On small solutions of delay equations in infinite dimensions

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Abstract - Let X be a Banach space and $1 \leqslant p < \infty$. Let L be a bounded linear operator from $L^p([-1,0],X)$ into X. Consider the delay differential equation $\dot{u}(t) = Lu_t$, $u(0) = x$, $u_0 = f$ on the state space $L^p([-1,0],X)$. We prove that a mild solution $u(t) = u(t;x,f)$ is a small solution if and only if the Laplace transform of $u(t;x,f)$ extends to an entire function. The same result holds for the state space $C([-1,0],X)$.

1991 Mathematics Subject Classification: 34K15, 34K25, 47D06

Let X be a Banach space and let $1 \leqslant p < \infty$. For a bounded linear operator L from $L^p([-1,0],X)$ into X, on the state space $L^p([-1,0],X)$ we consider the *delay differential equation*

$$\text{(DDE)} \qquad \begin{cases} \dot{u}(t) = Lu_t, \quad t \geqslant 0, \\ u(0) = x, \, u_0 = f. \end{cases}$$

Here, for a function $u \in L^p_{loc}([-1,\infty),X)$, the functions $u_t \in L^p([-1,0],X)$ are defined by $u_t(s) := u(t+s)$, $t \geqslant 0$, $-1 \leqslant s \leqslant 0$, and $f \in L^p([-1,0],X)$ is a given 'history'

[1]This paper was written while the authors were affiliated with the University of Tübingen. It is part of a research project supported by the Deutsche Forschungsgemeinschaft DFG. The authors warmly thank Professor Rainer Nagel and the AG Funktionalanalysis for the stimulating and enjoyable working environment.

[2]Support by DAAD is gratefully acknowledged.

[3]Support by an Individual Fellowship from the Human Capital and Mobility programme of the European Community is gratefully acknowledged.

function. A *mild solution* is function $u \in L^p([-1,\infty), X)$ such that $u(s) = f(s)$ for $-1 \leqslant s < 0$ and

$$u(t) = x + \int_0^t Lu_s\, ds, \quad t \geqslant 0.$$

A mild solution $u(t)$ of (DDE) is called a *small solution* if $\|u(t)\|$ decays to zero faster than any exponential.

For the theory of delay functional differential equations in finite-dimensional spaces $X = \mathbb{R}^n$ we refer to the books [HV] or [DGVW]. Delay equations in infinite-dimensional spaces have been considered, e.g., in [AS], [BHS], [En], [Kp], [Ma], [Nk1,2], [Pr].

The purpose of this paper is prove that a mild solution $u(t)$ of (DDE) is a small solution if and only if its Laplace transform extends to an entire function. In Hilbert space this is an easy result, but the general case where X is allowed to be an arbitrary Banach space depends the following individual stability theorem for C_0−semigroups [Ne].

Proposition 1. *Let* $\mathbf{T}$ *be a* C_0*−semigroup on a Banach space* X*, with generator* A*. Let* $x_0 \in X$ *be such that the local resolvent* $\lambda \mapsto R(\lambda, A)x_0$ *admits a bounded holomorphic extension to the open right half-plane* $\{\operatorname{Re}\lambda > 0\}$*. Then for every* $\lambda_0 \in \varrho(A)$ *there exists a constant* $M > 0$ *such that*

$$\|T(t)R(\lambda_0, A)x_0\| \leqslant M(1+t) \quad \textit{for all} \quad t \geqslant 0.$$

Here, as usual, $\varrho(A)$ denotes the set of all $\lambda \in \mathbb{C}$ such that the resolvent $R(\lambda, A) := (\lambda - A)^{-1}$ exists as a bounded linear operator on X. An improvement of this result for B-convex spaces is given in [HN].

