# CONDITIONS FOR STOCHASTIC INTEGRABILITY IN UMD BANACH SPACES

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Abstract. A detailed theory of stochastic integration in UMD Banach spaces has been developed recently in [14]. The present paper is aimed at giving various sufficient conditions for stochastic integrability.

## 1. Introduction

In the paper [14] we developed a detailed theory of stochastic integration in UMD Banach spaces and a number of necessary and sufficient conditions for stochastic integrability of processes with values in a UMD space were obtained. The purpose of the present paper is to complement these results by giving further conditions for stochastic integrability.

In Section 2, we prove a result announced in [14] on the strong approximation of stochastically integrable processes by elementary adapted processes. In Section 3 we prove two domination results. In Section 4 we state a criterium for stochastic integrability in terms of the smoothness of the trajectories of the process. This criterium is based on a recent embedding result due to Kalton and the authors [9]. In Section 5 we give an alternative proof of a special case of the embedding result from [9] and we prove a converse result which was left open there. In the final Section 6 we give square function conditions for stochastic integrability of processes with values in a Banach function spaces.

We follow the notations and terminology of the paper [14].

## 2. Approximation

Throughout this note, $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space endowed with a filtration $\mathbb{F} = (\mathscr{F}_t)_{t\in[0,T]}$ satisfying the usual conditions, $H$ is a separable real Hilbert space with inner product $[\cdot,\cdot]_H$, and $E$ is a real Banach space with norm $\|\cdot\|$. The dual of $E$ is denoted by $E^*$.

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We call an operator-valued stochastic process $\Phi : [0,T]\times\Omega \to \mathscr{L}(H,E)$ *elementary adapted* with respect to the filtration $\mathbb{F}$ if it is of the form

$$\Phi = \sum_{n=1}^{N}\sum_{m=1}^{M} \mathbf{1}_{(t_{n-1},t_n]\times A_{mn}} \sum_{k=1}^{K} h_k \otimes x_{kmn},$$

where $0 \leqslant t_0 < \cdots < t_N \leqslant T$ with the convention that $(t_{-1},t_0] = \{0\}$, the sets $A_{1n},\dots,A_{Mn} \in \mathscr{F}_{t_{n-1}}$ are disjoint for all $n = 1,\dots,N$, and the vectors $h_1,\dots,h_K \in H$ are orthonormal.

Let $W_H = (W_H(t))_{t\in[0,T]}$ be an *$H$-cylindrical Brownian motion*, i.e., each $W_H(t)$ is a bounded operator from $H$ to $L^2(\Omega)$, for all $h \in H$ the process $W_H h = (W_H(t)h)_{t\in[0,T]}$ is a Brownian motion, and for all $t_1,t_2 \in [0,T]$ and $h_1,h_2 \in H$ we have

$$\mathbb{E}(W_H(t_1)h_1\, W_H(t_2)h_2) = (t_1\wedge t_2)[h_1,h_2]_H.$$

We will always assume that $W_H$ is adapted to $\mathbb{F}$, i.e., each Brownian motion $W_H h$ is adapted to $\mathbb{F}$. The stochastic integral of an elementary adapted process $\Phi$ of the above form with respect to $W_H$ is defined in the obvious way as

$$\int_0^t \Phi\, dW_H = \sum_{n=1}^{N}\sum_{m=1}^{M} \mathbf{1}_{A_{mn}} \sum_{k=1}^{K} \big(W_H(t_n\wedge t)h_k - W_H(t_{n-1}\wedge t)h_k\big)\otimes x_{kmn}.$$

A process $\Phi : [0,T]\times\Omega \to \mathscr{L}(H,E)$ is called *$H$-strongly measurable* if for all $h \in H$, $\Phi h$ is strongly measurable. Similarly, $\Phi$ is *$H$-strongly adapted* if for all $h \in H$, $\Phi h$ is strongly adapted.

An $H$ strongly measurable and adapted process $\Phi : [0,T]\times\Omega \to \mathscr{L}(H,E)$ is called *stochastically integrable* with respect to $W_H$ if there exists a sequence of elementary adapted processes $\Phi_n : [0,T]\times\Omega \to \mathscr{L}(H,E)$ and a $\zeta : \Omega \to C([0,T];E)$ such that

(i) $\lim_{n\to\infty} \langle \Phi_n h, x^*\rangle = \langle \Phi h, x^*\rangle$ in measure for all $h \in H$ and $x^* \in E^*$;

(ii) $\lim_{n\to\infty} \int_0^{\cdot} \Phi_n\, dW_H = \zeta$ measure in $C([0,T];E)$.

The process $\zeta$ is uniquely determined almost surely. We call $\zeta$ the *stochastic integral* of $\Phi$, notation:

$$\zeta =: \int_0^{\cdot} \Phi\, dW_H.$$

It is an easy consequence of (i), (ii), and [8, Proposition 17.6] that if $\Phi$ is stochastically integrable, then for all $x^* \in E^*$ we have

$$\lim_{n\to\infty} \Phi_n^* x^* = \Phi^* x^* \ \text{ in } L^2(0,T;H) \text{ almost surely.}$$

For UMD spaces $E$ we show that in the definition of stochastic integrability it is possible to strengthen the convergence of the processes $\Phi_n h$ in (i) to strong convergence in measure. The main result of this section was announced without proof in [14] and is closely related to a question raised by McConnell [13, page 290].

**Theorem 2.1.** *Let $E$ be a UMD space. If the process $\Phi : [0,T]\times\Omega \to \mathscr{L}(H,E)$ is $H$-strongly measurable and adapted and stochastically integrable with respect to $W_H$, there exists a sequence of elementary adapted processes $\Phi_n : [0,T]\times\Omega \to \mathscr{L}(H,E)$ such that*

(i)′ $\lim_{n\to\infty} \Phi_n h = \Phi h$ *in measure for all $h \in H$;*

(ii) $\lim_{n\to\infty}\int_0^{\cdot} \Phi_n\,dW_H = \int_0^{\cdot} \Phi\,dW_H$ *in measure in* $C([0,T];E)$.

For the definition of the class of UMD Banach spaces and some of its applications in Analysis we refer to Burkholder's review article [4].

Let $\mathscr{H}$ be a separable real Hilbert space and let $(g_n)_{n\geqslant 1}$ be a sequence of independent standard Gaussian random variables on a probability space $(\Omega', \mathscr{F}', \mathbb{P}')$. A linear operator $R : \mathscr{H} \to E$ is said to be $\gamma$*-radonifying* if for some (every) orthonormal basis $(h_n)_{n\geqslant 1}$ of $\mathscr{H}$ the Gaussian sum $\sum_{n\geqslant 1} g_n\, Rh_n$ converges in $L^2(\Omega';E)$. The linear space of all $\gamma$-radonifying operators from $\mathscr{H}$ to $E$ is denoted by $\gamma(\mathscr{H},E)$. This is space is a Banach space endowed with the norm

$$\|R\|_{\gamma(\mathscr{H},E)} := \Big(\mathbb{E}'\Big\|\sum_{n\geqslant 1} g_n\, Rh_n\Big\|^2\Big)^{\frac12}.$$

For more information we refer to [3, 5, 10]. The importance of spaces of $\gamma$-radonifying operators in the theory of stochastic integration in infinite dimensions is well established; see [14, 15] and the references given therein.

An $H$-strongly measurable function $\Phi : [0,T] \to \mathscr{L}(H,E)$ is said to *represent* an element $R \in \gamma(L^2(0,T;H),E)$ if for all $x^* \in E^*$ we have $\Phi^* x^* \in L^2(0,T;H)$ and, for all $f \in L^2(0,T;H)$,

$$\langle Rf, x^*\rangle = \int_0^T [f(t), \Phi^*(t)x^*]\,dt.$$

Extending the above definition, we say that an $H$-strongly measurable process $\Phi : [0,T]\times\Omega \to \mathscr{L}(H,E)$ *represents* a random variable $X : \Omega \to \gamma(L^2(0,T;H),E)$ if for all $x^* \in E^*$ almost surely we have $\Phi^*x^* \in L^2(0,T;H)$ and, for all $f \in L^2(0,T;H)$,

$$\langle Xf, x^*\rangle = \int_0^T [f(t), \Phi^*(t)x^*]_H\,dt \quad \text{almost surely.}$$

For $H$-strongly measurable process we have the following simple result [14, Lemma 2.7].

**Lemma 2.2.** *Let* $\Phi : [0,T]\times\Omega \to \mathscr{L}(H,E)$ *be an* $H$*-strongly measurable process and let* $X : \Omega \to \gamma(L^2(0,T;H),E)$ *be strongly measurable. The following assertions are equivalent:*

(1) $\Phi$ *represents* $X$.
(2) $\Phi(\cdot,\omega)$ *represents* $X(\omega)$ *for almost all* $\omega \in \Omega$.

For a Banach space $F$ we denote by $L^0(\Omega;F)$ the vector space of all $F$-valued random variables on $\Omega$, identifying random variables if they agree almost surely. Endowed with the topology of convergence in measure, $L^0(\Omega;F)$ is a complete metric space. The following result is obtained in [14].

**Proposition 2.3.** *Let* $E$ *be a UMD space. For an* $H$*-strongly measurable and adapted process* $\Phi : [0,T]\times\Omega \to \mathscr{L}(H,E)$ *the following assertions are equivalent:*

(1) $\Phi$ *is stochastically integrable with respect to* $W_H$*;*
(2) $\Phi$ *represents a random variable* $X : \Omega \to \gamma(L^2(0,T;H),E)$.

