ASYMPTOTIC BEHAVIOUR OF C_0-SEMIGROUPS AND γ-BOUNDEDNESS OF THE RESOLVENT

JAN VAN NEERVEN

Dedicated to Rainer Nagel on the occasion of his 68th birthday

Abstract. Let X be a Banach space with type p and cotype q, and let A be the infinitesimal generator of a C_0-semigroup on X. We show that if the resolvent of A has a γ-bounded analytic extension to $\mathbb{C}_+$, then A has negative fractional growth bound of order $\frac{1}{p} - \frac{1}{q}$.

1. Introduction and preliminaries

A classical result of Gearhart, Herbst, and Prüss [9, 12, 29] states that if the resolvent of the generator A of a C_0-semigroup S on a Hilbert space H has a bounded analytic extension to the right half-plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$, then S is uniformly exponentially stable, i.e., $\omega_0(A) < 0$, where

$$\omega_0(A) := \inf \Big\{ \omega \in \mathbb{R} : \exists M \geqslant 0 \text{ such that } \|S(t)\| \leqslant M e^{\omega t} \ \forall t \geqslant 0 \Big\}$$

is the growth bound of A. For generators of C_0-semigroups on Banach spaces X one has the following result, due to Weis and Wrobel [31]: if X has Fourier type $1 \leqslant p \leqslant 2$ (the definition is recalled below) and the resolvent of A has a bounded analytic extension to $\mathbb{C}_+$, then $\omega_{\frac{1}{p}-\frac{1}{p'}}(A) < 0$, where

$$\omega_\theta(A) := \inf \Big\{ \omega \in \mathbb{R} : \exists M \geqslant 0 \text{ such that } \|S(t)x\| \leqslant M e^{\omega t} \|x\|_{X_\theta} \ \forall t \geqslant 0, \ x \in X_\theta \Big\}$$

is the fractional growth bound of A of order θ. Here, X_θ denotes the fractional domain of A of exponent θ. In the same paper it is shown that this result is the best possible, in the sense that in general it is not possible to replace the exponent $\theta = \frac{1}{p} - \frac{1}{p'}$ by a smaller one. By rescaling this follows from the following example [31] (see also [1]):

Example 1.1. Let $X = L^p(1,\infty) \cap L^{p'}(1,\infty)$ with $1 \leqslant p \leqslant 2$. This space has Fourier type p. Let A be the generator of the semigroup S defined on X by

$$S(t)f(x) := f(e^t x), \quad t \geqslant 0, \ x > 1.$$

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One has $\omega_0(A) = -\frac{1}{p'}$, $\omega_\theta(A) = -\frac{1}{p}$ for $\frac{1}{p} - \frac{1}{p'} \leqslant \theta \leqslant 1$, and $\omega_\theta(A)$ is linear on the interval $0 \leqslant \theta \leqslant \frac{1}{p} - \frac{1}{p'}$. On the other hand, the abscissa of boundedness of the resolvent of A equals $-\frac{1}{p}$.

It is a deep result of Bourgain [3] that a Banach space has nontrivial Fourier type if and only if it has nontrivial type. This leads naturally to the question whether a version of the Weis-Wrobel result still holds if we replace 'Fourier type' by 'type'. This is a nontrivial question, because there is no *a priori* relation between the Fourier type and the type of a Banach space. Our main result gives an affirmative partial answer:

Theorem 1.2. *Let X be a Banach space with type $1 \leqslant p \leqslant 2$ and cotype $2 \leqslant q \leqslant \infty$, and let A generate a C_0-semigroup on X. If the resolvent of A has a γ-bounded analytic extension to $\mathbb{C}_+$, then*

$$\omega_{\frac{1}{p}-\frac{1}{q}}(A) < 0.$$

The notion of γ-boundedness is a strengthening of the classical notion of uniform boundedness, and the two notions agree if the underlying Banach space is isomorphic to a Hilbert space. We refer to Section 2 for more details.

For $1 \leqslant p < \infty$ the spaces $L^p(\mu)$ have Fourier type $\min\{p, p'\}$, type $\min\{p, 2\}$ and cotype $\max\{p, 2\}$. This yields exponent $|\frac{1}{p} - \frac{1}{p'}|$ in the Weis-Wrobel result, whereas we obtain the exponent $|\frac{1}{p} - \frac{1}{2}|$ in Theorem 1.2. The price to pay for this improvement is an γ-boundedness assumption instead of a uniform boundedness assumption; it is an open problem whether this stronger assumption is really needed.

For positive semigroups on Banach lattices with finite cotype, the γ-boundedness of the resolvent follows from its uniform boundedness (see [10, Example 5.5(b)]) and we obtain the following corollary:

Corollary 1.3. *Let X be a Banach lattice with type $1 \leqslant p \leqslant 2$ and cotype $2 \leqslant q < \infty$, and let A generate a positive C_0-semigroup on X. If the resolvent of A has a bounded analytic extension to $\mathbb{C}_+$, then*

$$\omega_{\frac{1}{p}-\frac{1}{q}}(A) < 0.$$

Both in the theorem and its corollary, the cotype q assumption on X may be weakened to a type q' assumption on $X^\odot$, $\frac{1}{q} + \frac{1}{q'} = 1$.