Let $1 \leqslant p < \infty$ and let L be a bounded linear operator from $L^p([-1,0], X)$ into X. In order to treat the problem (DDE) by semigroup methods, we consider the following first order Cauchy problem on the product space $\mathcal{X} := X \times L^p([-1,0], X)$ (cf. [BHS], [Kp] and [En]):

$$\text{(ACP)} \qquad \begin{cases} \dfrac{d}{dt}\begin{pmatrix} v(t) \\ w(t) \end{pmatrix} = \mathcal{A}\begin{pmatrix} v(t) \\ w(t) \end{pmatrix} \\ v(0) = x, \ w(0) = f, \end{cases}$$

where the operator matrix $\mathcal{A}$ with domain

$$D(\mathcal{A}) := \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in X \times W^{1,p}([-1,0], X) : \ x = f(0) \right\}$$

is defined by

$$\mathcal{A}\begin{pmatrix} f(0) \\ f \end{pmatrix} := \begin{pmatrix} Lf \\ f' \end{pmatrix}, \quad f \in W^{1,p}([-1,0], X).$$

As shown in [BHS] (see also [En] and [Kp]), $\mathcal{A}$ generates a C_0-semigroup $\mathcal{T}$ on $\mathcal{X}$. It is easy to see that if $u(t) = u(t; x, f)$ is a mild solution of (DDE) then $\begin{pmatrix} u(t) \\ u_t \end{pmatrix}$ is a mild solution of (ACP) with initial value $\begin{pmatrix} x \\ f \end{pmatrix}$, i.e. we have

$$\begin{pmatrix} u(t) \\ u_t \end{pmatrix} = \mathcal{T}(t) \begin{pmatrix} x \\ f \end{pmatrix}. \tag{1}$$

If $u(t)$ is a small solution of (DDE), then an easy direct calculation shows that the map $t \mapsto \|u_t\|$ decays faster than any exponential. It follows that $u(t)$ is a small solution of (DDE) if and only if $\begin{pmatrix} u(t) \\ u_t \end{pmatrix}$ is a small solution of (ACP).

In order to describe the spectrum of $\mathcal{A}$ we define the *characteristic operators* on X by

$$L_\lambda x := \lambda x - L(\varepsilon_\lambda \otimes x), \quad x \in X; \quad \lambda \in \mathbb{C},$$

where $\varepsilon_\lambda(s) := e^{\lambda s}$, $s \in [-1, 0]$. Let H_λ be the bounded operator on $L^p([-1, 0], X)$ defined by

$$H_\lambda f(s) := \int_s^0 e^{\lambda(s-\tau)} f(\tau)\, d\tau, \quad -1 \leqslant s \leqslant 0; \quad f \in L^p([-1, 0], X).$$

It can be shown (see the proof of Lemma 2.3 in [BHS]; see also [Kp] or [En]) that $\lambda \in \varrho(\mathcal{A})$ if and only if the operator L_λ is invertible. In this case the resolvent of $\mathcal{A}$ is given by

$$R(\lambda, \mathcal{A}) \begin{pmatrix} x \\ f \end{pmatrix} = \begin{pmatrix} x_\lambda \\ f_\lambda \end{pmatrix}, \tag{2}$$

where

$$x_\lambda := L_\lambda^{-1}(x - LH_\lambda f) \tag{3}$$

and

$$f_\lambda(s) := e^{\lambda s} x_\lambda - H_\lambda f(s), \quad -1 \leqslant s \leqslant 0. \tag{4}$$

It is clear from (1) and (2) that for $\operatorname{Re}\lambda$ sufficiently large the maps $\lambda \mapsto x_\lambda$ and $\lambda \mapsto f_\lambda$ are the Laplace transforms of the maps $t \mapsto u(t)$ and $t \mapsto u_t$, respectively.

