A LIE–TROTTER PRODUCT FORMULA FOR ORNSTEIN–UHLENBECK SEMIGROUPS IN INFINITE DIMENSIONS

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Abstract. We prove a Lie–Trotter product formula for the Ornstein–Uhlenbeck semigroup associated with the stochastic linear Cauchy problem

$$\begin{aligned} dX(t) &= AX(t)\,dt + dW(t), \quad t \geqslant 0, \\ X(0) &= x_0. \end{aligned}$$

Here A is the generator of a C_0-semigroup on a separable real Banach space E and $\{W(t)\}_{t\geqslant 0}$ is an E-valued Brownian motion.

1. Introduction

In this paper we prove a Lie–Trotter product formula for the Ornstein–Uhlenbeck semigroup associated with the stochastic linear Cauchy problem

$$\begin{aligned} dX(t) &= AX(t)\,dt + dW(t), \quad t \geqslant 0, \\ X(0) &= x_0, \end{aligned} \tag{1.1}$$

where A is the generator of a C_0-semigroup $\{S(t)\}_{t\geqslant 0}$ on a separable real Banach space E and $\{W(t)\}_{t\geqslant 0}$ is an E-valued Brownian motion. A predictable E-valued process $\{X(t,x_0)\}_{t\geqslant 0}$ is called a *weak solution* of (1.1) if for all $x^* \in \mathscr{D}(A^*)$ the process $\{\langle X(t,x_0), A^*x^*\rangle\}_{t\geqslant 0}$ is locally integrable almost surely and for all $t \geqslant 0$ we have, almost surely,

$$\langle X(t,x_0), x^*\rangle = \langle x_0, x^*\rangle + \int_0^t \langle X(s,x_0), A^*x^*\rangle\,ds + \langle W(t), x^*\rangle.$$

It is well-known [4] that (1.1) has a unique weak solution $\{X(t,x_0)\}_{t\geqslant 0}$ for some (hence, for all) $x_0 \in E$ if and only if for all $t \geqslant 0$ the operator $Q_t \in \mathscr{L}(E^*, E)$ defined by

$$Q_t x^* := \int_0^t S(s)QS^*(s)x^*\,ds, \qquad x^* \in E^*, \tag{1.2}$$

is the covariance operator of a centred Gaussian measure on E; here $Q \in \mathscr{L}(E^*, E)$ is the covariance operator of the random variable $W(1)$. We then may define a one-parameter semigroup $\{\mathscr{P}(t)\}_{t\geqslant 0}$ of linear contractions on $C_b(E)$, the space of all bounded continuous real-valued functions on E, by

$$\mathscr{P}(t)f(x) := \mathbb{E}(f(X(t,x))), \qquad t \geqslant 0,\ x \in E.$$

This semigroup is usually referred to as the *transition semigroup* or the *Ornstein–Uhlenbeck semigroup* associated with equation (1.1). The random variables $X(t,x)$

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are Gaussian with mean $S(t)x$ and covariance Q_t. Denoting by μ_t the centred Gaussian measure with covariance Q_t, we have

$$\mathscr{P}(t)f(x) = \int_E f(S(t)x + y)\, d\mu_t(x), \qquad t \geqslant 0,\ x \in E.$$

In general, $\{\mathscr{P}(t)\}_{t\geqslant 0}$ fails to be strongly continuous in the supremum norm of $C_b(E)$. In this paper we consider $C_b(E)$ in its topology τ_c of uniform convergence on compact sets. In this topology, the Ornstein–Uhlenbeck semigroup is known to be strongly continuous [9]. We shall prove that under suitable conditions the following Lie–Trotter product formula holds:

$$\mathscr{P}(t)f = \tau_c\!-\lim_{m\to\infty} \big[\mathscr{T}(\tfrac{t}{m})\mathscr{S}(\tfrac{t}{m})\big]^m f$$

for all $f \in C_b(E)$, the convergence being uniform on every bounded time interval $[0, T]$. In this formula, $\{\mathscr{S}(t)\}_{t\geqslant 0}$ and $\{\mathscr{T}(t)\}_{t\geqslant 0}$ are the semigroups on $C_b(E)$ corresponding to the drift term and the diffusion term in (1.1). Thus,

$$\begin{aligned} \mathscr{S}(t)f(x) &= f(S(t)x), \\ \mathscr{T}(t)f(x) &= \int_E f(x + y)\, d\nu_t(y), \end{aligned} \qquad t \geqslant 0,\ x \in E,$$

where ν_t is distribution of the random variable $W(t)$.

