Characterization of exponential stability of a semigroup of operators in terms of its action by convolution on vector-valued function spaces over $\mathbb{R}_+$

J.M.A.M. van Neerven[1]

Mathematisches Institut, Universität Tübingen
Auf der Morgenstelle 10, D-72076, Tübingen, Germany
E-mail: jane@michelangelo.mathematik.uni-tuebingen.de

It is proved that a C_0-semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ of linear operators on a Banach space X is uniformly exponentially stable if and only if it acts boundedly on one of the spaces $L^p(\mathbb{R}_+, X)$ or $C_0(\mathbb{R}_+, X)$ by convolution. As an application, it is shown that $\mathbf{T}$ is uniformly exponentially stable if and only if

$$\sup_{s>0} \left\| \int_0^s T(t)g(t)\,dt \right\| < \infty, \qquad \forall g \in AP(\mathbb{R}_+, X),$$

where $AP(\mathbb{R}_+, X)$ is the space of X-valued almost periodic functions on $\mathbb{R}_+$.

1991 Mathematics Subject Classification: 47D03
Keywords & phrases: C_0-semigroup, exponentially stable, growth bound, convolution, Lebesgue-Bochner spaces, almost periodic functions.

0. Introduction

It is well-known that the asymptotic behaviour of a C_0-semigroup of linear operators $\mathbf{T} = \{T(t)\}_{t\geq 0}$ cannot be adequately described by the location of the spectrum $\sigma(A)$ of its infinitesimal generator A. If we define the *growth bound* $\omega_0(\mathbf{T})$ as the infimum of all $\omega \in \mathbb{R}$ for which a constant $M > 0$ exists such that

$$\|T(t)\| \leq Me^{\omega t}, \qquad \forall t \geq 0,$$

and the *spectral bound* $s(A)$ by

$$s(A) = \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(A)\},$$

[1] Supported by an Individual Fellowship in the Human Capital and Mobility programme of the European Communities

one has the inequality $s(A) \le \omega_0(\mathbf{T})$, but strict inequality may occur. In their recent paper [LM], Latushkin and Montgomery-Smith proved the following striking fact: If one tensors an arbitrary C_0-semigroup $\mathbf{T}$ on X with the translation group on $L^p(\mathbb{R})$, the growth bound and the spectral bound of the resulting C_0-semigroup $\mathbf{S}_p$ on $L^p(\mathbb{R}, X)$ coincide, and are equal to the growth bound of $\mathbf{T}$. In fact, they proved the stronger result that the spectral mapping theorem holds for $\mathbf{S}_p$. Independently, for $p = 2$ and X Hilbert, this result was obtained by Rau [Ra1]. These results also hold for tensoring with the rotation group on $L^p(\Gamma)$, where Γ is the unit circle. Furthermore, the analogous results hold for $C_0(\mathbb{R})$ and $C(\Gamma)$.

These results drew much attention and were extended into various directions; cf. [Ra2], [Ra3], [RS1], [Rs2], and the references given there. Also, L. Weis [W] recently announced a proof of the stability conjecture for positive C_0-semigroups on L^p which is based on them.

The present paper was inspired by the Latushkin-Montgomery-Smith results in the following way. Explicitly, $\mathbf{S}_p$ is given by

$$S_p(t)f(s) = T(t)f(s-t), \qquad s \in \mathbb{R}, \quad t \ge 0, \quad f \in L^p(\mathbb{R}, X).$$

Integrating this over $[0, s]$ with respect to t, the resulting integral can be interpreted as the convolution of $\mathbf{T}$ with the restriction $f|_{\mathbb{R}_+}$. Here, and in the rest of the paper, $\mathbb{R}_+ = [0, \infty)$. This observation motivates two questions: Firstly, do versions of the Latushkin-Montgomery-Smith theorem exist for the translation semigroups on $L^p(\mathbb{R}_+)$ and $C_0(\mathbb{R}_+)$, and secondly, can one study the asymptotic behaviour of $\mathbf{T}$ by looking at its convolution with functions in $L^p(\mathbb{R}_+, X)$ or $C_0(\mathbb{R}_+, X)$? In this paper, we answer the second question affirmatively as follows. For a function $f \in L^1_{loc}(\mathbb{R}_+, X)$ we define the convolution $\mathbf{T} * f$ by

$$(\mathbf{T} * f)(s) := \int_0^s T(t)f(s-t)\,dt, \qquad s \ge 0.$$

Theorem 0.1. *Let $\mathbf{T}$ be a C_0-semigroup on a Banach space X and let $1 \le p < \infty$. Then the following assertions are equivalent:*
(i) $\omega_0(\mathbf{T}) < 0$;
(ii) $\mathbf{T} * f \in L^p(\mathbb{R}_+, X)$ *for all* $f \in L^p(\mathbb{R}_+, X)$;
(iii) $\mathbf{T} * f \in C_0(\mathbb{R}_+, X)$ *for all* $f \in C_0(\mathbb{R}_+, X)$.

Loosely speaking, this means that $\omega_0(\mathbf{T}) < 0$ if and only if $\mathbf{T}$ acts on one of the spaces $L^p(\mathbb{R}_+, X)$ or $C_0(\mathbb{R}_+, X)$ by convolution. The proof, which is based on a partial answer to the first question, is given in Section 1. In Section 2, we apply the theorem to show that $\omega_0(\mathbf{T}) < 0$ if and only if

$$\sup_{s>0} \left\| \int_0^s T(t)f(t)\,dt \right\| < \infty, \qquad \forall f \in AP(\mathbb{R}_+, X),$$

where $AP(\mathbb{R}_+, X)$ is the space of X-valued almost-periodic functions. In fact, we show that the same result is true for a more general class of spaces of X-valued functions which contains $C_0(\mathbb{R}_+, X)$ and $AP(\mathbb{R}_+, X)$. Also, we improve (iii) of Theorem 0.1 by showing that $\omega_0(\mathbf{T}) < 0$ if $\mathbf{T} * f \in L^\infty(\mathbb{R}_+, X)$ for all f in one of the spaces in this class.

In Section 3, we discuss an extension of Theorem 0.1 to the action of $\mathbf{T}$ by convolution on vector-valued Banach function spaces over $\mathbb{R}_+$.

In Section 4, we deal with the weak analogue of Theorem 0.1. It is shown that the condition that $\langle x^*, \mathbf{T} * f\rangle \in L^1(\mathbb{R}_+)$ for all $x^* \in X^*$ and $f \in L^1(\mathbb{R}_+, X)$ is equivalent to $\omega_0(\mathbf{T}) < 0$ in Hilbert spaces. We also show that this need not be true in arbitrary Banach spaces.

1. Proof of Theorem 0.1

Let $\mathbf{T}$ be a C_0-semigroup on a Banach space X. Let $1 \le p < \infty$. We define a C_0-semigroup $\mathbf{S}_p$ on $L^p(\mathbb{R}_+, X)$ by tensoring $\mathbf{T}$ with the semigroup of right translations on $L^p(\mathbb{R}_+)$:

$$(S_p(t))f(s) = \begin{cases} T(t)f(s-t), & s-t \ge 0; \\ 0, & \text{else.} \end{cases} \tag{1.1}$$

We denote by B_p its infinitesimal generator. Similarly, by using the right translation semigroup on $C_{00}(\mathbb{R}_+)$, the subspace of all $f \in C_0(\mathbb{R}_+)$ such that $f(0) = 0$, we obtain a C_0-semigroup $\mathbf{S}_\infty$ on the corresponding X-valued space $C_{00}(\mathbb{R}_+, X)$. Its generator will be denoted by B_∞.