The next example shows that the corollary is optimal:

Example 1.4. Let $X = L^2(1, \infty) \cap L^q(1, \infty)$ with $2 \leqslant q < \infty$. Since X is isomorphic as a Banach space to $L^q(1, \infty)$ [19, Corollary 2.e.8], X has type 2 and cotype q. Defining A as in the previous example, one has $\omega_0(A) = -\frac{1}{q}$, $\omega_\theta(A) = -\frac{1}{2}$ for $\frac{1}{2} - \frac{1}{q} \leqslant \theta \leqslant 1$, and $\omega_\theta(A)$ is linear on the interval $0 \leqslant \theta \leqslant \frac{1}{2} - \frac{1}{q}$. On the other hand, the abscissa of boundedness of the resolvent of A equals $-\frac{1}{2}$.

Our semigroup notations are standard; see e.g., [7, 27]. For $\lambda \in \mathbb{C} \setminus \sigma(A)$ we write $R(\lambda, A) := (\lambda - A)^{-1}$ for the resolvent of A at λ. The closed subspace of X^* on which the adjoint semigroup S^* acts in a strongly continuous way is denoted by $X^\odot$. The part of A^* in $X^\odot$ is denoted by $A^\odot$; this operator is the generator of the restricted semigroup $S^\odot := S^*|_{X^\odot}$. We recall the easy fact that $X^\odot$ induces an equivalent norm on X; see, e.g., [23].

2. Preliminaries

In this section we collect some elementary facts about γ-bounded families of operators and spaces of γ-radonifying operators. These topics have been covered in detail in [4, 15, 17, 24]. We refer to these works for references to the literature and further information.

Let X and Y be Banach spaces and let $(\gamma_n)_{n\geqslant 1}$ be a sequence of independent standard Gaussian random variables on a probability space $(\Omega, \mathbb{P})$.

Definition 2.1. A family $\mathscr{T} \subseteq \mathscr{L}(X,Y)$ is called *γ-bounded* if there exists a finite constant $C \geqslant 0$ such that for all $N \geqslant 1$ and all choices $x_1, \dots, x_N \in X$ and $T_1, \dots, T_N \in \mathscr{T}$ we have

$$\mathbb{E}\Big\| \sum_{n=1}^{N} \gamma_n T_n x_n \Big\|^2 \leqslant C^2 \mathbb{E}\Big\| \sum_{n=1}^{N} \gamma_n x_n \Big\|^2.$$

The least admissible constant C is called the *γ-bound* of $\mathscr{T}$, notation $\gamma(\mathscr{T})$. Every γ-bounded family is uniformly bounded, and the converse holds if X and Y are isomorphic to Hilbert spaces.

Replacing the role of Gaussian variables by Rademacher variables, we obtain the related notion of *R-boundedness*. Every R-bounded family of operators is γ-bounded, and the converse holds if the range space Y has finite cotype.

Let H be a Hilbert space. For a finite rank operator $T \in \mathscr{L}(H,X)$ of the form $\sum_{n=1}^{N} h_n \otimes x_n$ with $h_1, \dots, h_N$ orthonormal in H and $x_1, \dots, x_N \in X$ we define

$$\|T\|^2_{\gamma(H,X)} := \mathbb{E}\Big\| \sum_{n=1}^{N} \gamma_n x_n \Big\|^2.$$

Definition 2.2. A linear operator $T \in \mathscr{L}(H,X)$ is called *γ-radonifying* if it belongs to the closure of the finite rank operators with respect to the norm $\|\cdot\|_{\gamma(H,X)}$.

If X is a Hilbert space, then $\gamma(H,X) = \mathscr{L}_2(H,X)$ isometrically, where $\mathscr{L}_2(H,X)$ is the space of all Hilbert-Schmidt operators from H to X.

We need two basic properties of γ-radonifying operators. The first, the so-called *right ideal property*, asserts that of $T \in \gamma(H,X)$ and $S \in \mathscr{L}(H',H)$ is a Hilbert space operator, then $TS \in \gamma(H',X)$ and

$$\|TS\|_{\gamma(H',X)} \leqslant \|T\|_{\gamma(H,X)} \|S\|_{\mathscr{L}(H',H)}.$$

The second, the so-called *covariance domination property*, asserts that if $T \in \gamma(H,X)$ and $T' \in \mathscr{L}(H',X)$ are operators such that $\|(T')^* x^*\| \leqslant \|T^* x^*\|$ for all $x^* \in X^*$, then $T \in \gamma(H',X)$ and

$$\|T'\|_{\gamma(H',X)} \leqslant \|T\|_{\gamma(H,X)}.$$

We refer to [24] for more details.

Let D be an open subset of $\mathbb{R}^n$. For a function $\phi \in L^1_{\rm loc}(D;X)$ we define a linear operator $T_\phi : C_{\rm c}(D) \to X$ by

$$T_\phi f := \int_D f(t)\phi(t)\, dt, \quad f \in C_{\rm c}(D). \tag{2.1}$$

In the situation that T_ϕ extends to a bounded operator from $L^2(D)$ to X we say that this operator is *represented* by ϕ. If X is a Hilbert space, then ϕ represents an operator in $\gamma(L^2(D), X)$ if and only if $\phi \in L^2(D; X)$, and in this case we have

$$\|T_\phi\|_{\gamma(L^2(D),X)} = \|\phi\|_{L^2(D;X)}.$$

We proceed with three results due to Kalton and Weis [15].

