APPROXIMATING BOCHNER INTEGRALS BY RIEMANN SUMS

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Abstract. Let μ be a tight Borel measure on a metric space Ω, let X be a Banach space, and let $f : (\Omega, \mu) \to X$ be Bochner integrable. We show that for every sequence of partitions $P^{(n)} = \{\Omega_1^{(n)}, \dots, \Omega_{N^{(n)}}^{(n)}\}$ of Ω satisfying $\lim_{n\to\infty} \text{mesh}\,(P^{(n)}) = 0$ there exists a sequence of sample point sets $S^{(n)} = \{s_1^{(n)}, \dots, s_{N^{(n)}}^{(n)}\}$ such that

$$\lim_{n\to\infty} \Big\| \sum_{j=1}^{N^{(n)}} \mu(\Omega_j^{(n)})\, f(s_j^{(n)}) - \int_\Omega f\, d\mu \Big\| = 0.$$

It is an old result of H. Lebesgue [5, pp. 30 ff.] that if $f : [0,1] \to \mathbb{R}$ is Lebesgue integrable, then there exist numbers $s_j^{(n)} \in \big[\frac{j-1}{n}, \frac{j}{n}\big]$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} f\big(s_j^{(n)}\big) = \int_a^b f(t)\, dt.$$

In this paper we will extend this result to Bochner integrable functions defined on an arbitrary metric space.

Let Ω be a metric space and let μ be a tight Borel measure on Ω; by definition, this is a finite Borel measure with the property that for every $\varepsilon > 0$ there exists a compact set $K \subseteq \Omega$ with $\mu(\Omega \setminus K) < \varepsilon$. A *partition* of (Ω, μ) is a finite collection $P = \{\Omega_1, \dots, \Omega_N\}$ of μ-measurable subsets of Ω with the following properties:

(1) $\mu(\Omega_j) > 0$ for all j;
(2) $\mu(\Omega_j \cap \Omega_k) = 0$ for all $j \neq k$;
(3) $\mu(\Omega \setminus \cup_j \Omega_j) = 0$.

The numbers $\max_j \big(\mu(\Omega_j)\big)$ and $\max_j \big(\text{diam}\,(\Omega_j)\big)$ (whenever this is finite) will be called the *measure* of P and the *mesh* of P, respectively. A finite subset $\{s_1, \dots, s_N\} \subseteq \Omega$ with $s_j \in \Omega_j$ for each j is called a set of *sample points* associated with the partition P.

Let X be a Banach space. For a μ-measurable function $f : \Omega \to X$ we define the *Riemann sum* of f relative to the partition $P = \{\Omega_1, \dots, \Omega_N\}$ and the associated sample point set $S = \{s_1, \dots, s_N\}$ by

$$\mathscr{R}(f; P, S) := \sum_{j=1}^{N} \mu(\Omega_j) f(s_j).$$

Our main result reads as follows.

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Theorem 1. *Let μ be a tight Borel measure on a metric space Ω. Let X be a Banach space and let $f : (\Omega, \mu) \to X$ be a Bochner integrable function. Then for every sequence of partitions $\big(P^{(n)}\big)$ of Ω satisfying $\lim_{n\to\infty} \mathrm{mesh}\,(P^{(n)}) = 0$ there exists a sequence of associated sample point sets $\big(S^{(n)}\big)$ such that*

$$\lim_{n\to\infty} \mathscr{R}\big(f; P^{(n)}, S^{(n)}\big) \;=\; \int_\Omega f\, d\mu$$

strongly in X.

Before starting with the proof we isolate some lemmas. The first is concerned with a topological property of tight measures on metric space.

Lemma 2. *Let μ be a tight Borel measure on a metric space Ω and let X be a Banach space. Let $f : \Omega \to X$ be μ-measurable. Then there exists a sequence (f_n) of bounded uniformly continuous functions on Ω such that $\lim_{k\to\infty} f_n = f$ μ-almost everywhere.*

Proof. By definition of μ-measurability there exists a sequence of simple μ-measurable functions (ϕ_n) converging to f μ-almost everywhere. In particular for every $m \geqslant 1$ there is an index N_m such that

$$\mu\{\phi_{N_m} \neq f\} < \frac{1}{m}.$$

Let us write $\phi_{N_m} = \sum_{j=1}^{K_m} \mathbf{1}_{\Omega_{j,m}} \otimes x_{j,m}$ with $x_{j,m} \in X$ and with the sets $\Omega_{j,m}$ pairwise disjoint and μ-measurable. Suppose for the moment that we are able to find scalar-valued bounded uniformly continuous functions $\psi_{j,m}$ such that

$$\mu\{\psi_{j,m} \neq \mathbf{1}_{\Omega_{j,m}}\} < \frac{1}{mK_m}, \qquad j = 1, \dots, K_m.$$

