On the stochastic Fubini theorem in infinite dimensions

JAN VAN NEERVEN* and MARK VERAAR
Delft Institute of Applied Mathematics, Technical University of Delft
P.O. Box 5031, 2600 GA Delft, The Netherlands

Abstract

Noting that every L^1-space satisfies a randomized version of the UMD property, we show how a general stochastic Fubini theorem for the stochastic integral of operator-valued processes with respect to cylindrical Brownian motions can be obtained as an application of the theory of stochastic integration developed recently by Lutz Weis and the authors.

1 Introduction

The stochastic Fubini theorem and stochastic processes indexed by a parameter have been studied by many authors, cf. [1, 4, 5, 8, 12, 19, 22]. A general version of the stochastic Fubini theorem, valid for real-valued semimartingales as integrators, is due to Doléans-Dade [8] and Jacod [12, Théorème 5.44]. Roughly speaking it can be formulated as follows. Let (S, Σ, μ) be a σ-finite measure space, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and let $\phi : S \times [0, T] \times \Omega \to \mathbb{R}$ be $\Sigma \otimes \mathscr{B}([0, T]) \otimes \mathscr{F}$-measurable. If $(t, \omega) \mapsto \phi_s(t, \omega) := \phi(s, t, \omega)$ is integrable with respect to a semimartingale X for all $s \in S$, if the process $(t, \omega) \mapsto \int_S \phi_s(t)\, d\mu(s)$ is well defined and integrable with respect to X, and if

$$\int_S \Big| \int_0^T \phi_s(t)\, dX(t) \Big|\, d\mu(s) < \infty \quad \text{almost surely,} \tag{1.1}$$

then, almost surely,

$$\int_S \int_0^T \phi_s(t)\, dX(t)\, d\mu(s) = \int_0^T \int_S \phi_s(t)\, d\mu(s)\, dX(t).$$

Motivated by applications to stochastic differential equations in infinite dimensions, it is desirable to have a version of the stochastic Fubini theorem for integrals of operator-valued processes with respect to cylindrical Hilbert space-valued semimartingales. Generalizing an earlier result of Chojnowska-Michalik [4], a stochastic Fubini theorem for $\mathscr{L}(H, H')$-valued processes with respect to H-cylindrical Brownian motions W_H was proved by Da Prato and Zabczyk [5]. Here H and H' are separable real Hilbert spaces. In this result the condition (1.1) is replaced by the condition

$$\phi \in L^1(S; L^2((0, T) \times \Omega; S; \mathscr{L}_2(H, H'))), \tag{1.2}$$

where $\mathscr{L}_2(H, H')$ denotes the Hilbert-Schmidt operators from H into H'.

The purpose of this paper is to prove a stochastic Fubini theorem for integration of $\mathscr{L}(H, E)$-valued processes with respect H-cylindrical Brownian motions under assumptions analogous to (1.1) but which may be easier to verify in concrete applications. Here, E is assumed to be a real Banach space. Since the special case $E = \mathbb{R}$ already exhibits all main ideas, we have written out our results in detail for H-valued processes only; here we identify

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$\mathscr{L}(H,\mathbb{R})$ with H. The extension to $\mathscr{L}(H,E)$-valued processes is sketched at the end of the paper. It turns out that condition (1.2) may be weakened to

$$\phi \in L^1(S; L^2(0,T;\mathscr{L}_2(H,H'))) \text{ almost surely.}$$

Our approach to the stochastic Fubini theorem is based on a straightforward extension of the theory of stochastic integration developed recently by Lutz Weis and the authors in [15] to processes with values in a UMD$^-$ space, together with the basic fact that L^1-spaces possess the UMD$^-$ property. The idea is to interpret the stochastic integral parametrized by S as a stochastic integral in the Banach space $L^1(S)$. The essence of the stochastic Fubini theorem is then nothing but the statement that a bounded linear functional may be moved into the stochastic integral:

$$\begin{aligned}\int_S \int_0^T \phi_s(t)\,dW_H(t)\,d\mu(s) &= \Big\langle \int_0^T \phi_{(\cdot)}(t)\,dW_H(t), \mathbf{1}\Big\rangle_{\langle L^1(S),L^\infty(S)\rangle} \\ &= \int_0^T \langle \phi_{(\cdot)}(t), \mathbf{1}\rangle_{\langle L^1(S;H),L^\infty(S)\rangle}\,dW_H(t) = \int_0^T \int_S \phi_s(t)\,d\mu(s)\,dW_H(t).\end{aligned}$$

In order to develop this simple idea in a rigorous way, some measurability problems have to be overcome. The main difficulty consists of lifting measurability properties of ϕ that hold pointwise in s to the corresponding $L^1(S)$-valued functions. This problem is discussed in Section 2. The main results of the paper are contained in Section 3.

In a forthcoming paper, the results of this paper will be applied to study stochastic evolution equations.

2 Measure theoretical preliminaries

Let (S,Σ) be a measurable space and let (Y,d) be a complete metric space. A function $\phi : S \to Y$ is called Σ-*simple* if it is of the form $\phi = \sum_{n=1}^N \mathbf{1}_{A_n} \otimes y_n$ with $A_n \in \Sigma$ and $y_n \in Y$ for $n = 1,\dots,N$. countably valued Σ-simple functions are defined similarly. A function $\phi : S \to Y$ is called *strongly* Σ-*measurable* if it is the pointwise limit in Y of a sequence of Σ-simple functions. It is well-known [18, Lemma V-2-4] that a function $\phi : S \to Y$ is strongly Σ-measurable if and only if the following two conditions are satisfied:

(i) the range of ϕ is separable;

(ii) we have $\phi^{-1}(B) \in \Sigma$ for all Borel sets B in Y.

This implies that the pointwise limit of a sequence of strongly Σ-measurable functions is strongly Σ-measurable again.

By covering the range of a strongly Σ-measurable function ϕ with countably many balls B_j^n with radius $\frac{1}{n}$ and centre y_j^n, and defining ϕ_n to have the constant value y_j^n on the set $\phi^{-1}(B_k^n \setminus \bigcup_{j<k} B_j^n)$ we obtain a countably valued Σ-simple function $\phi_n : S \to Y$ such that $\sup_{s\in S} d(\phi_n(s),\phi(s)) \leqslant \frac{1}{n}$. Thus every strongly Σ-measurable function is the uniform limit of a sequence of countably valued Σ-simple functions.

