A maximal inequality for stochastic convolutions in 2-smooth Banach spaces

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Abstract. Let $(e^{tA})_{t\geqslant 0}$ be a C_0-contraction semigroup on a 2-smooth Banach space E, let $(W_t)_{t\geqslant 0}$ be a cylindrical Brownian motion in a Hilbert space H, and let $(g_t)_{t\geqslant 0}$ be a progressively measurable process with values in the space $\gamma(H, E)$ of all γ-radonifying operators from H to E. We prove that for all $0 < p < \infty$ there exists a constant C, depending only on p and E, such that for all $T \geqslant 0$ we have

$$\mathbb{E} \sup_{0\leqslant t\leqslant T} \Big\| \int_0^t e^{(t-s)A} g_s \,\mathrm{d}W_s \Big\|^p \leqslant C\mathbb{E}\Big(\int_0^T \|g_t\|^2_{\gamma(H,E)} \,\mathrm{d}t\Big)^{\frac{p}{2}}.$$

For $p \geqslant 2$ the proof is based on the observation that $\psi(x) = \|x\|^p$ is Fréchet differentiable and its derivative satisfies the Lipschitz estimate $\|\psi'(x) - \psi'(y)\| \leqslant C(\|x\| + \|y\|)^{p-2}\|x - y\|$; the extension to $0 < p < 2$ proceeds via Lenglart's inequality.

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1. Introduction

Let $(e^{tA})_{t\geqslant 0}$ be a C_0-contraction semigroup on a 2-smooth Banach space E and let $(W_t)_{t\geqslant 0}$ be a cylindrical Brownian motion in a Hilbert space H. Let $(g_t)_{t\geqslant 0}$ be a progressively measurable process with values in the space $\gamma(H, E)$ of all γ-radonifying operators from H to E satisfying

$$\int_0^T \|g_t\|^2_{\gamma(H,E)} \,\mathrm{d}t < \infty \quad \mathbb{P}\text{-almost surely}$$

for all $T \geqslant 0$. As is well known (see [6, 15, 16]), under these assumptions the stochastic convolution process

$$X_t = \int_0^t e^{(t-s)A} g_s \,\mathrm{d}W_s, \quad t \geqslant 0,$$

is well-defined in E and provides the unique mild solution of the stochastic initial value problem

$$\mathrm{d}X_t = AX_t \,\mathrm{d}t + g_t \,\mathrm{d}W_t, \quad X_0 = 0.$$

In order to obtain the existence of a continuous version of this process, one usually proves a maximal estimate of the form

$$\mathbb{E} \sup_{0\leqslant t\leqslant T} \|X_t\|^p \leqslant C^p\mathbb{E}\Big(\int_0^T \|g_t\|^2_{\gamma(H,E)} \,\mathrm{d}t\Big)^{\frac{p}{2}}. \tag{1.1}$$

The first such estimate was obtained by Kotelenez [11, 12] for C_0-contraction semigroups on Hilbert spaces E and exponent $p = 2$. Tubaro [19] extended this result to exponents $p \geqslant 2$ by a different method of proof which applies Itô's formula to the C^2-mapping $x \mapsto \|x\|^p$. The case $p \in (0, 2)$ was covered subsequently by Ichikawa [10]. A very simple proof, still for C_0-contraction semigroups on Hilbert spaces, which works for all

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$p \in (0,\infty)$, was obtained recently by Hausenblas and Seidler [9]. It is based on the Sz.-Nagy dilation theorem, which is used to reduce the problem to the corresponding problem for C_0-contraction groups. Then, by using the group property, the maximal estimate follows from Burkholder's inequality. This proof shows, moreover, that the constant C in (1.1) may be taken equal to the constant appearing in Burkholder's inequality. In particular, this constant depends only on p.

The maximal inequality (1.1) has been extended by Brzeźniak and Peszat [4] to C_0-contraction semigroups on Banach spaces E with the property that, for some $p \in [2,\infty)$, $x \mapsto \|x\|^p$ is twice continuously Fréchet differentiable and the first and second Fréchet derivatives are bounded by constant multiples of $\|x\|^{p-1}$ and $\|x\|^{p-2}$, respectively. Examples of spaces with this property, which we shall call (C_p^2), are the spaces $L^q(\mu)$ for $q \in [p,\infty)$. Any (C_p^2) space is 2-smooth (the definition is recalled in Section 2), but the converse doesn't hold:

Example 1.1. Let F be a Banach space. The space $\ell^2(F)$ is 2-smooth whenever F is 2-smooth [8, Proposition 17]. On the other hand, the norm of $\ell^2(F)$ is twice continuously Fréchet differentiable away from the origin if and only if F is a Hilbert space [14, Theorem 3.9]. Thus, for $q \in (2,\infty)$, $\ell^2(\ell^q)$ and $\ell^2(L^q(0,1))$ are examples of 2-smooth Banach spaces which fail property (C_p^2) for all $p \in [2,\infty)$.

To the best of our knowledge, the general problem of proving the maximal estimate (1.1) for C_0-contraction semigroups on 2-smooth Banach space remains open. The present paper aims to fill this gap:

Theorem 1.2. *Let $(e^{tA})_{t\geqslant 0}$ be a C_0-contraction semigroup on a 2-smooth Banach space E, let $(W_t)_{t\geqslant 0}$ be a cylindrical Brownian motion in a Hilbert space H, and let $(g_t)_{t\geqslant 0}$ be a progressively measurable process in $\gamma(H,E)$. If*

$$\int_0^T \|g_t\|^2_{\gamma(H,E)}\,\mathrm{d}t < \infty \quad \mathbb{P}\text{-almost surely,}$$

then the stochastic convolution process $X_t = \int_0^t e^{(t-s)A} g_s \,\mathrm{d}W_s$ is well-defined and has a continuous version. Moreover, for all $0 < p < \infty$ there exists a constant C, depending only on p and E, such that

$$\mathbb{E} \sup_{0\leqslant t\leqslant T} \|X_t\|^p \leqslant C^p \mathbb{E}\Big(\int_0^T \|g_t\|^2_{\gamma(H,E)}\,\mathrm{d}t\Big)^{\frac{p}{2}}.$$

For $p \geqslant 2$, the proof of Theorem 1.2 is based on a version of Itô's formula (Theorem 3.1) which exploits the fact (proved in Lemma 2.1) that in 2-smooth Banach spaces the function $\psi(x) = \|x\|^p$ is Fréchet differentiable and satisfies the Lipschitz estimate

$$\|\psi'(x) - \psi'(y)\| \leqslant C(\|x\| + \|y\|)^{p-2}\|x-y\|.$$

The extension to exponents $0 < p < 2$ is obtained by applying Lenglart's inequality (see (4.1)).

We conclude this introduction with a brief discussion of some developments of the inequality (1.1) into different directions in the literature. Seidler [18] has proved the inequality (1.1) with optimal constant $C = O(\sqrt{p})$ as $p \to \infty$ for positive C_0-contraction semigroups on the (2-smooth) space $E = L^q(\mu)$, $q \geqslant 2$. He also proved that the same result holds if the assumption 'e^{tA} is a positive contraction semigroup' is replaced by '$-A$ has a bounded H^∞-calculus of angle strictly less than $\frac12\pi$'. The latter result was subsequently extended by Veraar and Weis [20] to arbitrary UMD spaces E with type 2. In the same paper, still under the assumption that $-A$ has a bounded H^∞-calculus of angle strictly less than $\frac12\pi$, the following stronger estimate is obtained for UMD spaces E with Pisier's property (α):

$$\mathbb{E} \sup_{0\leqslant t\leqslant T} \|X_t\|^p \leqslant C^p \mathbb{E}\|g\|^p_{\gamma(L^2(0,T;H),E)} \tag{1.2}$$

with a constant C depending only on p and E. If, in addition, E has type 2, then the mapping $f\otimes(h\otimes x) \mapsto (f\otimes h)\otimes x$ extends to a continuous embedding $L^2(0,T;\gamma(H,E)) \hookrightarrow \gamma(L^2(0,T;H),E)$ and (1.2) implies (1.1).