Then the functions $\psi_m = \sum_{j=1}^{K_m} \psi_{j,m} \otimes x_{j,m}$ are bounded and uniformly continuous, and we have

$$\mu\{\psi_m \neq f\} \leqslant \frac{1}{m} + K_m \cdot \frac{1}{mK_m} = \frac{2}{m}.$$

Hence $\lim_{m\to\infty} \psi_m = f$ in μ-measure, and the lemma follows by passing to a μ-almost everywhere convergent subseqence. This argument shows that it suffices to prove the following:

For every μ-measurable set $B \subseteq \Omega$ there exists a sequence (ψ_n) of bounded uniformly continuous functions on Ω such that $\lim_{n\to\infty} \psi_n = \mathbf{1}_B$ in μ-measure.

Choose a Borel subset $C \subseteq \Omega$ such that its symmetric difference with B satisfies $\mu(B \triangle C) = 0$. Then $\mathbf{1}_B = \mathbf{1}_C$ μ-almost everywhere. For each $n \geqslant 1$, by [6, Theorem 3.1] there is a compact subset $C_n \subseteq C$ such that $\mu(C \setminus C_n) < \frac{1}{n}$ and a compact set $\Omega_n \subseteq \Omega \setminus C$ with $\mu((\Omega \setminus C) \setminus \Omega_n) < \frac{1}{n}$. Define $\psi_n : \Omega \to [0, 1]$ by

$$\psi_n(\omega) := \frac{d(\omega, \Omega_n)}{d(\omega, \Omega_n) + d(\omega, C_n)}.$$

Then ψ_n is bounded and uniformly continuous, and we have $\psi_n|_{C_n \cup \Omega_n} = \mathbf{1}_C|_{C_n \cup \Omega_n}$. Hence $\psi_n|_{C_n \cup \Omega_n} = \mathbf{1}_B|_{C_n \cup \Omega_n}$ μ-almost everywhere. It follows that

$$\mu\{\psi_n \neq \mathbf{1}_B\} \leqslant \mu(\Omega \setminus (C_n \cup \Omega_n)) < \frac{2}{n}$$

and we conclude that $\lim_{n\to\infty} \psi_n = \mathbf{1}_B$ in μ-measure. $\square$

Without loss of generality we will assume from this point onwards that $\mu(\Omega) = 1$.

Lemma 3. *Let $f : (\Omega, \mu) \to X$ be Bochner integrable. Let $F \subseteq G \subseteq \Omega$ be μ-measurable sets such that $\mu(F) > \theta\mu(G)$ for some $\theta \in (0, 1)$. Then there exists a point $\omega \in F$ such that*

$$\mu(G)\|f(\omega)\| \leqslant \frac{1}{\theta}\int_G \|f\|\,d\mu.$$

Proof. Suppose the contrary. Then for all $\omega \in F$ we have

$$\mu(G)\|f(\omega)\| > \frac{1}{\theta}\int_G \|f\|\,d\mu.$$

Integrating over the set F gives

$$\mu(G)\int_F \|f\|\,d\mu \;\geqslant\; \frac{1}{\theta}\,\mu(F)\int_G \|f\|\,d\mu > \mu(G)\int_G \|f\|\,d\mu,$$

a contradiction. □

Lemma 4. *Let $f : (\Omega, \mu) \to X$ be Bochner integrable and let $P = \{\Omega_1, \dots, \Omega_N\}$ be a partition of Ω. Suppose A and B are μ-measurable subsets of Ω. Then there exists a set of sample points $S = \{s_1, \dots, s_N\}$ with the following properties:*

(i) $\mu\left(\bigcup_{s_j \in S\cap A} \Omega_j\right) \leqslant 3\mu(A)$;

(ii) $\mu\left(\bigcup_{s_j \in S\cap B} \Omega_j\right) \leqslant 3\mu(B)$;

(iii) *For all $s_j \in S \cap (A \cup B)$ we have*

$$\mu(\Omega_j)\|f(s_j)\| \leqslant 3\int_{\Omega_j} \|f\|\,d\mu. \tag{1}$$

Proof. We choose the sample points s_j as follows.

Case 1 - If there exists $s \in \Omega_j \setminus (A \cup B)$, then pick such an s and put $s_j := s$.

Suppose now such an s does not exist. Then we have $\Omega_j \subset A \cup B$.