2. Preliminaries

In this preliminary section we recall some well-known facts about Gaussian measures and reproducing kernel Hilbert spaces. For more details we refer to the books [2, 18].

2.1. **Gaussian measures.** Let E be a separable real Banach space. A *Gaussian measure* on E is a Borel probability measure μ on E with the property that for all $x^* \in E^*$ the image measure $\langle \mu, x^*\rangle := \mu \circ x^{*-1}$ is Gaussian on $\mathbb{R}$. The *mean* of μ is defined by

$$m := \int_E x\, d\mu(x);$$

this integral can be shown to converge absolutely in E. We call μ *centred* if $m = 0$. If μ is a Gaussian measure on E with mean m, there exists a unique compact operator $Q \in \mathscr{L}(E^*, E)$, the *covariance operator* of μ, with the property that

$$\langle Qx^*, y^*\rangle = \int_E \langle x - m, x^*\rangle\langle x - m, y^*\rangle\, d\mu(x), \qquad x^*, y^* \in E^*. \tag{2.1}$$

In terms of m and Q, the Fourier transform of μ is given by

$$\int_E \exp\big(-i\langle x, x^*\rangle\big)\, d\mu(x) = \exp\big(-i\langle m, x^*\rangle - \tfrac{1}{2}\langle Qx^*, x^*\rangle\big), \qquad x^* \in E^*. \tag{2.2}$$

Hence as a Gaussian measure, μ is determined uniquely by m and Q. Sometimes we shall use the notation $N(m, Q)$ to denote the Gaussian measure with mean m and covariance Q.

If $\{W(t)\}$ is an E-valued Brownian motion, then the distribution of the random variable $W(1)$ is a centred Gaussian measure on E. Denoting its covariance operator by Q, for all $s, t \geqslant 0$ and $x^*, y^* \in E^*$ we have

$$\mathbb{E}(\langle W(s), x^*\rangle\, \langle W(t), y^*\rangle) = (s \wedge t)\, \langle Qx^*, y^*\rangle, \qquad x^*, y^* \in E^*. \tag{2.3}$$

Conversely, if Q is the covariance operator of a Gaussian measure on E, there exist E-valued Brownian motions whose covariance is given by (2.3) [2, Proposition 7.2.3].

Let $\mathscr{M}(E)$ denote the set of all Borel probability measures on E. Every $\mu \in \mathscr{M}(E)$ determines a positive linear functional on $C_b(E)$ in a canonical way. The induced weak*-topology on $\mathscr{M}(E)$ is usually referred to as the *weak topology* of $\mathscr{M}(E)$.

Every measure $\mu \in \mathscr{M}(E)$ is *tight*, i.e., for every $\varepsilon > 0$ there exists a compact subset $K \subseteq E$ such that $\mu(K) \geqslant 1-\varepsilon$. A family $\mathscr{M} \subseteq \mathscr{M}(E)$ is said to be *tight* if for every $\varepsilon > 0$ there exists a compact subset $K \subseteq E$ such that $\mu(K) \geqslant 1-\varepsilon$ for all $\mu \in \mathscr{M}$. By Prohorov's theorem [2, Theorem 3.8.4], the family $\mathscr{M}$ is tight if and only if it is relatively compact with respect to the weak topology.