Proposition 2.3. *If the functions $\phi \in L^1_{\rm loc}(D; X)$ and $\psi \in L^1_{\rm loc}(D; X^*)$ represent operators $T_\phi \in \gamma(L^2(D); X)$ and $T_\psi \in \gamma(L^2(D); X^*)$, respectively, then $\langle \phi, \psi\rangle \in L^1(D)$ and*

$$\|\langle \phi, \psi\rangle\|_{L^1(D)} \leqslant \|T_\phi\|_{\gamma(L^2(D);X)}\|T_\psi\|_{\gamma(L^2(D);X^*)}.$$

If ϕ represents an operator in $\gamma(L^2(D), X)$ and m is a function in $L^\infty(D)$, then by covariance domination $m\phi$ represents an operator in $\gamma(L^2(D), X)$ and we have

$$\|T_{m\phi}\|_{\gamma(L^2(D),X)} \leqslant \|m\|_\infty \|T_\phi\|_{\gamma(L^2(D),X)}.$$

The following result extends this observation to operator-valued multipliers:

Proposition 2.4. *If $\phi \in L^1_{\rm loc}(D; X)$ represents an operator $T_\phi \in \gamma(L^2(D), X)$ and $M : D \to \mathscr{L}(X, Y)$ is strongly measurable (in the sense that $t \mapsto M(t)x$ is strongly measurable for all $x \in X$) and has γ-bounded range, say with γ-bound $\gamma(M)$, then $M\phi \in L^1_{\rm loc}(D; Y)$ represents an operator $T_{M\phi} \in \gamma(L^2(D), Y)$ and*

$$\|T_{M\phi}\|_{\gamma(L^2(D),Y)} \leqslant \gamma(M)\|T_\phi\|_{\gamma(L^2(D),X)}.$$

The importance of the class of γ-radonifying operators for our present purposes derives from the following fact:

Proposition 2.5. *Let H and H' be Hilbert spaces, and let $S \in \mathscr{L}(H', H)$ be a bounded operator. Let X be a Banach space. The formula $S^X T := T \circ S$ defines a bounded operator $S^X \in \mathscr{L}(\gamma(H, X), \gamma(H', X))$ of norm*

$$\|S^X\|_{\mathscr{L}(\gamma(H,X),\gamma(H',X))} = \|S\|_{\mathscr{L}(H',H)}.$$

By applying this to the Fourier-Plancherel isometry $\mathscr{F} \in \mathscr{L}(L^2(\mathbb{R}^n))$,

$$\mathscr{F} f(\xi) = \frac{1}{\sqrt{(2\pi)^n}} \int_{\mathbb{R}^n} \exp(-i\langle x, \xi\rangle) f(x)\, dx, \quad \xi \in \mathbb{R}^n,\ f \in L^2(\mathbb{R}^n),$$

we obtain an isometry $\mathscr{F}^X \in \mathscr{L}(\gamma(L^2(\mathbb{R}^n), X))$. This operator is given by

$$\mathscr{F}^X T_\phi = T_{\mathscr{F}\phi}.$$

Indeed, by the Fubini theorem,

$$\begin{aligned}
\mathscr{F}^X T_\phi f = T_\phi(\mathscr{F} f) &= \frac{1}{\sqrt{(2\pi)^n}} \int_{\mathbb{R}^n} \phi(x) \int_{\mathbb{R}^n} \exp(-i\langle x, \xi\rangle) f(\xi)\, d\xi\, dx \\
&= \frac{1}{\sqrt{(2\pi)^n}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp(-i\langle x, \xi\rangle)\phi(x)\, dx\, f(\xi)\, d\xi \\
&= \int_{\mathbb{R}^n} (\mathscr{F}\phi)(\xi) f(\xi)\, d\xi = T_{\mathscr{F}\phi} f.
\end{aligned}$$

In this way we overcome the problem that the Fourier-Plancherel isometry extends to $L^2(\mathbb{R}^n; X)$ only when X is isomorphic to a Hilbert space.

3. An individual stability result

Let W denote a standard real-valued Brownian motion on a probability space $(\Omega, \mathbb{P})$. If A is a closed, densely defined operator on X, we say that an X-valued stochastic process U on $(\Omega, \mathbb{P})$ is a *weak solution* of the abstract Cauchy problem

$$dU(t) = AU(t)\,dt + x\,dW(t), \quad U(0) = 0, \tag{3.1}$$

where $x \in X$ is given, if the trajectories of U are integrable on bounded intervals almost surely and for all $t > 0$ and $x^* \in \mathscr{D}(A^*)$ the following identity holds almost surely:

$$\langle U(t), x^*\rangle = \int_0^t \langle U(s), A^*x^*\rangle\,ds + \langle x, x^*\rangle\,W(t).$$

If A is the generator of a C_0-semigroup S on X, then the problem (3.1) admits a weak solution if and only if the orbit $t \mapsto S(t)x$ belongs to $\gamma(L^2(0,T), X)$ for some (equivalently, all) $T > 0$ [25, Theorem 7.1].