For an operator T, we denote by $A\sigma(T)$ the approximate point spectrum of T.

Lemma 1.1. *If $e^{i\theta} \in A\sigma(T(1))$ for some $\theta \in [0, 2\pi)$, then $0 \in A\sigma(B_p)$, $1 \le p \le \infty$.*

Proof: The result is essentially contained in [RS2]. For reasons of self-containedness, and since the setting in [RS2] is more general and slightly different, we give the proof in some detail. First assume $1 \le p < \infty$. Since $\theta \in A\sigma(T(1))$, for each $n = 1, 2, ...$ one can find a norm one vector $x_n \in X$ such that

$$\|T(k)x_n - e^{ik\theta}x_n\| \le \frac{1}{2}, \qquad k = 1, ..., n.$$

In particular, $\frac{1}{2} \le \|T(k)x_n\| \le 2$ for all n and $k = 1, ..., n$. Using the local boundedness of $\mathbf{T}$, it is easy to see that there are constants $0 < \alpha \le \beta < \infty$ such that

$$\alpha \le \|T(t)x_n\| \le \beta, \qquad t \in [0, n]; \quad n = 1, 2, ...$$

For each n, let $a_n : \mathbb{R}_+ \to [0, 1]$ be a continuously differentiable function such that $a_n = 0$ on $[0, \frac{1}{8}] \cup [n - \frac{1}{8}, \infty)$, $a_n = 1$ on $[\frac{1}{4}, n - \frac{1}{4}]$, and $a_n'(t) \le 10$ for all $t \ge 0$ and $n = 1, 2, ...$ Let $g_n := c_n^{-1} a_n(t)T(t)x_n$, where

$$c_n := \left(\int_0^n \|T(t)x_n\|^p \, dt\right)^{\frac{1}{p}}.$$

Then, using that $a_n = 1$ on $[\frac{1}{4}, n - \frac{1}{4}]$, we have

$$\frac{\alpha}{2^{\frac{1}{p}}\beta} \leq \frac{\alpha(n-\frac{1}{2})^{\frac{1}{p}}}{\beta n^{\frac{1}{p}}} \leq \|g_n\|_{L^p(\mathbb{R}_+,X)} \leq 1; \qquad n = 1, 2, ...$$

Also, by direct calculation one checks that

$$B_p g_n(t) = -c_n^{-1} a_n'(t) T(t) x_n, \qquad t \geq 0.$$

Since $|a_n'| \leq 10$ on $[\frac{1}{8}, \frac{1}{4}] \cup [n - \frac{1}{4}, n - \frac{1}{8}]$ and 0 elsewhere, we have

$$\|B_E g_n\|_{L^p(\mathbb{R}_+,X)} \leq \frac{10\beta \cdot 8^{-\frac{1}{p}} + 10\beta \cdot 8^{-\frac{1}{p}}}{\alpha n^{\frac{1}{p}}}; \qquad n = 1, 2, ...$$

Therefore $\lim_{n\to\infty} \|B_p g_n\|_{L^p(\mathbb{R}_+,X)} = 0$, which shows that $\left(\|g_n\|^{-1}_{L^p(\mathbb{R}_+,X)} \cdot g_n\right)_{n\geq 1}$ is an approximate eigenvector for B_p with approximate eigenvalue 0.

Next, we show how to modify this argument for $p = \infty$. In this case, we choose a C^1-function a_n that vanishes on $[0, \frac{1}{4}] \cup [n - \frac{1}{4}, \infty)$, and further satisfies $a_n(\frac{1}{2}n) = 1$, $\|a_n\|_\infty = 1$, and $\|a_n'\|_\infty \leq 5n^{-1}$. Then $g_n := a_n(t)T(t)x_n$, $n = 1, 2, ...$, defines an approximate eigenvector for B_∞ with approximate eigenvalue 0. ////

Lemma 1.2. *With the above notations, for all $1 \leq p \leq \infty$ we have $s(B_p) = \omega_0(\mathbf{S}_p) = \omega_0(\mathbf{T})$.*

Proof: Clearly, $s(B_p) \leq \omega_0(\mathbf{S}_p) \leq \omega_0(\mathbf{T})$. It remains to prove that $\omega_0(\mathbf{T}) \leq s(B_p)$. By rescaling, we may assume that $\omega_0(\mathbf{T}) = 0$, and it suffices to show that this implies that $s(B_p) \geq 0$. Since $r(T(1)) = e^{\omega_0(\mathbf{T})} = 1$, there is a $\theta \in [0, 2\pi)$ such that $e^{i\theta} \in \sigma(T(1))$, and in fact, $e^{i\theta} \in A\sigma(T(1))$ since the boundary spectrum is always contained in the approximate point spectrum. By Lemma 1.1, this implies that $0 \in \sigma(B_p)$. Therefore, $s(B_p) \geq 0$ and the proof is complete. ////

Theorem 1.3. *Let $\mathbf{T}$ be a C_0-semigroup on a Banach space X and let $1 \leq p < \infty$. Then the following assertions are equivalent:*
(i) $\omega_0(\mathbf{T}) < 0$;
(ii) $\mathbf{T} * f \in L^p(\mathbb{R}_+, X)$ *for all* $f \in L^p(\mathbb{R}_+, X)$;
(iii) $\mathbf{T} * f \in C_0(\mathbb{R}_+, X)$ *for all* $f \in C_0(\mathbb{R}_+, X)$.

Proof: First we prove (ii)⇒(i). By Lemma 1.2, we have to prove that $s(B_p) < 0$. For this, it is enough to prove that the resolvent of B_p exists and is uniformly bounded in the right half plane. Indeed, once this is established, a standard resolvent expansion argument shows that the resolvent exists and is uniformly bounded in a half plane $\{\mathrm{Re} z > -\epsilon\}$ for some $\epsilon > 0$; cf. [Hu] or [NSW]. Thus, $s(B_p) \leq -\epsilon$.

We start by observing that there is a constant $M > 0$ such that $\|\mathbf{T} * f\|_{L^p(\mathbb{R}_+,X)} \leq M\|f\|_{L^p(\mathbb{R}_+,X)}$ for all $f \in L^p(\mathbb{R}_+, X)$. To see this, we claim that the map $f \mapsto \mathbf{T} * f$ is closed as a map of $L^p(\mathbb{R}_+, X)$ into itself. Indeed, assume $f_n \to f$ and $\mathbf{T} * f_n \to g$ in

$L^p(\mathbb{R}_+, X)$. Then it is immediate that $(\mathbf{T} * f_n)(s) \to (\mathbf{T} * f)(s)$ for all $s > 0$. On the other hand, since a norm convergent sequence in $L^p(\mathbb{R}_+, X)$ contains a subsequence that converges pointwise a.e., it follows that $(\mathbf{T} * f_{n_k})(s) \to g(s)$ for some sequence (n_k) and almost all s. Therefore, $\mathbf{T} * f = g$, as was to be shown. The existence of the constant M now follows from the closed graph theorem.