As was mentioned in the introduction, it will be important to lift measurability properties of a process indexed by a parameter s to the corresponding $L^1(S)$-valued process. This problem is easily reduced to the following abstract question:

> If $(\Omega,\mathscr{F})$ is a measurable space, E is a Banach space, and $\phi : S\times\Omega \to E$ is a $\Sigma\otimes\mathscr{F}$-measurable function with the property that all sections ϕ_s are strongly $\mathscr{G}$-measurable, where $\mathscr{G}$ is some sub-σ-algebra of $\mathscr{F}$, does it follows that ϕ is strongly $\Sigma\otimes\mathscr{G}$-measurable?

In general the answer is negative even for indicator functions (cf. the example below). On the other hand, the answer is 'almost positive' if $(\Omega,\mathscr{F},\nu)$ is a σ-finite measure space, in the sense that ϕ has a modification which does have the required properties. We call two functions $\phi : S\times\Omega \to E$ and $\tilde{\phi} : S\times\Omega\to E$ *modifications* of each other if for all $s\in S$ we have $\phi(s,\omega) = \tilde{\phi}(s,\omega)$ for ν-almost all $\omega\in\Omega$.

Proposition 2.1 *Let (S,Σ) and $(\Omega,\mathscr{F},\nu)$ be as above, let $\mathscr{G}$ be a sub-σ-algebra of $\mathscr{F}$, and let E be a Banach space. If $\phi : S\times\Omega\to E$ is a strongly $\Sigma\otimes\mathscr{F}$-measurable function with the property that for all $s\in S$ the function ϕ_s is strongly $\mathscr{G}$-measurable, then ϕ admits a strongly $\Sigma\otimes\mathscr{G}$-measurable modification.*

We may assume without loss of generality that $\nu(\Omega)<\infty$. In fact, if $\nu(\Omega)=\infty$, pick sets $\Omega_n\in\mathscr{F}$ such that $\mu(\Omega_n)>0$ is strictly increasing with n and $\bigcup_{n\geqslant 1}\Omega_n=\Omega$. Put $Y_1:=\Omega_1$ and $Y_{n+1}=\Omega_{n+1}\setminus\Omega_n$ for $n\geqslant 1$, and define

$$\tilde{\nu}(A):=\sum_{n\geqslant 1}\frac{1}{2^n}\frac{\nu(A\cap Y_n)}{\nu(Y_n)},\quad A\in\mathscr{F}.$$

Then $\tilde{\nu}$ is a probability measure on $(\Omega,\mathscr{F})$ which has the same null sets as ν, and therefore we may replace ν by $\tilde{\nu}$ in Proposition 2.1.

From now on we assume that $\nu(\Omega)<\infty$. We denote by $L^0(\Omega;E)$ the space of strongly $\mathscr{F}$-measurable functions, identifying functions that are equal ν-almost everywhere. This is a complete metric space with respect to the translation invariant metric $\|\cdot\|_0$ defined by

$$\|f\|_0:=\int_\Omega \|f(\omega)\|\wedge 1\,d\nu(\omega).$$

A sequence in $L^0(\Omega;E)$ converges in the metric $\|\cdot\|_0$ if and only if it converges in ν-measure. If $\mathscr{G}$ is a sub-σ-algebra of $\mathscr{F}$, we denote by $L^0(\Omega,\mathscr{G};E)$ the closed subspace of $L^0(\Omega;E)$ consisting of all strongly $\mathscr{G}$-measurable functions, identifying again functions that are equal ν-almost everywhere.

For a sequence $(f_n)_{n\geqslant 1}$ in $L^0(\Omega;E)$ and $a:=(a_n)_{n\geqslant 1}\in l^1$, we make the following observation: if $\|f_n\|_0\leqslant a_n$ for all $n\geqslant 1$, then $\lim_{n\to\infty}f_n=0$ ν-almost everywhere. Indeed, define $g:\Omega\to[0,\infty]$ by $g(\omega):=\sum_{n\geqslant 1}\|f_n(\omega)\|\wedge 1$. We have

$$\int_\Omega g(\omega)\,d\nu(\omega)=\sum_{n\geqslant 1}\|f_n\|_0=\|a\|_{l^1}<\infty.$$

Hence g is ν-almost everywhere finite and the claim follows. The proof of the proposition follows the proof of the celebrated result of Dellacherie and Meyer on the existence of a progressively measurable version of adapted measurable processes [6, Theorem IV.30] with some simplifications due to the absence of a filtration, and is included for the reader's convenience.

Proof of Proposition 2.1. Assume that $\nu(\Omega)<\infty$. It follows from the Fubini theorem that for all $s\in S$, $\phi(s,\cdot)$ is a strongly $\mathscr{F}$-measurable function, so we may define $\psi:S\to L^0(\Omega;E)$ as $(\psi(s))(\omega):=\phi(s,\omega)$. We claim that ψ is strongly Σ-measurable. By a monotone class argument we can find a sequence of $\Sigma\otimes\mathscr{F}$-simple functions $\phi_n:S\times\Omega\to E$, each of which is a finite linear combination of functions of the form $\mathbf{1}_{A\times F}\otimes x$ with $A\in\Sigma$, $F\in\mathscr{F}$, $x\in E$, such that $\phi=\lim_{n\to\infty}\phi_n$ pointwise on $S\times\Omega$. Define $\psi_n:S\to L^0(\Omega;E)$ as $(\psi_n(s))(\omega):=\phi_n(s,\omega)$. Then each ψ_n is a Σ-simple function and for all $s\in S$ we have $\psi(s)=\lim_{n\to\infty}\psi_n(s)$ in $L^0(\Omega;E)$. This proves the claim.

Choose a sequence of countably valued Σ-simple functions $\eta_n:S\to L^0(\Omega;E)$, say

$$\eta_n(s)=\sum_k \mathbf{1}_{A_k^n}(s)h_k^n$$

with $A_k^n\in\Sigma$ and $h_k^n:\Omega\to E$ strongly $\mathscr{F}$-measurable, such that for all $s\in S$ we have

$$\|\psi(s)-\eta_n(s)\|_0\leqslant 2^{-n}.$$

For $n,k\geqslant 1$ let $s_k^n\in A_k^n$ be arbitrary and fixed. Then $\|\psi(s_k^n)-h_k^n\|_0\leqslant 2^{-n}$. Put

$$\tilde{\phi}_n(s,\omega):=\sum_k \mathbf{1}_{A_k^n}(s)\phi(s_k^n,\omega).$$

By the $\mathscr{G}$-measurability assumption on the sections of ϕ, we obtain a countably valued Σ-simple function $\tilde{\psi}_n : S \to L^0(\Omega, \mathscr{G}; E)$ by

$$(\tilde{\psi}_n(s))(\omega) := \tilde{\phi}_n(s, \omega),$$

and for all $s \in S$ we have

$$\|\psi(s) - \tilde{\psi}_n(s)\|_0 \leqslant \|\psi(s) - \eta_n(s)\|_0 + \|\eta_n(s) - \tilde{\psi}_n(s)\|_0 \leqslant 2^{-n+1}.$$