Lemma 2. *For every $r \in \mathbb{R}$, the set $\sigma(\mathcal{A}) \cap \{\operatorname{Re}\lambda \geqslant -r\}$ is compact, and for all $\varepsilon > 0$ we have the estimate*

$$\sup\{\|R(\lambda, \mathcal{A})\| : \ \operatorname{Re}\lambda \geqslant -r,\ \operatorname{dist}(\lambda, \sigma(\mathcal{A})) \geqslant \varepsilon\} < \infty. \tag{5}$$

Proof: From the estimate

$$\|L(\varepsilon_\lambda \otimes x)\| \leqslant \|L\| \cdot \|\varepsilon_\lambda \otimes x\| \leqslant \|L\| \cdot e^{\max\{0, -\operatorname{Re}\lambda\}} \cdot \|x\|, \quad x \in X,$$

we deduce that L_λ is invertible for all $\lambda \in \mathbb{C}$ with $|\lambda| > \|L\| \cdot e^{\max\{0,-\mathrm{Re}\,\lambda\}}$, and

$$\|L_\lambda^{-1}\| \leqslant \big(|\lambda| - \|L\| \cdot e^{\max\{0,-\mathrm{Re}\,\lambda\}}\big)^{-1}. \tag{6}$$

Since $\lambda \in \varrho(\mathcal{A})$ if and only if L_λ is invertible, it follows that the set $\{\lambda \in \sigma(\mathcal{A}) : \mathrm{Re}\,\lambda \geqslant -r\}$ is compact.

For H_λ we have the estimate

$$\begin{aligned}\|H_\lambda f(s)\| &\leqslant \int_s^0 e^{\mathrm{Re}\,\lambda(s-\tau)}\|f(\tau)\|\,d\tau \leqslant e^{\max\{0,-\mathrm{Re}\,\lambda\}} \int_s^0 \|f(\tau)\|\,d\tau \\ &\leqslant e^{\max\{0,-\mathrm{Re}\,\lambda\}} \left(\int_s^0 \|f(\tau)\|^p\,d\tau\right)^{1/p} \\ &\leqslant e^{\max\{0,-\mathrm{Re}\,\lambda\}} \cdot \|f\|_p.\end{aligned}$$

It follows that

$$\|H_\lambda f\|_p \leqslant e^{\max\{0,-\mathrm{Re}\,\lambda\}} \cdot \|f\|_p \tag{7}$$

for all $\lambda \in \mathbb{C}$ and $f \in L^p([-1,0],X)$. Hence the entire function $\lambda \mapsto H_\lambda$ is bounded in every right half-plane.

Now (5) follows from (3), (4), (6), and (7). ////

Lemma 3. *Let $u(t) = u(t;x,f)$ be the mild solution of (DDE). Assume that for some $\omega \in \mathbb{R}$ the Laplace transform of $u(t)$ extends to a holomorphic function F on a neighbourhood of the closed half-plane $\{\mathrm{Re}\,\lambda \geqslant -\omega\}$. Then,*

$$\lambda F(\lambda) - L(\varepsilon_\lambda \otimes F(\lambda)) = x - LH_\lambda f \tag{8}$$

for all $\mathrm{Re}\,\lambda \geqslant -\omega$, and we have

$$\lim_{t\to\infty} e^{\omega t}\|u(t)\| = 0. \tag{9}$$

Conversely, if F is holomorphic on $\{\mathrm{Re}\,\lambda > -\omega\}$ and satisfies (8), then F is a holomorphic extension of the Laplace transform of $u(t)$.

Proof: We start with the proof of (9). By Lemma 2, there exists an $r > \omega$ such that F is bounded in the half-plane $\{\mathrm{Re}\,\lambda > -r\}$. Using (4) and (7), it follows that the map $\lambda \mapsto f_\lambda$ admits a bounded holomorphic extension, which equals $\varepsilon_\lambda \otimes F(\lambda) - H_\lambda f$, in the half-plane $\{\mathrm{Re}\,\lambda > -r\}$. By Proposition 1 applied to the scaled semigroup $\{e^{rt}\mathcal{T}(t)\}_{t\geqslant 0}$,

$$\limsup_{t\to\infty} t^{-1}e^{rt}\left\|R(\lambda_0,\mathcal{A})\begin{pmatrix}u(t)\\u_t\end{pmatrix}\right\| = \limsup_{t\to\infty} t^{-1}e^{rt}\left\|\mathcal{T}(t)R(\lambda_0,\mathcal{A})\begin{pmatrix}x\\f\end{pmatrix}\right\| < \infty \tag{10}$$