Case 2 - If $\mu(\Omega_j \cap (A \cap B)) > \frac{1}{3}\mu(\Omega_j)$, then by Lemma 3 there exists an $s \in \Omega_j \cap (A \cap B)$ for which (1) holds. We take such an s and put $s_j := s$.

Suppose now that $\mu(\Omega_j \cap (A \cap B)) \leqslant \frac{1}{3}\mu(\Omega_j)$.

Case 3 - If there exists an $s \in \Omega_j \cap (B \backslash A)$ for which (1) holds, then we pick such an s and put $s_j := s$.

Case 4 - If there exists no $s \in \Omega_j \cap (B \backslash A)$ verifying (1), then by Lemma 3 we necessarily have $\mu(\Omega_j \cap (B \setminus A)) \leqslant \frac{1}{3}\mu(\Omega_j)$. This implies $\mu(\Omega_j \cap A) > \frac{2}{3}\mu(\Omega_j)$. Since we also have $\mu(\Omega_j \cap (A \cap B)) \leqslant \frac{1}{3}\mu(\Omega_j)$ it follows that $\mu(\Omega_j \cap (A \backslash B)) > \frac{1}{3}\mu(\Omega_j)$. By Lemma 3 there exists an $s \in \Omega_j \cap (A \backslash B)$ satisfying (1). We choose such an s and put $s_j := s$.

This rule defines a set of sample points $S = \{s_1, \dots, s_N\}$.

We can have $s_j \in A$ only in the cases 2 and 4. In both cases we have $\mu(\Omega_j \cap A) > \frac{1}{3}\mu(\Omega_j)$. Hence,

$$\mu\left(\bigcup_{s_j \in S\cap A} \Omega_j\right) = \sum_{s_j \in S\cap A} \mu(\Omega_j) \leqslant 3 \sum_{s_j \in S\cap A} \mu(\Omega_j \cap A) \leqslant 3\mu(A).$$

Moreover, in both cases we have chosen s_j in such a way that (1) holds.

We can have $s_j \in B$ only in the cases 2 and 3. In both cases we have $\mu(\Omega_j \cap B) > \frac{1}{3}\mu(\Omega_j)$. Hence,

$$\mu\left(\bigcup_{s_j \in S\cap B} \Omega_j\right) = \sum_{s_j \in S\cap B} \mu(\Omega_j) \leqslant 3 \sum_{s_j \in S\cap B} \mu(\Omega_j \cap B) \leqslant 3\mu(B).$$

Moreover, in both cases we have chosen s_j in such a way that (1) holds. □

We now turn to the proof of Theorem 1.

Step 1 - First we prove: *For every $\varepsilon > 0$ there exists an index N with the following property: for every $n \geqslant N$ there exists a sample point set $S^{(n)}$ associated with $P^{(n)}$ such that*

$$\left\| \mathscr{R}\big(f; P^{(n)}, S^{(n)}\big) - \int_\Omega f\, d\mu \right\| < \varepsilon. \tag{2}$$

By absolute continuity, we can choose $\eta > 0$ so small that for all μ-measurable sets $A \subseteq \Omega$ with $\mu(A) < \eta$ we have

$$\int_A \|f\|\, d\mu < \tfrac{1}{15}\varepsilon. \tag{3}$$

For $K \geqslant 1$ define $f_K : \Omega \to X$ by

$$f_K(\omega) := \begin{cases} f(\omega), & \text{if } \|f(\omega)\| \leqslant K; \\ 0, & \text{else.} \end{cases}$$

Define $A_K = \{\|f\| > K\}$. By dominated convergence there exists $K_0 \geqslant 1$ large enough such that

$$\int_\Omega \|f - f_{K_0}\|\, d\mu < \tfrac{1}{5}\varepsilon \tag{4}$$

and

$$3\mu(A_{K_0}) < \eta. \tag{5}$$

For notational convenience we put $g := f_{K_0}$ and $A := A_{K_0}$.