By the observation preceding the proof, for all $s \in S$ we have

$$\phi(s, \omega) = (\psi(s))(\omega) = \lim_{n\to\infty} (\tilde{\psi}_n(s))(\omega) = \lim_{n\to\infty} \tilde{\phi}_n(s, \omega) \quad \text{for } \nu\text{-almost all } \omega \in \Omega.$$

Let C be the set of all $(s, \omega) \in S \times \Omega$ for which the sequence $(\tilde{\phi}_n(s, \omega))$ converges. Then the function

$$\tilde{\phi}(s, \omega) := \lim_{n\to\infty} \mathbf{1}_C(s, \omega)\tilde{\phi}_n(s, \omega).$$

is a $\Sigma \otimes \mathscr{G}$-measurable modification of ϕ. $\square$

The following example was communicated to us by Klaas Pieter Hart. It shows that in general the strong $\mathscr{G}$-measurability of the sections ϕ_s of a jointly measurable function ϕ does not imply the strong $\Sigma \otimes \mathscr{G}$-measurability of ϕ.

Example. Let $(S, \Sigma) = (\Omega, \mathscr{F}) = (\omega_1, \mathscr{P})$, where ω_1 is the first uncountable ordinal and $\mathscr{P} = \mathscr{P}(\omega_1)$ is its power set. Let $\mathscr{G}$ be the sub-σ-algebra of $\mathscr{P}$ consisting of all sets that are either countable or have countable complement. Let

$$A := \{(\alpha, \beta) \in \omega_1 \times \omega_1 : \ \alpha < \beta\}.$$

It is well known that $\mathscr{P} \otimes \mathscr{P} = \mathscr{P}(\omega_1 \times \omega_1)$ [21], see also [13, Theorem 12.5], and therefore $A \in \mathscr{P} \otimes \mathscr{P}$. Moreover, for all $\alpha \in \omega_1$ the section $A_\alpha := \{\beta \in \omega_1 : (\alpha, \beta) \in A\}$ belongs to $\mathscr{G}$. We will show that $A \notin \mathscr{P} \otimes \mathscr{G}$. The example announced above is obtained by taking for ϕ the indicator function of A.

Define an increasing family of collections of subsets $(\mathscr{C}_\beta)_{\beta\in\omega_1}$ as follows. Let $\mathscr{C}_0$ denote the collection of all measurable rectangles in $\mathscr{P} \otimes \mathscr{G}$. If $\beta \in \omega_1$ is a successor ordinal, say $\beta = \alpha + 1$, let $\mathscr{C}_\beta$ be the collection of all sets obtained from $\mathscr{C}_\alpha$ by taking complements, intersections, and countable unions. If $\beta \in \omega_1$ is a limit ordinal, let $\mathscr{C}_\beta := \bigcup_{\alpha<\beta} \mathscr{C}_\alpha$. Note that $\mathscr{P} \otimes \mathscr{G} = \bigcup_{\beta\in\omega_1} \mathscr{C}_\beta$. With induction on β it is seen that every $C \in \mathscr{C}_\beta$ belongs to a σ-algebra generated by a countable family of measurable rectangles in $\mathscr{P} \otimes \mathscr{G}$.

Suppose now, for a contradiction, that $A \in \mathscr{P} \otimes \mathscr{G}$. Since there is a first ordinal $\beta \in \omega_1$ such that $A \in \mathscr{C}_\beta$, there exists a countable collection of measurable rectangles $P_n \times G_n \in \mathscr{P} \otimes \mathscr{G}$ such that $A \in \mathscr{P}_A := \sigma(P_n \times G_n;\ n = 1, 2, \dots)$. Choose $\alpha_1 \in \omega_1$ such that $G_n \subseteq [0, \alpha_1)$ for all countable G_n and $G_n \supseteq [\alpha_1, \omega_1)$ for all G_n whose complement is countable. For each n, $(F_n \times G_n) \cap (\omega_1 \times [\alpha_1, \omega_1))$ equals either $\varnothing$ or $F_n \times [\alpha_1, \omega)$. Hence if $B \in \mathscr{P}_A$, then $B \cap (\omega_1 \times [\alpha_1, \omega_1)) = P_B \times [\alpha_1, \omega)$ for some $P_B \subseteq \omega_1$. But obviously there exists no set $P_A \subseteq \omega_1$ such that $A \cap (\omega_1 \times [\alpha_1, \omega_1)) = P_A \times [\alpha_1, \omega_1)$. Thus $A \notin \mathscr{P}_A$, a contradiction.

Remark. Let ν be the probability measure on $(\omega_1, \mathscr{G})$ defined by

$$\nu(P) := \begin{cases} 0, & \text{if } P \text{ is countable,} \\ 1, & \text{if } P \text{ is uncountable.} \end{cases}$$

Any modification of the indicator function $\mathbf{1}_A$ fails to be $\mathscr{P} \otimes \mathscr{G}$-measurable equally well. This does not contradict Proposition 2.1, since ν cannot be extended to a measure on $(\omega_1, \mathscr{P})$.

We continue with a measurability result for functions having L^p-sections.

Proposition 2.2 *Let (S, Σ) and $(\Omega, \mathscr{F}, \nu)$ be as before, let $1 \leqslant p < \infty$, and let E be a Banach space. If $\phi : S \times \Omega \to E$ is a strongly $\Sigma \otimes \mathscr{F}$-measurable function such that for all $s \in S$ we have $\phi_s \in L^p(\Omega; E)$, then the function $\psi : S \to L^p(\Omega; E)$ defined by $\psi(s) := \phi_s$ is strongly Σ-measurable.*

Proof. Let (Ω_k) be an increasing sequence of sets in $\mathscr{F}$ with $\nu(\Omega_k) < \infty$ and $\bigcup_k \Omega_k = \Omega$. By approximating ϕ with the functions $\mathbf{1}_{\Omega_k \cap \{\|\phi\| \leqslant k\}}\phi$ and recalling that pointwise limits of strongly Σ-measurable functions are strongly Σ-measurable, we may assume that $\nu(\Omega) < \infty$ and that ϕ is uniformly bounded.