*In this case $\Phi h$ is stochastically integrable with respect to $W_H h$ for all $h \in H$, and for every orthonormal basis $(h_n)_{n\geqslant 1}$ of $H$ we have*

$$\int_0^{\cdot} \Phi \, dW_H = \sum_{n\geqslant 1} \int_0^{\cdot} \Phi h_n \, dW_H h_n,$$

*with almost sure unconditional convergence of the series expansion in $C([0,T];E)$. Moreover, for all $p \in (1,\infty)$*

$$\mathbb{E} \sup_{t\in[0,T]} \Big\| \int_0^t \Phi(s) \, dW_H(s) \Big\|^p \eqsim_{p,E} \mathbb{E} \|X\|^p_{\gamma(L^2(0,T;H),E)}.$$

*Furthermore, the mapping $X \mapsto \int_0^{\cdot} \Phi \, dW_H$ has a unique extension to a continuous mapping*

$$L^0(\Omega;\gamma(L^2(0,T;H),E)) \to L^0(\Omega;C([0,T];E)).$$

As we will show in a moment, the series expansion in Proposition 2.3 implies that in order to prove Theorem 2.1 it suffices to prove the following weaker version of the theorem:

**Theorem 2.4.** *Let $E$ be a UMD space. If the process $\phi : [0,T]\times\Omega \to E$ is strongly measurable and adapted and stochastically integrable with respect to a Brownian motion $W$, there exists a sequence of elementary adapted processes $\phi_n : [0,T]\times\Omega \to E$ such that*

(i)′ $\lim_{n\to\infty} \phi_n = \phi$ *in measure;*

(ii) $\lim_{n\to\infty} \int_0^{\cdot} \phi_n \, dW = \int_0^{\cdot} \phi \, dW$ *in measure in $C([0,T];E)$.*

This theorem may actually be viewed as the special case of Theorem 2.1 corresponding to $H = \mathbb{R}$, by identifying $\mathscr{L}(\mathbb{R},E)$ with $E$ and identifying $\mathbb{R}$-cylindrical Brownian motions with real-valued Brownian motions.

To see that Theorem 2.1 follows from Theorem 2.4 we argue as follows. Choose an orthonormal basis $(h_n)_{n\geqslant 1}$ of $H$ and define the processes $\Psi_n : [0,T]\times\Omega \to \mathscr{L}(H,E)$ by

$$\Psi_n h := \sum_{j=1}^n [h,h_j]_H \, \Phi h_j.$$

Clearly, $\lim_{n\to\infty} \Psi_n h = \Phi h$ pointwise, hence in measure, for all $h \in H$. In view of the identity

$$\int_0^{\cdot} \Psi_n \, dW_H = \sum_{j=1}^n \int_0^{\cdot} \Phi h_j \, dW_H h_j$$

and the series expansion in Proposition 2.3, we also have

$$\lim_{n\to\infty} \int_0^{\cdot} \Psi_n \, dW_H = \int_0^{\cdot} \Phi \, dW_H \quad \text{in measure in } C([0,T];E).$$

With Theorem 2.4, for each $n \geqslant 1$ we choose a sequence of elementary adapted processes $\phi_{j,n} : [0,T]\times\Omega \to E$ such that $\lim_{j\to\infty} \phi_{j,n} = \Phi h_n$ in measure and

$$\lim_{j\to\infty} \int_0^{\cdot} \phi_{j,n} \, dW_H h_n = \int_0^{\cdot} \Phi h_n \, dW_H h_n \quad \text{in measure in } C([0,T];E).$$

Given $k \geqslant 1$, choose $N_k \geqslant 1$ so large that

$$\mathbb{P}\Big\{\Big\| \int_0^\cdot \Phi - \Psi_{N_k}\, dW_H \Big\|_\infty > \frac{1}{k}\Big\} < \frac{1}{k}.$$

Let $\lambda$ be denoted for the Lebesgue measure on $[0,T]$. For each $n = 1,\dots,N_k$ choose $j_{k,n} \geqslant 1$ so large that

$$\lambda \otimes \mathbb{P}\Big\{\|\Phi h_n - \phi_{j_{k,n},n}\| > \frac{1}{kN_k}\Big\} < \frac{1}{kN_k}$$

and

$$\mathbb{P}\Big\{\Big\| \int_0^\cdot \Phi h_n - \phi_{j_{k,n},n}\, dW_H h_n \Big\|_\infty > \frac{1}{k}\Big\} < \frac{1}{kN_k}.$$

Define $\Phi_k : [0,T]\times\Omega \to \mathscr{L}(H,E)$ by

$$\Phi_k h := \sum_{n=1}^{N_k} [h, h_n]_H\, \phi_{j_{k,n},n}, \qquad h \in H.$$

Each $\Phi_k$ is elementary adapted. For all $h \in H$ with $\|h\|_H = 1$ and all $\delta > 0$ we have, for all $k \geqslant 1/\delta$,

$$\begin{aligned} &|\{\|\Phi h - \Phi_k h\| > 2\delta\}| \\ &\qquad \leqslant \lambda\otimes\mathbb{P}\{\|\Phi h - \Psi_{N_k} h\| > \delta\} + \lambda\otimes\mathbb{P}\{\|\Psi_{N_k} h - \Phi_k h\| > \delta\} \\ &\qquad < \lambda\otimes\mathbb{P}\{\|\Phi h - \Psi_{N_k} h\| > \delta\} + \frac{1}{k}. \end{aligned}$$

Hence, $\lim_{k\to\infty} \Phi_k h = \Phi h$ in measure for all $h \in H$. Also,

$$\begin{aligned} &\mathbb{P}\Big\{\Big\| \int_0^T \Phi - \Phi_k\, dW_H \Big\|_\infty > \frac{2}{k}\Big\} \\ &\quad \leqslant \mathbb{P}\Big\{\Big\| \int_0^T \Phi - \Psi_{N_k}\, dW_H \Big\|_\infty > \frac{1}{k}\Big\} + \mathbb{P}\Big\{\Big\| \int_0^T \Psi_{N_k} - \Phi_k\, dW_H \Big\|_\infty > \frac{1}{k}\Big\} \\ &\quad < \frac{1}{k} + \frac{1}{k} = \frac{2}{k}, \end{aligned}$$

and therefore $\displaystyle\lim_{k\to\infty} \int_0^\cdot \Phi_k\, dW_H = \int_0^\cdot \Phi\, dW_H$ in measure in $C([0,T];E)$. Thus the processes $\Phi_k$ have the desired properties.

This matter having been settled, the remainder of the section is aimed at proving Theorem 2.4. The following argument will show that it suffices to prove Theorem 2.4 for *uniformly bounded* processes $\phi$. To see why, for $n \geqslant 1$ define

$$\phi_n := \mathbf{1}_{\{\|\phi\| \leqslant n\}}\phi.$$

The processes $\phi_n$ are uniformly bounded, strongly measurable and adapted, and we have $\lim_{n\to\infty} \phi_n = \phi$ pointwise, hence also in measure.

We claim that each $\phi_n$ is stochastically integrable with respect to $W$ and

$$\lim_{n\to\infty} \int_0^\cdot \phi_n\, dW = \int_0^\cdot \phi\, dW \quad \text{in measure in } C([0,T];E).$$

To see this, let $X : \Omega \to \gamma(L^2(0,T),E)$ be the element represented by $\phi$. Put

$$X_n(\omega) f := X(\omega)(\mathbf{1}_{\{\|\phi(\cdot,\omega)\| \leqslant n\}} f), \qquad f \in L^2(0,T).$$

Then by [14, Proposition 2.4], $\lim_{n\to\infty} X_n = X$ almost surely in $\gamma(L^2(0,T),E)$. It is easily checked that $\phi_n$ represents $X_n$, and therefore $\phi_n$ is stochastically integrable by Proposition 2.3. The convergence in measure of the stochastic integrals now follows from the continuity assertion in Proposition 2.3. This completes the proof of the claim. A more general result in this spirit will be proved in Section 3.

It remains to prove Theorem 2.4 for uniformly bounded processes $\Phi$.

Let $\mathscr{D}_n$ denote the finite $\sigma$-field generated by the $n$-th dyadic equipartition of the interval $[0,T]$ and let $\mathscr{G}_n = \mathscr{D}_n \otimes \mathscr{F}$ be the product $\sigma$-field in $[0,T]\times\Omega$. Then $\mathbb{G} = \{\mathscr{G}_n\}_{n\geqslant 1}$ is a filtration in $[0,T]\times\Omega$ with $\bigvee_{n\geqslant 1}\mathscr{G}_n = \mathscr{B}\otimes\mathscr{F}$, where $\mathscr{B}$ is the Borel $\sigma$-algebra of $[0,T]$. In what follows with think of $[0,T]\times\Omega$ as probability space with respect to the product measure $\frac{dt}{T}\otimes\mathbb{P}$. Note that for all $f\in L^2([0,T]\times\Omega;E)$, for almost all $\omega\in\Omega$ we have

$$\mathbb{E}(f|\mathscr{G}_n)(\cdot,\omega) = \mathbb{E}(f(\cdot,\omega)|\mathscr{D}_n) \quad \text{in } L^2(0,T;E).$$

Define the operators $G_n$ on $L^2([0,T]\times\Omega;E)$ by

$$G_n f := \tau_n \mathbb{E}(f|\mathscr{G}_n),$$

where $\tau_n$ denotes the right translation operator over $2^{-n}T$ in $L^2([0,T]\times\Omega;E)$, i.e., $\tau_n f(t,\omega) = \mathbf{1}_{[2^{-n}T,T]} f(t-2^{-n}T,\omega)$.