The covariance operator Q of a Gaussian measure on E is always *positive*, i.e.,

$$\langle Qx^*, x^*\rangle \geqslant 0 \quad \text{for all } x^* \in E^*$$

and *symmetric*, i.e.,

$$\langle Qx^*, y^*\rangle = \langle Qy^*, x^*\rangle \quad \text{for all } x^*, y^* \in E^*.$$

The converse does not hold: not every positive symmetric operator $Q \in \mathscr{L}(E^*, E)$ is the covariance operator of some Gaussian measure. In this connection the following result, which is a special case of [2, Theorem 3.3.6], will be useful:

Proposition 2.1. *Let $R \in \mathscr{L}(E^*, E)$ be the covariance operator of a Gaussian measure on E. Let $\mathscr{Q} \subseteq \mathscr{L}(E^*, E)$ be a family of positive symmetric operators. If there exists a constant $C \geqslant 0$ such that*

$$\langle Qx^*, x^*\rangle \leqslant C\langle Rx^*, x^*\rangle$$

for all $x^* \in E^*$ and $Q \in \mathscr{Q}$, then every $Q \in \mathscr{Q}$ is the covariance of a centred Gaussian measure μ_Q on E. Moreover, the family $\{\mu_Q : Q \in \mathscr{Q}\}$ is tight.

The following result is concerned with weak convergence of sequences of Gaussian measures [2, Theorem 3.8.9].

Proposition 2.2. *Let (m_n) be a sequence in E and (Q_n) a sequence of covariance operators in $\mathscr{L}(E^*, E)$. For each n, put $\nu_n := N(0, Q_n)$ and $\mu_n := N(m_n, Q_n)$. Let further an element $m \in E$ and a covariance operator $Q \in \mathscr{L}(E^*, E)$ be given, and put $\nu := N(0, Q)$ and $\mu := N(m, Q)$. Then the following assertions are equivalent.*

1. $\lim_{n\to\infty} \mu_n = \mu$ *weakly.*
2. $\lim_{n\to\infty} m_n = m$ *strongly and* $\lim_{n\to\infty} \nu_n = \nu$ *weakly.*

In this situation, for all $x^*, y^* \in E^*$ we have

$$\lim_{n\to\infty} \langle Q_n x^*, y^*\rangle = \langle Qx^*, y^*\rangle. \tag{2.4}$$

Let us now assume that E is a separable real *Hilbert* space with inner product $[\,\cdot\,,\cdot\,]_E$. Identifying E^* with E in the canonical way, positive symmetric operators from E^* into E can be identified with positive selfadjoint operators on E. Under this identification, Such an operator Q is the covariance of a centred Gaussian measure μ on E if and only if it is a trace class operator. Moreover, if $\{e_k\}_{k=1}^\infty$ is an orthonormal basis for E, then by (2.1) we have

$$\operatorname{tr} Q = \sum_{k=1}^\infty [Qe_k, e_k]_E = \int_E \sum_{k=1}^\infty [x, e_k]_E^2 \, d\mu(x) = \int_E \|x\|^2 \, d\mu(x). \tag{2.5}$$

Proposition 2.3. *Let E be a separable real Banach space and let (μ_n) be a sequence of centred Gaussian measures on E with covariances (Q_n). Let μ be a centred Gaussian measure on E with covariance Q.*

(i) *If $\lim_{n\to\infty} \mu_n = \mu$ weakly, then:*
   (1) $\lim_{n\to\infty}[Q_n x, y]_E = [Qx, y]_E$ *for all* $x, y \in E$,
   (2) $\displaystyle\lim_{n\to\infty} \int_E \|x\|^2 \, d\mu_n(x) = \int_E \|x\|^2 \, d\mu(x)$.

(ii) *If E is a separable real Hilbert space, then conversely the conditions* (1) *and* (2) *imply that $\lim_{n\to\infty} \mu_n = \mu$ weakly.*

In (i), (1) follows by considering Fourier transforms and (2) is a special case of [2, Lemma 3.8.7]. In a formulation where (1) is replaced by a slightly stronger hypothesis, the converse assertion (ii) is proved in [2, Example 3.8.15]. For the convenience of the reader we include a proof of (ii) in its present formulation in the Appendix.

