SEPARATED SEQUENCES IN UNIFORMLY CONVEX BANACH SPACES

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ABSTRACT. We give a characterization of uniformly convex Banach spaces in terms of a uniform version of the Kadec-Klee property. As an application we prove that if (x_n) is a bounded sequence in a uniformly convex Banach space X which is ε-separated for some $0 < \varepsilon \leqslant 2$, then for all norm one vectors $x \in X$ there exists a subsequence (x_{n_j}) of (x_n) such that

$$\inf_{j \neq k} \big\| x - (x_{n_j} - x_{n_k}) \big\| \geqslant 1 + \delta_X(\tfrac{2}{3}\varepsilon),$$

where δ_X is the modulus of convexity of X. From this we deduce that the unit sphere of every infinite-dimensional uniformly convex Banach space contains a $(1 + \frac{1}{2}\delta_X(\frac{2}{3}))$-separated sequence.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

In this note we obtain a characterization of uniformly convex Banach spaces in terms of a uniform version of the Kadec-Klee property. Recall that a Banach space X with unit sphere S_X is called *uniformly convex* [2] if for all $0 < \varepsilon \leqslant 2$ we have $\delta_X(\varepsilon) > 0$, where

$$\varepsilon \mapsto \delta_X(\varepsilon) := \inf \big\{ 1 - \tfrac{1}{2}\|x + y\| : \ x, y \in S_X, \ \|x - y\| \geqslant \varepsilon \big\}$$

denotes the modulus of convexity of X. For general properties of this function we refer to [2, 3]. Before stating the main abstract result of this paper we formulate two applications. The first concerns the differences $x_j - x_k$ of a uniformly separated bounded sequence (x_n). To motivate the result let us first consider an arbitrary bounded sequence (x_n) in a Banach space X. It is easy to see that for all $x \in S_X$ there exists a subsequence (x_{n_j}) of (x_n) such that

$$\liminf_{j,k \to \infty} \big\| x - (x_{n_j} - x_{n_k}) \big\| \geqslant 1.$$

Indeed, let $x \in S_X$ be arbitrary and choose $x^* \in S_{X^*}$ such that $|\langle x, x^* \rangle| = 1$. If the subsequence (x_{n_j}) is chosen in such a way that the scalar sequence $(\langle x_{n_j}, x^* \rangle)$ is convergent, then

$$\liminf_{j,k \to \infty} \big\| x - (x_{n_j} - x_{n_k}) \big\| \geqslant \liminf_{j,k \to \infty} \big| \langle x - (x_{n_j} - x_{n_k}), x^* \rangle \big| = |\langle x, x^* \rangle| = 1.$$

Easy examples show that for $X = C[0, 1]$ the value of the constant 1 is the best possible, even if the sequence (x_n) is assumed to be ε-separated for some $\varepsilon > 0$, by which we mean that $\|x_j - x_k\| \geqslant \varepsilon$ for all $j \neq k$. For uniformly convex spaces X we obtain an improved constant in terms of the modulus of convexity δ_X:

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Theorem 1.1. *Let X be a uniformly convex Banach space and let (x_n) be a bounded sequence in X which is ε-separated for some $0 < \varepsilon \leqslant 2$. Then for all $x \in S_X$ there exists a subsequence (x_{n_j}) of (x_n) satisfying*

$$\inf_{j\neq k} \big\|x - (x_{n_j} - x_{n_k})\big\| \geqslant 1 + \delta_X(\tfrac{2}{3}\varepsilon).$$

A celebrated result of Elton and Odell [4] asserts that the unit sphere of every infinite-dimensional Banach space X contains a $(1+\mu)$-separated sequence for some $\mu > 0$. It was subsequently shown by Kryczka and Prus [7] that the unit sphere of every infinite-dimensional nonreflexive Banach space contains a $\sqrt[5]{4}$-separated sequence. For uniformly convex spaces X we use Theorem 1.1 to deduce a lower bound for the separation constant in terms of the modulus of convexity δ_X:

Theorem 1.2. *The unit sphere of every infinite-dimensional uniformly convex Banach space X contains a $(1+\frac{1}{2}\delta_X(\frac{2}{3}))$-separated sequence.*

Since uniformly convex spaces are reflexive, this result does not overlap with the result of Kryczka and Prus. Theorem 1.2 provides an affirmative answer, for the class of uniformly convex spaces, to a question of Diestel [3, page 254].