**Lemma 2.5.** *Let $\phi:[0,T]\times\Omega\to E$ be strongly measurable, adapted, uniformly bounded, and stochastically integrable with respect to $W$. Then the processes $\phi_n : [0,T]\times\Omega\to E$ defined by $\phi_n := G_n\phi$ are strongly measurable, adapted, uniformly bounded, and stochastically integrable with respect to $W$. Moreover, $\lim_{n\to\infty}\phi_n = \phi$ in measure and*

$$\lim_{n\to\infty}\int_0^{\cdot}\phi_n\,dW = \int_0^{\cdot}\phi\,dW \quad \textit{in measure in } C([0,T];E). \tag{2.1}$$

*Proof.* First note that each process $\phi_n$ is strongly measurable, uniformly bounded, strongly measurable and adapted. By the vector-valued martingale convergence theorem and the strong continuity of translations we have

$$\begin{aligned}
&\lim_{n\to\infty}\|\phi-\phi_n\|_{L^2([0,T]\times\Omega;E)}\\
&\quad\leqslant \lim_{n\to\infty}\|\phi-\tau_n\phi\|_{L^2([0,T]\times\Omega;E)} + \lim_{n\to\infty}\|\tau_n\phi-\tau_n\mathbb{E}(\phi|\mathscr{G}_n)\|_{L^2([0,T]\times\Omega;E)}\\
&\quad\leqslant \lim_{n\to\infty}\|\phi-\tau_n\phi\|_{L^2([0,T]\times\Omega;E)} + \lim_{n\to\infty}\|\phi-\mathbb{E}(\phi|\mathscr{G}_n)\|_{L^2([0,T]\times\Omega;E)} = 0.
\end{aligned}$$

It follows that $\lim_{n\to\infty}\phi_n = \phi$ in $L^2([0,T]\times\Omega;E)$, and therefore also in measure.

Let $X:\Omega\to\gamma(L^2(0,T),E)$ be the random variable represented by $\phi$. For all $n\geqslant 1$ let the random variable $X_n:\Omega\to\gamma(L^2(0,T),E)$ defined by

$$X_n(\omega) := X(\omega)\circ\tau_n^*\circ\mathbb{E}(\,\cdot\,|\mathscr{D}_n)$$

where $\tau_n^*\in\mathscr{L}(L^2(0,T))$ denotes the left translation operator. It is easily seen that for all $n\geqslant 1$, $X_n$ is represented by $\phi_n$, and therefore $\phi_n$ is stochastically integrable with respect to $W$ by Proposition 2.3. By [14, Proposition 2.4] we obtain $\lim_{j\to\infty} X_n = X$ almost surely in $\gamma(L^2(0,T),E)$. Hence, $\lim_{n\to\infty}X_n = X$ in measure in $\gamma(L^2(0,T),E)$, and (2.1) follows from the continuity assertion in Proposition 2.3. ◻

Now we can complete the proof of Theorem 2.4 for uniformly bounded processes $\phi$. The processes $\phi_n$ in Lemma 2.5 can be represented as

$$\phi_n = \sum_{j=1}^{2^n} \mathbf{1}_{I_j} \phi_{j,n},$$

where the $I_j$ is the $j$-th interval in the $n$-th dyadic partition of $[0,T]$ and the random variable $\phi_{j,n} : \Omega \to E$ is uniformly bounded and $\mathscr{F}_j$-measurable, where $\mathscr{F}_j = \mathscr{F}_{2^{-n}(j-1)T}$. The proof is completed by approximating the $\phi_{j,n}$ in $L^0(\Omega, \mathscr{F}_j; E)$ with simple random variables.

Let $E_1$ and $E_2$ be real Banach spaces. Theorem 2.1 can be strengthened for $\mathscr{L}(E_1, E_2)$-valued processes which are integrable with respect to an $E_1$-valued Brownian motion.

Let $\mu$ be a centred Gaussian Radon measure on $E_1$ and let $W_\mu$ be an $E_1$-valued Brownian motion with distribution $\mu$, i.e., for all $t \geqslant 0$ and $x^* \in E_1^*$ we have

$$\mathbb{E}\langle W_\mu(t), x^*\rangle^2 = t \int_{E_1} \langle x, x^*\rangle^2 \, d\mu(x).$$

Let $H_\mu$ denote the reproducing kernel Hilbert space associated with $\mu$ and let $i_\mu : H_\mu \hookrightarrow E_1$ be the inclusion operator. We can associate an $H_\mu$-cylindrical Brownian motion $W_{H_\mu}$ with $W_\mu$ by the formula

$$W_{H_\mu}(t) i_\mu^* x^* := \langle W_\mu(t), x^*\rangle.$$

We say $\Phi : [0,T] \times \Omega \to \mathscr{L}(E_1, E_2)$ is $E_1$-*strongly measurable and adapted* if for all $x \in E_1$, $\Phi x$ is strongly measurable and adapted. An $E_1$-strongly measurable and adapted process $\Phi : [0,T] \times \Omega \to \mathscr{L}(E_1, E_2)$ is called *stochastically integrable* with respect to the $E_1$-valued Brownian motion $W_\mu$ if the process $\Phi \circ i_\mu : [0,T] \times \Omega \to \mathscr{L}(H_\mu, E_2)$ is stochastically integrable with respect to $W_{H_\mu}$. In this case we write

$$\int_0^\cdot \Phi \, dW_\mu := \int_0^\cdot \Phi \circ i_\mu \, dW_{H_\mu}.$$

By the Pettis measurability theorem and the separability of $H_\mu$, the $E_1$-strong measurability of $\Phi$ implies the $H_\mu$-strong measurability of $\Phi \circ i_\mu$. We call $\Phi$ an *elementary adapted* process if $\Phi \circ i_\mu$ is elementary adapted.

**Theorem 2.6.** *Let $E_1$ be a Banach space and let $E_2$ be a UMD space and fix $p \in (1,\infty)$. Let $W_\mu$ be an $E_1$-valued Brownian motion with distribution $\mu$. If the process $\Phi : [0,T] \times \Omega \to \mathscr{L}(E_1, E_2)$ is $E_1$-strongly measurable and adapted and stochastically integrable with respect to $W_\mu$, there exists a sequence of elementary adapted processes $\Phi_n : [0,T] \times \Omega \to \mathscr{L}(E_1, E_2)$ such that*

(i)″ $\lim_{n\to\infty} \Phi_n x = \Phi x$ *in measure for $\mu$-almost all $x \in E_1$;*

(ii) $\lim_{n\to\infty} \int_0^\cdot \Phi_n \, dW_\mu = \int_0^\cdot \Phi \, dW_\mu$ *in measure in $C([0,T]; E_2)$.*

The proof depends on some well known facts about measurable linear extensions. We refer to [3, 6] for more details. If $\mu$ is a centred Gaussian Radon measure on $E_1$ with reproducing kernel Hilbert space $H_\mu$ and $(h_n)_{n\geqslant 1}$ is an orthonormal basis $(h_n)_{n\geqslant 1}$ for $H_\mu$, then the coordinate functionals $h \mapsto [h, h_n]_{H_\mu}$ can be extended to $\mu$-measurable linear mappings $\overline{h_n}$ from $E_1$ to $\mathbb{R}$. Moreover, these extensions are $\mu$-essentially unique in the sense that every two such extensions agree $\mu$-almost everywhere. Putting

$$\overline{P_n}x := \sum_{j=1}^{n} \overline{h_j}x\, h_j, \qquad x \in E_1,$$

we obtain a $\mu$-measurable linear extension of the orthogonal projection $P_n$ in $H_\mu$ onto the span of the vectors $h_1, \dots, h_n$. Again this extension is $\mu$-essentially unique, and we have

$$\lim_{n\to\infty} i_\mu \overline{P_n}x = \sum_{n\geqslant 1} \overline{h_j}x\, i_\mu h_j = x \ \text{ for } \mu\text{-almost all } x \in E_1. \tag{2.2}$$

*Proof of Theorem 2.6.* We will reduce the theorem to Theorem 2.4. Choose an orthonormal basis $(h_n)_{n\geqslant 1}$ of the reproducing kernel Hilbert space $H_\mu$ and define the processes $\Psi_n : [0,T]\times\Omega \to \mathscr{L}(E_1, E_2)$ by

$$\Psi_n x := \Phi i_\mu \overline{P_n}x, \qquad x \in E_1.$$

By (2.2),

$$\lim_{n\to\infty} \Psi_n x = \Phi x \ \text{ in measure for } \mu\text{-almost all } x \in E_1.$$

Also,

$$\begin{aligned}\lim_{n\to\infty} \int_0^{\cdot} \Psi_n\, dW_\mu &= \lim_{n\to\infty} \int_0^{\cdot} \Psi_n \circ i_\mu\, dW_{H_\mu} \\ &\overset{(*)}{=} \int_0^{\cdot} \Phi \circ i_\mu\, dW_{H_\mu} = \int_0^{\cdot} \Phi\, dW_\mu \quad \text{in measure in } C([0,T]; E_2),\end{aligned}$$

where the identity $(*)$ follows by series representation as in the argument following the statement of Theorem 2.4. The proof may now be completed along the lines of this argument; for $\Phi_k$ we take

$$\Phi_k x := \sum_{n=1}^{N_k} \overline{h_{N_k}}x\, \phi_{j_{k,n},n}, \qquad x \in E_1,$$

where the elementary adapted processes $\phi_{j,n}$ approximate $\Phi i_\mu h_n$ and the indices $N_k$ are chosen as before. $\square$

As a final comment we note that $L^p$-versions of the results of this section hold as well; for these one has to replace almost sure convergence by $L^p$-convergence in the proofs.