2.2. **Reproducing kernel Hilbert spaces.** We return to the situation where E is a separable real Banach space. Let $Q \in \mathscr{L}(E^*, E)$ be an arbitrary positive symmetric operator. The mapping

$$(Qx^*, Qy^*) \mapsto \langle Qx^*, y^* \rangle, \qquad x^*, y^* \in E^*,$$

defines an inner product on the range of Q. The completion of range$\,(Q)$ with respect to this inner product is a separable real Hilbert space H_Q, the *reproducing kernel Hilbert space* (RKHS) associated with Q. The inclusion mapping from range$\,(Q)$ into E extends to a continuous inclusion mapping $i_Q : H_Q \hookrightarrow E$. We have the operator identity

$$Q = i_Q \circ i_Q^*. \tag{2.6}$$

The following simple observation will be useful in the next section.

Lemma 2.4. *Let $Q, R \in \mathscr{L}(E^*, E)$ be positive and symmetric operators and assume that $H_Q \subseteq H_R$ as subsets of E. Then the inclusion mapping $i_{Q,R}$ from H_Q into H_R is bounded, and for all $x^* \in E^*$ we have*

$$\langle Qx^*, x^* \rangle \leqslant \|i_{Q,R}\|^2 \langle Rx^*, x^* \rangle.$$

Proof. First we claim that $i_{Q,R}$ is closed. Indeed, suppose that $h_n \to h$ in H_Q and $i_{Q,R} h_n \to \tilde{h}$ in H_R, Then $i_Q h_n \to i_Q h$ in E and also $i_Q h_n = i_R i_{Q,R} h_n \to i_R \tilde{h}$ in E. Hence $i_Q h = i_R \tilde{h}$ in E. But also, $i_Q h = i_R i_{Q,R} h$, and therefore $\tilde{h} = i_{Q,R} h$, by the injectivity of i_R. This proves the claim. Boundedness of $i_{Q,R}$ is now an immediate consequence of the closed graph theorem.

Next, for all $x^* \in E^*$ and all $h \in H_Q$ we have

$$\begin{aligned} |[h, i_Q^* x^*]_{H_Q}| &= |\langle i_Q h, x^* \rangle| = |\langle i_R i_{Q,R} h, x^* \rangle| = |[i_{Q,R} h, i_R^* x^*]_{H_R}| \\ &\leqslant \|i_{Q,R} h\|_{H_R} \|i_R^* x^*\|_{H_R} \leqslant \|i_{Q,R}\| \, \|h\|_{H_Q} \|i_R^* x^*\|_{H_R}. \end{aligned}$$

Taking the supremum over all $h \in H_Q$ with $\|h\|_{H_Q} \leqslant 1$ we obtain $\|i_Q^* x^*\|_{H_Q} \leqslant \|i_{Q,R}\| \, \|i_R^* x^*\|_{H_R}$, and hence

$$\langle Qx^*, x^* \rangle = \|i_Q^* x^*\|_{H_Q}^2 \leqslant \|i_{Q,R}\|^2 \, \|i_R^* x^*\|_{H_R}^2 = \|i_{Q,R}\|^2 \langle Rx^*, x^* \rangle.$$

$\square$

If E is a separable real Hilbert space and Q is a positive selfadjoint operator on E, then the RKHS associated with Q equals range $(Q^{\frac{1}{2}})$ with inner product

$$\big[Q^{\frac{1}{2}}x, Q^{\frac{1}{2}}y\big]_{H_Q} = [x,y]_E, \qquad x,y \in E.$$

3. The Lie–Trotter product formula

For the rest of the paper we will make the following standing assumption.

Assumption 3.1.

(1) *$\{S(t)\}_{t\geqslant 0}$ is a C_0-semigroup on a separable real Banach space E.*
(2) *$Q \in \mathscr{L}(E^*, E)$ is the covariance of a centred Gaussian measure ν on E.*
(3) *For all $t \geqslant 0$, the operator $Q_t \in \mathscr{L}(E^*, E)$ defined by*

$$Q_t x^* := \int_0^t S(s)QS^*(s)x^*\,ds, \qquad x^* \in E^*,$$

is the covariance of a centred Gaussian measure μ_t on E.

In the following situations, (3) automatically follows from (1) and (2).