By Lemma 2 there exists a sequence (g_k) of bounded uniformly continuous functions such that $\lim_{k\to\infty} g_k = g$ μ-almost everywhere. Replacing each g_k by its truncation between $-K_0$ and K_0, we may assume that

$$\sup_k \sup_{\omega\in\Omega} \|g_k(\omega)\| \leqslant K_0.$$

Define $B_k = \big\{\|g - g_k\| > \frac{1}{10}\varepsilon\big\}$. By dominated convergence there exists an index $k_0 \geqslant 1$ large enough such that

$$\int_\Omega \|g - g_{k_0}\|\, d\mu < \tfrac{1}{5}\varepsilon \tag{6}$$

and

$$\mu(B_{k_0}) < \frac{1}{60K_0}\varepsilon. \tag{7}$$

Again for notational convenience we put $B := B_{k_0}$ and $h := g_{k_0}$. Choose $\delta > 0$ small enough such that

$$\left\| \mathscr{R}(h; P, S) - \int_\Omega h \, d\mu \right\| < \tfrac{1}{5}\varepsilon. \tag{8}$$

whenever P is a partition of Ω with mesh $(P) < \delta$ and S is an associated sample point set. Such δ exists by the uniform continuity of h. From $\lim_{n\to\infty}$ mesh $(P^{(n)}) = 0$ we may choose N so large that mesh $\big(P^{(n)}\big) < \delta$ for all $n \geqslant N$.

For $n \geqslant N$ we apply Lemma 4 to the partitions $P^{(n)}$ and the sets A and B and obtain sample point sets $S^{(n)}$ verifying the conditions of the lemma. We fix $n \geqslant N$ and estimate:

$$\begin{aligned}
&\left\| \mathscr{R}\big(f; P^{(n)}, S^{(n)}\big) - \int_\Omega f \, d\mu \right\| \\
&\leqslant \left\| \mathscr{R}\big(f; P^{(n)}, S^{(n)}\big) - \mathscr{R}\big(g; P^{(n)}, S^{(n)}\big) \right\| + \left\| \mathscr{R}\big(g; P^{(n)}, S^{(n)}\big) - \mathscr{R}\big(h; P^{(n)}, S^{(n)}\big) \right\| \\
&\quad + \left\| \mathscr{R}\big(h; P^{(n)}, S^{(n)}\big) - \int_\Omega h \, d\mu \right\| + \left\| \int_\Omega h \, d\mu - \int_\Omega g \, d\mu \right\| + \left\| \int_\Omega h \, d\mu - \int_\Omega g \, d\mu \right\| \\
&=: \text{(I)} + \text{(II)} + \text{(III)} + \text{(IV)} + \text{(V)}.
\end{aligned}$$

We will prove (2) by showing that each of these five terms is smaller than $\frac{1}{5}\varepsilon$.

Distinguishing between points in A and $\Omega \setminus A$ respectively, noting that $g\big(s_j^{(n)}\big) = f\big(s_j^{(n)}\big)$ when $s_j^{(n)} \notin A$ and $g\big(s_j^{(n)}\big) = 0$ when $s_j^{(n)} \in A$, and using (1) we have

$$\begin{aligned}
\text{(I)} \;&\leqslant \sum_{s_j^{(n)} \in S^{(n)} \cap A} \mu\big(\Omega_j^{(n)}\big) \cdot \big\| f\big(s_j^{(n)}\big) - g\big(s_j^{(n)}\big) \big\| \\
&= \sum_{s_j^{(n)} \in S^{(n)} \cap A} \mu\big(\Omega_j^{(n)}\big) \cdot \big\| f\big(s_j^{(n)}\big) \big\| \;\leqslant\; 3 \int_{\bigcup_{s_j^{(n)} \in S^{(n)} \cap A} \Omega_j^{(n)}} \|f\| \, d\mu.
\end{aligned}$$

Thanks to (5) we have $\mu\left(\bigcup_{s_j^{(n)} \in S^{(n)} \cap A} \Omega_j^{(n)}\right) \leqslant 3\mu(A) < \eta$ and hence by (3) the integral on the right hand side is less than $\frac{1}{15}\varepsilon$. It follows that (I) $\leqslant 3 \cdot \frac{1}{15}\varepsilon = \frac{1}{5}\varepsilon$.

Next, distinguishing between points $s_j^{(n)} \in S^{(n)}$ belonging to B and $\Omega \setminus B$ respectively, noting that $\|g(\omega)\| \leqslant K_0$ and $\|h(\omega)\| \leqslant K_0$ for all $\omega \in \Omega$, and using (7) we have

$$\begin{aligned}
\text{(II)} &\leqslant \sum_{s_j^{(n)} \in S^{(n)} \cap B} \mu(\Omega_j) \cdot \big\| g\big(s_j^{(n)}\big) - h\big(s_j^{(n)}\big) \big\| + \sum_{s_j^{(n)} \notin S^{(n)} \cap B} \mu(\Omega_j) \cdot \big\| g\big(s_j^{(n)}\big) - h\big(s_j^{(n)}\big) \big\| \\
&\leqslant 2K_0 \cdot \mu\left(\bigcup_{s_j^{(n)} \in S^{(n)} \cap B} \Omega_j \right) + \frac{1}{10}\varepsilon \cdot \mu\left(\bigcup_{s_j^{(n)} \notin S^{(n)} \cap B} \Omega_j \right) \\
&\leqslant 2K_0 \cdot 3\mu(B) + \frac{1}{10}\varepsilon \cdot 1 < 2K_0 \cdot \frac{3}{60K_0}\varepsilon + \frac{1}{10}\varepsilon = \tfrac{1}{5}\varepsilon.
\end{aligned}$$