Let us finally mention that, for $p > 2$, a weaker version of (1.1) for arbitrary C_0-semigroups on Hilbert spaces has been obtained by Da Prato and Zabczyk [5]. Using the factorisation method they proved that

$$\mathbb{E} \sup_{0\leqslant t\leqslant T} \|X_t\|^p \leqslant C^p \mathbb{E} \int_0^T \|g_t\|^p_{\gamma(H,E)}\,\mathrm{d}t$$

with a constant C depending on p, E, and T. The proof extends *verbatim* to C_0-semigroups on martingale type 2 spaces. This relates to the above results for 2-smooth spaces through a theorem of Pisier [17, Theorem 3.1], which states that a Banach space has martingale type p if and only if it is p-smooth.

2. The Fréchet derivative of $\|\cdot\|^p$

Let $1 < q \leqslant 2$. A Banach space E is *q-smooth* if the modulus of smoothness

$$\rho_{\|\cdot\|}(t) = \sup\Big\{\tfrac{1}{2}(\|x+ty\| + \|x-ty\|) - 1 \;:\; \|x\| = \|y\| = 1\Big\}$$

satisfies $\rho_{\|\cdot\|}(t) \leqslant Ct^q$ for all $t > 0$.

It is known (see [17, Theorem 3.1]) that E is q-smooth if and only if there exists a constant $K \geqslant 1$ such that for all $x, y \in E$,

$$\|x+y\|^q + \|x-y\|^q \leqslant 2\|x\|^q + K\|y\|^q. \tag{2.1}$$

Lemma 2.1. *Let E be a Banach space and let $1 < q \leqslant 2$ be given. For $p \geqslant q$ set $\psi_p(x) := \|x\|^p$.*

1. *E is q-smooth if and only if the Fréchet derivative of ψ_q is globally $(q-1)$-Hölder continuous on E.*
2. *If E is q-smooth, then for $p > q$ the Fréchet derivative of ψ_p is locally $(q-1)$-Hölder continuous on E.*

Moreover, for all $p \geqslant q$ and $x, y \in E$ we have

$$\|\psi_p'(x) - \psi_p'(y)\| \leqslant C(\|x\| + \|y\|)^{p-q}\|x-y\|^{q-1}, \tag{2.2}$$

where C depends only on p, q and E.

Proof. If the Fréchet derivative of ψ_q is $(q-1)$-Hölder continuous on E, then by the mean value theorem we can find $0 \leqslant \theta, \rho \leqslant 1$ such that for all $x, y \in E$,

$$\begin{aligned}
\|x+y\|^q + \|x-y\|^q - 2\|x\|^q &= (\|x+y\|^q - \|x\|^q) + (\|x-y\|^q - \|x\|^q) \\
&\leqslant \|\psi_q'(x+\theta y) - \psi_q'(x-\rho y)\|\,\|y\| \\
&\leqslant L\|(x+\theta y) - (x-\rho y)\|^{q-1}\|y\| \leqslant 2^{q-1}L\|y\|^q.
\end{aligned}$$

Hence the Banach space E is q-smooth.

Suppose now that the norm of E is q-smooth. Then for all $x, y \in E$ with $\|x\|, \|y\| = 1$ and all $t > 0$ we have

$$\|x+ty\| + \|x-ty\| - 2\|x\| \leqslant K\|ty\|^q. \tag{2.3}$$

Thus

$$\lim_{t\to 0} \frac{\|x+ty\| + \|x-ty\| - 2\|x\|}{\|ty\|} = 0,$$

which by [7, Lemma I.1.3] means that $\|\cdot\|$ is Fréchet differentiable on the unit sphere. Hence, by homogeneity, $\|\cdot\|$ is Fréchet differentiable on $E\backslash\{0\}$. Let us denote by f_x its Fréchet derivative at the point $x \neq 0$.

We begin by showing the $(q-1)$-Hölder continuity of $x \mapsto f_x$ on the unit sphere of E, following the argument of [7, Lemma V.3.5]. We fix $x \neq y \in E$ such that $\|x\|, \|y\| = 1$ and $h \in E$ with $\|h\| = \|x-y\|$ and $x - y + h \neq 0$. Since the norm $\|\cdot\|$ is a convex function,

$$f_y(x-y) \leqslant \|x\| - \|y\|.$$

Similarly, we have

$$f_x(h) \leqslant \|x+h\| - \|x\|, \qquad f_y(y-x-h) \leqslant \|2y-x-h\| - \|y\|.$$

By using above inequalities and the linearity of the function f_x, we have

$$\begin{aligned}
f_x(h) - f_y(h) \leqslant \|x+h\| - \|x\| - f_y(h) &= \|x+h\| - \|y\| - f_y(x+h-y) + \|y\| - \|x\| + f_y(x-y) \\
&\leqslant \|x+h\| - \|y\| - f_y(x+h-y) \\
&= \|x+h\| - \|y\| + f_y(y-x-h) \\
&\leqslant \|x+h\| + \|2y-x-h\| - 2\|y\|
\end{aligned}$$

$$
\begin{aligned}
&= \Big\| y + \|x+h-y\| \cdot \frac{x+h-y}{\|x+h-y\|} \Big\| \\
&\qquad + \Big\| y - \|x+h-y\| \cdot \frac{x+h-y}{\|x+h-y\|} \Big\| - 2\|y\| \\
&\leqslant K\|x+h-y\|^q \leqslant K(\|x-y\| + \|h\|)^q = 2^q K\|x-y\|^q,
\end{aligned}
$$

where we also used (2.3). Since the roles of x and y may be reversed in this inequality, this implies

$$
\|f_x - f_y\| = \sup_{\|h\|=\|x-y\|} \frac{|f_x(h) - f_y(h)|}{\|x-y\|} \leqslant 2^q K \|x-y\|^{q-1}
$$

This proves the $(q-1)$-Hölder continuity of the norm $\|\cdot\|$ on the unit sphere.

We proceed with the proof of (2.2); the $(q-1)$-Hölder continuity of ψ_q as well as the local $(p-1)$-Hölder continuity of ψ_p follow from it. For all $x, y \in E$ with $x \neq 0$ and $y \neq 0$ we have $\psi_p'(x) = p\|x\|^{p-1} f_x$.

It is easy to check that $f_x = f_{\frac{x}{\|x\|}}$ and $\|f_x\| = 1$. Following once more the argument of [7, Lemma V.3.5], this gives

$$
\begin{aligned}
\|\psi_p'(x) - \psi_p'(y)\| &= p\big\|\|x\|^{p-1} f_x - \|y\|^{p-1} f_y\big\| \\
&\leqslant p\Big\|\|x\|^{p-1}(f_{\frac{x}{\|x\|}} - f_{\frac{y}{\|y\|}})\Big\| + p\Big\|(\|x\|^{p-1} - \|y\|^{p-1}) f_{\frac{y}{\|y\|}}\Big\| \\
&\leqslant p2^q K \|x\|^{p-1} \Big\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \Big\|^{q-1} + p\Big|\|x\|^{p-1} - \|y\|^{p-1}\Big| \\
&\leqslant p2^q K \|x\|^{p-q} \|y\|^{1-q} \Big\| x\|y\| - y\|x\| \Big\|^{q-1} + p\Big|\|x\|^{p-1} - \|y\|^{p-1}\Big| \qquad (2.4) \\
&= p2^q K \|x\|^{p-q} \|y\|^{1-q} \Big\| \|y\|(x-y) + y(\|y\| - \|x\|) \Big\|^{q-1} + p\Big|\|x\|^{p-1} - \|y\|^{p-1}\Big| \\
&\leqslant p2^q K \|x\|^{p-q} \|y\|^{1-q} (2\|y\|\|x-y\|)^{q-1} + p\Big|\|x\|^{p-1} - \|y\|^{p-1}\Big| \\
&= p2^{2q-1} K \|x\|^{p-q} \|x-y\|^{q-1} + p\Big|\|x\|^{p-1} - \|y\|^{p-1}\Big|.
\end{aligned}
$$