In $X = l^p$, the sequence of unit vectors is $2^{\frac{1}{p}}$-separated. On the other hand it was shown by Clarkson [1] and Hanner [6] that l^p is uniformly convex for $p \in (1,\infty)$ with modulus of convexity given by

$$\delta_{l^p}(\varepsilon) = 1 - \big(1 - (\tfrac{1}{2}\varepsilon)^p\big)^{\frac{1}{p}}$$

if $p \in [2,\infty)$, and by the equation

$$\big|1 - \delta_{l^p}(\varepsilon) + \tfrac{1}{2}\varepsilon\big|^p + \big|1 - \delta_{l^p}(\varepsilon) - \tfrac{1}{2}\varepsilon\big|^p = 2$$

if $p \in (1,2]$. Thus for the spaces l^p, Theorem 1.2 does not give the best possible separation constant. This raises the question whether Theorem 1.2 can be further improved.

Theorems 1.1 and 1.2 are obtained as consequences of the following characterization of uniformly convex spaces and a quantitative refinement stated in the next section.

Theorem 1.3. *For a Banach space X the following assertions are equivalent:*

(1) *X is uniformly convex;*
(2) *For all $\varepsilon > 0$ there exists a constant $\delta > 0$ such that for all $x \in S_X$, all $x' \in X$, and all linear contractions T from X into some Banach space Y satisfying*
   (i) $\big|1 - \|x'\|\big| < \delta$,
   (ii) $\|Tx\| > 1 - \delta$
   (iii) $\|Tx - Tx'\| < \delta$,
   we have $\|x - x'\| < \varepsilon$;
(3) *For all $\varepsilon > 0$ there exists a constant $\delta > 0$ such that for all $x \in S_X$, all $x' \in S_X$, and all $x^* \in S_{X^*}$ satisfying*
   (iv) $|\langle x, x^*\rangle| > 1 - \delta$,
   (v) $|\langle x - x', x^*\rangle| < \delta$,
   we have $\|x - x'\| < \varepsilon$.

Condition (3), when reformulated in terms of sequences, may be regarded as a uniform version of the Kadec-Klee property. Recall that X is said to have the *Kadec-Klee property* if for all $x \in S_X$ and all sequences $(x_n) \subseteq S_X$ such that $\lim_{n\to\infty}\langle x - x_n, x^*\rangle = 0$ for all $x^* \in X^*$ we have $\lim_{n\to\infty}\|x - x_n\| = 0$. Every uniformly convex space has the Kadec-Klee property; this fact is also contained as a special case in Theorem 1.3.

2. PROOFS

Proof of Theorem 1.3. (1)⇒(2): It is enough to prove this implication for $0 < \varepsilon \leqslant 2$. For such an ε, we will show that (2) holds for any $\delta > 0$ satisfying

$$\delta \leqslant \tfrac{1}{2}\delta_X(\varepsilon - \delta). \tag{2.1}$$

Such numbers δ exist since $\delta_X(\eta) > 0$ for all $0 < \eta \leqslant 2$.

We argue by contradiction. Suppose there exist $0 < \varepsilon \leqslant 2$, a number $\delta > 0$ satisfying (2.1), vectors $x \in S_X$ and $x' \in X$, and a linear contraction T from X into some Banach space Y such that the assumptions of (2) are satisfied while $\|x - x'\| \geqslant \varepsilon$. From $\delta \leqslant \frac{1}{2}\delta_X(\varepsilon - \delta) \leqslant \frac{1}{2}$ and (i) it follows that $x' \neq 0$. We estimate

$$\Big\|x - \frac{x'}{\|x'\|}\Big\| \geqslant \varepsilon - \Big\|x' - \frac{x'}{\|x'\|}\Big\| = \varepsilon - \big|\|x'\| - 1\big| > \varepsilon - \delta. \tag{2.2}$$

and

$$\tfrac{1}{2}\|x + x'\| \leqslant \tfrac{1}{2}\Big\|x + \frac{x'}{\|x'\|}\Big\| + \tfrac{1}{2}\Big\|x' - \frac{x'}{\|x'\|}\Big\| \overset{\text{(a)}}{<} \big(1 - \delta_X(\varepsilon - \delta)\big) + \tfrac{1}{2}\delta \overset{\text{(b)}}{\leqslant} 1 - \tfrac{3}{4}\delta_X(\varepsilon - \delta).$$

In (a), the first term is estimated using (2.2) and the definition of the modulus of convexity and the second term is estimated using assumption (i). The estimate (b) is immediate from (2.1). Thus, $\xi := x + x'$ satisfies $\|\xi\| < 2 - \frac{3}{2}\delta_X(\varepsilon - \delta)$. Next, from $\xi' := 2Tx - T\xi = Tx - Tx'$ and (iii) it follows that $\|\xi'\| < \delta$. Putting things together and using (ii), we obtain

$$2 - 2\delta < \|2Tx\| = \|T\xi + \xi'\| \leqslant \|T\xi\| + \|\xi'\| \leqslant \|\xi\| + \|\xi'\| < \big(2 - \tfrac{3}{2}\delta_X(\varepsilon - \delta)\big) + \delta.$$

Comparing left- and right hand sides, we have obtained a contradiction with (2.1).