For $T_0 > 0$, we define $\pi_{T_0} : L^p(\mathbb{R}_+, X) \to L^p([0, T_0], X)$ by restriction: $\pi_{T_0} f = f|_{[0,T_0]}$. For $T > 0$, $T_0 > 0$ and $f \in L^p(\mathbb{R}_+, X)$, we define the entire $L^p(\mathbb{R}_+, X)$-valued function $F_{T,f}$ and the entire $L^p([0, T_0], X)$-valued function $F_{T,T_0,f}$ by

$$\begin{aligned} F_{T,f}(z) &= \int_0^T e^{-zt} S_p(t) f \, dt, \\ F_{T,T_0,f}(z) &= \pi_{T_0}(F_{T,f}(z)). \end{aligned}$$

For each z, the map $f \mapsto F_{T,T_0,f}(z)$ is bounded as a map $L^p(\mathbb{R}_+, X) \to L^p([0, T_0], X)$. A trivial estimate shows that each of the functions $z \mapsto F_{T,f}(z)$ and $z \mapsto F_{T,T_0,f}(z)$ is bounded in each vertical strip $\{0 < \mathrm{Re} z < c\}$, $c > 0$.

For $\lambda \in \mathbb{R}$ and $f \in L^p(\mathbb{R}_+, X)$, let $f_\lambda(s) := e^{i\lambda s} f(s)$, $s \geq 0$. The restriction of $\mathbf{S}_p$ to the invariant subspace $C_{00}(\mathbb{R}_+, X) \cap L^p(\mathbb{R}_+, X)$ extends to the C_0-semigroup $\mathbf{S}_\infty$ on $C_{00}(\mathbb{R}_+, X)$. Since point evaluations on the latter space are continuous, for $f \in C_{00}(\mathbb{R}_+, X) \cap L^p(\mathbb{R}_+, X)$, $T \geq T_0$, and $0 \leq s \leq T_0$ we have

$$\begin{aligned} \left(\int_0^T e^{-i\lambda t} S_p(t) f \, dt \right)(s) &= \int_0^T e^{-i\lambda t} S_p(t) f(s) \, dt \\ &= \int_0^s e^{-i\lambda t} T(t) f(s-t) \, dt \\ &= e^{-i\lambda s} \int_0^s T(t) f_\lambda(s-t) \, dt. \end{aligned}$$

Therefore, for $T \geq T_0$,

$$\begin{aligned} \|F_{T,T_0,f}(i\lambda)\|_{L^p([0,T_0],X)} &= \left\| \pi_{T_0} \left(e^{-i\lambda(\cdot)} \int_0^{(\cdot)} T(t) f_\lambda(\cdot - t) \, dt \right) \right\|_{L^p([0,T_0],X)} \\ &\leq \|\pi_{T_0}\| \cdot \|\mathbf{T} * f_\lambda\|_{L^p(\mathbb{R}_+,X)} \leq \|\mathbf{T} * f_\lambda\|_{L^p(\mathbb{R}_+,X)} \\ &\leq M \|f_\lambda\|_{L^p(\mathbb{R}_+,X)} = M \|f\|_{L^p(\mathbb{R}_+,X)}. \end{aligned}$$

Since $C_{00}(\mathbb{R}_+, X) \cap L^p(\mathbb{R}_+, X)$ is dense in $L^p(\mathbb{R}_+, X)$, it follows that

$$\|F_{T,T_0,f}(i\lambda)\|_{L^p([0,T_0],X)} \leq M \|f\|_{L^p(\mathbb{R}_+,X)}, \qquad \forall \lambda \in \mathbb{R}, \quad T \geq T_0, \quad f \in L^p(\mathbb{R}_+, X). \tag{1.2}$$

Also, if we choose constants $N > 0$ and $\omega \geq 0$ such that $\|S_p(t)\| \leq N e^{\omega t}$ for all $t \geq 0$, then for $\mathrm{Re} z = \omega + 1$ we have

$$\begin{aligned} \|F_{T,T_0,f}(z)\|_{L^p([0,T_0],X)} &\leq \|\pi_{T_0}\| \int_0^T e^{-(\omega+1)t} N e^{\omega t} \|f\|_{L^p(\mathbb{R}_+,X)} \, dt \\ &\leq N(1 - e^{-T}) \|f\|_{L^p(\mathbb{R}_+,X)} \leq N \|f\|_{L^p(\mathbb{R}_+,X)}. \end{aligned}$$

It follows that for each $f \in L^p(\mathbb{R}_+, X)$ and $T_0 > 0$ fixed, the functions $z \mapsto F_{T,T_0,f}(z)$ are bounded on the line $\mathrm{Re} z = \omega + 1$, uniformly with respect to $T > 0$, with bound $N\|f\|_{L^p(\mathbb{R}_+,X)}$. Therefore, by (1.2) and the Phragmen-Lindelöf theorem [R, Thm. 12.8], for each f and T_0 fixed we have

$$\|F_{T,T_0,f}(z)\|_{L^p([0,T_0],X)} \le \max\{M, N\}\|f\|_{L^p(\mathbb{R}_+,X)}, \qquad \forall 0 < \mathrm{Re} z < \omega + 1, \quad T \ge T_0. \tag{1.3}$$

Also, for $\mathrm{Re} z > \omega$ we have

$$\lim_{T\to\infty} F_{T,T_0,f}(z) = \pi_{T_0} R(z, B_p)f.$$

Combining these facts, it follows from Vitali's theorem [HPh, Thm 3.14.1] that for each $f \in L^p(\mathbb{R}_+, X)$ and $T_0 > 0$ the function $z \mapsto \pi_{T_0} R(z, B_p)f$ has an analytic continuation $F_{\infty,T_0,f}$ to $\{0 < \mathrm{Re} z < \omega + 1\}$, and that for $0 < \mathrm{Re} z < \omega + 1$,

$$F_{\infty,T_0,f}(z) = \lim_{T\to\infty} F_{T,T_0,f}(z)$$

uniformly on compacta. Moreover, by (1.3),

$$\|F_{\infty,T_0,f}(z)\|_{L^p([0,T_0],X)} \le \max\{M, N\}\|f\|_{L^p(\mathbb{R}_+,X)}, \qquad \forall 0 < \mathrm{Re} z < \omega + 1, \quad T_0 > 0. \tag{1.4}$$

By regarding $L^p([0, T_0], X)$ as a closed subspace of $L^p(\mathbb{R}_+, X)$, for all $\omega < \mathrm{Re} z < \omega+1$ we have

$$\lim_{T_0\to\infty} F_{\infty,T_0,f}(z) = \lim_{T_0\to\infty} \pi_{T_0} R(z, B_p)f = R(z, B_p)f, \tag{1.5}$$

the convergence being with respect to the norm of $L^p(\mathbb{R}_+, X)$. Again by Vitali's theorem, now using (1.4), it follows that $R(z, B_p)f$ admits a holomorphic extension $F_{\infty,\infty,f}$ to $\{0 < \mathrm{Re} z < \omega + 1\}$, and that for all $0 < \mathrm{Re} z < \omega + 1$,

$$\lim_{T_0\to\infty} \lim_{T\to\infty} F_{T,T_0,f}(z) = \lim_{T_0\to\infty} F_{\infty,T_0,f}(z) = F_{\infty,\infty,f}(z) \tag{1.6}$$

uniformly on compacta. By an easy analytic continuation argument, we must have $\{0 < \mathrm{Re} z < \omega + 1\} \subset \varrho(B_p)$ and $F_{\infty,\infty,f}(z) = R(z, B_p)f$.

Therefore, by (1.4), (1.6), and the uniform boundedness theorem, it follows that $R(z, B_p)$ is uniformly bounded in $\{0 < \mathrm{Re} z < \omega + 1\}$. By the Hille-Yosida theorem, $R(z, B_p)$ is also uniformly bounded in $\{\mathrm{Re} z \ge \omega + 1\}$. Thus, $R(z, B_p)$ exists and is uniformly bounded in $\{\mathrm{Re} z > 0\}$. This completes the proof of (ii)⇒(i).