Our first result is a variation of results of Zwart [32] and Eisner and Zwart [5, 6]. Rather than imposing integrability conditions on the resolvents of A and $A^\odot$ we insist that certain stochastic Cauchy problems be solvable for $x \in X$ and $x^\odot \in X^\odot$. In the next section we shall only need the special case $\alpha = 0$.

Theorem 3.1. *Let A be the generator of a C_0-semigroup S on a Banach space X. Let $x \in X$ and $x^\odot \in X^\odot$ be such that the abstract Cauchy problems*

$$\begin{aligned} dU(t) &= AU(t)\,dt \quad + \quad x\,dW(t), \qquad U(0) = 0,\\ dU^\odot(t) &= A^\odot U^\odot(t)\,dt + x^\odot\,dW(t), \quad U^\odot(0) = 0, \end{aligned}$$

admit weak solutions.

(1) *If there exist real numbers $\sigma > 0$ and $0 \leqslant \alpha < 1$ such that the set*

$$\big\{s^\alpha R(s+it, A) :\ 0 < s < \sigma,\ t \in \mathbb{R}\big\}$$

is γ-bounded, there exist constants $M \geqslant 0$ and $\omega > 0$ such that

$$|\langle S(t)x, x^\odot\rangle| \leqslant Me^{-\omega t}, \quad t \geqslant 0.$$

(2) *If there exist real numbers $\sigma > 0$ and $\alpha \geqslant 1$ such that the set*

$$\big\{s^\alpha R(s+it, A) :\ 0 < s < \sigma,\ t \in \mathbb{R}\big\}$$

is γ-bounded, there exists a constant $M \geqslant 0$ such that

$$|\langle S(t)x, x^\odot\rangle| \leqslant M(1 + t^\alpha), \quad t \geqslant 0.$$

Proof. We start with the proof of (2), which combines the techniques of [26, Theorem 1.1] and [32, Theorem 2.1].

By assumption, $t \mapsto S(t)x$ defines an element of $\gamma(L^2(0,T), X)$ for some (all) $T > 0$. From [26, Proposition 4.5] we deduce that for all $w > \omega_0(A)$ the rescaled orbit $t \mapsto e^{-wt}S(t)x$ defines an element of $\gamma(L^2(\mathbb{R}_+), X)$. In the same way, $t \mapsto e^{-wt}S^\odot(t)x^\odot$ defines an element of $\gamma(L^2(\mathbb{R}_+), X^\odot)$. We denote their norms by $K_w(x)$ and $K_w(x^\odot)$, respectively.

Let us fix $w > \max\{0, \omega_0(A)\}$. By Proposition 2.5 the function $t \mapsto R(w+it, A)x$ defines an element of $\gamma(L^2(\mathbb{R}), X)$ of norm $\sqrt{2\pi}K_w(x)$. The resolvent identity

$$R(a+it, A)x = R(w+it, A)x + (w-a)R(a+it, A)R(w+it, A)x$$

and Proposition 2.4 imply that for $0 < a \leqslant \sigma$, the function $t \mapsto R(a+it, A)x$ defines an element of $\gamma(L^2(\mathbb{R}), X)$ of norm

$$\|R(a+i\cdot, A)x\|_{\gamma(L^2(\mathbb{R}),X)} \leqslant \sqrt{2\pi}K_w(x)\big(1 + \frac{|w-a|\Gamma}{a^\alpha}\big),$$

where Γ is the γ-bound of the set $\{s^\alpha R(s+it, A): \ 0 < s < \sigma, \ t \in \mathbb{R}\}$. By Proposition 2.5 this estimate implies that the rescaled orbit $t \mapsto e^{-at}S(t)x$ defines an element of $\gamma(L^2(\mathbb{R}_+), X)$ of norm

$$\|e^{-a(\cdot)}S(\cdot)x\|_{\gamma(L^2(\mathbb{R}_+),X)} = \frac{1}{\sqrt{2\pi}}\|R(a+i\cdot)x\|_{\gamma(L^2(\mathbb{R}),X)} \leqslant K_w(x)\big(1 + \frac{|w-a|\Gamma}{a^\alpha}\big).$$

Hence by Proposition 2.3,

$$\begin{aligned}(3.2)\quad & e^{-at}|\langle S(t)x, x^\odot\rangle| \\ &= \frac{w}{1-e^{-wt}}\int_0^t |\langle e^{-a(t-s)}S(t-s)x, e^{-(a+w)s}S^\odot(s)x^\odot\rangle|\,ds \\ &\leqslant \frac{w}{1-e^{-wt}}\|e^{-a(t-\cdot)}S(t-\cdot)x\|_{\gamma(L^2(0,t),X)}\|e^{-(a+w)(\cdot)}S^\odot(\cdot)x^\odot\|_{\gamma(L^2(0,t),X^\odot)} \\ &= \frac{w}{1-e^{-wt}}\|e^{-a(\cdot)}S(\cdot)x\|_{\gamma(L^2(0,t),X)}\|e^{-(a+w)(\cdot)}S^\odot(\cdot)x^\odot\|_{\gamma(L^2(0,t),X^\odot)} \\ &\leqslant \frac{w}{1-e^{-wt}}\|e^{-a(\cdot)}S(\cdot)x\|_{\gamma(L^2(\mathbb{R}_+),X)}\|e^{-w(\cdot)}S^\odot(\cdot)x^\odot\|_{\gamma(L^2(\mathbb{R}_+),X^\odot)} \\ &\leqslant \frac{w}{1-e^{-wt}}K_w(x)K_w(x^\odot)\big(1 + \frac{|w-a|\Gamma}{a^\alpha}\big).\end{aligned}$$