(2)⇒(3): Trivial.

(3)⇒(1): If X is not uniformly convex, there exist $\varepsilon > 0$ and sequences $(x_n) \subseteq S_X$, $(x_n') \subseteq S_X$ such that $\inf_n \|x_n - x_n'\| \geqslant \varepsilon$ and $\lim_{n\to\infty} \|x_n + x_n'\| = 2$. Choose a sequence $(x_n^*) \subseteq S_{X^*}$ such that $\langle x_n + x_n', x_n^*\rangle = \|x_n + x_n'\|$ for all n. Then $\lim_{n\to\infty}\langle x_n, x_n^*\rangle = \lim_{n\to\infty}\langle x_n', x_n^*\rangle = 1$ and $\lim_{n\to\infty}\langle x_n - x_n', x_n^*\rangle = 0$, and therefore by (3) we obtain $\lim_{n\to\infty} \|x_n - x_n'\| = 0$, a contradiction. ∎

For the proofs of Theorems 1.1 and 1.2 we need some quantitative information about the dependence of δ upon ε in the proof of (1)⇒$(2)_0$, where $(2)_0$ is obtained from (2) by replacing (ii) by the more restrictive condition

(ii)_0 $\|Tx\| = 1$.

Lemma 2.1. *Let X be uniformly convex and fix an arbitrary $0 < \varepsilon \leqslant 2$. Then the conclusion of* $(2)_0$ *holds if the assumptions* (i), (ii)_0, (iii) *are satisfied for* $\delta = \delta_X(\frac{2}{3}\varepsilon)$.

Proof. We first claim that the conclusion of $(2)_0$ holds if (i), (ii)_0, (iii) are satisfied for some $\delta > 0$ such that

$$\delta \leqslant \delta_X(\varepsilon - \delta). \tag{2.3}$$

Arguing by contradiction and proceeding as in the proof of (1)⇒(2) we first obtain

$$\tfrac{1}{2}\|x + y\| < 1 - \tfrac{1}{2}\delta_X(\varepsilon - \delta)$$

and then, with (ii)_0,

$$2 < \big(2 - \delta_X(\varepsilon - \delta)\big) + \delta.$$

This contradicts the choice of δ and the claim is proved.

It remains to check that (2.3) holds for $\delta = \delta_X(\frac{2}{3}\varepsilon)$. But from $\|x_1 + x_2\| \geqslant 2\|x_1\| - \|x_1 - x_2\|$ we have, for all $0 < \eta \leqslant 2$,

$$\delta_X(\eta) \leqslant \inf\big\{1 - \tfrac{1}{2}\|x_1 + x_2\| : \ x_1, x_2 \in S_X, \ \|x_1 - x_2\| = \eta\big\} \leqslant \tfrac{1}{2}\eta.$$

Hence if $\delta = \delta_X(\frac{2}{3}\varepsilon)$, then $\delta \leqslant \frac{1}{2}\cdot\frac{2}{3}\varepsilon = \frac{1}{3}\varepsilon$ and consequently, $\delta = \delta_X(\varepsilon - \frac{1}{3}\varepsilon) \leqslant \delta_X(\varepsilon - \delta)$ by the monotonicity of δ_X. ∎

In a similar way one checks that the conclusion of (2) holds if (i), (ii), (iii) are satisfied for $\delta = \frac{1}{2}\delta_X(\frac{4}{5}\varepsilon)$.

Proof of Theorem 1.1. Assume, for a contradiction, that the theorem were false. Then, for some $0 < \varepsilon \leqslant 2$ and some bounded ε-separated sequence (x_n), there exists an $x \in S_X$ such that every subsequence (x_{n_j}) of (x_n) contains two further subsequences $(x_{n^{(1)}_{j_k}})$ and $(x_{n^{(2)}_{j_k}})$, with $n^{(1)}_{j_k} \neq n^{(2)}_{j_k}$ for all k, satisfying

$$\big\|x - \big(x_{n^{(1)}_{j_k}} - x_{n^{(2)}_{j_k}}\big)\big\| < 1 + \delta_\varepsilon \quad \text{for all } k, \tag{2.4}$$

where $\delta_\varepsilon := \delta_X(\frac{2}{3}\varepsilon)$.