Next, we prove (i) ⇒(ii). Assume $\omega_0(\mathbf{T}) < 0$, and choose $\mu > 0$ and $M > 0$ such that $\|T(t)\| \le Me^{-\mu t}$ for all $t \ge 0$. Let $1 \le p < \infty$ be arbitrary and fixed. By applying Jensen's inequality [R, Thm. 3.3] to the probability measure $\mu(1 - e^{-\mu s})^{-1}e^{-\mu t}\, dt$ on

$[0, s]$, we have, noting that $\mu^{-(p-1)}(1 - e^{-\mu s})^{p-1} \le \mu^{-(p-1)}$ for all $s > 0$,

$$\begin{aligned}
\int_0^\infty \Big\| \int_0^s T(t) f(s-t)\, dt \Big\|^p ds &\le \int_0^\infty \Big(\int_0^s M e^{-\mu t} \|f(s-t)\|\, dt \Big)^p ds \\
&\le M^p \int_0^\infty \mu^{-p}(1 - e^{-\mu s})^p \int_0^s \|f(s-t)\|^p \mu (1 - e^{-\mu s})^{-1} e^{-\mu t}\, dt\, ds \\
&\le M^p \mu^{-(p-1)} \int_0^\infty \int_0^s e^{-\mu t} \|f(s-t)\|^p\, dt\, ds \\
&= M^p \mu^{-(p-1)} \int_0^\infty e^{-\mu t} \int_t^\infty \|f(s-t)\|^p\, ds\, dt \\
&\le M^p \mu^{-p} \|f\|^p.
\end{aligned}$$

This proves (i)⇒(ii).

The implication (i)⇒(iii) is proved as follows. Choose $M > 0$ and $\mu > 0$ such that $\|T(t)\| \le M e^{-\mu t}$ for all $t \ge 0$. Fix $\epsilon > 0$ and $f \in C_0(\mathbb{R}_+, X)$ arbitrary. Choose N so large that $s e^{-\mu s} \le \epsilon$ and $\|f(s)\| \le \epsilon \|f\|_{C_0(\mathbb{R}_+, X)}$ for all $s \ge N$. Then, for $s \ge 2N$,

$$\begin{aligned}
\Big\| \int_0^s T(t) f(s-t)\, dt \Big\| &\le \int_{\frac{s}{2}}^s M e^{-\mu \frac{s}{2}} \|f\|_{C_0(\mathbb{R}_+, X)}\, dt + \int_0^{\frac{s}{2}} M e^{-\mu t} \epsilon \|f\|_{C_0(\mathbb{R}_+, X)}\, dt \\
&\le M(1 + \mu^{-1}) \epsilon \|f\|_{C_0(\mathbb{R}_+, X)}.
\end{aligned}$$

Since $\mathbf{T} * f$ also is continuous, we obtain the desired conclusion.

It remains to prove (iii)⇒(i). We do this by modifying the proof of (ii)⇒(i). First we note that there exists a constant $M > 0$ such that

$$\|\mathbf{T} * f\|_{C_0(\mathbb{R}_+, X)} \le M \|f\|_{C_0(\mathbb{R}_+, X)} \qquad \forall f \in C_0(\mathbb{R}_+, X).$$

Indeed, this follows from applying the uniform boundedness theorem to the operators $T_s : f \mapsto \int_0^s T(t) f(s-t)\, dt$.

Let $f \in C_{00}(\mathbb{R}_+, X)$ arbitrary. Since $\mathbf{T} * f \in C_0(\mathbb{R}_+, X)$ by assumption and $(\mathbf{T} * f)(0) = 0$, it follows that $\mathbf{T}$ acts boundedly on $C_{00}(\mathbb{R}_+, X)$ by convolution, with norm at most M.

Let $C_{00}([0, T_0], X)$ be the closed subspace of $C_{00}(\mathbb{R}_+, X)$ consisting of all functions vanishing on $[T_0, \infty)$. For each $T_0 \ge 1$, define the piecewise linear function g on $\mathbb{R}_+$ by

$$g_{T_0}(t) = \begin{cases} 1, & 0 \le t \le T_0 - 1; \\ T_0 - t, & T_0 - 1 \le t \le T_0; \\ 0, & \text{else} \end{cases}$$

and the operator $\Pi_{T_0} : C_{00}(\mathbb{R}_+, X) \to C_{00}([0, T_0], X)$ by

$$(\Pi_{T_0} f)(t) = g_{T_0}(t) f(t), \qquad 0 \le t \le T_0.$$

Note that, for any function $f \in C_{00}(\mathbb{R}_+, X)$, $\|\Pi_{T_0} f\|_{C_{00}([0,T_0], X)} \le \|f\|_{C_{00}(\mathbb{R}_+, X)}$. With π_{T_0} replaced by Π_{T_0}, the proof of now proceeds along the lines of (ii)⇒(i). ////

The reason for introducing the operators Π_{T_0} is as follows. If we simply truncate a function in $C_{00}(\mathbb{R}_+, X)$ with π_{T_0}, the resulting function need not define an element of $C_{00}([0, T_0], X)$, so that we cannot perform the limiting operation (1.5). With the operator Π_{T_0}, this poses no problems.

In the next section, we will improve part of Theorem 1.3 by showing that $\omega_0(\mathbf{T}) < 0$ if (and only if) $\mathbf{T} * f$ is merely *bounded* for all $f \in C_0(\mathbb{R}_+, X)$.

2. Applications

The main result of this section is an application of Theorem 1.3 for the case of $C_0(\mathbb{R}_+, X)$. We need the following definitions in order to state the result. Let $BUC(\mathbb{R}_+, X)$ denote the space of all X-valued bounded uniformly continuous functions on $\mathbb{R}_+$. A linear subspace E of $BUC(\mathbb{R}_+, X)$ will be called *locally dense* in $BUC(\mathbb{R}_+, X)$ if for every $\epsilon > 0$, every bounded closed interval $I \subset \mathbb{R}_+$, and every $f \in C(I)$ there exists a function $f_{\epsilon,I} \in E$ such that

$$\sup_{t \in I} \|f(t) - f_{\epsilon,I}(t)\| \leq \epsilon.$$

If, in addition, there is a constant $K > 0$, independent of I and ϵ, such that for every $f \in C(I)$ the function $f_{\epsilon,I}$ can be chosen in such a way that $\|f_{\epsilon,I}\|_{BUC(\mathbb{R}_+,X)} \leq K\|f\|_{C(I)}$, we say that E is *boundedly locally dense* in $BUC(\mathbb{R}_+, X)$.

Theorem 2.1. *Let $\mathbf{T}$ be a C_0-semigroup a Banach space X and let E be a closed, boundedly locally dense subspace of $BUC(\mathbb{R}_+, X)$. Then the following assertions are equivalent:*

(i) $\omega_0(\mathbf{T}) < 0$;

(ii) $\sup_{s>0} \left\| \int_0^s T(t)g(t)\,dt \right\| < \infty$ *for all* $g \in E$.