Choose a sequence of Σ-simple functions $\phi_n : S \times \Omega \to E$, each of which is a finite linear combination of functions of the form $\mathbf{1}_{A\times F} \otimes x$ with $A \in \Sigma$, $F \in \mathscr{F}$, $x \in E$, such that $\phi = \lim_{n\to\infty} \phi_n$ pointwise on $S \times \Omega$. These functions may be chosen in such a way that in addition we have $\|\phi_n\|_\infty \leqslant 2\|\phi\|_\infty$. By the dominated convergence theorem, for all $s \in S$ we have $\phi(s, \cdot) = \lim_{n\to\infty} \phi_n(s, \cdot)$ in $L^p(\Omega; E)$. Define the Σ-simple functions $\psi_n : S \to L^p(\Omega; E)$ by $\psi_n(s) := \phi_n(s, \cdot)$. Then for all $s \in S$,

$$\psi(s) = \phi(s, \cdot) = \lim_{n\to\infty} \phi_n(s, \cdot) = \lim_{n\to\infty} \psi_n(s) \text{ in } L^p(\Omega; E).$$

This shows that ψ is strongly Σ-measurable. □

Note that we did not assume $L^p(\Omega; E)$ to be separable. If this is the case, the above proof can be simplified somewhat by using the Pettis measurability theorem.

By repeated application of Proposition 2.2 we obtain:

Proposition 2.3 *Let (S, Σ) be as before, let $(\Omega, \mathscr{F}, \nu)$ and $(\tilde\Omega, \tilde{\mathscr{F}}, \tilde\nu)$ be σ-finite measure spaces, let $1 \leqslant p, \tilde p < \infty$, and let E be a Banach space. If $\phi : S \times \Omega \times \tilde\Omega \to E$ is a strongly $\Sigma \otimes \mathscr{F} \otimes \tilde{\mathscr{F}}$-measurable function such that for all $s \in S$ we have $\phi_s \in L^p(\Omega; L^{\tilde p}(\tilde\Omega; E))$, then the function $\psi : S \to L^p(\Omega; L^{\tilde p}(\tilde\Omega; E))$ defined by $\psi(s) := \phi_s$ is strongly Σ-measurable.*

3 The stochastic Fubini theorem

Let H be a separable real Hilbert space. A family $W_H = \{W_H(t)\}_{t\in[0,T]}$ of bounded linear operators from H to $L^2(\Omega)$ is called a *H-cylindrical Brownian motion* if $W_H h = \{W_H(t)h\}_{t\in[0,T]}$ is an real Brownian motion for each $h \in H$ and

$$\mathbb{E}(W_H(s)g \cdot W_H(t)h) = (s \wedge t)\,[g, h]_H \qquad s, t \in [0, T],\ g, h \in H.$$

Let E be a real Banach space. A function $\Phi : [0, T] \to \mathscr{L}(H, E)$ *belongs to $L^2(0, T; H)$ scalarly* if for all $x^* \in E^*$ the function $t \mapsto \Phi^*(t)x^*$ belongs to $L^2(0, T; H)$. Note that by the separability of H and the Pettis measurability theorem [7], the strong measurability of $t \mapsto \Phi^*(t)x^*$ is equivalent to its weak measurability.

Definition. A function $\Phi : [0, T] \to \mathscr{L}(H, E)$ is called *stochastically integrable* with respect to W_H if Φ belongs to $L^2(0, T; H)$ scalarly and there exists there exists a sequence (Φ_n) of step functions such that:

(i) For all $x^* \in E^*$ we have $\lim_{n\to\infty} \Phi_n^* x^* = \Phi^* x^*$ in $L^2(0, T; H)$;

(ii) There exists a strongly measurable random variable $Y : \Omega \to E$ such that

$$Y = \lim_{n\to\infty} \int_0^T \Phi_n(t)\, dW_H(t) \quad \text{in } L^0(\Omega; E). \tag{3.1}$$

We then write $Y =: \int_0^T \Phi(t)\, dW_H(t)$.

Note that (i) and (ii) imply that for all $x^* \in E^*$,

$$\Big\langle \int_0^T \Phi(t)\, dW_H(t), x^* \Big\rangle = \int_0^T \Phi^*(t)x^*\, dW_H(t) \quad \text{almost surely.}$$

The stochastic integral for $L^2(0, T; H)$-functions on the right hand side of (3.1) is defined in the usual way: for step functions we put

$$\int_0^T \sum_{n=1}^N \mathbf{1}_{(t_{n-1},t_n]} \otimes h_n\, dW_H(t) := \sum_{n=1}^N W_H(t_n)h_n - W_H(t_{n-1})h_n$$

and this definition is extended to arbitrary $L^2(0,T;H)$-functions by approximation and using the Itô isometry.

It was shown in [16] that Φ is stochastically integrable with respect to W_H if and only if Φ belongs to $L^2(0,T;H)$ scalarly and there exists a γ-radonifying operator $I_\Phi : L^2(0,T;H) \to E$ such that

$$\langle I_\Phi g, x^* \rangle = \int_0^T [g(t), \Phi^*(t)x^*]_H \, dt, \quad g \in L^2(0,T;H), \ x^* \in E^*. \tag{3.2}$$

If (3.2) holds, we shall say that Φ *represents* the operator I_Φ. Recall that a bounded operator S from a separable real Hilbert space $\mathscr{H}$ into E is called *γ-radonifying* if for some (every) Gaussian sequence (γ_n) and some (every) orthonormal basis (h_n) of $\mathscr{H}$ the sum $\sum_n \gamma_n Sh_n$ converges in the L^2 sense. The vector space of all γ-radonifying operators from $\mathscr{H}$ to E is denoted by $\gamma(\mathscr{H},E)$. It is a Banach space with respect to the norm $\|\cdot\|_{\gamma(\mathscr{H},E)}$,

$$\|S\|^2_{\gamma(\mathscr{H},E)} = \mathbb{E}\Big\| \sum_n \gamma_n Sh_n \Big\|^2.$$

If $\Phi : [0,T] \to \mathscr{L}(H,E)$ is stochastically integrable, then for the operator I_Φ from (3.2) we have

$$\|I_\Phi\|^2_{\gamma(L^2(0,T;H),E)} = \mathbb{E}\Big\| \int_0^T \Phi(t)\, dW_H(t) \Big\|^2.$$

Let (S,Σ,μ) be a σ-finite measure space and fix an arbitrary $1 \leqslant p < \infty$. In the next two lemmas we consider a strongly $\Sigma \otimes \mathscr{B}([0,T])$-measurable function $\phi : S \times [0,T] \to H$ which has the property that for all $t \in [0,T]$ and $h \in H$, the function $s \mapsto [\phi(s,t),h]_H$ belongs to $L^p(S)$. We then define $\Phi : [0,T] \to \mathscr{L}(H, L^p(S))$ by

$$(\Phi(t)h)(s) := [\phi(s,t),h]_H. \tag{3.3}$$

As an application of Proposition 2.2 we have:

Lemma 3.1 *Let the function $\Phi : [0,T] \to \mathscr{L}(H,L^p(S))$ defined by* (3.3)*. For all $h \in H$ the function $\Phi h : [0,T] \to L^p(S)$ defined by $(\Phi h)(t) := \Phi(t)h$ is strongly $\mathscr{B}([0,T])$-measurable.*

The following lemma gives a necessary and sufficient condition for the stochastic integrability of the function Φ. It is a special case of [15, Proposition 4.1], which generalizes the case $H = \mathbb{R}$ considered in [16, Corollary 2.10]. See also [20, Corollary 4.3] and [2, Theorem 2.3] for related results.