## 3. Domination

In this section we present two domination results which were implicit in the arguments so far, and indeed some simple special cases of them have already been used.

The first comparison result extends [15, Corollary 4.4], where the case of functions was considered.

**Theorem 3.1** (Domination)**.** *Let $E$ be a UMD space. Let $\Phi, \Psi : [0,T]\times\Omega \to \mathscr{L}(H,E)$ be $H$-strongly measurable and adapted processes and assume that $\Psi$ is stochastically integrable with respect to $W_H$. If for all $x^* \in E^*$ we have*

$$\int_0^T \|\Phi^*(t)x^*\|_H^2\, dt \leqslant \int_0^T \|\Psi^*(t)x^*\|_H^2\, dt \quad \textit{almost surely,}$$

*then $\Phi$ is stochastically integrable and for all $p \in (1,\infty)$,*

$$\mathbb{E} \sup_{t\in[0,T]} \Big\| \int_0^t \Phi(s)\, dW_H(s) \Big\|^p \lesssim_{p,E} \mathbb{E} \sup_{t\in[0,T]} \Big\| \int_0^t \Psi(s)\, dW_H(s) \Big\|^p,$$

*whenever the right hand side is finite.*

*Proof.* Since $\Phi$ and $\Psi$ are $H$-strongly measurable and adapted, without loss of generality we may assume that $E$ is separable.

By Proposition 2.3, $\Psi$ represents a random variable $Y : \Omega \to \gamma(L^2(0,T;H),E)$. In particular, for all $x^* \in E^*$ we have $\Psi^* x^* \in L^2(0,T;H)$ almost surely. We claim that almost surely,

$$\int_0^T \|\Phi^*(t)x^*\|_H^2\, dt \leqslant \int_0^T \|\Psi^*(t)x^*\|_H^2\, dt \ \text{ for all } x^* \in E^*.$$

Indeed, by the reflexivity and separability of $E$ we may choose a countable, norm dense, $\mathbb{Q}$-linear subspace $F$ of $E^*$. Let $N_1$ be a null set such that

$$\int_0^T \|\Phi^*(t,\omega)x^*\|_H^2\, dt \leqslant \int_0^T \|\Psi^*(t,\omega)x^*\|_H^2\, dt \tag{3.1}$$

for all $\omega \in \complement N_1$ and all $x^* \in F$. By Lemma 2.2 there exists a null set $N_2$ such that $\Psi(\cdot,\omega)$ represents $Y(\omega)$ for all $\omega \in \complement N_2$. Fix $y^* \in E^*$ arbitrary and choose a sequence $(y_n^*)_{n\geqslant 1}$ in $F$ such that $\lim_{n\to\infty} y_n^* = y^*$ in $E^*$ strongly. Fix an arbitrary $\omega \in \complement(N_1 \cup N_2)$. We will prove the claim by showing that

$$\int_0^T \|\Phi^*(t,\omega)y^*\|_H^2\, dt \leqslant \int_0^T \|\Psi^*(t,\omega)y^*\|_H^2\, dt, \tag{3.2}$$

By the closed graph theorem there exists a constant $C_\omega$ such that

$$\|\Psi^*(\cdot,\omega)x^*\|_{L^2(0,T;H)} \leqslant C_\omega \|x^*\| \ \text{ for all } x^* \in E^*.$$

Hence, $\Psi^*(\cdot,\omega)y^* = \lim_{n\to\infty} \Psi^*(\cdot,\omega)y_n^*$ in $L^2(0,T;H)$, by the strong convergence of the $y_n^*$'s to $y^*$. It follows from (3.1), applied to the functionals $y_n^* - y_m^* \in F$, that $(\Phi^* y_n^*)_{n\geqslant 1}$ is a Cauchy sequence in $L^2(0,T;H)$. Identification of the limit shows that $\Phi^*(\cdot,\omega)y^* = \lim_{n\to\infty} \Phi^*(\cdot,\omega)y_n^*$ in $L^2(0,T;H)$. Now (3.2) follows from the corresponding inequality for $y_n^*$ by letting $n \to \infty$.

By the claim and [15, Theorem 4.2 and Corollary 4.4], almost every function $\Phi(\cdot,\omega)$ represents an element $X(\omega) \in \gamma(L^2(0,T;H),E)$ for which we have

$$\|X(\omega)\|_{\gamma(L^2(0,T;H),E)} \leqslant \|Y(\omega)\|_{\gamma(L^2(0,T;H),E)}.$$

By [14, Remark 2.8]) $X$ is strongly measurable as a $\gamma(L^2(0,T;H),E)$-valued random variable. Since $\Phi$ represents $X$, $\Phi$ is stochastically integrable by Proposition 2.3. Moreover, from Proposition 2.3 we deduce that

$$\begin{aligned}\mathbb{E} \sup_{t\in[0,T]} \Big\| \int_0^t \Phi(s)\, dW_H(s) \Big\|^p &\eqsim_{p,E} \mathbb{E}\|X\|^p_{\gamma(L^2(0,T;H),E)} \leqslant \mathbb{E}\|Y\|^p_{\gamma(L^2(0,T;H),E)} \\ &\eqsim_{p,E} \mathbb{E} \sup_{t\in[0,T]} \Big\| \int_0^t \Psi(s)\, dW_H(s) \Big\|^p.\end{aligned}$$

□

The next result extends [15, Theorem 6.2].

**Corollary 3.2** (Dominated convergence). *Let $E$ be a UMD space and fix $p \in (1,\infty)$. For $n \geqslant 1$, let $\Phi_n : [0,T]\times\Omega \to \mathscr{L}(H,E)$ be $H$-strongly measurable and adapted and stochastically integrable processes and assume that there exists an $H$-strongly measurable and adapted process $\Phi : [0,T]\times\Omega \to \mathscr{L}(H,E)$ such that for all $x^* \in E^*$,*

$$\lim_{n\to\infty} \Phi_n^* x^* = \Phi^* x^* \quad \textit{almost surely in } L^2(0,T;H). \tag{3.3}$$

*Assume further that there exists an $H$-strongly measurable and adapted process $\Psi : [0,T]\times\Omega \to \mathscr{L}(H,E)$ that is stochastically integrable and for all $n$ and all $x^* \in E^*$,*

$$\int_0^T \|\Phi_n^*(t)x^*\|_H^2\,dt \leqslant \int_0^T \|\Psi^*(t)x^*\|_H^2\,dt \quad \textit{almost surely.} \tag{3.4}$$

*Then $\Phi$ is stochastically integrable and*

$$\lim_{n\to\infty} \int_0^{\cdot} \Phi_n - \Phi\,dW_H = 0 \quad \textit{in measure in } C([0,T];E).$$

*Proof.* The assumptions (3.3) and (3.4) imply that for all $n$ and $x^* \in E^*$,

$$\int_0^T \|\Phi_n^*(t)x^*\|_H^2\,dt \leqslant \int_0^T \|\Psi^*(t)x^*\|_H^2\,dt \quad \text{almost surely.} \tag{3.5}$$

Theorem 3.1 therefore implies that each $\Phi_n$ is stochastically integrable, and by passing to the limit $n \to \infty$ in (3.5) we see that the same is true for $\Phi$. Let $Z_n : \Omega \to \gamma(L^2(0,T;H),E)$ be the element represented by $\Phi_n - \Phi$. By Proposition 2.3 it suffices to prove that

$$\lim_{n\to\infty} Z_n = 0 \quad \text{in measure in } \gamma(L^2(0,T;H),E). \tag{3.6}$$

As in the proof of Theorem 3.1, (3.4) implies that for almost all $\omega \in \Omega$,

$$\int_0^T \|\Phi_n^*(t,\omega)x^*\|_H^2\,dt \leqslant \int_0^T \|\Psi^*(t,\omega)x^*\|_H^2\,dt \quad \text{for all } n \geqslant 1 \text{ and } x^* \in E^*, \tag{3.7}$$

and

$$\int_0^T \|\Phi^*(t,\omega)x^*\|_H^2\,dt \leqslant \int_0^T \|\Psi^*(t,\omega)x^*\|_H^2\,dt \quad \text{for all } n \geqslant 1 \text{ and } x^* \in E^*. \tag{3.8}$$

Denoting by $Y : \Omega \to \gamma(L^2(0,T;H),E)$ the element represented by $\Psi$, we obtain that, for almost all $\omega \in \Omega$, for all $x^* \in E^*$,

$$\|Z_n^*(\omega)x^*\|_{L^2(0,T;H)} \leqslant 2\|Y^*(\omega)x^*\|_{L^2(0,T;H)} \tag{3.9}$$

Let $N_1$ be a null set such that (3.7) and (3.8) hold for all $\omega \in \complement N_1$. Then for all $\omega \in \complement N_1$ there is a constant $C(\omega)$ such that for all $x^* \in E^*$ and all $n \geqslant 1$,

$$\int_0^T \|\Phi^*(t,\omega) - \Phi_n^*(t,\omega)x^*\|_H^2\,dt \leqslant C^2(\omega)\|x^*\|^2. \tag{3.10}$$

Let $(x_j^*)_{j\geqslant 1}$ be a dense sequence in $E^*$. By (3.3) we can find a null set $N_2$ such that for all $\omega \in \complement N_2$ and all $j \geqslant 1$ we have

$$\lim_{n\to\infty} \Phi_n^*(\cdot,\omega)x_j^* = \Phi^*(\cdot,\omega)x_j^* \quad \text{in } L^2(0,T;H). \tag{3.11}$$

Clearly, (3.10) and (3.11) imply that for all $\omega \in \complement(N_1 \cup N_2)$ we have

$$\lim_{n\to\infty} \Phi_n^*(\cdot,\omega)x^* = \Phi^*(\cdot,\omega)x^* \text{ in } L^2(0,T;H) \quad \text{for all } x^* \in E^*,$$

hence for almost all $\omega \in \Omega$, for all $x^* \in E^*$,

$$\lim_{n\to\infty} Z_n^*(\omega)x^* = 0 \text{ in } L^2(0,T;H). \tag{3.12}$$

By (3.9) and (3.12) and a standard tightness argument as in [15, Theorem 6.2] we obtain that for almost all $\omega \in \Omega$, $\lim_{n\to\infty} Z_n(\omega) = 0$ in $\gamma(L^2(0,T;H),E)$. This gives (3.6). □

Again we leave it to the reader to formulate the $L^p$-version of these results.