If $q \leqslant p \leqslant 2$, then by the inequality $|t^r - s^r| \leqslant |t-s|^r$, valid for $0 < r \leqslant 1$ and $s, t \in [0, \infty)$, we have

$$
\big|\|x\|^{p-1} - \|y\|^{p-1}\big| \leqslant \big|\|x\| - \|y\|\big|^{p-1} \leqslant \|x-y\|^{p-1} \leqslant (\|x\| + \|y\|)^{p-q} \|x-y\|^{q-1}.
$$

If $p > 2$, by applying the mean value theorem, for some $\theta \in [0,1]$ we have

$$
\begin{aligned}
\big|\|x\|^{p-1} - \|y\|^{p-1}\big| &= (p-1)\Big\|\|\theta x + (1-\theta)y\|^{p-2} f_{\theta x + (1-\theta)y}(x-y)\Big\| \\
&\leqslant (p-1)(\|x\| + \|y\|)^{p-2} \|x-y\| \\
&\leqslant (p-1)(\|x\| + \|y\|)^{p-2} (\|x\| + \|y\|)^{2-q} \|x-y\|^{q-1} \\
&= (p-1)(\|x\| + \|y\|)^{p-q} \|x-y\|^{q-1}.
\end{aligned}
$$

Also, since $\psi_p'(0) = 0$, for $y \neq 0$ we have

$$
\|\psi_p'(0) - \psi_p'(y)\| = p\|y\|^{p-1} = p\|y\|^{p-1} \Big\| \frac{y}{\|y\|} \Big\|^{p-1} \leqslant p\|y\|^{p-1} \Big\| \frac{y}{\|y\|} \Big\|^{q-1} = p\|y\|^{p-q} \|y\|^{q-1}.
$$

□

The above lemma will be combined with the next one, which gives a first order Taylor formula with a remainder term involving the first derivative only.

Lemma 2.2. *Let E and F be Banach spaces, let $0 < \alpha \leqslant 1$, and let $\psi : E \to F$ be a Fréchet differentiable function whose Fréchet derivative $\psi' : E \to \mathscr{L}(E, F)$ is locally α-Hölder continuous. Then for all $x, y \in E$ we have*

$$
\psi(y) = \psi(x) + \psi'(x)(y-x) + R(x,y),
$$

where

$$R(x,y) = \int_0^1 (\psi'(x + r(y-x))(y-x) - \psi'(x)(y-x))\,\mathrm{d}r. \tag{2.5}$$

Proof. Pick $w \in E$ such that $\|w\| \leqslant 1$ and consider the function $f : \mathbb{R} \to F$ by

$$f(\theta) := \psi(x + \theta w).$$

For all $x^* \in F^*$, $\langle f', x^*\rangle$ is locally α-Hölder continuous. To see this, note that for $|\theta_1|, |\theta_2| \leqslant R$ and $\|x\| \leqslant R$ we have $\|x + \theta_1 w\|, \|x + \theta_2 w\| \leqslant 2R$, so by assumption there exists a constant C_{2R} such that

$$\begin{aligned}|\langle f'(\theta_1) - f'(\theta_2), x^*\rangle| &= |\langle \psi'(x+\theta_1 w)w, x^*\rangle - \langle \psi'(x+\theta_2 w)w, x^*\rangle| \\ &\leqslant \|\psi'(x+\theta_1 w) - \psi'(x+\theta_2 w)\|\,\|x^*\| \leqslant C_{2R}|\theta_1 - \theta_2|^\alpha \|x^*\|.\end{aligned}$$

Applying Taylor's formula and [1, Lemma 1, Theorem 3] to the function $\langle f, x^*\rangle$ we obtain

$$\langle f(t) - f(0), x^*\rangle = t\langle f'(0), x^*\rangle + \langle R_f(0,t), x^*\rangle,$$

where $R_f(0,t) = \int_0^1 t(f'(rt) - f'(0))\,\mathrm{d}r$. Now let $x, y \in E$ be given and set $t = \|y - x\|$ and $w = \frac{y-x}{\|y-x\|}$. With these choices we obtain

$$\begin{aligned}\langle \psi(y), x^*\rangle - \langle \psi(x), x^*\rangle - \langle \psi'(x)(y-x), x^*\rangle &= \langle \psi(x+tw), x^*\rangle - \langle \psi(x), x^*\rangle - t\langle \psi('x)w, x^*\rangle \\ &= \langle f(t) - f(0) - tf'(0), x^*\rangle \\ &= \int_0^1 t\langle f'(rt) - f'(0), x^*\rangle\,\mathrm{d}r \\ &= \int_0^1 \langle \psi'(x + r(y-x))(y-x) - \psi'(x)(y-x), x^*\rangle\,\mathrm{d}r.\end{aligned}$$

Since $x^* \in F^*$ was arbitrary, this proves the lemma. □

3. An Itô formula for $\|\cdot\|^p$

From now on we shall always assume that E is a 2-smooth Banach space. We fix $T \geqslant 0$ and let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space with a filtration $(\mathscr{F}_t)_{t\in[0,T]}$. Let H be a real Hilbert space, and denote by $\gamma(H,E)$ the Banach space of all γ-radonifying operators from H to E. We denote by $M([0,T];\gamma(H,E))$ the space of all progressively measurable processes $\xi : [0,T] \times \Omega \to \gamma(H,E)$ such that

$$\int_0^T \|\xi_t\|^2_{\gamma(H,E)}\,\mathrm{d}t < \infty \quad \mathbb{P}\text{-almost surely.}$$

The space of all such ξ which satisfy

$$\mathbb{E}\Big(\int_0^T \|\xi_t\|^2_{\gamma(H,E)}\,\mathrm{d}t\Big)^{\frac{p}{2}} < \infty$$

is denoted by $M^p([0,T];\gamma(H,E))$, $0 < p < \infty$.

On $(\Omega, \mathscr{F}, \mathbb{P})$, let $(W_t)_{t\in[0,T]}$ be an $(\mathscr{F}_t)_{t\in[0,T]}$-cylindrical Brownian motion in H. For adapted simple processes $\xi \in M([0,T];\gamma(H,E))$ of the form

$$\xi_t = \sum_{i=0}^{n-1} 1_{(t_i,t_{i+1}]}(t) \otimes A_i,$$

where $\Pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ is a partition of the interval $[0,T]$ and the random variables A_i are $\mathcal{F}_{t_i}$-measurable and take values in the space of all finite rank operators from H to E, we define the random variable $I(\xi) \in L^0(\Omega, \mathscr{F}_T; E)$ by

$$I(\xi) := \sum_{i=0}^{n-1} A_i(W_{t_{i+1}} - W_{t_i})$$

where $(h \otimes x)W_t := (W_t h) \otimes x$. It is well known that

$$\mathbb{E}\|I(\xi)\|^2 \leqslant C^2 \mathbb{E} \int_0^T \|\xi_t\|^2_{\gamma(H,E)} \, \mathrm{d}t,$$

where C depends on p and E only. It follows that I has a unique extension to a bounded linear operator $M^2([0,T];\gamma(H,E))$ to $L^2(\Omega, \mathscr{F}_T; E)$. By a standard localisation argument, I extends continuous linear operator from $M([0,T];\gamma(H,E))$ to $L^0(\Omega, \mathscr{F}_T; E)$. In what follows we write

$$\int_0^t \xi_s \, \mathrm{d}W_s := I(1_{(0,t]}\xi), \quad t \in [0,T].$$

This stochastic integral has the following properties:

1. For all $\xi \in M([0,T];\gamma(H,E))$ the process $t \to \int_0^t \xi_s \, \mathrm{d}W_s$ is an E-valued continuous local martingale, which is a martingale if $\xi \in M^2([0,T];\gamma(H,E))$.
2. For all $\xi \in M([0,T];\gamma(H,E))$ and stopping times τ with values in $[0,T]$,

$$\int_0^\tau \xi_t \, \mathrm{d}W_t = \int_0^T 1_{[0,\tau]}(t)\xi_t \, \mathrm{d}W_t \quad \mathbb{P}\text{-almost surely.} \tag{3.1}$$

3. For all $\xi \in M^2([0,T];\gamma(H,E))$ and $0 \leqslant u < t \leqslant T$,

$$\mathbb{E}\Big(\Big\| \int_u^t \xi_s \, \mathrm{d}W_s \Big\|^2 |\mathscr{F}_u\Big) \leqslant C\mathbb{E}\Big(\int_u^t \|\xi_s\|^2_{\gamma(H,E)} \, \mathrm{d}s \, |\mathscr{F}_u\Big). \tag{3.2}$$

4. (Burkholder's inequality [2, 6]) For all $0 < p < \infty$ there exists a constant C, depending only on p and E, such that for all $\xi \in M^p([0,T];\gamma(H,E))$ and $t \in [0,T]$,

$$\mathbb{E} \sup_{s\in[0,t]} \Big\| \int_0^s \xi_u \, \mathrm{d}W_u \Big\|^p \leqslant C\mathbb{E}\Big(\int_0^t \|\xi_s\|^2_{\gamma(H,E)} \, \mathrm{d}s\Big)^{\frac{p}{2}}. \tag{3.3}$$

An excellent survey of the theory of stochastic integration in 2-smooth Banach spaces with complete proofs is given in Ondreját's thesis [16], where also further references to the literature can be found.

In what follows we fix $p \geqslant 2$ and set $\psi(x) := \psi_p(x) = \|x\|^p$. Since we assume that E is 2-smooth, this function is Fréchet differentiable. Following the notation of Lemma 2.2 we set

$$R_\psi(x,y) := \int_0^1 (\psi'(x + r(y-x))(y-x) - \psi'(x)(y-x)) \, \mathrm{d}r.$$

We have the following version of Itô's formula.

Theorem 3.1 (Itô formula). *Let E be a 2-smooth Banach space and let $2 \leqslant p < \infty$. Let $(a_t)_{t\in[0,T]}$ be an E-valued progressively measurable process such that*

$$\mathbb{E}\Big(\int_0^T \|a_t\| \, \mathrm{d}t\Big)^p < \infty$$

and let $(g_t)_{t\in[0,T]}$ be a process in $M^p([0,T];\gamma(H,E))$. Fix $x \in E$ and let $(X_t)_{t\in[0,T]}$ be given by

$$X_t = x + \int_0^t a_s \, \mathrm{d}s + \int_0^t g_s \, \mathrm{d}W_s.$$

The process $s \mapsto \psi'(X_s)g_s$ is progressively measurable and belongs to $M^1([0,T];H)$, and for all $t \in [0,T]$ we have

$$\psi(X_t) = \psi(x) + \int_0^t \psi'(X_s)(a_s) \, \mathrm{d}s + \int_0^t \psi'(X_s)(g_s) \, \mathrm{d}W_s + \lim_{n\to\infty} \sum_{i=0}^{m(n)-1} R_\psi(X_{t_i^n \wedge t}, X_{t_{i+1}^n \wedge t}) \tag{3.4}$$

with convergence in probability, for any sequence of partitions $\Pi_n = \{0 = t_0^n < t_1^n < \cdots < t_{m(n)}^n = T\}$ *whose meshes* $\|\Pi_n\| := \max_{0\leqslant i\leqslant m(n)-1} |t_{i+1}^n - t_i^n|$ *tend to* 0 *as* $n \to \infty$. *Moreover, there exists a constant* C *and, for each* $\varepsilon > 0$, *a constant* C_ε, *both independent of* a *and* g, *such that*

$$\mathbb{E} \liminf_{n\to\infty} \sum_{i=0}^{m(n)-1} |R_\psi(X_{t_i^n\wedge t}, X_{t_{i+1}^n\wedge t})| \leqslant \varepsilon C \mathbb{E} \sup_{s\in[0,t]} \|X_s\|^p + C_\varepsilon \mathbb{E}\Big(\int_0^t \|g_s\|^2_{\gamma(H,E)}\,\mathrm{d}s\Big)^{\frac{p}{2}}. \tag{3.5}$$

The proof shows that we may take $C_\varepsilon = C'(\varepsilon^{1-\frac{2}{p}} + 1)$ for some constant C' independent of a, g, and ε.

Before we start the proof of the theorem we state some lemmas. The first is an immediate consequence of Burkholder's inequality (3.3).

Lemma 3.2. *Under the assumptions of Theorem* 3.1 *we have*

$$\mathbb{E} \sup_{0\leqslant t\leqslant T} \|X_t\|^p \leqslant C\mathbb{E}\Big(\int_0^T \|a_s\|\,\mathrm{d}s\Big)^p + C\mathbb{E}\Big(\int_0^T \|g_s\|^2_{\gamma(H,E)}\,\mathrm{d}s\Big)^{\frac{p}{2}}.$$

Lemma 3.3. *Under the assumptions of Theorem* 3.1, *the process* $t \mapsto \psi'(X_t)(g_t)$ *is progressively measurable and belongs to* $M^1([0,T];H)$.

Proof. By the identity $\|\psi'(x)\| = p\|x\|^{p-1}$ and Hölder's inequality,

$$\begin{aligned}
\mathbb{E}\Big(\int_0^T \|\psi'(X_t)(g_t)\|_H^2\,\mathrm{d}t\Big)^{\frac{1}{2}} &\leqslant \mathbb{E}\Big(\int_0^T \|\psi'(X_t)\|^2 \|g_t\|^2_{\gamma(H,E)}\,\mathrm{d}t\Big)^{\frac{1}{2}} \\
&\leqslant \mathbb{E} \sup_{t\in[0,T]} \|X_t\|^{p-1} \Big(\int_0^T \|g_t\|^2_{\gamma(H,E)}\,\mathrm{d}s\Big)^{\frac{1}{2}} \\
&\leqslant C\Big(\mathbb{E} \sup_{t\in[0,T]} \|X_t\|^p\Big)^{\frac{p-1}{p}} \Big(\mathbb{E}\Big(\int_0^T \|g_t\|^2_{\gamma(H,E)}\,\mathrm{d}s\Big)^{\frac{p}{2}}\Big)^{\frac{1}{p}},
\end{aligned}$$

and the right-hand side is finite by the previous lemma. The progressively measurability is clear. □

This lemma implies that the stochastic integral in (3.4) is well-defined.