Taking $a = 1/t$ with $t \geqslant \max\{w^{-1}, \sigma^{-1}\}$ we obtain the estimate

$$|\langle S(t)x, x^\odot\rangle| \leqslant \frac{ew}{1-e^{-1}}K_w(x)K_w(x^\odot)\big(1 + w\Gamma t^\alpha\big).$$

Next we prove (1). By the resolvent expansion argument of [26, Theorem 1.1] and [11, Theorem 3.2] (where the cases $\alpha = 0$ and $\alpha = \frac{1}{2}$ are considered, respectively), there exists $\delta > 0$ such that the resolvent of A has a γ-bounded analytic extension to $\{z \in \mathbb{C}: \ \mathrm{Re}\, z > -\delta\}$. By covariance domination, the stochastic Cauchy problems associated with x and $x^\odot$ are solvable for $A+\delta$ and $A^\odot + \delta$. Hence we may apply (2) to these rescaled operators to obtain the desired estimate. $\square$

This result has a number of ramifications which we discuss in a series of remarks.

Remark 3.2. In [5] it is shown that for $\alpha = 1$ the bound $M(1+t)$ is optimal even when X is a Hilbert space, in the sense that the bound cannot be improved to $M(1+t^\rho)$ for any $0 \leqslant \rho < 1$.

Remark 3.3. The assumptions of Theorem 3.1 imply that $S(\cdot)x$ and $S^\odot(\cdot)x^\odot$ belong to $\gamma(L^2(0,T), X)$ for all $T > 0$. If we make the stronger assumption that

$$\sup_{T>0}\|S(\cdot)x\|_{\gamma(L^2(0,T),X)} < \infty, \quad \sup_{T>0}\|S^\odot(\cdot)x^\odot\|_{\gamma(L^2(0,T),X^\odot)} < \infty, \tag{3.3}$$

which by [26, Proposition 4.4] is equivalent to assuming that the solutions U and $U^\odot$ of the associated stochastic Cauchy problems be bounded in probability, we may improve the bound in (2) to

$$|\langle S(t)x, x^\odot\rangle| \leqslant M/(1+t), \quad t \geqslant 0.$$

No γ-boundedness assumptions on the resolvent are needed here. To see this, just note that by Proposition 2.3,

$$\begin{aligned} t|\langle S(t)x, x^{\odot}\rangle| &= \int_0^t |\langle S(t-s)x, S^{\odot}(s)x^{\odot}\rangle|\, ds \\ &\leqslant \|S(t-\cdot)x\|_{\gamma(L^2(0,t),X)} \|S^{\odot}(\cdot)x^{\odot}\|_{\gamma(L^2(0,t),X^{\odot})} \\ &= \|S(\cdot)x\|_{\gamma(L^2(0,t),X)} \|S^{\odot}(\cdot)x^{\odot}\|_{\gamma(L^2(0,t),X^{\odot})}, \end{aligned}$$

and the right-hand side is bounded by a constant independent of $t \geqslant 0$.

Note that (3.3) is satisfied under the assumptions of part (1) of Theorem 3.1; this follows from [26, Theorem 1.1, Proposition 4.4].

Remark 3.4. If a Banach space Y has type 2 (see Section 4 for the definition), then every function $\phi \in L^2(D;Y)$ represents an operator $T_\phi \in \gamma(L^2(D), Y)$ [30]. Hence if $X^{\odot}$ has type 2, then in the Theorem 3.1 and the subsequent remarks, the solvability of the dual Cauchy problem is guaranteed for all $x^{\odot} \in X^{\odot}$ (since $S^{\odot}(\cdot)x^{\odot} \in L^2(0,T;X^{\odot})$). Recalling that $X^{\odot}$ induces an equivalent norm on X we obtain bounds on the growth of $\|S(t)x\|$. For example, if $X^{\odot}$ has type 2 and for some $x \in X$ the Cauchy problem

$$dU(t) = AU(t)\,dt + x\,dW(t), \quad U(0) = 0,$$

admits a weak solution U, then the γ-boundedness of the set

$$\{s^\alpha R(s+it, A) : \ 0 < s < \sigma, \ t \in \mathbb{R}\}$$

implies the bound $\|S(t)x\| \leqslant Me^{-\omega t}$ (for $0 \leqslant \alpha < 1$) and $\|S(t)x\| \leqslant M(1+t^\alpha)$ (for $\alpha \geqslant 1$). These results apply, e.g., to $X = L^p(\mu)$ with $1 < p \leqslant 2$.