The mesh of $P^{(n)}$ being smaller than δ, by (8) we have (III) $< \frac{1}{5}\varepsilon$. By (6) we have (IV) $< \frac{1}{5}\varepsilon$, and finally by (4) we have (V) $< \frac{1}{5}\varepsilon$.

Step 2 - By Step 1, for every $m \geqslant 1$ there exists an index N_m with the following property: for every $n \geqslant N_m$ there exists a sample set $S_m^{(n)}$ associated with $P^{(n)}$ such that

$$\left\| \mathscr{R}\big(f; P^{(n)}, S_m^{(n)}\big) - \int_\Omega f\,d\mu \right\| < \frac{1}{m}, \qquad n \geqslant N_m.$$

We may replace the numbers N_m by larger ones and thereby assume that $N_1 < N_2 < N_3 < \dots$. For $n \geqslant N_1$ we define

$$S^{(n)} := S_m^{(n)}, \qquad N_m \leqslant n < N_{m+1} \qquad (m = 1, 2, \dots)$$

an for $n < N_1$ we choose the sets $S^{(n)}$ in an arbitrary way. The resulting sequence $\big(S^{(n)}\big)$ has the desired properties. This concludes the proof of Theorem 1.

With little extra effort we obtain the following stronger version of Theorem 1:

Theorem 5. *Let μ be a tight Borel measure on a metric space Ω. Let X_i be Banach spaces and let $f_i : (\Omega, \mu) \to X_i$ be Bochner integrable functions $(i = 1, 2 \dots)$. Then for every sequence of partitions $\big(P^{(n)}\big)$ of Ω satisfying $\lim_{n\to\infty} \operatorname{mesh}\big(P^{(n)}\big) = 0$ there exists a sequence of associated sample point sets $\big(S^{(n)}\big)$ such that*

$$\lim_{n\to\infty} \mathscr{R}\big(f_i; P^{(n)}, S^{(n)}\big) = \int_\Omega f_i\,d\mu \qquad (i = 1, 2, \dots)$$

strongly in X.

Proof. By Theorem 1 applied to the Banach space $X_1 \oplus \cdots \oplus X_m$ and the function $F_m := (f_1, \dots, f_m)$, there exists an index $N_m \geqslant 1$ with the following property: for every $n \geqslant N_m$ there is a sample point set $S_m^{(n)}$ such that

$$\left\| \mathscr{R}\big(F_m; P^{(n)}, S_m^{(n)}\big) - \int_\Omega F_m\,d\mu \right\| < \frac{1}{m}, \qquad n \geqslant N_m.$$

This immediately implies

$$\left\| \mathscr{R}\big(f_i; P^{(n)}, S_m^{(n)}\big) - \int_\Omega f_i\,d\mu \right\| < \frac{1}{m}, \qquad n \geqslant N_m \quad (i = 1, \dots, m).$$

The desired sequence of sample point sets $\big(S^{(n)}\big)$ is finally obtained as in Step 2 of the proof of Theorem 1. $\square$

Definition 6. Let $f : \Omega \to X$ be an arbitrary function. We call a vector $I \in X$ a *Riemann sum limit* of f if the following holds: for every sequence of partitions $\big(P^{(n)}\big)$ with $\lim_{n\to\infty} \operatorname{mesh}(P^{(n)}) = 0$ there exists a sequence of sample point sets $\big(S^{(n)}\big)$ such that

$$\lim_{n\to\infty} \left\| \sum_{j=1}^{N^{(n)}} \mu\big(\Omega_j^{(n)}\big)\, f\big(s_j^{(n)}\big) - I \right\| = 0.$$

Using this terminology, Theorem 1 states that if $f : (\Omega, \mu) \to X$ is Bochner integrable, then $\int_\Omega f\,d\mu$ is a Riemann sum limit of f. If Ω is totally bounded and $f : (\Omega, \mu)$ is essentially bounded, then $\int_\Omega f\,d\mu$ is the *unique* Riemann sum limit of f; this follows from the following theorem.