## 4. Smoothness - I

Extending a result of Rosiński and Suchanecki (who considered the case $H = \mathbb{R}$), it was shown in [15] (for arbitrary Banach spaces $E$ and functions $\Phi$) and [14] (for UMD Banach spaces and processes $\Phi$) that if $E$ is a Banach space with type 2, then every $H$-strongly measurable and adapted process $\Phi : [0,T]\times\Omega \to \mathscr{L}(H,E)$ with trajectories in $L^2(0,T;\gamma(H,E))$ is stochastically integrable with respect to an $H$-cylindrical Brownian motion $W_H$. Moreover, for $H = \mathbb{R}$ this property characterises the spaces $E$ with type 2. Below (Theorem 4.2) we shall obtain an extension of this result for processes in UMD spaces with type $p \in [1,2)$.

The results will be formulated in terms of vector valued Besov spaces. We briefly recall the definition. We follow the approach of Peetre; see [19, Section 2.3.2] (where the scalar-valued case is considered) and [1, 7, 18]. The Fourier transform of a function $f \in L^1(\mathbb{R}^d;E)$ will be normalized as

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} f(x)e^{-ix\cdot\xi}\,dx, \quad \xi \in \mathbb{R}^d.$$

Let $\phi \in \mathscr{S}(\mathbb{R}^d)$ be a fixed Schwartz function whose Fourier transform $\widehat{\phi}$ is nonnegative and has support in $\{\xi \in \mathbb{R}^d : \frac12 \leqslant |\xi| \leqslant 2\}$ and which satisfies

$$\sum_{k\in\mathbb{Z}} \widehat{\phi}(2^{-k}\xi) = 1 \quad \text{for } \xi \in \mathbb{R}^d\setminus\{0\}.$$

Define the sequence $(\varphi_k)_{k\geqslant 0}$ in $\mathscr{S}(\mathbb{R}^d)$ by

$$\widehat{\varphi_k}(\xi) = \widehat{\phi}(2^{-k}\xi) \quad \text{for} \quad k = 1,2,\dots \quad \text{and} \quad \widehat{\varphi_0}(\xi) = 1 - \sum_{k\geqslant 1}\widehat{\varphi_k}(\xi), \quad \xi \in \mathbb{R}^d.$$

For $1 \leqslant p,q \leqslant \infty$ and $s \in \mathbb{R}$ the *Besov space* $B^s_{p,q}(\mathbb{R}^d;E)$ is defined as the space of all $E$-valued tempered distributions $f \in \mathscr{S}'(\mathbb{R}^d;E)$ for which

$$\|f\|_{B^s_{p,q}(\mathbb{R}^d;E)} := \Big\|\big(2^{ks}\varphi_k * f\big)_{k\geqslant 0}\Big\|_{l^q(L^p(\mathbb{R}^d;E))}$$

is finite. Endowed with this norm, $B^s_{p,q}(\mathbb{R}^d;E)$ is a Banach space, and up to an equivalent norm this space is independent of the choice of the initial function $\phi$. The sequence $(\varphi_k * f)_{k\geqslant 0}$ is called the *Littlewood-Paley decomposition* of $f$ associated with the function $\phi$.

Next we define the Besov space for domains. Let $D$ be a nonempty bounded open domain in $\mathbb{R}^d$. For $1 \leqslant p,q \leqslant \infty$ and $s \in \mathbb{R}$ we define

$$B^s_{p,q}(D;E) = \{f|_D : f \in B^s_{p,q}(\mathbb{R}^d;E)\}.$$

This space is a Banach space endowed with the norm

$$\|g\|_{B^s_{p,q}(D;E)} = \inf_{f|_D = g}\|f\|_{B^s_{p,q}(\mathbb{R}^d;E)}.$$

See [20, Section 3.2.2] (where the scalar case is considered) and [2].

We have the following embedding result, which is a straightforward extension of [9, Theorems 1.1 and 3.2] where the case $H = \mathbb{R}$ was considered:

**Proposition 4.1.** *Let $E$ be a Banach space and $H$ be a non-zero separable Hilbert space. Let $D \subseteq \mathbb{R}^d$ be an open domain and let $p \in [1,2]$. Then $E$ has type $p$ if and only if we have a continuous embedding*

$$B^{\frac{d}{p}-\frac{d}{2}}_{p,p}(D;\gamma(H,E)) \hookrightarrow \gamma(L^2(D;H);E).$$

If we combine this result with Proposition 2.3 we obtain the following condition for stochastic integrability of processes.

**Theorem 4.2.** *Let $H$ be a separable Hilbert space and let $E$ be a UMD Banach space with type $p \in [1,2]$. If $\Phi : [0,T]\times\Omega \to \mathscr{L}(H,E)$ is an $H$-strongly measurable process and adapted process with trajectories in $B^{\frac{1}{p}-\frac{1}{2}}_{p,p}(0,T;\gamma(H,E))$ almost surely, then $\Phi$ is stochastically integrable with respect $W_H$. Moreover, for all $q \in (1,\infty)$,*

$$\mathbb{E} \sup_{t\in[0,T]} \Big\| \int_0^t \Phi(s)\,dW_H(s)\Big\|^q \eqsim_{p,E} \mathbb{E}\|\Phi\|^q_{B^{\frac{1}{p}-\frac{1}{2}}_{p,p}(0,T;\gamma(H,E))}.$$

A similar result can be given for processes with Hölder continuous trajectories. In particular, invoking [9, Corollary 3.4] we see that Theorem 4.2 may be applied to functions in $C^\alpha([0,1];\gamma(H,E))$ and, if $E$ is a UMD space, to processes with paths almost surely in $C^\alpha([0,1];\gamma(H,E))$, where $\alpha > \frac{1}{p}-\frac{1}{2}$. Since UMD spaces always have non-trivial type, there exists an $\varepsilon > 0$ such that every $H$-strongly measurable and adapted process with paths in $C^{\frac{1}{2}-\varepsilon}([0,1];\gamma(H,E))$ is stochastically integrable with respect to $W_H$. In the converse direction, [9, Theorem 3.5] implies that if $E$ is a Banach space failing type $p \in (1,2)$, then for any $0 < \alpha < \frac{1}{p}-\frac{1}{2}$ there exist examples of functions in $C^\alpha([0,1];E)$ which fail to be stochastically integrable with respect to scalar Brownian motions.

## 5. Smoothness - II

In this section we give an alternative proof of Proposition 4.1 in the case $D$ is a finite interval. The argument uses the definition of the Besov space from [11] instead of the Fourier analytic definition of Peetre.

For $s \in (0,1)$ and $p,q \in [1,\infty]$ we will recall the definition of the Besov space $\Lambda^s_{p,q}(0,T;E)$ from [11]. Since it is not obvious that this space is equal to the Besov space of Section 4 we use the notation $\Lambda^s_{p,q}(0,T;E)$ instead of $B^s_{p,q}(0,T;E)$.

Let $I = (0,T)$. For $h \in \mathbb{R}$ and a function $\phi : I \to E$ we define the function $T(h)\phi : I \to E$ as the translate of $\phi$ by $h$, i.e.

$$(T(h)\phi)(t) := \begin{cases} \phi(t+h) & \text{if } t+h \in I, \\ 0 & \text{otherwise.} \end{cases}$$

For $h \in \mathbb{R}$ put

$$I[h] := \Big\{ r \in I : \ r+h \in I \Big\}.$$

For a strongly measurable function $\phi \in L^p(I;E)$ and $t > 0$ let

$$\varrho_p(\phi,t) := \sup_{|h|\leqslant t} \Big( \int_{I[h]} \|T(h)\phi(r) - \phi(r)\|^p\,dr \Big)^{\frac{1}{p}}.$$

We use the obvious modification if $p=\infty$.

Now define

$$\Lambda^s_{p,q}(I;\mathscr{L}(E,F)) := \{\phi \in L^p(I;E) : \|\phi\|_{\Lambda^s_{p,q}(I;E)} < \infty\},$$

where

$$\|\phi\|_{\Lambda^s_{p,q}(I;E)} = \Big(\int_0^T \|\phi(t)\|^p\,dt\Big)^{\frac1p} + \Big(\int_0^1 \big(t^{-s}\varrho_p(\phi,t)\big)^q\,\frac{dt}{t}\Big)^{\frac1q} \tag{5.1}$$

with the obvious modification for $q=\infty$. Endowed with the norm $\|\cdot\|_{\Lambda^s_{p,q}(I;E)}$, $\Lambda^s_{p,q}(I;E)$ is a Banach space. The following continuous inclusions hold:

$$\Lambda^s_{p,q_1}(I;E) \hookrightarrow \Lambda^s_{p,q_2}(I;E),\ \Lambda^{s_1}_{p,q}(I;E) \hookrightarrow \Lambda^{s_2}_{p,q}(I;E),$$

and

$$\Lambda^s_{p_1,q}(I;E) \hookrightarrow \Lambda^s_{p_2,q}(I;E)$$

for all $s,s_1,s_2\in(0,1)$, $p,p_1,p_2,q,q_1,q_2\in[1,\infty]$ with $1\leqslant p_2\leqslant p_1\leqslant\infty$, $q_1\leqslant q_2$, $s_2\leqslant s_1$.

