing and real-valued, and hence its inverse Φ^{-1} is well-defined. The fundamental function E_Φ is then given by $\varphi_{E_\Phi}(t) = \left(\Phi^{-1}\left(\frac{1}{t}\right)\right)^{-1}$.

A Young function Φ is said to satisfy a Δ_2-*condition* if there exists a constant $K > 0$ such that $\Phi(2t) \leqslant K\Phi(t)$ for all $t \geqslant 0$. An Orlicz space satisfying a Δ_2-condition has order continuous norm [Z, Thm. 133.3].

Let $1 \leqslant p < q < \infty$ and let $\Phi : (0,\infty) \to (0,\infty)$ be any Young function such that:

(i) $\Phi(t/s) \geqslant cs^{-p}\Phi(t)$ for all $0 < t \leqslant 1$, all $s \geqslant 1$, and some $c > 0$;
(ii) $\Phi(t) \geqslant Ct^q$ for all $t \geqslant 1$ and some $C > 0$.
(iii) Φ satisfies a Δ_2-condition.

Note that (i) and (ii) imply that $0 < \phi(t) < \infty$ for all $t > 0$. By multiplying Φ with an appropriate positive real number, we may furthermore assume that $\Phi(1) = 1$ and hence $\varphi_{E_\Phi}(1) = 1$.

Since the norm of E_Φ is defined in terms of decreasing rearrangements, (4.1) also gives the norm of the translated space $E_{\Phi_{p,q}}(1,\infty)$ over $(1,\infty)$. In that space, we have the following result.

Theorem 4.1. *Under the above assumptions, the semigroup $\mathbf{T}_{E_\Phi}$ is strongly continuous on $E_\Phi(1,\infty)$ and satisfies*

$$s(A_{E_\Phi}) \leqslant -\frac{1}{p} < -\frac{1}{q} \leqslant \omega_0(\mathbf{T}_{E_\Phi}).$$

Proof: Strong continuity follows from Lemma 2.2 and the fact that E_Φ has order continuous norm.

First we prove the inequality concerning the growth bound. Rather than calculating the lower Boyd index, we argue directly as follows. For all $t > 0$ sufficiently large we have

$$\|T_{E_\Phi}(t)\| \geqslant \|T_{E_\Phi}(t)\chi_{(e^t,e^t+1)}\| = \|\chi_{(1,1+e^{-t})}\| = \varphi_{E_\Phi}(e^{-t}) = \left(\Phi^{-1}(e^t)\right)^{-1} \geqslant C^{-1}e^{-\frac{t}{q}}.$$

Therefore,

$$\omega_0(\mathbf{T}_{E_\Phi}) = \lim_{t\to\infty} \frac{\ln\|T_{E_\Phi}(t)\|}{t} \geqslant -\frac{1}{q}.$$

For the inequality concerning to spectral bound, we verify the conditions of Theorem 2.5 for $\beta = \frac{1}{p}$. Let $f \in E_\Phi(0,\infty) \cap L^\infty(0,\infty)$ satisfy $f^* \geqslant \|f\|_\infty \chi_{(0,1)}$. Noting that $\|f\| \geqslant \|f\|_\infty \varphi_{E_\Phi}(1) = \|f\|_\infty$, we have $f^*(t) \leqslant \|f\|$ for all $t > 0$. Therefore, for all $s \geqslant 1$ we have $f^*(t) \leqslant s^{\frac{1}{p}}\|f\|$ for all $t > 0$, and

$$\int_0^\infty \Phi\left(\frac{f^*(t)}{s^{\frac{1}{p}}\|f\|}\right) dt \geqslant \frac{c}{s}\int_0^\infty \Phi\left(\frac{f^*(t)}{\|f\|}\right) dt.$$

But it is an easy consequence of the Δ_2-condition (cf. [Z, Thm 131.8]) that

$$\int_0^\infty \Phi\left(\frac{f^*(t)}{\|f\|}\right) dt = 1.$$

Therefore,

$$\int_0^\infty \Phi\left(\frac{(D_{\frac{1}{s}} f^*)(t)}{s^{\frac{1}{p}}\|f\|}\right) dt = s\int_0^\infty \Phi\left(\frac{f^*(t)}{s^{\frac{1}{p}}\|f\|}\right) dt \geqslant c.$$

Choose $n \in \mathbb{N}$ so large that $2^n c \geqslant 1$. Noting that

$$\Phi(2t) = \int_0^{2t} \phi(s)\, ds \geqslant 2\int_0^t \phi(s)\, ds = 2\Phi(t)$$

since ϕ is non-decreasing, it follows that

$$\int_0^\infty \Phi\left(\frac{2^n(D_{\frac{1}{s}} f^*)(t)}{s^{\frac{1}{p}}\|f\|}\right) dt \geqslant 2^n c \geqslant 1.$$

By the definition of $\|\cdot\|$, it follows that $\|D_{\frac{1}{s}} f\| \geqslant 2^{-n} s^{\frac{1}{p}}\|f\|$ for all $s \geqslant 1$. The desired inequality $s(A_{E_\Phi}) \leqslant -\frac{1}{p}$ now follows from Theorem 2.5. ////

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