Theorem 7. *Let Ω be totally bounded and let $f : (\Omega, \mu) \to X$ be an arbitrary function. Suppose $x^* \in X^*$ is such that $\langle f, x^*\rangle$ is μ-integrable and μ-essentially bounded. If I is a Riemann sum limit of f, then*

$$\langle I, x^*\rangle = \int_\Omega \langle f, x^*\rangle \, d\mu.$$

Proof. Let $K > 0$ be a constant such that the set $N = \big\{|\langle f, x^*\rangle| > K\big\}$ is μ-null. Fix $n \geqslant 1$ and divide the interval $[-K, K]$ into disjoint subintervals $J_1, \dots, J_M$ of diameter at most $\frac{1}{n}$. For $m = 1, \dots, M$ let $V_m := \{\omega \in \Omega : \ f(\omega) \in J_m\}$. Since Ω is assumed to be totally bounded it is possible to subdivide each V_m of positive μ-measure into finitely many disjoint subsets having positive μ-measure and diameter $\leqslant \frac{1}{n}$. In this way we obtain a partition $P^{(n)} = \big\{\Omega_1^{(n)}, \dots, \Omega_{N^{(n)}}^{(n)}\big\}$ of Ω of mesh $\leqslant \frac{1}{n}$. Note that

$$\big|\langle f(s), x^*\rangle - \langle f(t), x^*\rangle\big| \leqslant \frac{1}{n}$$

for all $s, t \in \Omega_j^{(n)}$; $j = 1, \dots, N^{(n)}$.

By assumption, associated to the resulting sequence $\big(P^{(n)}\big)$ there exists a sequence of sample point sets $\big(T^{(n)}\big)$ such that

$$\lim_{n\to\infty} \mathscr{R}\big(f; P^{(n)}, T^{(n)}\big) = I.$$

Then,

$$\lim_{n\to\infty} \mathscr{R}\big(\langle f, x^*\rangle; P^{(n)}, T^{(n)}\big) = \langle I, x^*\rangle.$$

On the other hand, since $\langle f, x^*\rangle$ is μ-integrable, by Theorem 1 there exists a sequence of sample point sets $\big(S^{(n)}\big)$ such that

$$\lim_{n\to\infty} \mathscr{R}\big(\langle f, x^*\rangle; P^{(n)}, S^{(n)}\big) = \int_\Omega \langle f, x^*\rangle \, d\mu.$$

We estimate:

$$\begin{aligned}
\left|\int_\Omega \langle f, x^*\rangle \, d\mu - \langle I, x^*\rangle\right| &\leqslant \left|\int_\Omega \langle f, x^*\rangle \, d\mu - \mathscr{R}\big(\langle f, x^*\rangle; P^{(n)}, S^{(n)}\big)\right| \\
&\qquad + \sum_{j=1}^{N^{(n)}} \mu\big(\Omega_j^{(n)}\big) \cdot \big|\big\langle f\big(s_j^{(n)}\big), x^*\big\rangle - \big\langle f\big(t_j^{(n)}\big), x^*\big\rangle\big| \\
&\qquad\quad + \left|\mathscr{R}\big(\langle f, x^*\rangle; P^{(n)}, T^{(n)}\big) - \langle I, x^*\rangle\right| \\
&\leqslant \left|\int_\Omega \langle f, x^*\rangle \, d\mu - \mathscr{R}\big(\langle f, x^*\rangle; P^{(n)}, S^{(n)}\big)\right| \\
&\qquad + \frac{1}{n} + \left|\mathscr{R}\big(\langle f, x^*\rangle; P^{(n)}, T^{(n)}\big) - \langle I, x^*\rangle\right|.
\end{aligned}$$

Passing to the limit $n \to \infty$ in the right hand side gives the desired result. □

The following example shows that the converse of Theorem 1 does not hold: a strongly measurable function with a Riemann sum limit need not be Bochner integrable, even if this limit is unique.

Example 8. Let $(e_k)_{k=1}^\infty$ denote the standard unit basis of the Hilbert space ℓ^2 and define $f : (0, 1] \to \ell^2$ by

$$f(t) := \tfrac{1}{k} 2^k \, e_k, \qquad t \in I_k := (2^{-k}, 2^{-k+1}].$$

Clearly f is strongly measurable and

$$\int_{(0,1]} \|f(t)\|\,dt = \sum_{k=1}^{\infty} |I_k|\cdot\frac{2^k}{k} = \sum_{k=1}^{\infty}\frac{1}{k} = \infty.$$

Thus f fails to be Bochner integrable. Next we will show that

$$I := \sum_{k=1}^{\infty} \tfrac{1}{k}\, e_k$$

is a Riemann sum limit integral of f.