Lemma 3.4. *Let* $0 \leqslant u \leqslant t \leqslant T$ *be arbitrary and fixed. Under the assumptions of Theorem* 3.1, *the process* $s \mapsto \psi'(X_u)(g_s)$ *is progressively measurable and belongs to* $M^1([0,T];H)$. *Moreover,* $\mathbb{P}$*-almost surely,*

$$\psi'(X_u)\int_u^t g_s\,\mathrm{d}W_s = \int_u^t \psi'(X_u)(g_s)\,\mathrm{d}W_s.$$

Proof. By similar estimates as in the previous lemma,

$$\mathbb{E}\Big(\int_u^t \|\psi'(X_u)(g_s)\|_H^2\,\mathrm{d}s\Big)^{\frac{1}{2}} \leqslant C(\mathbb{E}\|X_u\|^p)^{\frac{p-1}{p}} \Big(\mathbb{E}\Big(\int_u^t \|g_s\|^2_{\gamma(H,E)}\,\mathrm{d}s\Big)^{\frac{p}{2}}\Big)^{\frac{1}{p}}.$$

The progressively measurability is again clear. To prove the identity we first assume that g is a simple adapted process of the form

$$g_s = \sum_{i=0}^{n-1} 1_{(t_i,t_{i+1}]}(s) A_i,$$

where $\Pi = \{u = t_0 < t_1 < \cdots < t_n = t\}$ is a partition of the interval $[0,T]$ and the random variables are $\mathcal{F}_{t_i}$-measurable and take values in the space of all finite rank operators from H to E. Then,

$$\begin{aligned}
\psi'(X_u)\int_u^t g_s\,\mathrm{d}W_s &= \psi'(X_u)\Big(\sum_{i=0}^{n-1} A_i(W_{t_{i+1}} - W_{t_i})\Big) \\
&= \sum_{i=0}^{n-1} \psi'(X_u)(A_i(W_{t_{i+1}} - W_{t_i})) = \int_u^t \psi'(X_u)(g_s)\,\mathrm{d}W_s.
\end{aligned}$$

For general progressively measurable $g \in L^p(\Omega; L^2([0,T];\gamma(H,E)))$, the identity follows by a routine approximation argument. □

Proof of Theorem 3.1. The proof of the theorem proceeds in two steps. All constants occurring in the proof may depend on E and p, even where this is not indicated explicitly, but not on T. The numerical value of the constants may change from line to line.

Step 1 – Applying Lemma 2.2 to the function $\psi(x) = \|x\|^p$ and the process X, we have, for every $t \in [0,T]$,

$$\begin{aligned}\psi(X_t) - \psi(x) &= \sum_{i=0}^{m(n)-1} \Big(\psi(X_{t^n_{i+1}\wedge t}) - \psi(X_{t^n_i\wedge t})\Big)\\ &= \sum_{i=0}^{m(n)-1} \psi'(X_{t^n_i\wedge t})(X_{t^n_{i+1}\wedge t} - X_{t^n_i\wedge t}) + \sum_{i=0}^{m(n)-1} R_\psi(X_{t^n_i\wedge t}, X_{t^n_{i+1}\wedge t}).\end{aligned}$$

We shall prove the identity (3.4) by showing that

$$\lim_{n\to\infty} \sum_{i=0}^{m(n)-1} \psi'(X_{t^n_i\wedge t})(X_{t^n_{i+1}\wedge t} - X_{t^n_i}) = \int_0^t \psi'(X_s)(a_s)\,\mathrm{d}s + \int_0^t \psi'(X_s)(g_s)\,\mathrm{d}W_s$$

with convergence in probability. In view of the definition of X_t, it is enough to show that

$$\lim_{n\to\infty} \Big| \sum_{i=0}^{m(n)-1} \psi'(X_{t^n_i\wedge t})\Big(\int_{t^n_i\wedge t}^{t^n_{i+1}\wedge t} a_s\,\mathrm{d}s\Big) - \int_0^t \psi'(X_s)(a_s)\,\mathrm{d}s\Big| = 0 \ \ \mathbb{P}\text{-almost surely}$$

and

$$\lim_{n\to\infty} \sum_{i=0}^{m(n)-1} \psi'(X_{t^n_i\wedge t})\Big(\int_{t^n_i\wedge t}^{t^n_{i+1}\wedge t} g_s\,\mathrm{d}W_s\Big) - \int_0^t \psi'(X_s)(g_s)\,\mathrm{d}W_s = 0 \ \text{ in probability.} \tag{3.6}$$

By (2.2), $\mathbb{P}$-almost surely we have

$$\begin{aligned}&\limsup_{n\to\infty}\Big| \sum_{i=0}^{m(n)-1} \psi'(X_{t^n_i\wedge t})\Big(\int_{t^n_i\wedge t}^{t^n_{i+1}\wedge t} a_s\Big) - \int_0^t \psi'(X_s)(a_s)\,\mathrm{d}s\Big|\\ &\leqslant \limsup_{n\to\infty} \sum_{i=0}^{m(n)-1} \Big| \int_{t^n_i\wedge t}^{t^n_{i+1}\wedge t} (\psi'(X_{t^n_i\wedge t}) - \psi'(X_s))(a_s)\,\mathrm{d}s\Big|\\ &\leqslant C \sup_{s\in[0,T]} \|X_s\|^{p-2} \times \limsup_{n\to\infty} \sum_{i=0}^{m(n)-1} \int_{t^n_i\wedge t}^{t^n_{i+1}\wedge t} \|X_{t^n_i\wedge t} - X_s\|\,\|a_s\|\,\mathrm{d}s\\ &\leqslant C \sup_{s\in[0,T]} \|X_s\|^{p-2} \times \limsup_{n\to\infty}\Big(\sup_{0\leqslant i\leqslant m(n)-1}\ \sup_{s\in[t^n_i\wedge t, t^n_{i+1}\wedge t]} \|X_{t^n_i\wedge t} - X_s\|\Big) \times \Big(\sum_{i=0}^{m(n)-1} \int_{t^n_i\wedge t}^{t^n_{i+1}\wedge t} \|a_s\|\,\mathrm{d}s\Big)\\ &= 0,\end{aligned}$$

where we used the continuity of the process X in the last line.

Next, by Lemma 3.4 and the inequalities (3.2) and (2.2),

$$\begin{aligned}&\sum_{i=0}^{m(n)-1} \psi'(X_{t^n_i\wedge t})\Big(\int_{t^n_i\wedge t}^{t^n_{i+1}\wedge t} g_s\,\mathrm{d}W_s\Big) - \int_0^t \psi'(X_s)(g_s)\,\mathrm{d}W_s\\ &\quad= \sum_{i=0}^{m(n)-1} \int_{t^n_i\wedge t}^{t^n_{i+1}\wedge t} \psi'(X_{t^n_i\wedge t})(g_s)\,\mathrm{d}W_s - \int_0^t \psi'(X_s)(g_s)\,\mathrm{d}W_s\\ &\quad= \int_0^t \sum_{i=0}^{m(n)-1} 1_{(t^n_i, t^n_{i+1}]}(s)(\psi'(X_{t^n_i\wedge t}) - \psi'(X_s))(g_s)\,\mathrm{d}W_s.\end{aligned}$$

Recall that the localized stochastic integral is continuous from $M([0,t];\gamma(H,E)))$ into $L^0(\Omega,\mathscr{F}_t;E)$. Hence, in order to prove that the right-hand side converges to 0 in probability, it suffices to prove that

$$\lim_{n\to\infty}\Big\|s\mapsto\sum_{i=0}^{m(n)-1}1_{(t_i^n,t_{i+1}^n]}(s)(\psi'(X_{t_i^n\wedge t})-\psi'(X_s))(g_s)\Big\|_{L^2([0,t];H)}=0\ \text{ in probability.}$$

For this, in turn, it suffices to observe that $\mathbb{P}$-almost surely

$$\begin{aligned}&\lim_{n\to\infty}\Big\|\sum_{i=0}^{m(n)-1}1_{(t_i^n,t_{i+1}^n]}(s)(\psi'(X_{t_i^n\wedge t})-\psi'(X_s))\Big\|_{L^\infty([0,t];E^*)}\\&=\lim_{n\to\infty}\sup_{0\leqslant i\leqslant n-1}\sup_{s\in[t_i^n\wedge t,t_{i+1}^n\wedge t]}\|\psi'(X_{t_i^n\wedge t})-\psi'(X_s)\|=0\end{aligned}$$

by the path continuity of X.

