Weak measurability of the orbits of an adjoint semigroup [1]

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Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a positive C_0-semigroup on a Banach lattice E, let $E^\odot$ denote the maximal subspace of the dual space E^* on which the adjoint semigroup $\mathbf{T}^*$ acts in a strongly continuous way, and let $(E^\odot)^{dd}$ denote the band g enerated by $E^\odot$. Assuming Martin's Axiom, we prove the followig result. If, for some $x^* \in E^*$, the map $t \mapsto T^*(t)x^*$ is weakly measurable, then $T^*(t)x \in (E^\odot)^{dd}$ for all $t > 0$. If moreover $\mathbf{T}^*$ is a lattice semigroup, then $x \in (E^\odot)^{dd}$. As a consequen ce, translation of a finite Borel measure μ on $\mathbb{R}$ is weakly measurable if and only if μ is absolutely continuous with respect to the Lebesgue measure.

0. INTRODUCTION

Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a C_0-semigroup on a Banach space X. As usual, we denote by $\mathbf{T}^* = \{T^*(t)\}_{t\geq 0}$ the adjoint semigroup on the dual space X^*, defined by $T^*(t) := (T(t))^*$, and by $X^\odot$ the maximal subspace of X^* on which $\mathbf{T}^*$ ac ts in a strongly continuous way.

Let Y be a Banach space. We say that a map $f : \mathbb{R}_+ \to Y$ is weakly measurable if for each $y^* \in Y^*$ the map $f_{y^*}(t) := \langle y^*, f(t)\rangle$ is Lebesgue measurable. The purpose of this paper is to prove the following result, assuming Martin's Axiom (MA).

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$T^*(t)x \in (E^{\odot})^{dd}$ for all $t > 0$.

Here $(E^{\odot})^{dd}$ denotes the band generated by $E^{\odot}$. This theorem implies for example that translating a finite Borel measure μ on $\mathbb{R}$ is weakly measurable if and only if μ is absolutely continuous with respect to the Lebesgue measure; cf. Cor ollary 3.7 below.

In order to motivate our result, we will briefly sketch its history. Let us agree to call $\mathbf{T}^*$ weakly measurable if each of its orbits $t \mapsto T^*(t)x^*$ is. In Feller (1953) it is proved that the adjoint of the translation group $\mathbf{T}$ on $C_0(\mathbb{R})$ fa ils to be weakly measurable. Explicitly, he shows that the map $t \mapsto T^*(t)\delta_0 = \delta_t$, the Dirac measure at t, is not weakly measurable. The proof of this is very short and is reproduced at the beginning of Section 3, for it is essentially t his argument that we will generalize. In Van Neerven (1992) the following is shown: if $\mathbf{T}$ is a positive C_0-semigroup on a space $E = C(K)$, K compact Hausdorff, then $\mathbf{T}^*$ is weakly Borel measurable if and only if $\mathbf{T}^*$ is strongly continuous for $t > 0$. Moreover, in this setting it is known that $E^{\odot} = (E^{\odot})^{dd}$. Thus, apart from the subtlety concerning Borel measurability, this result is implied by our Theorem 0.1. Both the proofs in Feller and Van Neerven avoid the use of set-theoretical axioms. Finally, Talagrand (1982) shows, assuming Martin's Axiom, that translation of a function $f \in L^\infty(G)$, where G is a compact abelian group, is weakly measurable if and only if f is equal a.e. to a Riemann measurable function. In fact, it was this result which motivated us to use Martin's Axiom. Of course, his conclusion for the concrete case of translation in $L^\infty(G)$ is much stronger than what is implied by our general result.

1. PRELIMINARIES

In this section we recall some of the basic facts about adjoint semigroups which will be used in the sequel. Proofs of these facts can be found in Van Neerven (1992).