Remark 3.5. The previous remark may be used to prove similar results for the stochastic Cauchy problem

$$dU(t) = AU(t)\,dt + B\,dW_H(t), \quad U(0) = 0, \tag{3.4}$$

where W_H is a cylindrical Brownian motion over a Hilbert space H and $B : H \to X$ is a bounded operator. We refer to [25] for a discussion of this problem and unexplained terminology. If a weak solution exists (which happens if and only if $S(\cdot)B$ defines an operator $\gamma(L^2(0,T;H), X)$ for some (all) $T > 0$), then so does the problem

$$dU(t) = AU(t)\,dt + x\,dW(t), \quad U(0) = 0,$$

for every $x \in \mathscr{R}(B)$. Indeed, if $x = Bh$, then the claim follows from

$$\|S(\cdot)Bh\|_{\gamma(L^2(0,T),X)} \leqslant \|S(\cdot)B\|_{\gamma(L^2(0,T;H),X)} \|h\|.$$

To give a sample result, if $X^{\odot}$ has type 2 and the Cauchy problem (3.4) admits a weak solution U, then the γ-boundedness of the set

$$\{s^\alpha R(s+it, A) : \ 0 < s < \sigma, \ t \in \mathbb{R}\}$$

implies the bound $\|S(t)B\| \leqslant Me^{-\omega t}$ (for $0 \leqslant \alpha < 1$) and $\|S(t)B\| \leqslant M(1+t^\alpha)$ (for $\alpha \geqslant 1$). This follows from the corresponding result of the previous remark and the uniform boundedness theorem.

As an immediate consequence of Theorem 3.1(1) we obtain the following Banach space analogue of the Gearhart-Herbst-Prüss theorem:

Corollary 3.6. *Let A be the generator of a C_0-semigroup S on a Banach space X. Suppose there exists a real number $\sigma > 0$ such that the set*

$$\{R(s+it, A) : \ 0 < s < \sigma, \ t \in \mathbb{R}\}$$

is γ-bounded. If for all $x \in X$ and $x^{\odot} \in X^{\odot}$ the abstract Cauchy problems

$$\begin{aligned} dU(t) &= AU(t)\,dt \quad + \quad x\,dW(t), \qquad U(0) = 0, \\ dU^{\odot}(t) &= A^{\odot}U^{\odot}(t)\,dt + x^{\odot}\,dW(t), \quad U^{\odot}(0) = 0, \end{aligned}$$

admit weak solutions, then S is uniformly exponentially stable.

Proof. By the resolvent expansion argument which has already been used in the proof of Theorem 3.1(1), the resolvent of A is γ-bounded on some half-plane $\{\lambda \in \mathbb{C} : \ \mathrm{Re}\,\lambda > -\delta\}$ with $\delta > 0$. Fix $w > \omega_0(A)$. By a closed graph argument there exists a constant K_w such that $K_w(x) \leqslant K_w\|x\|$ and $K_w(x^{\odot}) \leqslant K_w\|x^{\odot}\|$ for all $x \in X$ and $x^{\odot} \in X^{\odot}$. Since $X^{\odot}$ induces an equivalent norm on X, Theorem 3.1 applied to $A + \delta$ implies that $\omega_0(A) \leqslant -\delta$. $\square$

Of course the assumptions of this corollary are difficult to check in practice. Its interest is mainly theoretical, in that it exhibits the obstructions to extending the Gearhart-Herbst-Prüss theorem from Hilbert spaces to Banach spaces. It shows that if the Gearhart-Herbst-Prüss theorem fails, then either the resolvent of A fails to be γ-bounded or the stochastic Cauchy problems fails to be solvable for some $x \in X$ or $x^{\odot} \in X^{\odot}$. Note that in Hilbert spaces these obstructions disappear.

4. Fractional growth bounds

In this section we present our main application of Theorem 3.1, which is based upon the observation that the solvability conditions are satisfied for sufficiently 'regular' x and $x^{\odot}$.

We start by reviewing some standard material on Fourier type, type, and cotype. For a detailed overview of the theory relating to these notions we refer to the review articles [8, 21] where also references to the literature can be found.

Let $1 \leqslant p \leqslant 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$. The Hausdorff-Young inequality asserts that the Fourier-Plancherel transform $\mathscr{F}$ extends to a bounded operator from $L^p(\mathbb{R})$ to $L^{p'}(\mathbb{R})$. As a result, we may define a linear operator $\mathscr{F} \otimes I$ from $L^p(\mathbb{R}) \otimes X$ to $L^{p'}(\mathbb{R}) \otimes X$ by the formula

$$(\mathscr{F} \otimes I)(f \otimes x) := \mathscr{F}f \otimes x.$$

Definition 4.1. A Banach space X has *Fourier type* $1 \leqslant p \leqslant 2$ if $\mathscr{F} \otimes I$ extends to a bounded operator from $L^p(\mathbb{R}; X)$ to $L^{p'}(\mathbb{R}; X)$.