To this end let $\big(P^{(n)}\big)$ be an arbitrary sequence of partitions of $(0,1]$ satisfying $\lim_{n\to\infty} \mathrm{mesh}\,(P^{(n)}) = 0$. Fix $\varepsilon > 0$ arbitrary and fix an integer K such that $2^{-K} < \frac12\varepsilon$ and $\sum_{k=K+1}^{\infty}\frac{1}{k^2} < \frac14\varepsilon^2$. Choose n_0 so large that $\mathrm{mesh}\,(P^{(n)}) \leqslant 2^{-K}$ for all $n \geqslant n_0$.

Fix an index $n \geqslant n_0$ and write $P^{(n)} =: P = \{\Omega_1, \dots, \Omega_N\}$. We define a sample point set $S^{(n)} =: S = \{s_1, \dots, s_N\}$ by the following rule: for $j = 1, \dots, N$ let

$$k(j) = \min\{k \geqslant 1 :\ \Omega_j \cap I_k \neq \varnothing\}$$

and let s_j be an arbitrary point in $\Omega_j \cap I_{k(j)}$.

For $k \geqslant 1$ let J_k be the set of all $j \in \{1, \dots, N\}$ for which we have $s_j \in I_k$. From $\mathrm{diam}\,(\Omega_j) < 2^{-K}$ and $\Omega_j \cap I_m = \varnothing$ $(m = 1, \dots, k-1)$ we have, for all $k = 1, \dots, K$,

$$(2^{-k}, 2^{-k+1} - 2^{-K}] \subseteq \bigcup_{j\in J_k} \Omega_j \subseteq (2^{-k} - 2^{-K}, 2^{-k+1}].$$

For these k it follows that

$$2^{-k} - 2^{-K} \leqslant \sum_{j\in J_k} |\Omega_j| \leqslant 2^{-k} + 2^{-K}.$$

Hence,

$$\begin{aligned}\Big\| \sum_{j\in J_k} |\Omega_j|\, f(s_j) \ - \ \tfrac{1}{k}\, e_k \Big\|_{\ell^2} &= \Big\| \sum_{j\in J_k} |\Omega_j|\, \tfrac{1}{k} 2^k\, e_k \ - \ \tfrac{1}{k}\, e_k \Big\|_{\ell^2} \\ &= \tfrac{1}{k} \Big| 2^k \sum_{j\in J_k} |\Omega_j| - 1 \Big| \leqslant \tfrac{1}{k}\, 2^{-K+k}.\end{aligned}$$

Summing over $k = 1, \dots, K$ we obtain

$$\Big\| \sum_{k=1}^{K} \sum_{j\in J_k} |\Omega_j|\, f(s_j) \ - \ \sum_{k=1}^{K} \tfrac{1}{k}\, e_k \Big\|_{\ell^2} \leqslant \sum_{k=1}^{K} \tfrac{1}{k}\, 2^{-K+k} \leqslant 2^{-K}.$$

Next fix $k \geqslant K+1$. If $s_j \in I_k$, then $\Omega_j \cap I_m = \varnothing$ $(m = 1, \dots, k-1)$ and therefore

$$\bigcup_{j\in J_k} \Omega_j \subseteq (0, 2^{-k+1}].$$

Put

$$\beta_k := \tfrac{1}{k} 2^k \sum_{j\in J_k} |\Omega_j|.$$

Noting that

$$0 \leqslant 2^k \sum_{j\in J_k} |\Omega_j| \leqslant 2$$

we have

$$|\beta_m - \tfrac{1}{k}| = \tfrac{1}{k}\left|2^k \sum_{j\in J_k} |\Omega_j| - 1\right| \leqslant \tfrac{1}{k}.$$

Hence,

$$\Big\| \sum_{k=K+1}^{\infty} \sum_{j\in J_k} |\Omega_j|\, f(s_j) \;-\; \sum_{k=K+1}^{\infty} \tfrac{1}{k}\, e_k \Big\|_{\ell^2}^2 = \Big\| \sum_{k=K+1}^{\infty} (\beta_k - \tfrac{1}{k})\, e_k \Big\|_{\ell^2}^2 \leqslant \sum_{k=K+1}^{\infty} \frac{1}{k^2}.$$