For all $p\in[1,\infty)$ we have

$$\Lambda^s_{p,q}(I;E) = B^s_{p,q}(I;E)$$

with equivalent norms. Here $B^s_{p,q}(I;E)$ is the space defined in Section 4. Since we could not find a reference for this, we include the short argument. If $I=\mathbb{R}$ this follows from [16, Proposition 3.1] (also see [18, Theorem 4.3.3]). Therefore, for general $I$ the inclusion "$\supseteq$" follows from the definitions. For the other inclusion notice that by [11, Theorem 3.b.7] one has

$$\Lambda^s_{p,q}(I;E) = (L^p(I;E), W^{1,p}(I;E))_{s,q}.$$

It is well-known that there is a common extension operator from the spaces $L^p(I;E)$ and $W^{1,p}(I;E)$ into $L^p(\mathbb{R};E)$ and $W^{1,p}(\mathbb{R};E)$ for all $p\in[1,\infty]$. Therefore, by interpolation we obtain an extension operator from $(L^p(I;E),W^{1,p}(I;E))_{s,q}$ into $(L^p(\mathbb{R};E),W^{1,p}(\mathbb{R};E))_{s,q}$. Now the latter is again equal to $B^s_{p,q}(\mathbb{R};E)$ and therefore "$\subseteq$" holds as well.

We put, for $t>0$,

$$\varphi^s_p(\phi,t) := t^{-s}\varrho_p(\phi,t)$$

and observe for later use the easy fact that there is a constant $c_{q,s}>0$ such that for all $\phi\in\Gamma^s_{p,q}(I;E)$ we have

$$c^{-1}_{q,s}\|\varphi^s_p(\phi,\cdot)\|_{L^q(0,1;\frac{dt}{t})} \leqslant \big\|\big(\varphi^s_p(\phi,2^{-n})\big)_{n\geqslant0}\big\|_{l^q} \leqslant c_{q,s}\|\varphi^s_p(\phi,\cdot)\|_{L^q(0,1;\frac{dt}{t})}. \tag{5.2}$$

**Theorem 5.1.** *Let $H$ be a separable Hilbert space, $E$ a Banach space, and let $p\in[1,2)$. Then $E$ has type $p$ if and only if $\Lambda^{\frac1p-\frac12}_{p,p}(0,T;\gamma(H,E)) \hookrightarrow \gamma(L^2(0,T;H),E)$ continuously.*

*Proof.* For the proof that $E$ has type $p$ if the inclusion holds we refer to [9, Theorem 3.3]. To prove the converse we may assume $T=1$. Let $(g_{00},g_{nk} : n\geqslant0, k=1,\dots,2^n)$ be the $L^2$-normalized Haar system on $[0,1]$, i.e. $g_{00}\equiv1$ and for all other $n$ and $k$ let

$$g_{nk} = \begin{cases} 2^{\frac n2} & \text{on } [(k-1)2^{-n},(k-\frac12)2^{-n}) \\ -2^{\frac n2} & \text{on } [(k-\frac12)2^{-n},k2^{-n}) \\ 0 & \text{otherwise.}\end{cases}$$

Let $(h_i)_{i\geqslant 1}$ be an orthonormal basis for $H$. Then $(g_{nk}\otimes h_i)_{m,k,i}$ is an orthonormal basis for $L^2(0,1;H)$. Let $(\gamma_i)_i$, $(\gamma_{nki})_{n,k,i}$ be Gaussian sequences and let $(r_{nk})_{n,k}$ be an independent Rademacher sequence. Let $\Phi \in \Lambda^{\frac1p-\frac12}_{p,p}(0,T;\gamma(H,E))$ be arbitrary. Since $E$ has type $p$, $L^2(\Omega;E)$ has type $p$ with $T_p(L^2(\Omega;E)) = T_p(E)$ (cf. [5]) and we have

$$\begin{aligned}
&\Big(\mathbb{E}\Big\|\sum_{i\geqslant 1}\gamma_{nki}I_\Phi g_{00}\otimes h_i + \sum_{n\geqslant 0}\sum_{k=1}^{2^n}\sum_{i\geqslant 1}\gamma_{nki}I_\Phi g_{nk}\otimes h_i\Big\|^2\Big)^{\frac12}\\
&= \Big(\mathbb{E}_r\mathbb{E}\Big\|\sum_{i\geqslant 1}\gamma_{nki}I_\Phi g_{00}\otimes h_i + \sum_{n\geqslant 0}\sum_{k=1}^{2^n}\sum_{i\geqslant 1}r_{nk}\gamma_{nki}I_\Phi g_{nk}\otimes h_i\Big\|^2\Big)^{\frac12}\\
&\leqslant \Big\|\sum_{i\geqslant 1}\gamma_i I_\Phi g_{00}\otimes h_i\Big\|_{L^2(\Omega;E)} + T_p(E)\Big(\sum_{n\geqslant 0}\sum_{k=1}^{2^n}\Big\|\sum_{i\geqslant 1}\gamma_i I_\Phi g_{nk}\otimes h_i\Big\|^p_{L^2(\Omega;E)}\Big)^{\frac1p}
\end{aligned}$$

Now one easily checks that

$$\Big\|\sum_{i\geqslant 1}\gamma_i I_\Phi g_{00}\otimes h_i\Big\|_{L^2(\Omega;E)} \leqslant \|\Phi\|_{L^p(0,1;\gamma(H,E))}.$$

For the other term note that

$$I_\Phi g_{nk}\otimes h_i = 2^{\frac n2}\int_{(k-1)2^{-n}}^{(k-\frac12)2^{-n}} (\Phi(s)-\Phi(s+2^{-n-1}))h_i\,ds.$$

Therefore,

$$\begin{aligned}
&\sum_{k=1}^{2^n}\Big\|\sum_{i\geqslant 1}\gamma_i I_\Phi g_{nk}\otimes h_i\Big\|^p_{L^2(\Omega;E)}\\
&\qquad = 2^{\frac{np}{2}}\sum_{k=1}^{2^n}\Big\|\int_{(k-1)2^{-n}}^{(k-\frac12)2^{-n}}\Phi(s)-\Phi(s+2^{-n-1})\,ds\Big\|^p_{\gamma(H,E)}\\
&\qquad \leqslant 2^{\frac{np}{2}}2^{(n+1)(1-p)}\sum_{k=1}^{2^n}\int_{(k-1)2^{-n}}^{(k-\frac12)2^{-n}}\|\Phi(s)-\Phi(s+2^{-n-1})\|^p_{\gamma(H,E)}\,ds\\
&\qquad \leqslant 2^{-p+1}2^{n(1-\frac p2)}\int_0^{1-2^{-n-1}}\|\Phi(s)-\Phi(s+2^{-n-1})\|^p_{\gamma(H,E)}\,ds
\end{aligned}$$

We conclude that

$$\begin{aligned}
&\Big(\sum_{n\geqslant 0}\sum_{k=1}^{2^n}\mathbb{E}\Big\|\sum_{i\geqslant 1}\gamma_i I_\Phi g_{nk}\otimes h_i\Big\|^p\Big)^{\frac1p}\\
&\qquad \leqslant 2^{-1+\frac1p}\Big(\sum_{n\geqslant 0}2^{n(1-\frac p2)}\int_0^{1-2^{-n-1}}\|\Phi(s)-\Phi(s+2^{-n-1})\|^p_{\gamma(H,E)}\,ds\Big)^{\frac1p}\\
&\qquad \lesssim_p \|\Phi\|_{\Lambda^{\frac1p-\frac12}_{p,p}(0,T;\gamma(H,E))},
\end{aligned}$$

where the last inequality follows from (5.2). □

If $(0,T)$ is replaced with $\mathbb{R}$, one can use the Haar basis on each interval $(j,j+1)$ to obtain the analogous embedding result for $\mathbb{R}$. More generally, the proof can be adjusted to the case of finite or infinite rectangles $D\subseteq\mathbb{R}^d$. Furthermore, using extension operators one can extend the embedding result to bounded regular domains.

As a consequence of Theorem 5.1 we recover a Hölder space embedding result from [9]. Using [15, Theorem 2.3] this can be reformulated as follows.

**Proposition 5.2.** *Let $E$ be a Banach space and let $p \in [1,2)$. If $E$ type $p$, then for all $\alpha > \frac{1}{p} - \frac{1}{2}$ it holds that $\phi \in C^\alpha([0,1];E)$ implies that $\phi$ is stochastically integrable with respect to $W$. Moreover, there exists a constant $C$ only depending on the type $p$ constant of $E$ such that*

$$\mathbb{E}\Big\| \int_0^1 \phi \, dW \Big\|^2 \leqslant C^2 \|\phi\|^2_{C^\alpha([0,1];E)}$$

In [9] a converse to this result is obtained as well: if all functions in $C^\alpha([0,1];E)$ are stochastically integrable, then $E$ has type $p$ for all $p \in [1,2)$ satisfying $\alpha < \frac{1}{p} - \frac{1}{2}$. However, the case that $\alpha = \frac{1}{p} - \frac{1}{2}$ is left open there and will be considered in the following theorem. For the definition of stable type $p$ we refer to [12].