- If $\{S(t)\}_{t\geqslant 0}$ restricts to a C_0-semigroup on the RKHS H_Q associated with Q. This is an easy consequence of Proposition 2.1; cf. [12].
- If E has type 2 (in particular, if E is a Hilbert space) [14]. For the special case of M-type 2 spaces a more general result was proved by Brzeźniak [3, Section 2].

Let us pause to make a number of simple observations. First, by the positivity of Q, for all $0 \leqslant s \leqslant t$ and $x^* \in E^*$ we have

$$\begin{aligned} 0 \leqslant \langle Q_s x^*, x^*\rangle &= \int_0^s \langle QS^*(\sigma)x^*, S^*(\sigma)x^*\rangle\,d\sigma \\ &\leqslant \int_0^t \langle QS^*(\sigma)x^*, S^*(\sigma)x^*\rangle\,d\sigma = \langle Q_t x^*, x^*\rangle. \end{aligned} \tag{3.1}$$

Next, for all $s,t \geqslant 0$ and $x^* \in E^*$ we have

$$Q_{t+s}x^* = Q_t x^* + S(t)Q_s S^*(t)x^*$$

and therefore

$$\mu_{t+s} = \mu_t * S(t)\mu_s, \tag{3.2}$$

where the $*$ denotes convolution and $S(t)\mu_s := \mu_s \circ S(t)^{-1}$ denotes the image measure.

We define linear contractions $\mathscr{P}(t)$ on $C_b(E)$ by

$$\mathscr{P}(t)f(x) := \int_E f(S(t)x + y)\,d\mu_t(y), \qquad x \in E, \quad t \geqslant 0. \tag{3.3}$$

It is an easy consequence of (3.2) that the family $\{\mathscr{P}(t)\}_{t\geqslant 0}$ is a semigroup on $C_b(E)$. In general, this semigroup fails to be strongly continuous in the supremum norm, even on the closed invariant subspace $BUC(E)$ of bounded uniformly continuous functions on E. In fact, $\{\mathscr{P}(t)\}_{t\geqslant 0}$ is strongly continuous on $BUC(E)$ if and only if $A = 0$, i.e., if the drift term is trivial [15, 13]. For this reason many authors have studied strong continuity of $\{\mathscr{P}(t)\}_{t\geqslant 0}$ in various locally convex topologies on $C_b(E)$, cf. [5], [6], [8], [11], [16]. They only consider the situation where E is a

Hilbert space, in which case Itô calculus may be applied. Using analytic methods, the Banach space case was studied in [10], [12], [13], [9].

We will need the following result from [13], which is an easy consequence of Proposition 2.1 and (3.1).

Proposition 3.2. *We have* $\lim_{t\downarrow 0}\mu_t = \mu_0 = \delta_0$ *weakly, where* δ_0 *is the Dirac measure on* E *concentrated at* 0.

For the proof of the Lie–Trotter product formula it will be necessary to study tightness of a family of measures that is obtained by 'discretizing' the covariance operators of the measures μ_t.

Let $P = \{t_0, \dots, t_N\}$ be a partition of the interval $[0,t]$; i.e., $0 = t_0 < \dots < t_N = t$. We define positive symmetric operators $Q_t^P \in \mathscr{L}(E^*, E)$ by

$$Q_t^P x^* := \sum_{j=1}^N (t_j - t_{j-1})\, S(t_j) Q S^*(t_j) x^*, \qquad x^* \in E^*. \tag{3.4}$$

Note that the sum defining $Q_t^P x^*$ is the Riemann sum for the integral

$$Q_t x^* = \int_0^t S(s) Q S^*(s) x^* \, ds$$

corresponding with the right endpoints of the partition intervals.