Note that in the double sum on the left hand side only finitely many terms are non-zero. Putting everything toghether we obtain

$$\begin{aligned}
&\Big\| \sum_{j=1}^{N} |\Omega_j|\, f(s_j) \;-\; I \Big\|_{\ell^2} = \Big\| \sum_{j=1}^{N} |\Omega_j|\, f(s_j) \;-\; \sum_{k=1}^{\infty} \tfrac{1}{k}\, e_k \Big\|_{\ell^2} \\
&\leqslant \Big\| \sum_{k=1}^{K} \sum_{j\in J_k} |\Omega_j|\, f(s_j) - \sum_{k=K+1}^{\infty} \tfrac{1}{k}\, e_k \Big\|_{\ell^2} + \Big\| \sum_{k=K+1}^{\infty} \sum_{j\in J_k} |\Omega_j|\, f(s_j) - \sum_{k=K+1}^{\infty} \tfrac{1}{k}\, e_k \Big\|_{\ell^2} \\
&\leqslant 2^{-K} + \Big(\sum_{k=K+1}^{\infty} \frac{1}{k^2} \Big)^{\frac12} < \tfrac12\varepsilon + \tfrac12\varepsilon = \varepsilon.
\end{aligned}$$

This shows that $I = \sum_{k=1}^{\infty} \frac{1}{k}\, e_k$ is a Riemann sum limit of f. The uniqueness of I as Riemann sum limit of f follows from Theorem 7 applied to the coordinate functionals e_k.

To conclude we compare our results with two theories of generalized Riemann integration: the theories of McShane integration and Henstock-Kurzweil integration. For simplicity we confine the discussion to the case where Ω is the unit interval $[0,1]$. For more details we refer to the book [4] and the papers [1], [3].

Let P be a partition of $[0,1]$ and let S be a finite subset of $[0,1]$. Let us call a pair (P,S) a *McShane pair* if P is of the form $P = \{[p_{j-1},p_j] :\; j=1,\dots,N\}$ with $0 = p_0 < \dots p_N = 1$ and if $S = \{s_1,\dots,s_N\} \subseteq [0,1]$.

A *gauge* is a strictly positive function δ on $[0,1]$. We call the McShane pair (P,S) *subordinate* to the gauge δ if

$$\big[p_{j-1},p_j\big] \subseteq \big(s_j - \delta(s_j), s_j + \delta(s_j)\big), \qquad j = 1,\dots N.$$

A function $f : [0,1] \to X$ is called *McShane-integrable*, with integral $I \in X$, if for every $\varepsilon > 0$ there exists a gauge δ such that

$$\|\mathscr{R}(f;P,S) - I\| < \varepsilon$$

for every McShane pair (P,S) subordinate to δ. Here, of course,

$$\mathscr{R}(f;P,S) := \sum_{j=1}^{N} (p_j - p_{j-1}) \cdot f(s_j).$$

By a *Henstock-Kurzweil pair* we mean a McShane pair (P,S) with the property that S is a sample point set for P, i.e. we have $s_j \in [p_{j-1},p_j]$ for all j. A function

$f : [0,1] \to X$ is called *Henstock-Kurzweil-integrable*, with integral $I \in X$, if for every $\varepsilon > 0$ there exists a gauge δ such that

$$\|\mathscr{R}(f; P, S) - I\| < \varepsilon \tag{9}$$

for every Henstock-Kurzweil pair (P, S) subordinate to δ. Clearly every McShane integrable function is Henstock-Kurzweil integrable, with the same integral I.

The following is proved in [3, Theorem 16]:

Theorem 9. *Every Bochner integrable function $f : [a,b] \to X$ is McShane integrable, and therefore Henstock-Kurzweil integrable, and the integrals coincide.*

Let us compare this result with Theorem 1 in the case $\Omega = [a,b]$. By taking $\varepsilon = \frac{1}{n}$ in (9), from Theorem 9 it follows immediately that the integral of a Bochner integrable $f : [a,b] \to \mathbb{R}$ can be realized as the limit of *certain* Riemann sums. The point of Theorem 1, however, is that partitions allowed there need not be subordinate to the gauges used in (9).

In the converse direction, the existence of a Riemann sum limit for f in the sense of Definition 6 does not produce a sequence of gauges in any obvious way, which leaves open the possibility that such a function fails to be Henstock-Kurzweil integrable. Let us point out in this connection that by [3, Theorem 15], the function f in Example 8 is indeed McShane integrable, and hence Henstock-Kurzweil integrable; cf. [1, Example 3E].

Finally we mention the paper [2], where McShane- and Henstock-Kurzweil integrability of vector-valued functions on metric spaces Ω is studied. In contract to the approach just described, in this paper the partitions of Ω consist of countably many sets.

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