Let $\mathbf{T}$ be a C_0-semigroup (i.e., a strongly continuous semigroup) on a Banach space X. Its generator will be denoted by A with domain $D(A)$. Considering the adjoint semigroup $\mathbf{T}^*$ on the dual space X^*, we define

$$X^{\odot} = \{x^* \in X^* : \lim_{t \downarrow 0} \|T^*(t)x^* - x^*\| = 0\},$$

the domain of strong continuity of $\mathbf{T}^*$. Then $X^{\odot}$ is a $\mathbf{T}^*$-invariant, norm closed, weak*-dense subspace of X^* (hence $X^{\odot} = X^*$ if X is reflexive). The space $X^{\odot}$ is precisely the norm closure of $D(A^*)$, the domain of the adjoint of A. In particular, for $\lambda \in \varrho(A) = \varrho(A^*)$ we have $R(\lambda, A^*)x^* \in X^{\odot}$ for all $x^* \in X^*$, where $R(\lambda, A^*) = R(\lambda, A)^* = (\lambda - A^*)^{-1}$ is the resolvent.

If $\mathbf{T}$ extends to a C_0-group, then the space $X^{\odot}$ with respect to the semigroup $\{T(t)\}_{t \geq 0}$ is equal to the domain of strong continuity of the group $\{T^*(t)\}_{t \in \mathbb{R}}$. In particular we mention that if $\mathbf{T}$ is the translation group on $X = C_0(\mathbb{R})$, then $X^{\odot}$ is precisely the subspace of all finite regular Borel measures on $\mathbb{R}$ which are absolutely continuous with respect to the Lebesgue measure. In this way $X^{\odot}$ can be identified canonically with $L^1(\mathbb{R})$. This classical result is due to Plessner (1929).

$R(\lambda, A) \geq 0$ (for the general theory of positive semigroups we refer to Nagel (1986)). We denote by $(E^{\odot})^d$ the disjoint complement of $E^{\odot}$ in E^*, i.e.,

$$(E^{\odot})^d = \{x^* \in E^* : x^* \perp y^{\odot} \text{ for all } y^{\odot} \in E^{\odot}\}.$$

Here $x^* \perp y^{\odot}$ means that $|x^*| \wedge |y^{\odot}| = 0$. Then $(E^{\odot})^{dd}$, the disjoint complement of $(E^{\odot})^d$, is equal to the band generated by $E^{\odot}$. We will need the following results concerning the adjoint of a positive C_0-semigroups (Van Neerven and De Pagter, to appear). Let $\mathbf{T}$ be a positive C_0-semigroup on a Banach lattice E, let ω be as above and fix $\lambda > \omega$.

Lemma 1.1 (Van Neerven and De Pagter, Section 1). *The band $(E^{\odot})^{dd}$ generated by $E^{\odot}$ is $\mathbf{T}^*$-invariant. If, moreover, $\mathbf{T}^*$ is a lattice semigroup, then also $(E^{\odot})^d$ is $\mathbf{T}^*$-invariant.*

Lemma 1.2 (Van Neerven and De Pagter, Lemma 3.1). *Let $0 \leq x \in E$ and $0 \leq x^*, y^* \in E^*$. If $\langle R(\lambda, A^*)x^* \wedge y^*, x\rangle = 0$, then $\langle T^*(t)x^* \wedge y^*, x\rangle = 0$ for almost all $t \geq 0$.*

Lemma 1.3 (Van Neerven and De Pagter, Theorem 3.5). *For any $0 \leq x^* \in E^*$ we have $x^* \perp E^{\odot}$ if and only if $x^* \perp R(\lambda, A^*)x^*$.*

We assume the reader to be familiar with the elementary theory of Banach lattices, and refer to Aliprantis and Burkinshaw (1985) or Meyer-Nieberg (1991) for more details. Our results are valid both for real and complex Banach lattices.

2. SOME CONSEQUENCES OF MARTIN'S AXIOM

In this section we collect some useful consequences of Martin's Axiom (MA). We let m and m^* denote the Lebesgue measure and outer Lebesgue measure, respectively. The following result can be found in Fremlin (1984), Exercise 32P(d).

Lemma 2.1 (MA). *Let $S \subset [0,1] \times [0,1]$ be Lebesgue measurable, $m(S) = 1$. Then there is a subset $H \subset [0,1]$, $m^*(H) = 1$, such that for all $t, s \in H$, $t \neq s$, we have $(t,s) \in S$.*

In other words, $(H \times H) \backslash \Delta \subset S$, where Δ is the diagonal. The set H need not be Lebesgue measurable in general.