Step 2 – In this step we prove the estimate (3.5). By (2.2), for all $x,y\in E$ and $r\in[0,1]$ we have

$$|\psi'(x+r(y-x))-\psi'(x)|\leqslant(\|x\|^{p-2}\|x-y\|+\|x-y\|^{p-1}).$$

Combining this with (2.5) we obtain

$$|R_\psi(X_{t_i^n\wedge t},X_{t_{i+1}^n\wedge t})|\leqslant C\|X_{t_i^n\wedge t}\|^{p-2}\|X_{t_{i+1}^n\wedge t}-X_{t_i^n\wedge t}\|^2+C\|X_{t_{i+1}^n\wedge t}-X_{t_i^n\wedge t}\|^p. \tag{3.7}$$

We shall estimate the two terms on the right hand of (3.7) side separately.

For the first term, using the inequality $|a+b|^2\leqslant 2|a|^2+2|b|^2$ we obtain

$$\begin{aligned}&\sum_{i=0}^{m(n)-1}\|X_{t_i^n\wedge t}\|^{p-2}\|X_{t_{i+1}^n\wedge t}-X_{t_i^n\wedge t}\|^2\\&\leqslant 2\sum_{i=0}^{m(n)-1}\|X_{t_i^n\wedge t}\|^{p-2}\Big\|\int_{t_i^n\wedge t}^{t_{i+1}^n\wedge t}a_s\,\mathrm{d}s\Big\|^2+2\sum_{i=0}^{m(n)-1}\|X_{t_i^n\wedge t}\|^{p-2}\Big\|\int_{t_i^n\wedge t}^{t_{i+1}^n\wedge t}g_s\,\mathrm{d}W_s\Big\|^2=:I_1^n+I_2^n.\end{aligned}$$

For the first term we have

$$\begin{aligned}I_1^n&\leqslant 2C\sup_{s\in[0,t]}\|X_s\|^{p-2}\times\sup_i\Big\|\int_{t_i^n\wedge t}^{t_{i+1}^n\wedge t}a_s\,\mathrm{d}s\Big\|\times\sum_{i=0}^{m(n)-1}\Big\|\int_{t_i^n\wedge t}^{t_{i+1}^n\wedge t}a_s\,\mathrm{d}s\Big\|\\&\leqslant 2C\sup_{s\in[0,t]}\|X_s\|^{p-2}\times\sup_i\Big\|\int_{t_i^n\wedge t}^{t_{i+1}^n\wedge t}a_s\,\mathrm{d}s\Big\|\times\int_0^t\|a_s\|\,\mathrm{d}s.\end{aligned}$$

By letting $n\to\infty$ we have $\max_{0\leqslant i\leqslant m(n)-1}(t_{i+1}^n-t_i^n)\to 0$, so

$$\sup_{0\leqslant i\leqslant m(n)-1}\Big\|\int_{t_i^n\wedge t}^{t_{i+1}^n\wedge t}a_s\,\mathrm{d}s\Big\|\to 0$$

as $n\to\infty$. Therefore,

$$\lim_{n\to\infty}I_1^n=0,\ \mathbb{P}\text{-almost surely.}$$

To estimate I_2 we use (3.2) and Young's inequality with $\varepsilon>0$ to infer

$$\begin{aligned}\mathbb{E}\liminf_n I_2^n\leqslant\liminf_n\mathbb{E}I_2^n&=\liminf_n\mathbb{E}\sum_{i=0}^{m(n)-1}\|X_{t_i^n\wedge t}\|^{p-2}\Big\|\int_{t_i^n\wedge t}^{t_{i+1}^n\wedge t}g_s\,\mathrm{d}W_s\Big\|^2\\&=\liminf_n\sum_{i=0}^{m(n)-1}\mathbb{E}\Big(\|X_{t_i^n\wedge t}\|^{p-2}\mathbb{E}\Big(\Big\|\int_{t_i^n\wedge t}^{t_{i+1}^n\wedge t}g_s\,\mathrm{d}W_s\Big\|^2\Big|\mathscr{F}_{t_i^n\wedge t}\Big)\Big)\\&\leqslant C\liminf_n\sum_{i=0}^{m(n)-1}\mathbb{E}\Big(\|X_{t_i^n\wedge t}\|^{p-2}\mathbb{E}\Big(\int_{t_i^n\wedge t}^{t_{i+1}^n\wedge t}\|g_s\|^2_{\gamma(H,E)}\,\mathrm{d}s\Big|\mathscr{F}_{t_i^n\wedge t}\Big)\Big)\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\liminf_n \sum_{i=0}^{m(n)-1}\mathbb{E}\Big(\|X_{t_i^n\wedge t}\|^{p-2}\int_{t_i^n\wedge t}^{t_{i+1}^n\wedge t}\|g_s\|^2_{\gamma(H,E)}\,\mathrm{d}s\Big)\\
&\leqslant C\liminf_n \mathbb{E}\Big(\sup_{s\in[0,t]}\|X_s\|^{p-2}\sum_{i=0}^{m(n)-1}\int_{t_i^n\wedge t}^{t_{i+1}^n\wedge t}\|g_s\|^2_{\gamma(H,E)}\,\mathrm{d}s\Big)\\
&= C\mathbb{E}\Big(\sup_{s\in[0,t]}\|X_s\|^{p-2}\int_0^t\|g_s\|^2_{\gamma(H,E)}\,\mathrm{d}s\Big)\\
&\leqslant C\varepsilon\mathbb{E}\Big(\sup_{s\in[0,t]}\|X_s\|^{p}\Big)+C\varepsilon^{1-\frac{p}{2}}\mathbb{E}\Big(\int_0^t\|g_s\|^2_{\gamma(H,E)}\,\mathrm{d}s\Big)^{\frac{p}{2}}.
\end{aligned}
$$

Next we estimate the second term in (3.7). We have

$$
\sum_{i=0}^{m(n)-1}\|X_{t_{i+1}^n\wedge t}-X_{t_i^n\wedge t}\|^p \leqslant C\sum_{i=0}^{m(n)-1}\Big\|\int_{t_i^n\wedge t}^{t_{i+1}^n\wedge t}a_s\,\mathrm{d}s\Big\|^p + C\sum_{i=0}^{m(n)-1}\Big\|\int_{t_i^n\wedge t}^{t_{i+1}^n\wedge t}g_s\,\mathrm{d}W_s\Big\|^p =: I_3^n+I_4^n.
$$

A similar consideration as before yields

$$
\lim_{n\to\infty} I_3^n \leqslant C\lim_{n\to\infty}\sup_{0\leqslant i\leqslant m(n)-1}\Big\|\int_{t_i^n\wedge t}^{t_{i+1}^n\wedge t}a_s\,\mathrm{d}s\Big\|^{p-1}\times\int_0^t\|a_s\|\,\mathrm{d}s = 0.
$$

Moreover, by Burkholder's inequality (3.3),

$$
\begin{aligned}
\mathbb{E}\liminf_n I_4^n \leqslant \liminf_n \mathbb{E}I_4^n &= C\liminf_n\sum_{i=0}^{m(n)-1}\mathbb{E}\Big\|\int_{t_i^n\wedge t}^{t_{i+1}^n\wedge t}g_s\,\mathrm{d}W_s\Big\|^p\\
&\leqslant C\liminf_n\sum_{i=0}^{m(n)-1}\mathbb{E}\Big(\int_{t_i^n\wedge t}^{t_{i+1}^n\wedge t}\|g_s\|^2_{\gamma(H,E)}\,\mathrm{d}s\Big)^{\frac{p}{2}}\\
&\leqslant C\liminf_n\mathbb{E}\Big(\sum_{i=0}^{m(n)-1}\int_{t_i^n\wedge t}^{t_{i+1}^n\wedge t}\|g_s\|^2_{\gamma(H,E)}\,\mathrm{d}s\Big)^{\frac{p}{2}}\\
&= C\mathbb{E}\Big(\int_0^t\|g_s\|^2_{\gamma(H,E)}\,\mathrm{d}s\Big)^{\frac{p}{2}}.
\end{aligned}
$$