For every partition P of $[0,t]$, the operator Q_t^P is the covariance of a centred Gaussian measure μ_t^P on E. To see this, first note that for all $\lambda_j > 0$, the operator $R_j := \lambda_j Q$ is the covariance of the scaled measure $\nu_j(B) := \nu(B/\sqrt{\lambda_j})$, $B \subseteq E$ Borel. Next, if R_j is the covariance of a centred Gaussian measure ν_j on E and if $S_1, \dots, S_N$ are bounded operators on E, then $\sum_{j=1}^N S_j R_j S_j^*$ is the covariance of the centred Gaussian measure $S_1\nu_1 * \dots * S_N\nu_N$. We finally apply this with $\lambda_j = t_j - t_{j-1}$ and $S_j = S(t_j)$.

The *mesh* of a partition P is the number $\mathrm{mesh}\,(P) := \max_{j=1,\dots,N}(t_j - t_{j-1})$.

Lemma 3.3. *Let* (t_n) *be a sequence of strictly positive real numbers satisfying* $\lim_{n\to\infty} t_n = t$. *For each* n *let* P_n *be a partition of* $[0, t_n]$, *and assume that* $\lim_{n\to\infty} \mathrm{mesh}\,(P_n) = 0$. *Then, for all* $x^*, y^* \in E^*$, *we have*

$$\lim_{n\to\infty} \langle Q_{t_n}^{P_n} x^*, y^* \rangle = \langle Q_t x^*, y^* \rangle.$$

Proof. Fix $x^*, y^* \in E$. Being a Gaussian covariance operator, Q is compact and therefore the function

$$\phi(s) := \langle S(s) Q S^*(s) x^*, y^* \rangle = \langle Q S^*(s) x^*, S^*(s) y^* \rangle, \qquad s \in [0,\infty),$$

is continuous for all $x^* \in E^*$. Indeed, this follows from the weak*-continuity of the adjoint semigroup $\{S^*(t)\}_{t\geqslant 0}$ which is uniform on compact subsets of E.

Fix $\varepsilon > 0$ arbitrary and fix $T > 0$ large enough such that $0 \leqslant t_n \leqslant T$ for all n. The uniform continuity of ϕ on $[0,T]$ enables us to find $\delta > 0$ small enough such that $|\phi(s) - \phi(s')| < \varepsilon/T$ for all $s, s' \in [0,T]$ with $|s - s'| < \delta$. Choose N so large that $\mathrm{mesh}\,(P_n) < \delta$ for all $n \geqslant N$. Then, for all $n \geqslant N$ we have

$$\left| \langle Q_{t_n}^{P_n} x^*, y^* \rangle - \int_0^{t_n} \phi(s)\, ds \right| < \varepsilon.$$

Therefore,

$$\Big|\langle Q_{t_n}^{P_n} x^*, y^*\rangle - \langle Q_t x^*, y^*\rangle\Big| = \Big|\langle Q_{t_n}^{P_n} x^*, y^*\rangle - \int_0^t \phi(s)\,ds\Big| < \varepsilon + |t - t_n| \cdot \sup_{s\in[0,T]} |\phi(s)|.$$

From this we conclude that

$$\limsup_{n\to\infty} \Big|\langle Q_{t_n}^{P_n} x^*, y^*\rangle - \langle Q_t x^*, y^*\rangle\Big| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this proves the lemma. □

We define semigroups $\{\mathscr{S}(t)\}_{t\geqslant 0}$ and $\{\mathscr{T}(t)\}_{t\geqslant 0}$ on $C_b(E)$ by

$$\begin{aligned} \mathscr{S}(t)f(x) &:= f(S(t)x), \\ \mathscr{T}(t)f(x) &:= \int_E f(x+y)\,d\nu_t(y), \end{aligned} \qquad t \geqslant 0,\ x \in E,$$

where ν_t denotes the centred Gaussian measure on E with covariance tQ. The first semigroup, $\{\mathscr{S}(t)\}_{t\geqslant 0}$, can be interpreted as the transition semigroup corresponding to the deterministic equation

$$dX(t) = AX(t)\,dt. \tag{3.5}$$

The second semigroup, $\{\mathscr{T}(t)\}_{t\geqslant 0}$, can be interpreted as the transition semigroup corresponding to the equation

$$dX(t) = dW(t), \tag{3.6}$$

assuming that $\{W(t)\}_{t\geqslant 0}$ is an E-valued Brownian motion such that for all $t \geqslant 0$ the random variable $W(t)$ has distribution ν_t. Comparing this with (1.1), we see that equations (3.5) and (3.6) correspond to the drift term and the diffusion term in (1.1), respectively.