The next result is implicit in Talagrand (1984), p. 99-102. (actually, this reference contains an extension of a combination of Lemmas 2.1 and 2.2).

Lemma 2.2 (MA). *Let $H \subset [0,1]$, $m^*(H) = 1$. Then $H = H_0 \cup H_1$ with $H_0 \cap H_1 = \emptyset$ and $m^*(H_0) = m^*(H_1) = 1$.*

Under assumption of the Continuum Hypothesis, Lemma 2.2 had been proved earlier by Sierpinski (1950). If H is Lebesgue measurable, then the lemma can be proved without set-theoretical axioms. This is not very relevant for our purposes, however, since we will be interested in the case where H is non-measurable.

$g|_H = f$, then g is not Lebesgue measurable.

Proof: Let $H = H_0 \cup H_1$ as in Lemma 2.2 and let f be the characteristic function of H_0. Let $g : [0,1] \to \mathbb{C}$ be any function such that $g|_H = f$, and consider the set $G = \{|g| \leq \frac{1}{2}\}$. Then $H_1 \subset G$, so $m^*(G) = 1$. Also, $H_0 \subset [0,1]\backslash G$, so $m^*([0,1]\backslash G) = 1$. Therefore, G cannot be Lebesgue measurable. ∎

3. PROOF OF THE MAIN THEOREM

We start with Feller's proof that translation of a Dirac measure is not weakly measurable in $M(\mathbb{R})$, the space of finite Borel measures on $\mathbb{R}$. Denote by $\mathbf{T} = \{T(t)\}_{t\in\mathbb{R}}$ the translation group defined by $T(t)f(\omega) = f(\omega+t)$ in the space $C_0(\mathbb{R})$ of all continuous functions on $\mathbb{R}$ vanishing at infinity. Let $g : \mathbb{R} \to [0,1]$ be any non Lebesgue measurable function. Let $Z = span\{\delta_t : \ t \in \mathbb{R}\}$ and define a linear form Φ on Z by $\Phi(\delta_t) := g(t)$. Because

$$\Big|\Phi\Big(\sum_{j=1}^{n}\alpha_j\delta_{t_j}\Big)\Big| = \Big|\sum_{j=1}^{n}\alpha_j g(t_j)\Big| \leq \sum_{j=1}^{n}|\alpha_j| = \Big\|\sum_{j=1}^{n}\alpha_j\delta_{t_j}\Big\|,$$

it follows that Φ extends to a bounded linear functional on the closure of Z. Letting $x^{**} \in (M(\mathbb{R}))^*$ be any Hahn-Banach extension of Φ, we have

$$\langle x^{**}, T^*(t)\delta_0\rangle = \Phi(T^*(t)\delta_0) = \Phi(\delta_t) = g(t).$$

This shows that $t \mapsto T^*(t)\delta_0$ is not weakly Lebesgue measurable.

Inspection of the proof shows that it depends on three facts:

(i) For any two $t \neq s$, we have $\delta_t \perp \delta_s$;
(ii) For any two t, s we have $\|\delta_t\| = \|\delta_s\|$;
(iii) $M(\mathbb{R})$ is an AL-space, and therefore $\Big\|\sum_{j=1}^{n}\alpha_j\delta_{t_j}\Big\| = \sum_{j=1}^{n}|\alpha_j|$.