**Theorem 5.3.** *Let $E$ be a Banach space, let $\alpha \in (0, \frac{1}{2}]$ and let $p \in [1,2)$ be such that $\alpha = \frac{1}{p} - \frac{1}{2}$. If every function in $C^\alpha([0,1];E)$ is stochastically integrable with respect to $W$, then $E$ has stable type $p$.*

Since $l^p$ spaces for $p \in [1,2)$ do not have stable type $p$, it follows from Theorem 5.3 that there exists a $(\frac{1}{p} - \frac{1}{2})$-Hölder continuous function $\phi : [0,1] \to l^p$ that is not stochastically integrable with respect to $W$. An explicit example can be obtained from the construction below. This extends certain examples in [17, 21]

*Proof. Step 1:* Fix an integer $N \geqslant 1$. First we construct an certain function with values in $l^p_N$. Let $\varphi_{00}, \varphi_{nk}$ for $n \geqslant 0, k = 1, \ldots, 2^n$ be the Schauder functions on $[0,1]$, i.e., $\varphi_{nk}(x) = \int_0^x g_{nk}(t)\, dt$ where $g_{nk}$ are the $L^2$-normalized Haar functions. Let $(e_n)_{n=1}^N$ be the standard basis in $l^p_N$. Let $\psi : [0,1] \to l^p$ be defined as

$$\psi(t) = \sum_{n=0}^N \sum_{k=1}^{2^n} 2^{\frac{(p-1)n}{p}} \varphi_{nk}(t) e_{2^n+k}.$$

Then $\psi$ is stochastically integrable and

$$\begin{aligned}
\mathbb{E}\Big\| \int_0^1 \psi \, dW \Big\|^p &= \mathbb{E}\Big\| \sum_{n=0}^N \sum_{k=1}^{2^n} 2^{\frac{(p-1)n}{p}} \int_0^1 \varphi_{nk} \, dW e_{2^n+k} \Big\|^p \\
&= \sum_{n=0}^N \sum_{k=1}^{2^n} 2^{(p-1)n} \mathbb{E}\Big| \int_0^1 \varphi_{nk} \, dW \Big|^p \\
&= m_p^p \sum_{n=0}^N \sum_{k=1}^{2^n} 2^{(p-1)n} \Big( \mathbb{E}\Big| \int_0^1 \varphi_{nk} \, dW \Big|^2 \Big)^{\frac{p}{2}} \\
&= m_p^p \sum_{n=0}^N \sum_{k=1}^{2^n} 2^{(p-1)n} \Big( \int_0^1 \varphi_{nk}^2(t) \, dt \Big)^{\frac{p}{2}} \\
&= m_p^p \sum_{n=0}^N \sum_{k=1}^{2^n} 2^{(p-1)n} \Big( \frac{2^{-2n-2}}{3} \Big)^{\frac{p}{2}} = m_p^p \frac{N}{12^{\frac{p}{2}}},
\end{aligned}$$

where $m_p = (\mathbb{E}|W(1)|^p)^{\frac{1}{p}}$. Therefore,

$$\Big(\mathbb{E}\Big\|\int_0^1 \psi\,dW\Big\|^2\Big)^{\frac12} \geqslant \Big(\mathbb{E}\Big\|\int_0^1 \psi\,dW\Big\|^p\Big)^{\frac1p} = K_p N^{\frac1p}, \tag{5.3}$$

with $K_p = m_p/\sqrt{12}$.

On the other hand $\psi$ is $\alpha$-Hölder continuous with

$$\|\psi\|_{C^\alpha([0,1];E)} = \sup_{t\in[0,1]} \|\psi(t)\| + \sup_{0\leqslant s<t\leqslant 1} \frac{\|\psi(t)-\psi(s)\|}{(t-s)^\alpha} \leqslant C_p, \tag{5.4}$$

where $C_p$ is a constant only depending on $p$. Indeed, for each $t\in[0,1]$, we have

$$\begin{aligned}\|\psi(t)\| &= \Big(\sum_{n=0}^N \sum_{k=1}^{2^n} 2^{(p-1)n}|\varphi_{nk}(t)|^p\Big)^{\frac1p} \leqslant \Big(\sum_{n=0}^N 2^{(p-1)n}2^{-\frac{(n+2)p}{2}}\Big)^{\frac1p}\\ &= \Big(\sum_{n=0}^N 2^{-(1-p/2)n-p}\Big)^{\frac1p} \leqslant \frac12\Big(\frac{2^{1-p/2}}{2^{1-p/2}-1}\Big)^{\frac1p}.\end{aligned}$$

Now fix $0\leqslant s<t\leqslant 1$. Let $n_0$ be the largest integer such that there exists an integer $k$ with the property that $s,t\in[(k-1)2^{-n_0},(k+1)2^{-n_0}]$. Then[1]

$$(k-1)2^{-n_0}\leqslant s\leqslant k2^{-n_0}\leqslant t\leqslant (k+1)2^{-n_0}.$$

Indeed, otherwise one can replace $n_0$ with $n_0+1$ using the point $(k-1/2)2^{-n_0} = (2k-1)2^{-(n_0+1)}$ or $(k+1/2)2^{-n_0} = (2k+1)2^{-(n_0+1)}$ as the middle of the dyadic interval with length $2^{-(n_0+1)}$. Moreover, $2^{-(n_0+1)}\leqslant (t-s)\leqslant 2^{-n_0+1}$. The upper estimate is clear. For the lower estimate assume that $(t-s)<2^{-(n_0+1)}$. Then $(t-k2^{-n})<2^{-(n_0+1)}$ and $(k2^{-n}-s)<2^{-(n_0+1)}$. Therefore, $t,s\in[(k-1/2)2^{-n_0},(k+1/2)2^{-n_0}] = [(2k-1)2^{-(n_0+1)},(2k+1)2^{-(n_0+1)}]$. This contradicts the choice of $n_0$.

Now for each $0\leqslant n\leqslant n_0$ let $k_n$ be the unique integer such that $s\in[(k_n-1)2^{-n},k_n2^{-n})$. Now two cases occur: (i) $t\in[(k_n-1)2^{-n},k_n2^{-n}]$ or (ii) $t\in[k_n2^{-n},(k_n+1)2^{-n}]$.

In case (i) it follows that

$$|\varphi_{nk_n}(t)-\varphi_{nk_n}(s)| \leqslant 2^{\frac n2}(t-s)\leqslant 2^{\frac n2}2^{(-n_0+1)(1-\alpha)}(t-s)^\alpha.$$

In case (ii) it follows that

$$\begin{aligned}|\varphi_{nk_n}(t)-\varphi_{nk_n}(s)| &= |\varphi_{nk_n}(k_n2^{-n})-\varphi_{nk_n}(s)|\leqslant 2^{\frac n2}(k_n2^{-n}-s)\\ &\leqslant 2^{\frac n2}(t-s)\leqslant 2^{\frac n2}2^{(-n_0+1)(1-\alpha)}(t-s)^\alpha\end{aligned}$$

and in the same way

$$|\varphi_{nk_n+1}(t)-\varphi_{nk_n+1}(s)| = |\varphi_{nk_n+1}(t)-\varphi_{nk_n+1}(k_n2^{-n})|\leqslant 2^{\frac n2}2^{(-n_0+1)(1-\alpha)}(t-s)^\alpha.$$

For $n_0<n\leqslant N$ let $k_n>\ell_n$ be the unique integers such that $t\in[(k_n-1)2^{-n},k_n2^{-n}]$ and $s\in[(\ell_n-1)2^{-n},\ell_n2^{-n}]$. Then

$$|\varphi_{nk_n}(t)-\varphi_{nk_n}(s)| = |\varphi_{nk_n}(t)|\leqslant 2^{-\frac n2-1},$$

$$|\varphi_{n\ell_n}(t)-\varphi_{n\ell_n}(s)| = |\varphi_{n\ell_n}(s)|\leqslant 2^{-\frac n2-1}.$$

[1]This argument corrects a minor mistake in the proof in the published version of the paper.

We conclude that

$$\begin{aligned}
&\|\psi(t)-\psi(s)\|^p \\
&\quad\leqslant \sum_{n=0}^{n_0}\sum_{k=1}^{2^n} 2^{(p-1)n}|\varphi_{nk}(t)-\varphi_{nk}(s)|^p + \sum_{n=n_0+1}^{N}\sum_{k=1}^{2^n} 2^{(p-1)n}|\varphi_{nk}(t)-\varphi_{nk}(s)|^p \\
&\quad\leqslant 2\sum_{n=0}^{n_0} 2^{(p-1)n}2^{\frac{np}{2}}2^{(-n_0+1)(1-\alpha)p}(t-s)^{\alpha p} + \sum_{n=n_0+1}^{N} 2^{(p-1)n}2^{-\frac{np}{2}} \\
&\quad\leqslant \frac{2^{2-\alpha}}{2^{\frac{3}{2}p-1}-1}(t-s)^{\alpha p} + \frac{2^{-(n_0+1)(1-\frac{p}{2})}}{1-2^{-(1-\frac{p}{2})}}
\end{aligned}$$

Noting that $2^{-(n_0+1)} \leqslant (t-s)$ and $(1-\frac{p}{2}) = \alpha p$ it follows that

$$\|\psi(t)-\psi(s)\| \leqslant \Big(\frac{2^{2-\alpha}}{2^{\frac{3}{2}p-1}-1} + \frac{1}{1-2^{-(1-\frac{p}{2})}}\Big)^{\frac{1}{p}}(t-s)^{\alpha}.$$

Therefore, (5.4) follows.