Our main abstract result relates the transition semigroup $\{\mathscr{P}(t)\}_{t\geqslant 0}$ to the semigroups $\{\mathscr{S}(t)\}_{t\geqslant 0}$ and $\{\mathscr{T}(t)\}_{t\geqslant 0}$.

Theorem 3.4. *Let (t_n) be a sequence of strictly positive real numbers satisfying $\lim_{n\to\infty} t_n = t$. For each n let P_n be a partition of $[0, t_n]$, and assume that $\lim_{n\to\infty} \mathrm{mesh}\,(P_n) = 0$. Write $P_n = \{t_{0,n}, \dots, t_{N_n,n}\}$, and for $j = 1, \dots, N_n$ put $\Delta t_{j,n} := t_{j,n} - t_{j-1,n}$ and*

$$\mathscr{V}(\Delta t_{j,n}) := \mathscr{T}(\Delta t_{j,n}) \circ \mathscr{S}(\Delta t_{j,n}).$$

If

$$\lim_{n\to\infty} \mu_{t_n}^{P_n} = \mu_t \quad \textit{weakly}, \tag{3.7}$$

then for all $f \in C_b(E)$ and all sequences (x_n) in E with $\lim_{n\to\infty} x_n = x$ we have

$$\mathscr{P}(t)f(x) = \lim_{n\to\infty} \left[\mathscr{V}(\Delta t_{N_n,n}) \circ \cdots \circ \mathscr{V}(\Delta t_{1,n})\right] f(x_n). \tag{3.8}$$

Remark 3.5. In Sections 4 and 5 below we will show that condition (3.7) is automatically satisfied in each of the following two situations:

- $\{S(t)\}_{t\geqslant 0}$ restricts to a C_0-semigroup on the RKHS H_Q.
- E is a Hilbert space.

Proof of Theorem 3.4. Fix $f \in C_b(E)$ and $\xi \in E$. For all $s \geqslant 0$ we have

$$[\mathscr{T}(s) \circ \mathscr{S}(s)]\, f(\xi) = \int_E \mathscr{S}(s) f(\xi + y)\, d\nu_s(y) = \int_E f(S(s)\xi + S(s)y)\, d\nu_s(y).$$

Writing ν_s^t for the image measure $S(t)\nu_s$, for all $s_1, s_2 \geqslant 0$ we obtain

$$\begin{aligned}
&[\mathscr{T}(s_2) \circ \mathscr{S}(s_2)] \circ [\mathscr{T}(s_1) \circ \mathscr{S}(s_1)]\, f(\xi) \\
&\quad = \int_E [\mathscr{T}(s_1) \circ \mathscr{S}(s_1)]\, f(S(s_2)\xi + S(s_2)y)\, d\nu_{s_2}(y) \\
&\quad = \int_E \int_E f(S(s_1+s_2)\xi + S(s_1+s_2)y + S(s_1)z)\, d\nu_{s_1}(z)\, d\nu_{s_2}(y) \\
&\quad = \int_E \int_E f(S(s_1+s_2)\xi + \eta + \zeta)\, d\nu_{s_1}^{s_1}(\zeta)\, d\nu_{s_2}^{s_1+s_2}(\eta) \\
&\quad = \int_E f(S(s_1+s_2)\xi + \varrho)\, d(\nu_{s_1}^{s_1} * \nu_{s_2}^{s_1+s_2})(\varrho).
\end{aligned}$$

By induction, for all $s_1, \dots, s_N \geqslant 0$ we obtain

$$\begin{aligned}
&[\mathscr{T}(s_N) \circ \mathscr{S}(s_N)] \circ \cdots \circ [\mathscr{T}(s_1) \circ \mathscr{S}(s_1)]\, f(\xi) \\
&\quad = \int_E f(S(s_1 + \cdots + s_N)\xi + \varrho)\, d(\nu_{s_1}^{s_1} * \cdots * \nu_{s_N}^{s_1+\cdots+s_N})(\varrho).
\end{aligned}$$