By (i), it is possible to define the form Φ and by (ii) and (iii), it is bounded. Our proof of Theorem 0.1 will be a generalization of the above argument. However, we run into a number of obstructions. The first problem is to find an analogue of (i). This is done in Lemma 3.3 below. There we obtain a certain subset of $\mathbb{R}_+ \times \mathbb{R}_+$. In order to get from this something similar to (i), this subset should contain a sufficiently large set of the form $(H \times H)\backslash\Delta$, where $H \subset \mathbb{R}_+$ and Δ is the diagonal of $\mathbb{R}_+ \times \mathbb{R}_+$. At this point we use Lemma 2.1 and Martins Axiom comes in. Secondly, in general $\mathbf{T}$ need not be isometric. In Lemma 3.4 we establish a substitute for (ii), which at the same time takes care of the fact that in general $(E^{\odot})^d$ need not be $\mathbf{T}^*$-invariant. Finally, not every dual Banach lattice is an AL-space. This leads to the difficulty of proving that the linear form Φ is bounded. We overcome this problem by identifying an isomorphic copy of the AL-space $l^1(H)$ in the closed span of any orbit which does not lie in $(E^{\odot})^{dd}$ for all $t > 0$. For this we use a variant of the lattice identity $|\phi + \psi| = |\phi| + |\psi|$, $\phi \perp \psi$; see Lemma 3.5.

semigroup with generator A on a Banach lattice E. We fix constants M and ω such that $\|T(t)\| \le Me^{\omega t}$ and fix some $\lambda > \omega$. By P we will denote the band projection onto the band $(E^{\odot})^d$, the disjoint complement of $E^{\odot}$.

Lemma 3.1. *Let $0 \le x^* \in E^*$ and let Q denote the band projection onto the band generated by $R(\lambda, A^*)x^*$. Then for all $t \ge 0$ we have $(I-P)T^*(t)x^* = QT^*(t)x^*$.*

Proof: Clearly $\{R(\lambda, A^*)x^*\}^{dd} \subset (E^{\odot})^{dd}$, so $Q \le I-P$ and $QT^*(t)x^* \le (I-P)T^*(t)x^*$.

To prove the reverse inequality, fix $t \ge 0$ and put $y^* := (I-Q)(I-P)T^*(t)x^*$. Then $y^* \perp R(\lambda, A^*)x^*$. Also, from $0 \le y^* \le T^*(t)x^*$ we infer that

$$0 \le R(\lambda, A^*)y^* \le R(\lambda, A^*)T^*(t)x^* = e^{\lambda t}\int_t^{\infty} e^{-\lambda s}T^*(s)x^* ds \le e^{\lambda t}R(\lambda, A^*)x^*,$$

th e integral being in the weak* sense. It follows that $y^* \perp R(\lambda, A^*)y^*$. By Lemma 1.3, $y^* \perp E^{\odot}$. But by its definition, we also have $y^* \in (E^{\odot})^{dd}$, so $y^* = 0$. Hence, $(I-P)T^*(t)x^* = Q(I-P)T^*(t)x^* \le QT^*(t)x^*$. ∎

Lemma 3.2. *Let $0 \le x^* \in E^*$, $0 \le y^* \in E^*$ and $0 \le x \in E$. Then the map $(t,s) \mapsto \langle T^*(t)x^* \wedge PT^*(s)t^*, x\rangle$ is Borel measurable on $\mathbb{R}_+ \times \mathbb{R}_+$.*

Proof: First,

$$\langle T^*(t)x^* \wedge T^*(s)y^*, x\rangle = \inf\{\langle T^*(t)x^*, u\rangle + \langle T^*(s)y^*, v\rangle : \ u, v \in [0,x]; u+v = x\},$$

so the map $(t,s) \mapsto \langle T^*(t)x^* \wedge T^*(s)y^*, x\rangle$ is Borel measurable, being the pointwise infimum of continuous functions. Therefore it remains to show that the map

$$(t,s) \mapsto \langle T^*(t)x^* \wedge (I-P)T^*(s)y^*, x\rangle$$

is Borel. Let Q be the band projection onto the band generated by $R(\lambda, A^*)y^*$. By Lemma 3.1 and general vector lattice theory,

$$(I-P)T^*(s)y^* = QT^*(s)y^* = \sup_n\{T^*(s)y^* \wedge nR(\lambda, A^*)y^*\}.$$