*Step 2:* Assume that every function in $C^\alpha([0,1];E)$ is stochastically integrable. It follows from the closed graph theorem that there exists a constant $C$ such that for all $\phi \in C^\alpha([0,1];E)$ we have

$$\Big(\mathbb{E}\Big\|\int_0^1 \phi\,dW\Big\|^2\Big)^{\frac{1}{2}} \leqslant C\|\phi\|_{C^\alpha([0,1];E)} \tag{5.5}$$

Now assume that $E$ does not have stable type $p$. By the Maurey-Pisier theorem [12, Theorem 9.6] it follows that $l^p$ is finitely representable in $E$. In particular it follows that for each integer $N$ there exists an operator $T_N : l^p_N \to E$ such that $\|x\| \leqslant \|T_N x\| \leqslant 2\|x\|$ for all $x \in l^p_N$. Now let $\phi : [0,1] \to E$ be defined as $\phi(t) = T_N\psi_N(t)$, where $\psi_N : [0,1] \to l^p_N$ is the function constructed in Step 1. Then it follows from (5.3), (5.4) and (5.5) that

$$\begin{aligned}
K_p N^{\frac{1}{p}} &\leqslant \Big(\mathbb{E}\Big\|\int_0^1 \psi\,dW\Big\|^2\Big)^{\frac{1}{2}} \leqslant \Big(\mathbb{E}\Big\|\int_0^1 \phi\,dW\Big\|^2\Big)^{\frac{1}{2}} \\
&\qquad\leqslant C\|\phi\|_{C^\alpha([0,1];E)} \leqslant 2C\|\psi\|_{C^\alpha([0,1];l^p_N)} \leqslant 2CC_p.
\end{aligned}$$

This cannot hold for $N$ large and therefore $E$ has stable type $p$. □

As a corollary we obtain that the set of all $\alpha \in (0,\frac{1}{2}]$ such that every $f \in C^\alpha([0,1];E)$ is stochastically integrable is relatively open.

**Corollary 5.4.** *Let $E$ be a Banach space and let $\alpha \in (0,\frac{1}{2}]$ and let $p \in [1,2)$ be such that $\alpha = \frac{1}{p}-\frac{1}{2}$. If every function in $C^\alpha([0,1];E)$ is stochastically integrable with respect to $W$, then $E$ has (stable) type $p_1$ for some $p_1 > p$. In particular, there exists an $\varepsilon \in (0,\alpha)$ such that every function in $C^{\alpha-\varepsilon}([0,1];E)$ is stochastically integrable.*

*Proof.* The first part follows from Theorem 5.3 and [12, Corollary 9.7, Proposition 9.12]. The last statement is a consequence of this and Proposition 5.2, where $\varepsilon > 0$ may be taken such that $\alpha - \varepsilon = \frac{1}{p_1} - \frac{1}{2}$ □

## 6. Banach function spaces

In this section we prove a criterium (Theorem 6.2) for stochastic integrability of a process in the case $E$ is a UMD Banach function space which was stated without proof in [14]. It applies to the spaces $E = L^p(S)$, where $p \in (1,\infty)$ and $(S,\Sigma,\mu)$ is a $\sigma$-finite measure space.

We start with the case where $\Phi$ is a function with values in $\mathscr{L}(H,E)$. The following proposition extends [15, Corollary 2.10], where the case $H = \mathbb{R}$ was considered.

**Proposition 6.1.** *Let $E$ be Banach function space with finite cotype over a $\sigma$-finite measure space $(S,\Sigma,\mu)$. Let $\Phi : [0,T] \to \mathscr{L}(H,E)$ be an $H$-strongly measurable function and assume that there exists a strongly measurable function $\phi : [0,T]\times S \to H$ such that for all $h \in H$ and $t \in [0,T]$,*

$$(\Phi(t)h)(\cdot) = [\phi(t,\cdot), h]_H \quad \text{in } E.$$

*Then $\Phi$ is stochastically integrable if and only if*

$$\Big\| \Big( \int_0^T \|\phi(t,\cdot)\|_H^2 \, dt \Big)^{\frac12} \Big\|_E < \infty. \tag{6.1}$$

*In this case we have*

$$\Big(\mathbb{E}\Big\| \int_0^T \Phi \, dW_H \Big\|_E^2 \Big)^{\frac12} \eqsim_E \Big\| \Big( \int_0^T \|\phi(t,\cdot)\|_H^2 \, dt \Big)^{\frac12} \Big\|_E.$$

*Proof.* First assume that $\Phi$ is stochastically integrable. Let $\mathscr{N} = \{n \in \mathbb{N} : 1 \leqslant n < \dim(H) + 1\}$, let $(e_m)_{m\in\mathscr{N}}$ be the standard unit basis for $L^2(\mathscr{N},\tau)$, where $\tau$ denotes the counting measure on $\mathscr{N}$. Choose orthonormal bases $(f_n)_{n\geqslant1}$ for $L^2(0,T)$ and $(h_n)_{n\in\mathscr{N}}$ for $H$. Define $\Psi : [0,T]\times\mathscr{N} \to E$ by $\Psi(t,n) := \Phi(t)h_n$ and define the integral operator $I_\Psi : L^2([0,T]\times\mathscr{N}, dt\times\tau) \to E$ by

$$I_\Psi f := \int_{\mathscr{N}} \int_{[0,T]} f(t,n)\Psi(t,n)\, dt\, d\tau(n) = \sum_{n\in\mathscr{N}} \int_0^T f(t,n)\Phi(t)h_n \, dt.$$

Note that the integral on the right-hand side is well defined as a Pettis integral. Let $I_\Phi \in \gamma(L^2(0,T;H),E)$ be the operator representing $\Phi$ as in Proposition 2.3 (the special case for functions). Then $I_\Psi \in \gamma(L^2([0,T]\times\mathscr{N}, dt\times\tau),E)$ and

$$\Big(\mathbb{E}\Big\| \int_0^T \Phi \, dW_H \Big\|_E^2 \Big)^{\frac12} = \|I_\Phi\|_{\gamma(L^2(0,T;H),E)} = \|I_\Psi\|_{\gamma(L^2([0,T]\times\mathscr{N}, dt\times\tau),E)}.$$

On the other hand, by a similar calculation as in [15, Corollary 2.10] one obtains, with $(r_{mn})$ denoting a doubly indexed sequence of Rademacher variables on a probability space $(\Omega',\mathscr{F}',\mathbb{P}')$,

$$\begin{aligned}
\|I_\Psi\|_{\gamma(L^2([0,T]\times\mathscr{N}, dt\times\tau),E)} &\eqsim_E \Big(\mathbb{E}'\Big\| \sum_{m,n} r_{mn} \int_0^T \sum_k \Psi(t,k)e_m(k)f_n(t)\, dt \Big\|_E^2 \Big)^{\frac12} \\
&\eqsim_E \Big\| \Big( \int_0^T \sum_k \big|\Psi(t,k)(\cdot)\big|^2 \, dt \Big)^{\frac12} \Big\|_E \\
&= \Big\| \Big( \int_0^T \|\phi(t,\cdot)\|_H^2 \, dt \Big)^{\frac12} \Big\|_E.
\end{aligned}$$

For the converse one can read all estimates backwards, but we have to show that $\Phi$ belongs to $L^2(0,T;H)$ scalarly if (6.1) holds. For all $x^* \in E^*$ we have

$$\begin{aligned}\|\Phi^* x^*\|^2_{L^2(0,T;H)} &= \Big(\sum_{m,n}\Big(\int_0^T [\Phi^*(t)x^*, h_m]_H f_n(t)\,dt\Big)^2\Big)^{\frac12}\\ &= \Big(\sum_{n,m}\Big(\int_0^T \sum_k \langle \Psi(t,k), x^*\rangle e_m(k) f_n(t)\,dt\Big)^2\Big)^{\frac12}\\ &\leqslant \Big(\mathbb{E}'\Big\|\sum_{n,m} r_{mn}\int_0^T \sum_k \Psi(t,k)e_m(k)f_n(t)\,dt\Big\|_E^2\Big)^{\frac12}\|x^*\|.\end{aligned}$$

□

By combining this proposition with Proposition 2.3 and recalling the fact that UMD spaces have finite cotype, we obtain:

**Theorem 6.2.** *Let $E$ be UMD Banach function space over a $\sigma$-finite measure space $(S,\Sigma,\mu)$ and let $p \in (1,\infty)$. Let $\Phi : [0,T]\times\Omega \to \mathscr{L}(H,E)$ be an $H$-strongly measurable and adapted process and assume that there exists a strongly measurable function $\phi : [0,T]\times\Omega\times S \to H$ such that for all $h \in H$ and $t \in [0,T]$,*

$$(\Phi(t)h)(\cdot) = [\phi(t,\cdot), h]_H \quad \text{in } E.$$

*Then $\Phi$ is stochastically integrable if and only if*

$$\Big\|\Big(\int_0^T \|\phi(t,\cdot)\|_H^2\,dt\Big)^{\frac12}\Big\|_E < \infty \quad \text{almost surely.}$$

*In this case for all $p \in (1,\infty)$ we have*

$$\mathbb{E}\sup_{t\in[0,T]}\Big\|\int_0^t \Phi(t)\,dW_H(t)\Big\|^p \eqsim_{p,E} \mathbb{E}\Big\|\Big(\int_0^T \|\phi(t,\cdot)\|_H^2\,dt\Big)^{\frac12}\Big\|_E^p.$$

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