Collecting terms, for any $\varepsilon>0$ we obtain the estimate

$$
\mathbb{E}\liminf_{n\to\infty}\sum_{i=0}^{m(n)-1}|R_\psi(X_{t_i^n\wedge t},X_{t_{i+1}^n\wedge t})| \leqslant C\varepsilon\mathbb{E}\Big(\sup_{s\in[0,t]}\|X_s\|^p\Big)+C(\varepsilon^{1-\frac{p}{2}}+1)\mathbb{E}\Big(\int_0^t\|g_s\|^2_{\gamma(H,E)}\,\mathrm{d}s\Big)^{\frac{p}{2}}.
$$

□

In the proof of Theorem 1.2 we will also need the following simple observation.

Lemma 3.5. $\mathbb{P}$-*Almost surely we have*

$$
\liminf_{n\to\infty}\sup_{t\in[0,T]}\sum_{i=0}^{m(n)-1}|R_\psi(X_{t_i^n\wedge t},X_{t_{i+1}^n\wedge t})| \leqslant \liminf_{n\to\infty}\sum_{i=0}^{m(n)-1}|R_\psi(X_{t_i^n},X_{t_{i+1}^n})|. \tag{3.8}
$$

Proof. Fix $t\in(0,T]$ and let $k(n)$ be the unique index such that $t\in(t_{k(n)}^n,t_{k(n)+1}^n]$. Then

$$
\begin{aligned}
\sum_{i=0}^{m(n)-1}|R_\psi(X_{t_i^n\wedge t},X_{t_{i+1}^n\wedge t})| &= \sum_{i=0}^{k(n)-1}|R_\psi(X_{t_i^n},X_{t_{i+1}^n})|+|R_\psi(X_{t_{k(n)}^n},X_t)|\\
&\leqslant \sum_{i=0}^{m(n)-1}|R_\psi(X_{t_i^n},X_{t_{i+1}^n})|+|R_\psi(X_{t_{k(n)}^n},X_t)|
\end{aligned}
$$

$$\leqslant \sum_{i=0}^{m(n)-1} |R_\psi(X_{t_i^n}, X_{t_{i+1}^n})| + C\|X_{t_{k(n)}^n}\|^{p-2}\|X_t - X_{t_{k(n)}^n}\|^2 + C\|X_t - X_{t_{k(n)}^n}\|^p$$
$$\leqslant \sum_{i=0}^{m(n)-1} |R_\psi(X_{t_i^n}, X_{t_{i+1}^n})| + C\sup_{s\in[0,T]}\|X_s\|^{p-2}\|X_t - X_{t_{k(n)}^n}\|^2 + C\|X_t - X_{t_{k(n)}^n}\|^p.$$

Now (3.8) follows by taking the limes inferior for $n\to\infty$ and using path continuity. □

4. Proof of Theorem 1.2

We proceed in four steps. In Steps 1 and 2 we establish the estimate in the theorem for $g \in M^p([0,T];\gamma(H,E))$ with $2 \leqslant p < \infty$. In order to be able to cover exponents $0 < p < 2$ in Step 3, we need a stopped version of the inequalities proved in Steps 1 and 2. For reasons of economy of presentations, we therefore build in a stopping time τ from the start. In Step 4 we finally consider the case where $g \in M([0,T];\gamma(H,E))$.

We shall apply (a special case of) Lenglart's inequality [13, Corollaire II] which states that if $(\xi_t)_{t\in[0,T]}$ and $(a_t)_{t\in[0,T]}$ are continuous non-negative adapted processes, the latter non-decreasing, such that $\mathbb{E}\xi_\tau \leqslant \mathbb{E}a_\tau$ for all stopping times τ with values in $[0,T]$, then for all $0 < r < 1$ one has

$$\mathbb{E}\sup_{0\leqslant t\leqslant T}\xi_t^r \leqslant \frac{2-r}{1-r}\mathbb{E}a_T^r. \tag{4.1}$$

Step 1 – Fix $p \geqslant 2$ and suppose first that $g \in M^p([0,T];\gamma(H,\mathsf{D}(A)))$. As is well known (see [16]), under this condition the process $X_t = \int_0^t e^{(t-s)A}g_s\,\mathrm{d}W_s$ is a strong solution to the equation

$$dX_t = AX_t\,\mathrm{d}t + g_t\,\mathrm{d}W_t, \quad t \geqslant 0; \quad X_0 = 0.$$

In other words, X satisfies

$$X_t = \int_0^t AX_s\,\mathrm{d}s + \int_0^t g_s\,\mathrm{d}W_s \quad \forall t\in[0,T] \;\; \mathbb{P}\text{-almost surely}.$$

Hence if τ is a stopping time with values in $[0,T]$, then by (3.1),

$$X_{t\wedge\tau} = \int_0^t 1_{[0,\tau]}(s)AX_s\,\mathrm{d}s + \int_0^t 1_{[0,\tau]}(s)g_s\,\mathrm{d}W_s \quad \forall t\in[0,T], \quad \mathbb{P}\text{-almost surely}.$$

Let us check next that $a_t := 1_{[0,\tau]}(t)AX_t$ satisfies the assumptions of Theorem 3.1. Indeed, with $h_t := 1_{[0,\tau]}(t)Ag_t$ we have, using the contractivity of the semigroup S and Burkholder's inequality (3.3),

$$\begin{aligned}
\mathbb{E}\Big(\int_0^T \|a_t\|\,\mathrm{d}t\Big)^p &\leqslant \mathbb{E}\Big(\int_0^T \Big\|\int_0^t e^{(t-s)A}h_s\,\mathrm{d}W_s\Big\|\,\mathrm{d}t\Big)^p \\
&\leqslant CT^{p-1}\mathbb{E}\int_0^T \Big\|\int_0^t e^{(t-s)A}h_s\,\mathrm{d}W_s\Big\|^p\,\mathrm{d}t \\
&\leqslant CT^{p-1}\mathbb{E}\int_0^T \Big(\int_0^t \|e^{(t-s)A}h_s\|_{\gamma(H,E)}^2\,\mathrm{d}s\Big)^{\frac{p}{2}}\,\mathrm{d}t \\
&\leqslant CT^p\mathbb{E}\Big(\int_0^T \|h_s\|_{\gamma(H,E)}^2\,\mathrm{d}s\Big)^{\frac{p}{2}} < \infty.
\end{aligned}$$