Hence,

$$\langle T^*(t)x^* \wedge (I-P)T^*(s)t^*, x\rangle = \sup_n\{\langle T^*(t)x^* \wedge T^*(s)y^* \wedge nR(\lambda, A^*)y^*, x\rangle\}$$

and

$$\begin{aligned}&\langle T^*(t)x^* \wedge T^*(s)y^* \wedge nR(\lambda, A^*)y^*, x\rangle = \\ &\inf\{\langle T^*(t)x^*, u\rangle + \langle T^*(s)y^*, v\rangle + \langle nR(\lambda, A^*)y^*, w\rangle : \ u, v, w \in [0,x]; u+v+w = x\}.\end{aligned}$$

Since the latter is clearly a Borel function, the lemma follows. ∎

$$N := \{(t,s) \in \mathbb{R}_+ \times \mathbb{R}_+ : \langle |T^*(t)x^*| \wedge |PT^*(s)x^*|, x\rangle > 0\}$$

is a Lebesgue measurable set of measure zero.

Proof: Put $M := \{(t,s) \in \mathbb{R}_+ \times \mathbb{R}_+ : \langle T^*(t)|x^*| \wedge PT^*(s)|x^*|, x\rangle > 0\}$. Then $N \subset M$, so it suffices to show that M is a Borel set of measure zero. By Lemma 3.2, M is a Borel set. Fix $s \geq 0$. We have $PT^*(s)|x^*| \in (E^{\odot})^d$, hence c ertainly $PT^*(s)|x^*| \wedge R(\lambda, A^*)|x^*| = 0$. Therefore, by Lemma 1.2 we see that $\langle T^*(t)|x^*| \wedge PT^*(s)|x^*|, x\rangle = 0$ for almost all $t \geq 0$. Hence, for all $s \geq 0$, the set $\{t \geq 0 : (t,s) \in M\}$ is a zero set. Therefore M is a zero set by the Fubini theorem. ∎

The next lemma describes a certain weak*-continuity property of orbits which do not lie in $(E^{\odot})^{dd}$ for all $t > 0$.

Lemma 3.4. *Let $x^* \in E^*$, $0 \leq x \in E$ and $s_0 > 0$ be such that $\langle |PT^*(s_0)x^*|, x\rangle > 0$. Then there exists $0 < \epsilon < s_0$ such that*

$$\langle |PT^*(s)x^*|, x\rangle \geq \frac{1}{2}\langle |PT^*(s_0)x^*|, x\rangle, \quad \forall s \in [s_0 - \epsilon, s_0].$$

Proof: Let $C > 0$ be such that $\|PT^*(s)x^*\| \leq C$ for all $s \in [0, s_0]$. For $s \in [0, s_0]$ we write

$$\begin{aligned} T^*(s_0)x^* &= T^*(s_0 - s)T^*(s)x^* \\ &= T^*(s_0 - s)PT^*(s_0)x^* + T^*(s_0 - s)(I - P)T^*(s)x^*, \end{aligned}$$

so

$$\begin{aligned} PT^*(s_0)x^* &= PT^*(s_0 - s)PT^*(s)x^* + PT^*(s_0 - s)(I - P)T^*(s)x^* \\ &= PT^*(s_0 - s)PT^*(s)x^*. \end{aligned}$$

In the last identity we used the fact that $(E^{\odot})^{dd}$ is $\mathbf{T}^*$-invariant (Lemma 1.1). It follows that $|PT^*(s_0)x^*| \leq T^*(s_0 - s)|PT^*(s)x^*|$. Hence,

$$\begin{aligned} \langle |PT^*(s_0)x^*|, x\rangle &\leq \langle |PT^*(s)x^*|, T(s_0 - s)x\rangle \\ &= \langle |PT^*(s)x^*|, x\rangle + \langle |PT^*(s)x^*|, T(s_0 - s)x - x\rangle \\ &\leq \langle |PT^*(s)x^*|, x\rangle + C\|T(s_0 - s)x - x\|. \end{aligned}$$

By taking $\epsilon > 0$ so small that $\|T(s_0 - s)x - x\| \leq (2C)^{-1}\langle |PT^*(s_0)x^*|, x\rangle$ for all $0 \leq s_0 - s \leq \epsilon$, the lemma now follows. ∎

Finally we need the following substitute for the norm additivity of disjoint vectors.