Lemma 3.2 *The function $\Phi : [0,T] \to \mathscr{L}(H,L^p(S))$ defined by* (3.3) *is stochastically integrable in $L^p(S)$ with respect to an H-cylindrical Brownian motion W_H if and only if ϕ defines an element of $L^p(S;L^2(0,T;H))$. In this case we have*

$$\mathbb{E}\Big\| \int_0^T \Phi(t)\, dW_H(t) \Big\|^2 = \|\Phi\|^2_{\gamma(L^2(0,T;H),L^p(S))} \eqsim_p \|\phi\|^2_{L^p(S;L^2(0,T;H))}.$$

Here '$\eqsim_p$' means that we have a two-sided estimate with constants depending only on p.

In order to extend the notions introduced above to processes $\Phi : [0,T] \times \Omega \to \mathscr{L}(H,E)$ we need to introduce some terminology. Throughout, $\mathbb{F} = (\mathscr{F}_t)_{t\in[0,T]}$ denotes a filtration satisfying the usual conditions. We assume that the H-cylindrical Brownian motion W_H is adapted to $\mathbb{F}$, by which we mean that for all $h \in H$ the real-valued Brownian motion $W_H h$ is adapted to $\mathbb{F}$.

A process $\Phi : [0,T] \times \Omega \to \mathscr{L}(H,E)$ *belongs to $L^0(\Omega;L^2(0,T;H))$ scalarly* if for all x^* the process $\Phi^* x^*$ belong to $L^0(\Omega;L^2(0,T;H))$. Such a process is said to *represent* an element X_Φ of $L^0(\Omega;\gamma(L^2(0,T;H);E))$ if for all $f \in L^2(0,T;H)$ and $x^* \in E^*$ we have

$$\langle X_\Phi f, x^* \rangle = \int_0^T [f(t), \Phi^*(t)x^*]_H \, dt \ \text{ almost surely.}$$

A process $\Phi : [0,T] \times \Omega \to \mathscr{L}(H,E)$ is called *scalarly progressively measurable* with respect to $\mathbb{F}$ if for all $h \in H$ and $x^* \in E^*$ the process $\Phi^* x^*$ is progressively measurable with respect to $\mathbb{F}$. By the Pettis measurability theorem, this happens if and only if for all $h \in H$ and $x^* \in E^*$ the process $\langle \Phi h, x^* \rangle$ is progressively measurable with respect to $\mathbb{F}$.

A process $\Phi : [0,T] \times \Omega \to \mathscr{L}(H,E)$ is called *elementary progressive* with respect to $\mathbb{F}$ if it is of the form

$$\Phi(t,\omega) = \sum_{n=1}^{N} \sum_{m=1}^{M} 1_{(t_{n-1},t_n] \times A_{mn}}(t,\omega) \sum_{k=1}^{K} h_k \otimes x_{kmn},$$

where $0 \leqslant t_0 < \cdots < t_N \leqslant T$, $A_{mn} \in \mathscr{F}_{t_{n-1}}$, $x_{knm} \in E$, and $(h_k)_{k\geqslant 1}$ is a fixed orthonormal basis for H. Clearly, every elementary progressive process is scalarly progressive and represents an element of $L^0(\Omega; \gamma(L^2(0,T;H);E))$.

Definition. A process $\Phi : [0,T] \times \Omega \to \mathscr{L}(H,E)$ is called *stochastically integrable* with respect to W_H if Φ belongs to $L^0(\Omega; L^2(0,T;H))$ scalarly and there exists a sequence (Φ_n) of elementary progressive processes such that:

(i) For all $x^* \in E^*$ we have $\lim_{n\to\infty} \Phi_n^* x^* = \Phi^* x^*$ in $L^0(\Omega; L^2(0,T;H))$;

(ii) There exists a strongly measurable random variable $Y : \Omega \to E$ such that

$$Y = \lim_{n\to\infty} \int_0^T \Phi_n(t)\, dW_H(t) \quad \text{in } L^0(\Omega;E).$$

We then write $Y =: \int_0^T \Phi(t)\, dW_H(t)$. It is easy to check that if Φ is stochastically integrable, then Φ is scalarly progressively measurable and for all $x^* \in E^*$ we have

$$\Big\langle \int_0^T \Phi(t)\, dW_H(t), x^* \Big\rangle = \int_0^T \Phi^*(t) x^*\, dW_H(t) \quad \text{almost surely.} \tag{3.4}$$

Remark. In [15] a slightly narrower definition of stochastic integrability is used and a correspondingly stronger version of Proposition 3.4 is proved. Since the proposition is used only as a technical tool in the proof of Theorem 3.5, where it is applied to elementary progressive processes Φ_n, the simpler definition given above is sufficient for our present purposes. We refer to [15] for a fuller explanation on this point.

Let $(r_n)_{n\geqslant 1}$ be a Rademacher sequence. A Banach space E is called a *UMD$^-$ space* if for some (every) $1 < p < \infty$ there exists a constant β_p such that for every finite E-valued martingale difference sequence $(d_n)_{n=1}^N$ independent of $(r_n)_{n\geqslant 1}$ we have

$$\mathbb{E}\Big\|\sum_{n=1}^N d_n\Big\|^p \leqslant \beta_p^p\, \mathbb{E}\Big\|\sum_{n=1}^N r_n d_n\Big\|^p.$$

The class of *UMD$^+$ spaces* is defined by reversing the estimate. By a simple randomization argument it is seen that a Banach space is a UMD space if and only if it is both UMD$^-$ and UMD$^+$. The classes of UMD$^-$ and UMD$^+$ space were introduced by Garling [11] who proved among other things:

- If E is a UMD$^+$ space, then its dual E^* is a UMD$^-$ space. If E^* is a UMD$^+$ space, then its predual E is a UMD$^-$ space[1].
- Every UMD$^-$ space has finite cotype. Every UMD$^+$ space is superreflexive.
- E is a UMD space if and only if E is both UMD$^-$ and UMD$^+$.