Hence we may apply Theorem 3.1 and infer that

$$\begin{aligned}
\|X_{t\wedge\tau}\|^p = &\int_0^t 1_{[0,\tau]}(s)\psi'(X_s)(AX_s)\,\mathrm{d}s \\
&+ \int_0^t 1_{[0,\tau]}(s)\psi'(X_s)(g_s)\,\mathrm{d}W_s + \lim_{n\to\infty}\sum_{i=0}^{m(n)-1} R_\psi(X_{t_i^n\wedge t\wedge\tau}, X_{t_{i+1}^n\wedge t\wedge\tau})
\end{aligned}$$

$$\leqslant \int_0^t 1_{[0,\tau]}(s)\psi'(X_s)(g_s)\,\mathrm{d}W_s + \lim_{n\to\infty}\sum_{i=0}^{m(n)-1} R_\psi(X_{t_i^n\wedge t\wedge\tau}, X_{t_{i+1}^n\wedge t\wedge\tau})$$

since $\psi'(x)(Ax) \leqslant 0$ for all $x \in \mathsf{D}(A)$ by the contractivity of e^{tA} (see, e.g., [3, Lemma 4.2]). Hence, by Lemma 3.5,

$$\begin{aligned}
\mathbb{E}\sup_{t\in[0,T]}\|X_{t\wedge\tau}\|^p &\leqslant \mathbb{E}\sup_{t\in[0,T]}\int_0^t 1_{[0,\tau]}(s)\psi'(X_s)(g_s)\,\mathrm{d}W_s + \mathbb{E}\sup_{t\in[0,T]}\liminf_{n\to\infty}\sum_{i=0}^{m(n)-1}|R_\psi(X_{t_i^n\wedge t\wedge\tau}, X_{t_{i+1}^n\wedge t\wedge\tau})| \\
&\leqslant \mathbb{E}\sup_{t\in[0,T]}\int_0^t 1_{[0,\tau]}(s)\psi'(X_s)(g_s)\,\mathrm{d}W_s + \mathbb{E}\liminf_{n\to\infty}\sum_{i=0}^{m(n)-1}|R_\psi(X_{t_i^n\wedge\tau}, X_{t_{i+1}^n\wedge\tau})| \\
&\leqslant C\mathbb{E}\sup_{t\in[0,T]}\int_0^t 1_{[0,\tau]}(s)\psi'(X_s)(g_s)\,\mathrm{d}W_s \\
&\qquad + \varepsilon C\mathbb{E}\sup_{s\in[0,T]}\|X_{s\wedge\tau}\|^p + C_\varepsilon\mathbb{E}\Big(\int_0^T 1_{[0,\tau]}(s)\|g_s\|^2_{\gamma(H,E)}\,\mathrm{d}s\Big)^{\frac{p}{2}}.
\end{aligned}$$

By Burkholder's inequality (3.3) and the identity $\|\psi'(y)\| = p\|y\|^{p-1}$,

$$\begin{aligned}
\mathbb{E}\sup_{t\in[0,T]}\Big|\int_0^t 1_{[0,\tau]}(s)\psi'(X_s)(g_s)\,\mathrm{d}W_s\Big| &\leqslant C\mathbb{E}\Big(\int_0^T 1_{[0,\tau]}(s)\big\|\psi'(X_s)\big\|^2\|g_s\|^2_{\gamma(H,E)}\,\mathrm{d}s\Big)^{\frac12} \\
&= C\mathbb{E}\Big(\int_0^T 1_{[0,\tau]}(s)\big\|X_s\big\|^{2(p-1)}\|g_s\|^2_{\gamma(H,E)}\,\mathrm{d}s\Big)^{\frac12} \\
&\leqslant C\mathbb{E}\Big(\sup_{t\in[0,T]}\|X_{t\wedge\tau}\|^{p-1}\Big(\int_0^T 1_{[0,\tau]}(s)\|g_s\|^2_{\gamma(H,E)}\,\mathrm{d}s\Big)^{\frac12}\Big) \\
&\leqslant Cp^p\Big(\mathbb{E}\sup_{t\in[0,T]}\|X_{t\wedge\tau}\|^p\Big)^{\frac{p-1}{p}}\Big(\mathbb{E}\Big(\int_0^T 1_{[0,\tau]}(s)\|g_s\|^2_{\gamma(H,E)}\,\mathrm{d}s\Big)^{\frac{p}{2}}\Big)^{\frac1p} \\
&\leqslant C\varepsilon\mathbb{E}\sup_{t\in[0,T]}\|X_{t\wedge\tau}\|^p + C_\varepsilon\mathbb{E}\Big(\int_0^T 1_{[0,\tau]}(s)\|g_s\|^2_{\gamma(H,E)}\,\mathrm{d}s\Big)^{\frac{p}{2}},
\end{aligned}$$

where we also used the Hölder's inequality and Young's inequality.

Combining these estimates and taking $\varepsilon > 0$ small enough, we infer that

$$\mathbb{E}\sup_{t\in[0,T]}\|X_{t\wedge\tau}\|^p \leqslant C\mathbb{E}\Big(\int_0^T 1_{[0,\tau]}(s)\|g_s\|^2_{\gamma(H,E)}\,\mathrm{d}s\Big)^{\frac{p}{2}}.$$

Step 2 – Now let $g \in M^p([0,T];\gamma(H,E)$ be arbitrary. Set $g^n = n(nI-A)^{-1}g$, $n \geqslant 1$. These processes satisfy the assumptions of Step 1 and we have $\|g^n\|_{\gamma(H,E)} \leqslant \|g\|_{\gamma(H,E)}$ pointwise. Define $X_t^n = \int_0^t e^{(t-s)A}g_s^n\,\mathrm{d}s$. From Step 1 we know that for any stopping time τ in $[0,T]$ we have

$$\mathbb{E}\sup_{t\in[0,T]}\|X^n_{t\wedge\tau}\|^p \leqslant C\mathbb{E}\Big(\int_0^T 1_{[0,\tau]}(s)\|g^n_s\|^2_{\gamma(H,E)}\,\mathrm{d}s\Big)^{\frac{p}{2}}.$$

In particular, as $n, m \to \infty$,

$$\mathbb{E}\sup_{t\in[0,T]}\|X_t^n - X_t^m\|^p \to 0.$$

In these circumstances there is a process $\bar{X}$ such that $\lim_{n\to\infty}\mathbb{E}\sup_{t\in[0,T]}\|\bar{X}_t^n - X_t\|^p = 0$ and

$$\mathbb{E}\sup_{t\in[0,T]}\|\bar{X}_{t\wedge\tau}\|^p \leqslant C\mathbb{E}\Big(\int_0^T 1_{[0,\tau]}(s)\|g_s\|^2_{\gamma(H,E)}\,\mathrm{d}s\Big)^{\frac{p}{2}}. \tag{4.2}$$

Also, notice that for every $t \in [0,T]$, we have

$$\mathbb{E}\|X_t^n - X_t\|^p = \mathbb{E}\Big\| \int_0^t e^{(t-s)A} g_s^n \, \mathrm{d}s - \int_0^t e^{(t-s)A} g_s \, \mathrm{d}s \Big\|^p \leqslant C\big(\mathbb{E}\int_0^t \|g_s^n - g_s\|^2_{\gamma(H,E)} \, \mathrm{d}s\big)^p.$$

Hence $X_t^n \to X_t$ in $L^p(\Omega; E)$. Therefore, $\bar{X}$ is a modification of X. This concludes the proof for $p \geqslant 2$.

Step 3 – In this step we extend the result to exponents $0 < p < 2$. First consider the case where $g \in M^2([0,T]; \gamma(H,E))$. By (4.2), for all stopping times τ in $[0,T]$ we have

$$\mathbb{E}\|X_\tau\|^2 \leqslant C\mathbb{E} \int_0^\tau \|g_s\|^2_{\gamma(H,E)} \, \mathrm{d}s.$$

It then follows from Lenglart's inequality (4.1) that for all $0 < p < 2$,

$$\mathbb{E} \sup_{t \in [0,T]} \|X_t\|^p \leqslant C\mathbb{E}\Big(\int_0^T \|g_s\|^2_{\gamma(H,E)} \, \mathrm{d}s\Big)^{\frac{p}{2}}.$$

For $g \in M^p([0,T]; \gamma(H,E))$ the result follows by approximation.

Step 4 – Finally, the existence of a continuous version for the process X under the assumption $g \in M([0,T]; \gamma(H,E))$ follows by a standard localisation argument.

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