Lemma 3.5. *Suppose $\phi_1, ...\phi_n \in E^*$ and $0 \leq x \in E$ are such that $\langle |\phi_j| \wedge |\phi_k|, x\rangle = 0$ for all $j \neq k$. Then*

$$\Big\langle \Big|\sum_{j=1}^{n} \phi_j\Big|, x\Big\rangle = \sum_{j=1}^{n} \langle |\phi_j|, x\rangle.$$

Proof: Let $\phi, \psi \in E^*$ such that $\langle |\phi| \wedge |\psi|, x\rangle = 0$. Because of the lattice identity $2(|\phi| \wedge |\psi|) = |\phi| + |\psi| - \big||\phi| - |\psi|\big|$ we have $\langle \big||\phi| - |\psi|\big|, x\rangle = \langle |\phi|, x\rangle + \langle |\psi|, x\rangle$. Since $\big||\phi| - |\psi|\big| \leq |\phi + \psi| \leq |\phi| + |\psi|$ it follows that $\langle |\phi + \psi|, x\rangle = \langle |\phi|, x\rangle + \langle |\psi|, x\rangle$. Now proceed by induction on n. ∎

to be weakly measurable.

Choose $0 \leq x \in E$, $\|x\| = 1$, such that $\langle |PT^*(s_0)x^*|, x\rangle > 0$. Let $\epsilon > 0$ be as in Lemma 3.4 and put

$$S := \{(t,s) \in [s_0 - \epsilon, s_0] \times [s_0 - \epsilon, s_0] : \ \langle |T^*(t)x^*| \wedge |PT^*(s)x^*|, x\rangle = 0\}.$$

By Lemma 3.3 we know that $m(S) = \epsilon^2$. Hence by Lemma 2.1, there is a subset $H \subset [s_0 - \epsilon, s_0]$ with $m^*(H) = \epsilon$ such that for all $t, s \in H$, $t \neq s$, we have $\langle |T^*(t)x^*| \wedge |PT^*(s)x^*|, x\rangle = 0$.

We claim that there exists a constant $K > 0$ such that for all $n \in \mathbb{N}$, scalars $\alpha_1, ..., \alpha_n$ and $s_1, ..., s_n \in H$ we have

$$\Big\|\sum_{j=1}^n \alpha_j T^*(s_j)x^*\Big\| \geq K \sum_{j=1}^n |\alpha_j|.$$

To see this, first observe that for all $t, s \geq 0$ we have $|T^*(t)x^*| \wedge |PT^*(s)x^*| \in (E^{\odot})^d$. Therefore, for any two $s_j, s_k \in H$, $s_j \neq s_k$, we have

$$\begin{aligned}\langle |PT^*(s_j)x^*| \wedge |PT^*(s_k)x^*|, x\rangle &= \big\langle P\big(|T^*(s_j)x^*| \wedge |PT^*(s_k)x^*|\big), x\big\rangle \\ &= \langle |T^*(s_j)x^*| \wedge |PT^*(s_k)x^*|, x\rangle = 0.\end{aligned}$$

Hence Lemma 3.5 applies, and by using it in tandem with the choice of ϵ we have

$$\begin{aligned}\Big\|\sum_{j=1}^n \alpha_j T^*(s_j)x^*\Big\| &\geq \Big\|\sum_{j=1}^n \alpha_j PT^*(s_j)x^*\Big\| \geq \Big\langle\Big|\sum_{j=1}^n \alpha_j PT^*(s_j)x^*\Big|, x\Big\rangle \\ &= \sum_{j=1}^n |\alpha_j|\ \langle |PT^*(s_j)x^*|, x\rangle \geq \frac{1}{2}\langle |PT^*(s_0)x^*|, x\rangle \cdot \sum_{j=1}^n |\alpha_j|.\end{aligned}$$

This proves the claim.