[1] This corrects a misprint in the published version

For the theory of UMD spaces we refer to the review article by Burkholder [3] and the references given therein.

By [11, Theorem 2] and the Lévy-Octaviani inequalities one easily sees that a Banach space E is a UMD$^-$ space if and only if for some (every) $p \in [1, \infty)$ there exists a constant $\tilde{\beta}^-_{p,E} \geqslant 0$ such that for all E-valued martingale difference sequences $(d_n)_{n=1}^N$ we have

$$\mathbb{E} \sup_{1 \leqslant n \leqslant N} \Big\| \sum_{k=1}^n d_k \Big\|^p \leqslant (\tilde{\beta}^-_{p,E})^p \, \mathbb{E}\tilde{\mathbb{E}} \Big\| \sum_{k=1}^N \tilde{r}_k d_k \Big\|^p.$$

This may be used to prove:

Proposition 3.3 *If (S, Σ, μ) is σ-finite and E is a UMD$^-$ space, then for all $p \in [1, \infty)$ the space $L^p(S; E)$ is a UMD$^-$ space.*

The fact that $L^1(S)$, and more generally every space which is finitely representable in l^1, is a UMD$^-$ space is proved in [11, Theorem 3]. Apart from the trivial case where (S, Σ, μ) consists of finitely many atoms, the space $L^1(S)$ is an example of a UMD$^-$ space that is not a UMD space.

The following proposition is proved in the same way as [15, Theorem 3.7] and generalizes of a result of McConnell [14] for $H = \mathbb{R}$ and UMD spaces E. It uses an obvious one-sided generalization of [10, Theorem 2′] to UMD$^-$ spaces.

Proposition 3.4 *Let E be a UMD$^-$ space and let $\Phi : [0, T] \times \Omega \to \mathscr{L}(H, E)$ be a scalarly progressively measurable process. If Φ represents an element X_Φ of $L^0(\Omega; \gamma(L^2(0, T; H), E))$, then Φ is stochastically integrable with respect to W_H, and there exists a sequence of elementary progressive processes $\Phi_n : [0, T] \times \Omega \to \mathscr{L}(H, E)$ such that:*

(i) $\lim_{n \to \infty} X_{\Phi_n} = X_{\Phi_n}$ *in* $L^0(\Omega; \gamma(L^2(0, T; H), E))$*;*

(ii) $\lim_{n \to \infty} \int_0^T \Phi_n(t) \, dW_H(t) = \int_0^T \Phi(t) \, dW_H(t)$ *in* $L^0(\Omega; E)$*.*

Below we shall apply the proposition to the space $E = L^1(S)$. By Lemma 3.2, the space $L^0(\Omega; \gamma(L^2(0, T; H), L^1(S)))$ can be identified with $L^0(\Omega; L^1(S; L^2(0, T; H)))$ isomorphically.

After these preparations we are in a position to state and prove our first main result.

Theorem 3.5 (Stochastic Fubini theorem, first version) *Let $\phi : S \times [0, T] \times \Omega \to H$ be a process satisfying the following assumptions:*

(i) ϕ is strongly $\Sigma \otimes \mathscr{B}([0, T]) \otimes \mathscr{F}$-measurable;

(ii) For all $s \in S$, the section ϕ_s is progressively measurable;

(iii) For almost all $\omega \in \Omega$, $(s, t) \mapsto \phi(s, t, \omega)$ belongs to $L^1(S; L^2(0, T; H))$.

Then:

1. *For almost all $s \in S$, the process ϕ_s is stochastically integrable with respect to W_H;*
2. *For almost all $(t, \omega) \in [0, T] \times \Omega$, $s \mapsto \phi_s(t, \omega)$ defines an element of $L^1(S; H)$ and there exists a progressively measurable process $\langle \phi, \mu \rangle : [0, T] \times \Omega \to H$, stochastically integrable with respect to W_H, such that*

$$\langle \phi, \mu \rangle(t, \omega) = \int_S \phi_s(t, \omega) \, d\mu(s)$$

for almost all $(t, \omega) \in [0, T] \times \Omega$;

3. For almost all $\omega \in \Omega$, $s \mapsto \big(\int_0^T \phi_s(t)\, dW_H(t)\big)(\omega)$ belongs to $L^1(S)$ and

$$\int_S \Big(\int_0^T \phi_s(t)\, dW_H(t)\Big)(\omega)\, d\mu(s) = \Big(\int_0^T \langle \phi, \mu\rangle(t)\, dW_H(t)\Big)(\omega).$$

If in (iii) we make the stronger assumption that $\phi \in L^p(\Omega; L^1(S; L^2(0,T;H)))$ for some $p \in [1,\infty)$, then it follows from similar estimates as in [15] that

$$\mathbb{E}\Big|\int_0^T \langle \phi, \mu\rangle(t)\, dW_H(t)\Big|^p \leqslant C_p \mathbb{E}\|\phi\|^p_{L^1(S;L^2(0,T;H))},$$

for some universal constant C_p, and the equality in (3) may be interpreted in $L^p(\Omega)$.

Proof. By Proposition 2.1 (where we replace Ω by $[0,T]\times\Omega$ and for $\mathscr{G}$ we take the progressive σ-algebra $\mathscr{P}$ of $[0,T]\times\Omega$) we may choose a version of ϕ which is $\Sigma\otimes\mathscr{P}$-measurable.

(1): For almost all $\omega \in \Omega$ we have $\phi(s,\cdot,\omega) \in L^2(0,T;H)$ for almost all $s \in S$. Hence by Fubini's theorem, for almost all $s \in S$ the process $(t,\omega) \mapsto \phi(s,t,\omega)$ has trajectories in $L^2(0,T;H)$ almost surely. By standard results it follows that for almost all $s \in S$ the process ϕ_s is stochastically integrable with respect to W_H.

(2): Using the embedding

$$L^1(S; L^2(0,T;H)) \hookrightarrow L^1(S; L^1(0,T;H)) \eqsim L^1(0,T; L^1(S,H))$$

and the Fubini theorem, (iii) implies that for almost all $(t,\omega) \in [0,T]\times\Omega$ the function $s \mapsto \phi(s,t,\omega)$ defines an element of $L^1(S;H)$. The exceptional set N being progressively measurable, we may redefine $\phi(\cdot,t,\omega)$ to be 0 for $(t,\omega) \in N$ and thereby assume that $\phi(\cdot,t,\omega)$ defines an element of $L^1(S;H)$ for all $(t,\omega) \in [0,T]\times\Omega$. Now define an operator-valued process $\Phi : [0,T]\times\Omega \to \mathscr{L}(H, L^1(S))$ by

$$(\Phi(t,\omega)h)(s) := [\phi(s,t,\omega), h]_H.$$

Since by (ii) the process $(t,\omega) \mapsto [\phi(s,t,\omega), h]_H$ is progressively measurable for all $s \in S$, it follows by Proposition 2.2 that Φh is strongly progressively measurable for all $h \in H$. In particular, Φ is scalarly progressively measurable.