Proof: The implication (i)⇒(ii) is trivial. We will prove (ii)⇒(i). By the uniform boundedness theorem, there is a constant $C > 0$ such that

$$\sup_{s>0} \left\| \int_0^s T(t)g(t)\,dt \right\| \leq C\|g\|_{BUC(\mathbb{R}_+,X)}, \qquad \forall g \in E. \tag{2.1}$$

For a given $f \in C_0(\mathbb{R}_+, X)$ and $s > 0$, let $M_s = \sup_{0 \leq t \leq s} \|T(t)\|$ and let $f_s \in E$ be any function such that

$$\sup_{0 \leq t \leq s} \|f(s-t) - f_s(t)\| \leq \frac{1}{sM_s}\|f\|_{C_0(\mathbb{R}_+,X)},$$

and

$$\|f_s\|_{BUC(\mathbb{R}_+,X)} \leq K\|f\|_{C_0(\mathbb{R}_+,X)}.$$

Such an f_s exists by the definition of a boundedly locally dense subspace; K is the constant from the definition. Then, by (2.1),

$$\begin{aligned}
\|(\mathbf{T} * f)(s)\| &= \left\| \int_0^s T(t)\,(f(s-t) - f_s(t) + f_s(t))\,dt \right\| \\
&\le \|f\|_{C_0(\mathbb{R}_+,X)} + \left\| \int_0^s T(t) f_s(t)\,dt \right\| \\
&\le \|f\|_{C_0(\mathbb{R}_+,X)} + C\|f_s\|_{BUC(\mathbb{R}_+,X)} \le (1+CK)\|f\|_{C_0(\mathbb{R}_+,X)}.
\end{aligned} \tag{2.2}$$

It follows that $\mathbf{T} * f$ is a bounded continuous function. If we can prove that

$$\lim_{s\to\infty} \int_0^s T(t) f(s-t)\,dt = 0,$$

it follows that convolution with $\mathbf{T}$ maps $C_0(\mathbb{R}_+,X)$ into $C_0(\mathbb{R}_+,X)$. Then we can apply Theorem 1.3 to obtain that $\omega_0(\mathbf{T}) < 0$.

Fix $\epsilon > 0$ arbitrary and choose N so large that $\|f(s)\| \le (1+CK)^{-1}\epsilon$ for all $s \ge N$. Write $f = f_0 + f_1$, where $f_0 \in C_0(\mathbb{R}_+,X)$ is chosen in such a way that $f_0(s) = f(s)$ for all $s \ge N$ and $\|f_0\|_{C_0(\mathbb{R}_+,X)} \le (1+CK)^{-1}\epsilon$. Note that the support of f_1 is contained in the interval $[0,N]$. By (2.2), for all $s \ge 0$ we have

$$\left\| \int_0^s T(t) f_0(s-t)\,dt \right\| \le (1+CK)\|f_0\|_{C_0(\mathbb{R}_+,X)} \le \epsilon.$$

It follows that it is sufficient to prove that

$$\lim_{s\to\infty} T(s-N) \left(\int_0^N T(t) f_1(N-t)\,dt \right) = \lim_{s\to\infty} \int_0^s T(t) f_1(s-t)\,dt = 0. \tag{2.3}$$

Since $\lim_{\lambda\to\infty} \lambda^2 R(\lambda,A)^2 f_1(\cdot) \to f_1(\cdot)$ uniformly on $[0,N]$, and hence on all of $\mathbb{R}_+$, (2.2) shows that it is even sufficient to prove that

$$\lim_{s\to\infty} T(s-N) \left(\int_0^N T(t) \lambda^2 R(\lambda,A)^2 f_1(N-t)\,dt \right) = 0 \tag{2.4}$$

for all λ sufficiently large. Note that, for each such λ,

$$\int_0^N T(t)\lambda^2 R(\lambda,A)^2 f_1(N-t)\,dt \in D(A^2). \tag{2.5}$$

In order to prove (2.4), we claim that the resolvent of A exists and is uniformly bounded in the right halfplane.

To prove this, fix $\mu > 0$, $g \in E$, and $s > 0$ arbitrary. Choose a function $g_{\mu,s} \in E$ such that

$$\sup_{0\le t\le s} \|g_{\mu,s}(t) - e^{-\mu t} g(t)\| \le \frac{1}{sM_s} \|g\|_{BUC(\mathbb{R}_+,X)}$$

and $\|g_{\mu,s}\|_{BUC(\mathbb{R}_+,X)} \le K\|g\|_{BUC(\mathbb{R}_+,X)}$. Then, by (2.1),

$$\begin{aligned}\left\|\int_0^s e^{-\mu t}T(t)g(t)\,dt\right\| &\le \|g\|_{BUC(\mathbb{R}_+,X)} + C\|g_{\mu,s}\|_{BUC(\mathbb{R}_+,X)} \\ &\le (1+CK)\|g\|_{BUC(\mathbb{R}_+,X)}.\end{aligned} \tag{2.6}$$

Next, let $\nu \in \mathbb{R}$ and $x \in X$ be arbitrary and fixed. Let $g_{\nu,x,s} \in E$ be a function such that

$$\sup_{0\le t\le s} \|g_{\nu,x,s}(t) - e^{-i\nu t}\otimes x\| \le \frac{1}{sM_s}\|x\|$$

and $\|g_{\nu,x,s}\|_{BUC(\mathbb{R}_+,X)} \le K\|x\|$. Then, by (2.6) applied to $g_{\nu,x,s}$,

$$\begin{aligned}\left\|\int_0^s e^{-(\mu+i\nu)t}T(t)x\,dt\right\| &\le \|x\| + \left\|\int_0^s e^{-\mu t}T(t)g_{\nu,x,s}(t)\,dt\right\| \\ &\le \|x\| + (1+CK)\|g_{\nu,x,s}\|_{BUC(\mathbb{R}_+,X)} \\ &\le (1+(1+CK)K)\|x\|.\end{aligned} \tag{2.7}$$

Since $s > 0$ and $\mu > 0, \nu > 0$ are arbitrary, (2.7) shows that the entire X-valued functions $z \mapsto \int_0^s e^{-zt}T(t)x\,dt$ are bounded in the right half plane, uniformly in $s > 0$. Moreover, for $\operatorname{Re} z$ sufficiently large, they converge to $R(z,A)x$ as $s \to \infty$. An application of Vitali's theorem and an analytic continuation argument show that the right half plane is contained in $\varrho(A)$ and that the resolvent of A is uniformly bounded there. This concludes the proof of the claim.

By a well-known theorem of Slemrod [Sl], the uniform boundedness of the resolvent in the right half plane implies that $\omega_2(\mathbf{T}) < 0$, i.e. there are constants $K > 0$ and $\mu > 0$ such that $\|T(t)y\| \le Ke^{-\mu t}\|y\|_{D(A^2)}$ for all $y \in D(A^2)$ and $t \ge 0$. Therefore, (2.4) is a consequence of (2.5). ////

Since $C_0(\mathbb{R}_+,X)$ is a closed, boundedly locally dense subspace of $BUC(\mathbb{R}_+,X)$, Theorem 2.1 implies:

Corollary 2.2. *Let $\mathbf{T}$ be a C_0-semigroup on a Banach space X. Then $\omega_0(\mathbf{T}) < 0$ if and only if*

$$\sup_{s>0}\left\|\int_0^s T(t)g(t)\,dt\right\| < \infty, \qquad \forall g \in C_0(\mathbb{R}_+,X).$$

As another application, we define $AP(\mathbb{R}_+,X)$ as the closure in $BUC(\mathbb{R}_+,X)$ of the set of all functions $\{e^{i\lambda(\cdot)}\otimes x :\ \lambda \in \mathbb{R},\ x \in X\}$. It is easy to see that $AP(\mathbb{R}_+,X)$ is boundedly locally dense in $BUC(\mathbb{R}_+,X)$. Indeed, if $I \subset \mathbb{R}_+$ is a bounded closed interval and $f \in C(I)$ is given, we choose N so large that $I \subset [0,N]$ and fix an arbitrary continuous function $f_N \in C([0,N+1])$ that coincides with f on I and satisfies $f(0) = f(N+1)$. Then we approximate f_N uniformly in $[0,N+1]$ by linear combinations of functions $e^{i\delta t}\otimes x$, $\delta \in \{2\pi k/(N+1) :\ k \in \mathbb{Z}\}$, $x \in X$. Since these functions are $N+1$-periodic, their sup-norms on $\mathbb{R}_+$ are the same as their sup-norms in $[0,N+1]$. Therefore, $AP(\mathbb{R}_+,X)$ is boundedly locally dense in $BUC(\mathbb{R}_+,X)$. Since $AP(\mathbb{R}_+,X)$ is also closed in $BUC(\mathbb{R}_+,X)$, we obtain:

Corollary 2.3. *Let* $\mathbf{T}$ *be a* C_0*-semigroup on a Banach space* X*. Then* $\omega_0(\mathbf{T}) < 0$ *if and only if*

$$\sup_{s>0} \left\| \int_0^s T(t)g(t)\,dt \right\| < \infty, \qquad \forall g \in AP(\mathbb{R}_+, X). \tag{2.8}$$

One should compare this theorem to the following result of [Ne]: if

$$\sup_{\lambda\in\mathbb{R}} \sup_{s>0} \left\| \int_0^s e^{i\lambda t} T(t)x\,dt \right\| < \infty, \tag{2.9}$$

then $\omega_1(\mathbf{T}) < 0$. Here, $\omega_1(\mathbf{T})$ denotes the growth bound of orbits originating from $D(A)$, i.e. the infimum of all $\omega \in \mathbb{R}$ for which there is an $M > 0$ such that $\|T(t)x\| \le Me^{\omega t}\|x\|_{D(A)}$ for all $x \in D(A)$. The difference between (2.8) and (2.9) is that in the latter we consider one function $e^{i\lambda(\cdot)} \otimes x \in AP(\mathbb{R}_+, X)$ at a time, whereas in (2.8) we consider *all* functions in $AP(\mathbb{R}_+, X)$. The supremum over $\lambda \in \mathbb{R}$ in (2.9) accounts for the fact that the sup-norms of $e^{i\lambda(\cdot)} \otimes x$ are uniform in λ.

We conclude this section with an improvement of Theorem 1.3 for the case $C_0(\mathbb{R}_+, X)$.

Theorem 2.4. *Let* $\mathbf{T}$ *be a* C_0*-semigroup a Banach space* X *and let* E *be a closed, boundedly locally dense subspace of* $BUC(\mathbb{R}_+, X)$*. Then the following assertions are equivalent:*
(i) $\omega_0(\mathbf{T}) < 0$;
(ii) $\mathbf{T} * f \in L^\infty(\mathbb{R}_+, X)$ *for all* $f \in E$.

Proof: The implication (i)⇒(ii) is trivial. We will prove (ii)⇒(i). By the uniform boundedness theorem applied to the operators $T_s : f \mapsto \int_0^s T(t)f(s-t)\,dt$, there is a constant $C > 0$ such that

$$\sup_{s>0} \left\| \int_0^s T(t)f(s-t)\,dt \right\| \le C\|f\|_{BUC(\mathbb{R}_+,X)}, \qquad \forall f \in E. \tag{2.10}$$

For a given $f \in E$ and $s > 0$, let $M_s = \sup_{0\le t\le s} \|T(t)\|$ and let $f_s \in E$ be any function such that

$$\sup_{0\le t\le s} \|f(s-t) - f_s(t)\| \le \frac{1}{sM_s}\|f\|_{BUC(\mathbb{R}_+,X)},$$

and

$$\|f_s\|_{BUC(\mathbb{R}_+,X)} \le K\|f\|_{BUC(\mathbb{R}_+,X)}.$$

Then, by (2.10),

$$\begin{aligned}\left\| \int_0^s T(t)f(t)\,dt \right\| &\le \|f\|_{BUC(\mathbb{R}_+,X)} + \left\| \int_0^s T(t)f_s(s-t)\,dt \right\| \\ &\le \|f\|_{BUC(\mathbb{R}_+,X)} + C\|f_s\|_{BUC(\mathbb{R}_+,X)} \le (1+CK)\|f\|_{BUC(\mathbb{R}_+,X)}.\end{aligned}$$

Since $s > 0$ is arbitrary, $\omega_0(\mathbf{T}) < 0$ by Theorem 2.1. ////

3. Generalization to Banach function spaces

Let E be a rearrangement invariant Banach function space over $\mathbb{R}_+$. We will adopt the terminology of the book [BS]. Although the definition of a Banach function space given there is very restrictive, it is not difficult to show that in the rearrangement invariant case the assumptions (P4) and (P5) of [BS], Def. I.1.1, are redundant provided one assumes that E is *carried* by $\mathbb{R}_+$. This means that for each measurable $H \subset \mathbb{R}_+$ of positive measure there exists a function $f \in E$ that is not identically zero a.e. on H.

The *fundamental function* is the function φ_E defined by

$$\varphi_E(t) := \|\chi_H\|_E,$$

where H is a subset of measure t. One can show that for every $t \geq 0$, such a set exists. By the rearrangement invariance, the function φ_E is well-defined.

By [BS, Lemma III.6.3], the right translation semigroup is strongly continuous on E if and only if $\varphi_E(0+) = 0$, provided the simple functions are dense in E. This is the case if E has order continuous norm, which in turn is the case if E is separable (this follows from [BS, Chapters I.4 and I.5]). Examples of rearrangement invariant Banach function spaces with order continuous norm satisfying $\varphi_E(0+) = 0$ are the spaces $L^p(\mathbb{R}_+)$, $1 \leq p < \infty$, and all reflexive spaces rearrangement invariant Banach function spaces.

If $E = E(\mathbb{R}_+)$ is a rearrangement invariant Banach function space over $\mathbb{R}_+$ with order continuous norm satisfying $\varphi_E(0+) = 0$, the operators $S_E(t)$ defined by

$$S_E(t)f(s) := \begin{cases} T(t)f(s-t), & s-t \geq 0; \\ 0, & \text{else}, \end{cases}$$

define a C_0-semigroup $\mathbf{S}_E$ on $E(\mathbb{R}_+, X)$. It generator will be denoted by B_E. Here, $E(\mathbb{R}_+, X)$ is the space of all Bochner measurable functions $f : \mathbb{R}_+ \to X$ such that $\|f(\cdot)\| \in E(\mathbb{R}_+)$. This space is a Banach space, as can be seen by a modification of the proof of the completeness of L^p.

It is not hard to see that the proofs of Lemmas 1.1 and 1.2 carry over almost verbatim to $\mathbf{S}_E$. One has to distiguish between the cases that $\lim_{t\to\infty} \varphi_E(t)$ is infinite or finite. In the first case, one argues as for $L^p(\mathbb{R}_+)$ and in the second case as for $C_0(\mathbb{R}_+)$.

Summarizing, we have:

Lemma 3.1. *Let $\mathbf{T}$ be a C_0-semigroup on X and let $E(\mathbb{R}_+)$ be a rearrangement invariant Banach function space with order continuous norm satisfying $\varphi_E(0+) = 0$. Then, for the semigroup $\mathbf{S}_E$ we have $s(B_E) = \omega_0(\mathbf{S}_E) = \omega_0(\mathbf{T})$.*

Theorem 3.2. *Let $\mathbf{T}$ be a C_0-semigroup on X and let $E(\mathbb{R}_+)$ be a rearrangement invariant Banach function space with order continuous norm satisfying $\varphi_E(0+) = 0$. Let $\mathbf{T}$ be a C_0-semigroup on a Banach space X. Then the following assertions are equivalent:*

(i) $\omega_0(\mathbf{T}) < 0$;

(ii) $\mathbf{T} * f \in E(\mathbb{R}_+, X)$ *for all* $f \in E(\mathbb{R}_+, X)$.