for all $\lambda_0 \in \varrho(\mathcal{A})$. Fix $\lambda_0 \in \varrho(\mathcal{A})$. Then, by (2) and (3) and using that $r > \omega$,

$$\lim_{t\to\infty} e^{\omega t}\|L_{\lambda_0}^{-1}(u(t) - LH_{\lambda_0}u_t)\| = 0$$

and hence, by multiplying with the bounded operator L_{λ_0},

$$\lim_{t\to\infty} e^{\omega t}\|u(t) - LH_{\lambda_0}u_t\| = 0. \tag{11}$$

Also, by (2), (3), (4), (10), and (11),

$$\lim_{t\to\infty} e^{\omega t}\|H_{\lambda_0}u_t\| = 0. \tag{12}$$

Noting that $\|LH_{\lambda_0}u_t\| \leqslant \|L\| \cdot \|H_{\lambda_0}u_t\|$, it follows from (11) and (12) that (9) holds.

The remaining assertions are an obvious consequence of the fact that by (3), the Lapalce transform of $u(t)$ satisfies (8) for all $\operatorname{Re}\lambda$ large. ////

Putting these results together we have established the following characterization of small solutions of (DDE) in $L^p([-1,0],X)$.

Theorem 4. *Let X be a Banach space, let $1 \leqslant p < \infty$, and let L be a bounded linear operator from $L^p([-1,0],X)$ into X. Consider the problem* (DDE) *on the space $L^p([-1,0],X)$. Then, for a mild solution $u(t) := u(t;x,f)$ the following assertions are equivalent:*

(i) $u(t)$ is a small solution;
(ii) The Laplace transform of $u(t)$ extends to an entire function;
(iii) There exists an entire function $F : \mathbb{C} \to X$ such that

$$\lambda F(\lambda) - L(\varepsilon_\lambda \otimes F(\lambda)) = x - LH_\lambda f, \quad \lambda \in \mathbb{C}.$$

Proof: It is obvious that (i) implies (ii). The implication (ii)⇒(iii) follows from (8) in Lemma 3. Finally, if (iii) holds for some entire function F, then by Lemma 3 F must be the Laplace transform of $u(t)$, and then (9) shows that $u(t)$ is a small solution. ////

The above approach also works for the delay differential equation

$$\text{(DDE)} \qquad \begin{cases} \dot u(t) = Lu_t, \quad t \geqslant 0, \\ u(0) = f(0),\ u_0 = f. \end{cases}$$

in the state space $C([-1,0],X)$, with L a bounded operator from $C([-1,0],X)$ into X. Parallel to Theorem 4 we obtain:

Theorem 5. *Let X be a Banach space and let L be a bounded linear operator for $C([-1,0],X)$ into X. Consider the problem (*DDE*) in the state space $C([-1,0],X)$. For a mild solution $u(t) := u(t;f)$ the following assertions are equivalent:*

(i) $u(t)$ is a small solution;
(ii) The Laplace transform of $u(t)$ extends to an entire function;
(iii) There exists an entire function $F : \mathbb{C} \to X$ such that

$$\lambda F(\lambda) - L(\varepsilon_\lambda \otimes F(\lambda)) = f(0) - LH_\lambda f, \quad \lambda \in \mathbb{C}.$$

We refer to [Na, pp. 219-231] or [Kp] for more details of the basic theory of (DDE) in $C([-1,0],X)$.

Remark 6.
(i) In Hilbert space, Theorems 4 and 5 are considerably easier to prove.
(ii) In finite dimensions more complete results are known; in particular, small solutions can be characterized in terms of the coefficients of the equation.

From Lemma 3 we obtain the following characterization of uniform exponential stability of mild solutions.

Theorem 7. *Consider the problems (DDE) in $E = L^p([-1,0],X)$, $1 \leqslant p < \infty$, and $E = C([-1,0],X)$, respectively, and let $L : E \to X$ be bounded. Let $\omega \in \mathbb{R}$. If the operators L_λ are invertible for all $\operatorname{Re}\lambda \geqslant -\omega$, then there exists constant $M > 0$ such that*

$$\|u(t;x,f)\| \leqslant Me^{-\omega t}\|(x,f)\|$$

for all $t \geqslant 0$ and all initial values (x,f).