Let $f : H \to [0,1]$ be a function as Lemma 2.3. Define

$$Z := span\{T^*(s)x^* : \ s \in H\}.$$

It follows in particular from the above that the set $\{T^*(s)x^* : \ s \in H\}$ is linearly independent, so it makes sense to define a linear form Φ on Z by

$$\Phi\big(T^*(s)x^*\big) := f(s).$$

We show that Φ is bounded. Indeed, by the claim we have

$$\Big|\Phi\Big(\sum_{j=1}^n \alpha_j T^*(s_j)x^*\Big)\Big| = \Big|\sum_{j=1}^n \alpha_j f(s_j)\Big| \leq \sum_{j=1}^n |\alpha_j| \leq \frac{1}{K}\Big\|\sum_{j=1}^n \alpha_j T^*(s_j)x^*\Big\|.$$

Therefore, Φ extends to a bounded linear functional on the closure of Z in E^*. Let $x^{**} \in E^{**}$ be any Hahn-Banach extension of Φ and let the function g be defined by $g(t) = \langle x^{**}, T^*(t)x^*\rangle$. Clearly $g|_H = f$, so $g|_{[s_0-\epsilon,s_0]}$ is not Lebesgue measurable by Lemma 2.3. ∎

Corollary 3.6 (MA). *Suppose* $\mathbf{T}$ *is a positive* C_0*-semigroup on* E *such that* $\mathbf{T}^*$ *is a lattice semigroup. If for some* $x^* \in E^*$ *the map* $t \mapsto T^*(t)x^*$ *is weakly measurable, then* $x^* \in (E^\odot)^{dd}$.

Proof: Write $x^* = x_0^* + x_1^*$ with $x_0^* \in (E^\odot)^d$, $x_1^* \in (E^\odot)^{dd}$. By Lemma 1.1, we have for any $t \geq 0$ that $T^*(t)x_0^* \in (E^\odot)^d$ and $T^*(t)x_1^* \in (E^\odot)^{dd}$. Also, by Theorem 0.1, $T^*(t)x^* \in (E^\odot)^{dd}$ for $t > 0$. Therefore we must have $T^*(t)x_0^* = 0$ for all $t > 0$. From the weak*-continuity of $t \mapsto T^*(t)x_0^*$, we conclude that $x_0^* = 0$. ∎

If the dual space E^* has order continuous norm, then $E^\odot = (E^\odot)^{dd}$ holds for any positive C_0-semigroup on E (see Van Neerven and De Pagter (to appear), Theorem 2.1 or De Pagter (1992), Theorem 2.1). This holds in particular if $E = C_0(\Omega)$, Ω locally compact Hausdorff. In combination with the above results, this yields the following corollary.

Corollary 3.7 (MA). *Suppose that* $\mathbf{T}$ *is a positive* C_0*-semigroup on a Banach lattice* E *whose dual* E^* *has order continuous norm. If, for some* $x^* \in E^*$, *the map* $t \mapsto T^*(t)x^*$ *is weakly measurable, then* $t \mapsto T^*(t)x^*$ *is strongly continuous for* $t > 0$. *If moreover,* $\mathbf{T}^*$ *is a lattice semigroup, then* $t \mapsto T^*(t)x^*$ *is strongly continuous for* $t \geq 0$, *i.e.* $x^* \in E^\odot$.

For the translation group in $E = C_0(\mathrm{I\!R})$ we know that $E^\odot = (E^\odot)^{dd} = L^1(\mathrm{I\!R})$ (cf. Section 1), which gives the following result.

Corollary 3.8 (MA). *Let* μ *be a finite Borel measure on* $\mathrm{I\!R}$. *Then translation of* μ *is weakly measurable in* $M(\mathrm{I\!R})$ *if and only if* μ *is absolutely continuous with respect to the Lebesgue measure.*

Finally, we mention one more consequence of Corollary 3.6.

Corollary 3.9 (MA). *Let* $\mathbf{T}$ *be a positive* C_0*-group with unbounded generator on a space* $C_0(\Omega)$, Ω *locally compact Hausdorff. Then* $\mathbf{T}^*$ *is not weakly measurable.*

Proof: From Van Neerven and De Pagter (to appear), Corollary 3.18, we know that $E^\odot = (E^\odot)^{dd} \neq E^*$, so the result follows from Corollary 3.6. ∎

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