Proof: (ii)⇒(i): Armed with Lemma 3.1, we can copy the proof of (ii)⇒(i) of Theorem 1.3 almost verbatim. In order to establish the boundedness of convolution with $\mathbf{T}$ as a map of $E(\mathbb{R}_+, X)$ to itself, we now use the fact that a norm convergent sequence in $E(\mathbb{R}_+, X)$ has a subsequence that converges pointwise a.e. This in turn follows from the fact that $f_n \to f$ in $E(\mathbb{R}_+, X)$ implies that $f_n|_{[0.k]} \to f|_{[0,k]}$ in the norm of $L^1[0,k]$ for all $k = 1, 2, \ldots$ [BS, Cor. II.6.7] and a diagonal argument.

The proof that (i) implies (ii) proceeds as follows. For $\mu > 0$ and $g \in L^1_{loc}(\mathbb{R}_+)$, define

$$(T_\mu(g))(s) := \int_0^s e^{-\mu t} g(s-t)\, dt, \qquad s \geq 0.$$

The proof of Theorem 1.3, (i)⇒(ii), shows that this defines a bounded operator $T_\mu : L^1(\mathbb{R}_+) \to L^1(\mathbb{R}_+)$ of norm $\leq \mu^{-1}$. Also, it is trivial that T_μ is bounded as an operator $L^\infty(\mathbb{R}_+) \to L^\infty(\mathbb{R}_+)$, of norm $\leq \mu^{-1}$. By a well-known theorem of Calderón [C] (see also [BS, Thm. III.2.12]), every rearrangement invariant Banach function space over $\mathbb{R}_+$ is an exact interpolation space between $L^1(\mathbb{R}_+)$ and $L^\infty(\mathbb{R}_+)$. Therefore, T_μ is bounded as an operator $E(\mathbb{R}_+) \to E(\mathbb{R}_+)$, of norm $\leq \mu^{-1}$.

Let $f \in E(\mathbb{R}_+, X)$ be arbitrary. Since $\omega_0(\mathbf{T}) < 0$, there are constants $M > 0$ and $\mu > 0$ such that $\|T(t)\| \leq Me^{-\mu t}$ for all $t \geq 0$. Since $\|f(\cdot)\| \in E(\mathbb{R}_+)$, we have

$$\begin{aligned}
\|\mathbf{T} * f\|_{E(\mathbb{R}_+, X)} &= \Big\| \|(\mathbf{T} * f)(\cdot)\| \Big\|_{E(\mathbb{R}_+)} \leq \left\| \int_0^{(\cdot)} Me^{-\mu t} \|f(\cdot - t)\|\, dt \right\|_{E(\mathbb{R}_+)} \\
&= M \Big\| T_\mu(\|f\|)(\cdot) \Big\|_{E(\mathbb{R}_+)} \leq M\mu^{-1} \Big\| \|f(\cdot)\| \Big\|_{E(\mathbb{R}_+)} \\
&= M\mu^{-1} \|f\|_{E(\mathbb{R}_+, X)}.
\end{aligned}$$

////

4. The weak case

In this section, we study the weak analogue of Theorem 1.3. If $E(\mathbb{R}_+)$ is a given function space over $\mathbb{R}_+$, we want to characterize those semigroups $\mathbf{T}$ on X for which $\langle x^*, \mathbf{T} * f\rangle$ defines an element of $E(\mathbb{R}_+)$ for all $x^* \in X^*$ and $f \in E(\mathbb{R}_+, X)$. Here, and in the following, for a $g \in L^1_{loc}(\mathbb{R}_+, X)$ and a functional $x^* \in X^*$, the function $\langle x^*, g\rangle \in L^1_{loc}(\mathbb{R}_+)$ is defined in the natural way: $\langle x^*, g\rangle(s) = \langle x^*, g(s)\rangle$; $s \geq 0$.

For $E = L^1$ we solve this problem as follows. A semigroup $\mathbf{T}$ is said to be *weakly* L^1 if

$$\int_0^\infty |\langle x^*, T(t)x\rangle|\, dt < \infty, \qquad \forall x \in X, x^* \in X^*.$$

Theorem 4.1. *Let $\mathbf{T}$ be a C_0-semigroup on a Banach space X. Then the following assertions are equivalent:*
(i) $\mathbf{T}$ is weakly L^1;
(ii) $\langle x^*, \mathbf{T} * f\rangle \in L^1(\mathbb{R}_+)$ for all $f \in L^1(\mathbb{R}_+, X)$ and $x^* \in X^*$.

Proof: Assume (ii). Let $S : Y_0 \times Y_1 \to Z$ be a separately continuous bilinear map. For $y_0 \in Y_0$ define $S_0 : Y_1 \to Z$, $S_{y_0}(y_1) := S(y_0, y_1)$. Then each S_{y_0} is bounded by the continuity in the Y_1-variable. Using the continuity in the Y_0-variable, it is easy to see that the map $y_0 \mapsto S_{y_0}$ is closed, and hence bounded by the closed graph theorem. It follows that there is an $M > 0$ such that

$$\|S(y_0, y_1)\| \le \|S_{y_0}\| \, \|y_1\| \le M \|y_0\| \, \|y_1\|.$$

Applying this to the separately continuous bilinear map $T : X^* \times L^1(\mathbb{R}_+, X) \to L^1(\mathbb{R}_+)$ defined by $T(x^*, f) = \langle x^*, \mathbf{T} * f \rangle$, it follows that there exists an $M > 0$ such that

$$\|\langle x^*, \mathbf{T} * f \rangle\|_{L^1(\mathbb{R}_+)} \le M \|f\|_{L^1(\mathbb{R}_+, X)} \|x^*\|, \qquad \forall f \in L^1(\mathbb{R}_+, X), x^* \in X^*.$$

Fix $x \in X$, $x^* \in X^*$ and $s_0 > 1$ arbitrary. Choose $0 < \tau_0 < 1$ such that

$$\frac{1}{\tau_0} \left| \int_0^{\tau_0} \langle x^*, T(s-t)x \rangle \, dt \right| \ge \frac{1}{2} |\langle x^*, T(s)x \rangle|, \qquad \forall 1 \le s \le s_0.$$

Then,

$$\begin{aligned}
\int_1^{s_0} |\langle x^*, T(s)x \rangle| \, ds &\le 2 \int_1^{s_0} \frac{1}{\tau_0} \left| \int_0^{\tau_0} \langle x^*, T(s-t)x \rangle \, dt \right| \, ds \\
&= 2 \int_1^{s_0} \frac{1}{\tau_0} \left| \int_0^{s} \langle x^*, T(t)x \rangle \, \chi_{[0,\tau_0]}(s-t) \, dt \right| \, ds \\
&\le \frac{2}{\tau_0} \|\langle x^*, \mathbf{T} * (\chi_{[0,\tau_0]} \otimes x) \rangle\|_{L^1(\mathbb{R}_+)} \\
&\le \frac{2M}{\tau_0} \|\chi_{[0,\tau_0]} \otimes x\|_{L^1(\mathbb{R}_+, X)} \|x^*\| = 2M \|x\| \, \|x^*\|.
\end{aligned}$$

Since $s_0 > 1$ is arbitrary, it follows that

$$\int_1^{\infty} |\langle x^*, T(s)x \rangle| \, ds \le 2M \|x\| \, \|x^*\|, \qquad \forall x \in X, x^* \in X^*.$$

Therefore, $\int_0^\infty |\langle x^*, T(s)x \rangle| \, ds < \infty$ for all $x \in X$ and $x^* \in X^*$, which proves (i).