The notion of Fourier type was introduced by Peetre [28] and further studied by Bourgain [2, 3]. Every Banach space has Fourier type 1, and Hilbert spaces have Fourier type 2. Kwapień [18] has shown that every Banach space with Fourier type 2 is isomorphic to a Hilbert space. For $1 \leqslant p < \infty$ the spaces $L^p(\mu)$ have Fourier type $\min\{p, p'\}$. Furthermore, X has Fourier type p if and only if the dual space X^* has Fourier type p.

Let $(r_n)_{n=1}^{\infty}$ be a Rademacher sequence, i.e., a sequence of independent random variables defined on some probability space $(\Omega, \mathbb{P})$ which satisfy $\mathbb{P}(r_n = \pm 1) = \frac{1}{2}$.

Definition 4.2. A Banach space X has *type* $1 \leqslant p \leqslant 2$ if there exists a finite constant $C \geqslant 0$ such that for all $N \geqslant 1$ and all choices $x_1, \dots, x_N \in X$ we have

$$\Big(\mathbb{E}\Big\|\sum_{n=1}^{N} r_n x_n\Big\|^2\Big)^{\frac12} \leqslant C\Big(\sum_{n=1}^{N}\|x_n\|^p\Big)^{\frac1p}.$$

A Banach space X has *cotype* $2 \leqslant q \leqslant \infty$ if there exists a finite constant $C \geqslant 0$ such that for all $N \geqslant 1$ and all choices $x_1, \dots, x_N \in X$ we have

$$\Big(\sum_{n=1}^{N}\|x_n\|^q\Big)^{\frac1q} \leqslant C\Big(\mathbb{E}\Big\|\sum_{n=1}^{N} r_n x_n\Big\|^2\Big)^{\frac12},$$

with the obvious modification for $q = \infty$.

In this definition, the role of Rademacher variables may be replaced by Gaussian variables; this only affects the numerical values of the constants C.

The notions of type and cotype were introduced by Hoffman-Jørgensen [13] and have subsequently been studied by Maurey and Pisier [22] and many others. Every Banach space has type 1 and cotype ∞, and Hilbert spaces have type 2 and cotype 2. Kwapień [18] has shown that every Banach space with type 2 and cotype 2 is isomorphic to a Hilbert space. For $1 \leqslant p < \infty$ the spaces $L^p(\mu)$ have type $\min\{p, 2\}$ and cotype $\max\{p, 2\}$. If X has type p, then the dual X^* has type p', $\frac1p + \frac1{p'} = 1$. The analogue for cotype is not true: l^1 has cotype 2, but its dual l^∞ has only type 1. However, if X has type $1 < p \leqslant 2$ and cotype q, then X^* has type q', where $\frac1q + \frac1{q'} = 1$.

Let us say that a Banach space has *nontrivial* (Fourier) type if it has (Fourier) type $1 < p \leqslant 2$. It is a deep result of Bourgain [3] (see also [8, Corollary 8.9]) that a Banach space X has nontrivial Fourier type if and only if it has nontrivial type. In the same paper it is shown that it not possible to give a general expression relating the Fourier type of X to its type and cotype; one also has to take into account the numerical values of the constants C appearing in Definition 4.2. On the other hand, if X has Fourier type p, then X has type p (see [8, Theorem 8.2]).

Let X have type $1 \leqslant p \leqslant 2$. For arbitrary C_0-semigroups on X, the solvability assumption on x of Theorem 3.1 is satisfied if $S(\cdot)x$ belongs to the Besov space $B^{\frac1p-\frac12}_{p,p}(0,T;X)$ for some (all) $T > 0$, and a similar result holds on the dual side. Indeed, if X has type p, then for all $T > 0$ we have a continuous inclusion

$$B^{\frac1p-\frac12}_{p,p}(0,T;X) \hookrightarrow \gamma(L^2(0,T), X) \tag{4.1}$$

given by $\phi \mapsto T_\phi$ as in (2.1); this is the main result of [14]. We refer to [16] for the definition of vector-valued Besov spaces and references to the literature. What matters for us is the real interpolation identity

$$(L^p(0,T;X), W^{1,p}(0,T;X))_{\alpha,q} = B^{\alpha}_{p,q}(0,T;X), \tag{4.2}$$

valid for $1 \leqslant p, q < \infty$ and $0 < \alpha < 1$.

Lemma 4.3. *Let X and $X^\odot$ have type $1 \leqslant p < 2$ and $1 \leqslant p^\odot < 2$, respectively. Let S be a C_0-semigroup on X with generator A, and suppose there exists a real number $\sigma > 0$ such that the set*

$$\big\{R(s+it, A) :\ 0 < s < \sigma,\ t \in \mathbb{R}\big\}$$

is γ-bounded. If

$$x \in (X, \mathscr{D}(A))_{\frac{1}{p}-\frac{1}{2},p}, \quad x^\odot \in (X^\odot, \mathscr{D}(A^\odot))_{\frac{1}{p^\odot}-\frac{1}{2},p^\odot},$$

then there exist $M \geqslant 0$ and $\omega > 0$ such that

$$|\langle S(t)x, x^\odot\rangle| \leqslant Me^{-\omega t}, \quad t \geqslant 0.$$