Choose $x^* \in S_{X^*}$ with $\langle x, x^*\rangle = 1$. Since (x_n) is bounded we may pass to a subsequence (x_{n_j}) for which the limit $\lim_{j\to\infty}\langle x_{n_j}, x^*\rangle$ exists. We extract two further subsequences $(x_{n^{(1)}_{j_k}})$ and $(x_{n^{(2)}_{j_k}})$ of (x_{n_j}) satisfying (2.4) and put

$$\xi_k := x - \big(x_{n^{(1)}_{j_k}} - x_{n^{(2)}_{j_k}}\big).$$

Then $\|\xi_k\| < 1 + \delta_\varepsilon$ for all k and

$$\lim_{k\to\infty} \langle x - \xi_k, x^*\rangle = 0.$$

Hence $\lim_{k\to\infty}\langle \xi_k, x^*\rangle = 1$. In particular, $\|\xi_k\| > 1 - \delta_\varepsilon$ for large k. Thus,

$$\big|1 - \|\xi_k\|\big| < \delta_\varepsilon \quad \text{for large } k.$$

By Lemma 2.1 the conclusion of $(2)_0$ applies with $T := x^*$. As a result, for large k we obtain

$$\big\|x_{n^{(1)}_{j_k}} - x_{n^{(2)}_{j_k}}\big\| = \|x - \xi_k\| < \varepsilon.$$

But this contradicts the fact that (x_n) is ε-separated. ∎

Proof of Theorem 1.2. We start from an arbitrary 1-separated sequence $(\xi_n)_{n\in\mathbb{N}} \subseteq S_X$; a short and elementary construction of such sequences is given in the notes of [3, Chapter 1].

Let $\delta_1 := \delta_X(\frac{2}{3})$. By Ramsey's theorem [5], (ξ_n) has a subsequence (ξ_{n_j}) such that either

$$\|\xi_{n_j} - \xi_{n_k}\| \in [1, 1 + \tfrac{1}{2}\delta_1] \quad \text{for all } j \neq k$$

or

$$\|\xi_{n_j} - \xi_{n_k}\| \in (1 + \tfrac{1}{2}\delta_1, 2] \quad \text{for all } j \neq k.$$

In the second case we are done (take $x_j = \xi_{n_j}$ and recall that $\xi_{n_j} \in S_X$). Hence, after relabeling we may assume that

$$\|\xi_j - \xi_k\| \in [1, 1 + \tfrac{1}{2}\delta_1] \quad \text{for all } j \neq k. \tag{2.5}$$

Let $\phi : \mathbb{N} \to (\mathbb{N} \times \mathbb{N}) \setminus \mathbb{D}$ be a bijection, where $\mathbb{D} = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \ j = k\}$, and write $\phi(n) = (\phi_1(n), \phi_2(n))$. Put

$$x_0 := \frac{\xi_{\phi_1(0)} - \xi_{\phi_2(0)}}{\|\xi_{\phi_1(0)} - \xi_{\phi_2(0)}\|}.$$

Suppose next that integers $0 =: n_0 < \cdots < n_{m-1}$ have been chosen subject to the condition that the vectors

$$x_j := \frac{y_j}{\|y_j\|}, \quad 0 \leqslant j \leqslant m-1$$

where $y_j := \xi_{\phi_1(n_j)} - \xi_{\phi_2(n_j)}$, satisfy

$$\|x_j - x_k\| \geqslant 1 + \tfrac{1}{2}\delta_1 \quad \text{for all } 0 \leqslant j < k \leqslant m-1.$$

By Theorem 1.1, applied consecutively to the vectors $x = x_j$, $0 \leqslant j \leqslant m-1$, there exists an integer $n_m > n_{m-1}$ such that

$$\|x_j - y_m\| \geqslant 1 + \delta_1 \quad \text{for all } \ 0 \leqslant j \leqslant m-1,$$

where $y_m := \xi_{\phi_1(n_m)} - \xi_{\phi_2(n_m)}$. With

$$x_m := \frac{y_m}{\|y_m\|}$$

we have, for all $0 \leqslant j \leqslant m-1$,

$$\begin{aligned} \|x_j - x_m\| &\geqslant (1+\delta_1) - \Big\| y_m - \frac{y_m}{\|y_m\|} \Big\| \\ &= (1+\delta_1) - \big| \|y_m\| - 1 \big| \geqslant (1+\delta_1) - \tfrac{1}{2}\delta_1 = 1 + \tfrac{1}{2}\delta_1, \end{aligned}$$

where the last inequality follows from (2.5). Continuing this way we obtain a sequence $(x_n)_{n\in\mathbb{N}}$ with the desired properties. ∎

A Banach space X is called *locally uniformly rotund* [2] if for all $x \in S_X$ and all sequences $(x_n) \subseteq S_X$ with $\lim_{n\to\infty} \|x + x_n\|_X = 2$ we have $\lim_{n\to\infty} \|x - x_n\|_X = 0$. A characterization of locally uniformly rotund Banach spaces analogous to Theorem 1.3 holds; the numbers δ in (2) and (3) will now depend on ε and x. As a result, Theorem 1.1 remains true with a separation constant depending on x.

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