By Proposition 2.3, the random variable $\omega \mapsto \phi(\cdot,\cdot,\omega)$ is strongly $\mathscr{F}$-measurable from Ω to $L^1(S; L^2(0,T;H))$. Thus ϕ defines an element of $L^0(\Omega; L^1(S; L^2(0,T;H)))$. By Proposition 3.4 and the remark following it, Φ is stochastically integrable with respect to W_H.

Identifying integration with respect to μ with a bounded linear operator T_μ acting from $\mathscr{L}(H, L^1(S))$ to H in the canonical way, we have $\langle \phi, \mu\rangle = T_\mu \circ \Phi$. Since $T_\mu\circ\Phi$ is stochastically integrable with respect to W_H the result follows.

(3): By what has been proved in Step 2, Φ is scalarly progressive and represents an element of $L^0(\Omega; \gamma(L^2(0,T;H), L^1(S)))$. Hence by Proposition 3.4 there exists a sequence of elementary progressive processes $\Phi_n : [0,T]\times\Omega \to \mathscr{L}(H, L^1(S))$ such that $\lim_{n\to\infty} X_{\Phi_n} = X_\Phi$ in $L^0(\Omega; \gamma(L^2(0,T;H), E))$. Upon passing to a subsequence we may assume that

$$\begin{aligned}\Big(\Big(\int_0^T \Phi(t)\, dW_H(t)\Big)(\omega)\Big)(s) &= \lim_{n\to\infty} \Big(\Big(\int_0^T \Phi_n(t)\, dW_H(t)\Big)(\omega)\Big)(s) \\ &= \lim_{n\to\infty} \Big(\int_0^T (\Phi_n(t))(s)\, dW_H(t)\Big)(\omega) \\ &\qquad\qquad \text{for almost all } (s,\omega) \in S\times\Omega.\end{aligned} \tag{3.5}$$

For each n, let ϕ_n be the element of $L^0(\Omega; (L^1(S; L^2(0,T;H))))$ corresponding to the process Φ_n. By passing to a further subsequence we may also assume that

$$\lim_{n\to\infty} \phi_n(s,\cdot,\omega) = \phi(s,\cdot,\omega) \text{ in } L^2(0,T;H) \text{ for almost all } (s,\omega) \in S\times\Omega. \tag{3.6}$$

Defining $\phi_{n,s}(t,\omega) := \phi_n(s,t,\omega)$, by (3.6) and the Fubini theorem for almost all $s \in S$ we have $\phi_s(\cdot,\omega) = \lim_{n\to\infty} \phi_{n,s}(\cdot,\omega)$ in $L^2(0,T;H)$ for almost all $\omega \in \Omega$. This implies that

$\phi_s(\cdot) = \lim_{n\to\infty} \phi_{n,s}(\cdot)$ in $L^2(0,T;H)$ in probability. By standard results on stochastic integration, from this it follows that for almost all $s \in S$,

$$\int_0^T \phi_s(t)\, dW_H(t) = \lim_{n\to\infty} \int_0^T \phi_{n,s}(t)\, dW_H(t) \text{ in probability.} \tag{3.7}$$

Comparing limits in (3.5) and (3.7), for almost all $s \in S$ we obtain

$$\Big(\int_0^T \phi_s(t)\, dW_H(t)\Big)(\omega) = \Big(\Big(\int_0^T \Phi(t)\, dW_H(t)\Big)(\omega)\Big)(s) \text{ for almost all } \omega \in \Omega.$$

But then by the Fubini theorem, for almost all $\omega \in \Omega$ we have

$$\Big(\int_0^T \phi_s(t)\, dW_H(t)\Big)(\omega) = \Big(\Big(\int_0^T \Phi(t)\, dW_H(t)\Big)(\omega)\Big)(s) \text{ for almost all } s \in S. \tag{3.8}$$

Since $\int_0^T \Phi(t)\, dW_H(t)$ is a random variable with values in $L^1(S)$, this proves the μ-integrability assertion. The final identity follows by integrating (3.8) with respect to μ. This gives, for almost all $\omega \in \Omega$,

$$\begin{aligned}
\int_S \Big(\int_0^T \phi_s(t)\, dW_H(t)\Big)(\omega)\, d\mu(s) &= \int_S \Big(\Big(\int_0^T \Phi(t)\, dW_H(t)\Big)(\omega)\Big)(s)\, d\mu(s) \\
&\overset{\text{(i)}}{=} \Big\langle \Big(\int_0^T \Phi(t)\, dW_H(t)\Big)(\omega), \mathbf{1} \Big\rangle \\
&\overset{\text{(ii)}}{=} \Big(\int_0^T \Phi^*(t)\mathbf{1}\, dW_H(t)\Big)(\omega) \\
&\overset{\text{(iii)}}{=} \Big(\int_0^T \langle \phi, \mu\rangle(t)\, dW_H(t)\Big)(\omega).
\end{aligned}$$

In (i) the brackets denote the duality between $L^1(S)$ and $L^\infty(S)$, in (ii) we used the identity (3.4), and in (iii) we used (2) and the Fubini theorem to the effect that for almost all $t \in [0,T]$ we have, almost surely,

$$[\Phi^*(t)\mathbf{1}, h]_H = \int_S [\phi(s,t,\cdot), h]_H\, d\mu(s) = [\langle \phi, \mu\rangle(t), h]_H \text{ for all } h \in H.$$

□

Theorem 3.5 can easily be extended to the more general situation where ϕ is a process with values in $\mathscr{L}(H,H')$. In this way, a generalization of the result by Da Prato and Zabczyk [5] as stated in the Introduction is obtained. More generally, one can replace the rôle of H' by an arbitrary real Banach space E. The condition from [5] that ϕ should take values in $L^0(\Omega; L^2(0,T;\mathscr{L}_2(H,H')))$ is then replaced by the condition that ϕ should take values in $L^0(\Omega;\gamma(L^2(0,T;H),E))$. The latter condition reduces to the former if $E = H'$ since $\gamma(L^2(0,T;H),H') = L^2(0,T;\mathscr{L}_2(H,H'))$ isometrically. In order to be able to give a precise statement of the theorem we need to introduce some notations from [15].