Proof. Interpolating the mapping $x \mapsto S(\cdot)x$, from (4.2) we deduce that $x \in (X, \mathscr{D}(A))_{\frac{1}{p}-\frac{1}{2},p}$ implies $S(\cdot)x \in B^{\frac{1}{p}-\frac{1}{2}}_{p,p}(0,T;X)$. By a similar argument, $x^\odot \in (X^\odot, \mathscr{D}(A^\odot))_{\frac{1}{p^\odot}-\frac{1}{2},p^\odot}$ implies that $S^\odot(\cdot)x^\odot \in B^{\frac{1}{p^\odot}-\frac{1}{2}}_{p^\odot,p^\odot}(0,T;X^\odot)$. By (4.1), $S(\cdot)x$ and $S^\odot(\cdot)x^\odot$ belong to $\gamma(L^2(0,T),X)$ and $\gamma(L^2(0,T),X^\odot)$, respectively, and the lemma follows from Theorem 3.1(1). □

Proof of Theorem 1.2. Recall (e.g., from [20]) that for all $0 < \eta' < \eta < 1$ and $1 \leqslant r < \infty$,

$$X_\eta \hookrightarrow (X, \mathscr{D}(A))_{\eta,\infty} \hookrightarrow (X, \mathscr{D}(A))_{\eta',r}. \tag{4.3}$$

We start with the case $1 \leqslant p < 2$ and $2 < q \leqslant \infty$. We may assume that $\frac{1}{p} - \frac{1}{q} < 1$, since otherwise we have $p = 1$ and $q = \infty$, in which case the inequality $\omega_1(A) < 0$ follows from the uniform boundedness of the resolvent [31].

Fix $\frac{1}{p} - \frac{1}{q} < \theta < 1$ and write $\theta = \beta + \beta'$ with $\beta > \frac{1}{p} - \frac{1}{2}$ and $\beta' > \frac{1}{2} - \frac{1}{q}$. Let $x \in X_\theta$ and $x^\odot \in X^\odot$ be arbitrary. Fix $w > \omega_0(A)$ and write $x = (w-A)^{-\theta}y$. By (4.3), $(w-A)^{-\beta}y \in (X, \mathscr{D}(A))_{\frac{1}{p}-\frac{1}{2},p}$ and $(w-A^\odot)^{-\beta'}x^\odot \in (X^\odot, \mathscr{D}(A^\odot))_{\frac{1}{2}-\frac{1}{q},q'} = (X^\odot, \mathscr{D}(A^\odot))_{\frac{1}{q'}-\frac{1}{2},q'}$. Since $X^\odot$ has type q', it follows from Lemma 4.3 that

$$|\langle S(t)x, x^\odot\rangle| = |\langle S(t)(w-A)^{-\beta}y, (w-A^\odot)^{-\beta'}x^\odot\rangle| \leqslant Me^{-\omega t}$$

for some $M \geqslant 0$, $\omega > 0$, and all $t \geqslant 0$. Since $x^\odot \in X^\odot$ was arbitrary and $X^\odot$ induces an equivalent norm on X, this proves that $\omega_\theta(A) < 0$ for all $\theta > \frac{1}{p} - \frac{1}{q}$. By [31], the function $\theta \mapsto \omega_\theta(A)$ is convex on $(0,1)$, and therefore continuous on $(0,1)$. It follows that $\omega_{\frac{1}{p}-\frac{1}{q}}(A) \leqslant 0$.

Next suppose that $p = 2$ and $2 < q \leqslant \infty$. Since X has type p' for all $1 \leqslant p' < 2$, the above reasoning gives us that $\omega_{\frac{1}{p'}-\frac{1}{q}}(A) \leqslant 0$ for all $1 \leqslant p' < 2$. Using once more the convexity of $\theta \mapsto \omega_\theta(A)$ we find that $\omega_{\frac{1}{p}-\frac{1}{q}}(A) \leqslant 0$. The same argument works in the case $1 \leqslant p < 2$ and $q = 2$.

If $p = 2$ and $q = 2$, then by Kwapień's theorem X is isomorphic to a Hilbert space and the inequality $\omega_0(A) < 0$ follows from the Gearhart-Herbst-Prüss theorem.

Summarising what has been proved so far, the assumptions stated in the theorem imply that $\omega_{\frac{1}{p}-\frac{1}{q}}(A) \leqslant 0$. Finally, the γ-boundedness of the resolvent of A on $\mathbb{C}_+$ implies the γ-boundedness of the resolvent of the shifted operator $A + \delta$ on $\mathbb{C}_+$ for some $\delta > 0$. Applying what has just been proved to $A + \delta$, we see that $\omega_{\frac{1}{p}-\frac{1}{q}}(A+\delta) \leqslant 0$, that is, $\omega_{\frac{1}{p}-\frac{1}{q}}(A) \leqslant -\delta$. □

Remark 4.4. We have stated the result in terms of the type p and cotype q of X. What is really used in the proof is that X has type p and $X^\odot$ has type q'.

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Delft Institute of Applied Mathematics, Delft University of Technology, P.O. Box 5031, 2600 GA Delft, The Netherlands

E-mail address: J.M.A.M.vanNeerven@tudelft.nl