Now assume (i). As is well-known and easy to see, there exists a constant C such that

$$\int_0^{\infty} |\langle x^*, T(t)x \rangle| \, dt \le C \|x\| \, \|x^*\|, \qquad \forall x \in X, x^* \in X^*.$$

Let $N := \sup_{0 \le s \le 1} \|T(s)\|$. Fix $x \in X$, $x^* \in X^*$ and real numbers $0 \le t_0 < t_1$

with $t_1 - t_0 \le 1$. Then,

$$
\begin{aligned}
&\int_0^\infty \left| \left\langle x^*, \int_0^s T(\tau)(\chi_{[t_0,t_1]} \otimes x)(s-\tau)\, d\tau \right\rangle \right| ds \\
&\quad = \int_0^\infty \left| \left\langle x^*, \int_{\max\{s-t_1,0\}}^{\max\{s-t_0,0\}} T(\tau)x\, d\tau \right\rangle \right| ds \\
&\quad \le \int_{t_0}^{t_1} \left| \left\langle x^*, \int_0^{s-t_0} T(\tau)x\, d\tau \right\rangle \right| ds + \int_{t_1}^\infty \left| \left\langle x^*, \int_{s-t_1}^{s-t_0} T(\tau)x\, d\tau \right\rangle \right| ds \\
&\quad \le \int_{t_0}^{t_1} (s-t_0)N\|x\|\,\|x^*\|\, ds + \int_0^\infty \left| \left\langle x^*, T(s) \int_0^{t_1-t_0} T(\tau)x\, d\tau \right\rangle \right| ds \\
&\quad \le (t_1-t_0)N\|x\|\,\|x^*\| + C \left\| \int_0^{t_1-t_0} T(\tau)x\, d\tau \right\| \|x^*\| \\
&\quad \le (t_1-t_0)N(1+C)\|x\|\,\|x^*\|.
\end{aligned}
$$

Therefore, with $M = N(1+C)$, we have

$$\|\langle x^*, \mathbf{T} * (\chi_{[t_0,t_1]} \otimes x)\rangle\|_{L^1(\mathbb{R})_+} \le M(t_1-t_0)\|x\|\,\|x^*\|.$$

Next, let f be a stepfunction of the form $f = \sum_{k=0}^{n-1} \chi_{[t_k,t_{k+1}]} \otimes x_k$. By splitting large intervals into finitely many smaller ones, we may assume that $0 < t_{k+1} - t_k \le 1$ for all $k = 0, ..., n-1$. By the above estimate we have

$$
\begin{aligned}
\|\langle x^*, \mathbf{T} * f\rangle\|_{L^1(\mathbb{R}_+)} &\le \sum_{k=0}^{n-1} \|\langle x^*, \mathbf{T} * (\chi_{[t_k,t_{k+1}]} \otimes x_k)\rangle\|_{L^1(\mathbb{R}_+)} \\
&\le M \sum_{k=0}^{n-1} (t_{k+1}-t_k)\|x_k\|\,\|x^*\| = M\|f\|_{L^1(\mathbb{R}_+,X)}\|x^*\|.
\end{aligned}
\tag{4.1}
$$

Since the stepfunctions supported by finite unions of intervals of length ≤ 1 are dense in $L^1(\mathbb{R}_+, X)$, (4.1) holds for arbitrary $f \in L^1(\mathbb{R}_+, X)$. This proves the implication (i)$\Rightarrow$ (ii). ////

Since there exist weakly L^1 semigroups with positive growth bound, the theorem shows that condition (ii) does not characterize exponential stability. In Hilbert space however, a C_0-semigroup $\mathbf{T}$ is weakly L^1 if and only if $\omega_0(\mathbf{T}) < 0$ [HK], [We]. In combination with Theorem 4.1, this leads to the following result.

Corollary 4.2. *Let $\mathbf{T}$ be a C_0-semigroup on a Hilbert space H. Then the following assertions are equivalent:*

(i) $\omega_0(\mathbf{T}) < 0$;

(ii) $\langle y, \mathbf{T} * f\rangle \in L^1(\mathbb{R}_+)$ *for all* $f \in L^1(\mathbb{R}_+, H)$ *and* $y \in H$.

A positive C_0-semigroup on a Banach lattice X is weakly L^1 if and only if $s(A) < 0$. This is more or less folklore; a proof can be found in [NSW].

Corollary 4.3. *Let $\mathbf{T}$ be a positive C_0-semigroup with generator A on a Banach lattice X. Then the following assertions are equivalent:*

(i) $s(A) < 0$*;*

(ii) $\langle x^*, \mathbf{T} * f\rangle \in L^1(\mathbb{R}_+)$ *for all* $f \in L^1(\mathbb{R}_+, X)$ *and* $x^* \in X$.

5. References

[BS] C. Bennett, R. Sharpley, *Interpolation of Operators*, Pure and Applied Mathematics 129, Academic Press (1988).

[C] A.P. Calderón, Spaces between L^1 and L^∞ and the theorem of Marcinkiewicz, Studia Math. 26, 273-299 (1966).

[HPh] E. Hille and R.S. Phillips, *Functional Analysis and Semi-Groups*, Coll. Publ. Amer. Math. Soc. XXXI, Providence, R.I. (1957).

[Hu] Huang Falun, Characteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces, Ann. Diff. Eq. 1, 43-56 (1985).

[HK] Huang Falun and Liu Kangsheng, A problem of exponential stability for linear dynamical systems in Hilbert spaces, *Kexue Tongbao* 33, 460-462 (1988).

[LM] Y. Latushkin and S. Montgomery-Smith, Lyapunov theorems in Banach spaces, Bull. Am. Math. Soc 31, 44-49 (1994).

[NSW] J.M.A.M.van Neerven, B. Straub, and L. Weis, On the asymptotic behaviour of a semigroup of linear operators, submitted.

[Ne] J.M.A.M. van Neerven, Individual stability of C_0-semigroups with uniformly bounded local resolvent, submitted.

[P] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, Berlin (1983).

[RS1] F. Räbiger and R. Schnaubelt, A spectral mapping characterization of exponentially dichotomic and hyperbolic evolution families, submitted; as preprint in: Tübinger Berichte zur Funktionalanalysis 3, Tübingen (1994).

[RS2] F. Räbiger and R. Schnaubelt, The spectral mapping theorem for evolution semigroups on spaces of vector-valued functions, submitted; as preprint in: Tübinger Berichte zur Funktionalanalysis 3, Tübingen (1994).

[Ra1] R.T. Rau, *Hyperbolic Evolution Semigroups*, Ph.D. Dissertation, Universität Tübingen (1992).

[Ra2] R.T. Rau, Hyperbolic evolution semigroups on vector-valued function spaces, Semigroup Forum 48, 107-118 (1994).

[Ra3] R.T. Rau, Hyperbolic evolution groups and dichotomic evolution families, J. Dyn. Diff. Eq. 6, 335-350 (1994).

[R] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, 3rd Ed. (1986).

[Sl] M. Slemrod, Asymptotic behavior of C_0-semigroups as determined by the spectrum of the generator, Indiana Univ. Math. J. 25, 783-792 (1976).

[W] L. Weis, The stability of positive semigroups on L_p spaces, to appear in: Proc. Am. Math. Soc.

[We] G. Weiss, Weak L^p-stability of a linear semigroup on a Hilbert space implies exponential stability, *J. Diff. Eq.* 76, 269-285 (1988).