Every functional $x^* \in E^*$ induces a bounded operator $x^* : \gamma(L^2(0,T;H),E)) \to L^2(0,T;H)$ by

$$x^*(S) := S^*x^*.$$

We shall write $\langle S, x^*\rangle$ instead of $x^*(S)$. Applying this operator pointwise, we obtain an operator $x^* : L^0(\Omega;\gamma(L^2(0,T;H),E)) \to L^0(\Omega; L^2(0,T;H))$ by

$$(x^*(X))(\omega) := X^*(\omega)x^*.$$

In what follows we shall write $\langle X, x^*\rangle$ for $x^*(X)$. Let $L^0_{\mathbb{F}}(\Omega;\gamma(L^2(0,T;H),E))$ denote the closed subspace of all of $L^0(\Omega;\gamma(L^2(0,T;H),E))$ of all elements X such that for all $x^* \in E^*$, $\langle X, x^*\rangle$ is a progressively measurable as an H-valued process with respect to the filtration $\mathbb{F}$. Here, we identify the elements $\langle X, x^*\rangle \in L^0(\Omega; L^2(0,T;H))$ with processes

$\langle X, x^* \rangle : [0,T] \times \Omega \to H$. Note that if $X = X_\Phi$ is represented by a process Φ, then $\langle X, x^* \rangle = \Phi^* x^*$.

Since every elementary progressive process is representable, the subspace of representable elements is dense in $L^0_{\mathbb{F}}(\Omega; \gamma(L^2(0,T;H),E))$. It is shown as in [15] that the linear operator $X_\Phi \mapsto \int_0^T \Phi(t)\, dW_H(t)$, which is well defined for representable processes by Proposition 3.4, has a unique extension to a continuous linear operator

$$I^{W_H} : L^0_{\mathbb{F}}(\Omega; \gamma(L^2(0,T;H),E)) \to L^0(\Omega; E).$$

We shall write $\int_0^T X\, dW_H$ for Itô(X).

The proof of Theorem 3.5 can be adapted to obtain the following result.

Theorem 3.6 (Stochastic Fubini theorem, second version) *Let E be a UMD$^-$ space and let $\phi : S \times [0,T] \times \Omega \to \mathscr{L}(H,E)$ be a process satisfying the following assumptions:*

(i) For all $h \in H$, ϕh is strongly $\Sigma \otimes \mathscr{B}([0,T]) \otimes \mathscr{F}$-measurable;

(ii) For all $s \in S$, the section ϕ_s is progressively measurable for all $h \in H$;

(iii) For almost all $(s,\omega) \in S \times \Omega$, the function $t \mapsto \phi(s,t,\omega)$ represents an element $U_{s,\omega} \in \gamma(L^2(0,T;H),E)$, and for almost all $\omega \in \Omega$, $s \mapsto U_{s,\omega}$ defines an element of $L^1(S; \gamma(L^2(0,T;H),E))$.

Then:

1. *For almost all $s \in S$, ϕ_s is stochastically integrable with respect to W_H;*
2. *For all $x^* \in E^*$, $s \mapsto \phi_s^*(t,\omega)x^*$ defines an element of $L^1(S;H)$ for almost all $(t,\omega) \in [0,T] \times \Omega$, and there exists an element $\langle \phi, \mu \rangle \in L^0_{\mathbb{F}}(\Omega; \gamma(L^2(0,T;H),E))$ such that for all $x^* \in E^*$ we have*

$$\big\langle \langle \phi, \mu \rangle, x^* \big\rangle(t,\omega) = \int_S \phi_s^*(t,\omega) x^* \, d\mu(s)$$

for almost all $(t,\omega) \in [0,T] \times \Omega$;

3. *For almost all $\omega \in \Omega$, $s \mapsto \big(\int_0^T \phi_s(t)\, dW_H(t)\big)(\omega)$ belongs to $L^1(S;E)$ and we have*

$$\int_S \Big(\int_0^T \phi_s(t)\, dW_H(t) \Big)(\omega)\, d\mu(s) = \Big(\int_0^T \langle \phi, \mu \rangle(t)\, dW_H(t) \Big)(\omega).$$

If E has type 2 we have a continuous embedding $L^2(0,T;\gamma(H,E)) \hookrightarrow \gamma(L^2(0,T;H),E)$, cf. [17]. Condition (iii) is then implied by the stronger condition

(iii)$'$ *For almost all $\omega \in \Omega$, $(s,t) \mapsto \phi(s,t,\omega)$ defines an element of $L^1(S; L^2(0,T;\gamma(H,E)))$.*

If E has cotype 2 we have a continuous embedding $\gamma(L^2(0,T;H),E) \hookrightarrow L^2(0,T;\gamma(H,E))$, cf. [17]. Because of this, every $X \in L^0_{\mathbb{F}}(\Omega; \gamma(L^2(0,T;H),E))$ can be identified with a progressively measurable process in $L^0(\Omega; L^2(0,T;\gamma(H,E)))$ and the use of the abstract Itô operator can be avoided. Moreover it can be shown that in this situation, (2) can be strengthened as follows:

(2)$'$ *For almost all $(t,\omega) \in [0,T] \times \Omega$, $s \mapsto \phi(s,t,\omega)h$ belongs to $L^1(S;E)$ for all $h \in H$ and there exists a process $\langle \phi, \mu \rangle : [0,T] \times \Omega \to \mathscr{L}(H,E)$, stochastically integrable with respect to W_H, such that for almost all $(t,\omega) \in [0,T] \times \Omega$ we have*

$$\langle \phi, \mu \rangle(t,\omega)h = \int_S \phi(s,t,\omega)h \, d\mu(s)$$

for all $h \in H$.

Both remarks apply if $E = H'$ is a Hilbert space, in which case we have $\gamma(H,E) = \gamma(H,H') = \mathscr{L}_2(H,H')$.

Finally, in (iii) we may replace the almost sure conditions by moment conditions to obtain random variables with finite moments in (3). For example, in the case $E = H'$ we could assume that $\phi \in L^p(\Omega; L^1(S; L^2(0,T;\mathscr{L}_2(H,H'))))$ for some $p \in [1,\infty)$, in which case we obtain

$$\mathbb{E}\Big\|\int_0^T \langle \phi, \mu\rangle(t)\, dW_H(t)\Big\|^p \leqslant C_p \mathbb{E}\|\phi\|^p_{L^1(S;L^2(0,T;\mathscr{L}_2(H,H')))},$$

and the equality in (3) may be interpreted in $L^p(\Omega; E)$.

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