In terms of the generator $\mathcal{A}$, this result can be restated as asserting that *the growth bound and the spectral bound of $\mathcal{A}$ coincide.*

We conclude with some remarks concerning the finite-dimensional setting. For $X = \mathbb{C}^n$, Henry's theorem [He] (see also [HV, pp. 74-85] and [V]) asserts that there exists a constant $t_0 > 0$ such that if $u(t) = u(t;f)$ is a small solution for (DDE) in $C([-1,0],\mathbb{C}^n)$, then $u(t) = 0$ for all $t \geqslant t_0$. This result is no longer true if the space X has infinite dimension; this can be seen from an easy direct sum construction such that on the n-th summand we have a small solution which do not vanishes for some $t \geqslant n$.

For $X = \mathbb{C}^n$, Theorem 6 shows that the following are equivalent (cf. [HV, p.32]):
(i) All mild solutions $u(t) = u(t;x,f)$ of (DDE) in the state space $L^p([-1,0],\mathbb{C}^n)$ or $C([-1,0],\mathbb{C}^n)$ are uniformly exponentially stable;
(ii) All roots of the characteristic equation $\det L_\lambda = 0$ have strictly negative real parts.

References

[AS] O. Arino and E. Sanchez, Linear theory of abstract functional-differential equations of retarded type, J. Math. Anal. Appl. **191** (1995), 547-571.

[BHS] J.A. Burns, T.L. Herdman, and H.W. Stech, Linear functional-differential equations as semigroups on product spaces, SIAM J. Math. Anal. **14** (1983), 98-116.

[DGVW] O. Diekmann, S.A. van Gils, S.M. Verduyn Lunel, and H.O. Walther, *Delay Equations: Complex, Functional and Nonlinear Analysis*, Springer-Verlag (1995).

[En] K.-J. Engel, Spectral theory and generator property for one-sided coupled operator matrices, as preprint in Tübinger Berichte zur Funktionalanalysis, Heft 4 (1994/95).

[He] D. Henry, Small solutions of linear autonomous functional differential equations, J. Diff. Eq. **8** (1970), 494-501.

[HN] S.-Z. Huang and J.M.A.M. van Neerven, Individual stability of C_0−semigroups in B−convex Banach spaces and a Tauberian theorem, J. Math. Anal. Appl., to appear.

[HV] J.K. Hale and S.M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer-Verlag (1993).

[Kp] F. Kappel, Semigroups and delay equations, in: H. Brezis, M. Crandall and F. Kappel (eds.): *Semigroups, Theory and Applications, Vol. II* (Trieste 84), 136-176, Pitman Res. Notes Math. Ser. 152, Pitman (1986).

[Ma] M. Mastinšek, Structural operators for abstract functional-differential equations, *Semigroups, Theory and Applications* (Trieste, 1987), 303-314, Lecture Notes in Pure and Appl. Math. 116, Dekker, New York (1989).

[Nk1] S. Nagakiri, Optimal control of linear retarded systems in Banach spaces, J. Math. Anal. Appl. **120** (1986), 169-210.

[Nk2] S. Nakagiri, Structural properties of functional-differential equations in Banach spaces, Osaka J. Math. **25** (1988), 353-398.

[Na] R. Nagel (Ed.), *One-parameter Semigroups of Positive Operators*, Springer Lect. Notes in Math. 1184 (1986).

[Ne] J.M.A.M. van Neerven, Individual stability of C_0−semigroups and uniform boundedness of the local resolvent, Semigroup Forum **53** (1996), 155-161.

[Pr] J. Prüss, Bounded solutions of Volterra equations, SIAM J. Math. Anal. **19** (1988), 133-149.

[V] S.M. Verduyn Lunel, A sharp version of Henry's theorem on small solutions, J. Diff. Eq. **62** (1986), 266-274.