Let us now fix a partition $P = \{\tau_0, \dots, \tau_N\}$ of an interval $[0, \tau]$, take $s_j = \Delta\tau_j := \tau_j - \tau_{j-1}$ in the identity above and note that $\Delta\tau_1 + \cdots + \Delta\tau_k = \tau_k$ for $k = 1, \dots, N$. The covariance operator of $\nu_{\Delta\tau_1}^{\tau_1} * \cdots * \nu_{\Delta\tau_N}^{\tau_N}$ equals

$$\sum_{j=1}^N S(\tau_j)\big(\Delta\tau_j Q\big) S^*(\tau_j) \;=\; Q_\tau^P.$$

Thus, we obtain

$$\begin{aligned}
[\mathscr{V}(\Delta\tau_N) \circ \cdots \circ \mathscr{V}(\Delta\tau_1)]\, f(\xi) &= \int_E f(S(\tau_N)\xi + \varrho)\, d(\nu_{\Delta\tau_1}^{\tau_1} * \cdots * \nu_{\Delta\tau_N}^{\tau_N})(\varrho) \\
&= \int_E f(S(\tau)\xi + \varrho)\, d\mu_\tau^P(\varrho).
\end{aligned}$$

After these preparations we turn to the proof of (3.8). Let $N(m, R)$ denote the Gaussian measure on E with mean m and covariance R. If $\lim_{n\to\infty} x_n = x$ in E, then by (3.7) and Proposition 2.2, we have

$$\lim_{n\to\infty} N\big(S(t_n)x_n, Q_{t_n}^{P_n}\big) = N\big(S(t)x, Q_t\big) \quad \text{weakly.}$$

It follows that

$$\begin{aligned}
\lim_{m\to\infty} [\mathscr{V}(\Delta t_{N_n,n}) \circ \cdots \circ \mathscr{V}(\Delta t_{1,n})]\, f(x_n) &= \lim_{n\to\infty} \int_E f(S(t_n)x_n + \varrho)\, d\mu_{t_n}^{P_n}(\varrho) \\
&= \int_E f(S(t)x + \varrho) d\mu_t(\varrho) = \mathscr{P}(t) f(x).
\end{aligned}$$

This proves (3.8). $\square$

From this result we deduce the following Lie–Trotter product formula for the semigroup $\{\mathscr{P}(t)\}_{t\geqslant 0}$:

Theorem 3.6. *For $t > 0$, let $\mu_{t,n} := \mu_t^{\pi_n}$, where π_n is the equipartition of $[0,t]$ into n subintervals of equal length. If for all $t > 0$ we have*

$$\lim_{n\to\infty} \mu_{t,n} = \mu_t \quad \textit{weakly}, \tag{3.9}$$

then for all $f \in C_b(E)$ and all $t \geqslant 0$ and $x \in E$ we have

$$\mathscr{P}(t)f(x) = \lim_{n\to\infty} \left[\mathscr{T}(\tfrac{t}{n}) \circ \mathscr{S}(\tfrac{t}{n})\right]^n f(x), \tag{3.10}$$

the convergence being uniform on finite time intervals $[0,T]$ and compact subsets $K \subseteq E$.

Proof. Suppose (3.9) holds but (3.10) fails. We will deduce a contradiction as follows.

By assumption there exist an $\varepsilon > 0$, a compact set $K \subseteq E$, a real number $T > 0$, and a subsequence (n_k) such that

$$\sup_{(t,x)\in[0,T]\times K} \left|\mathscr{P}(t)f(x) - \left[\mathscr{T}(\tfrac{t}{n_k}) \circ \mathscr{S}(\tfrac{t}{n_k})\right]^{n_k} f(x)\right| \geqslant \varepsilon$$

for all k. Thus, we can choose points $(t_k, x_k) \in [0,T] \times K$ such that

$$\left|\mathscr{P}(t_k)f(x_k) - \left[\mathscr{T}(\tfrac{t_k}{n_k}) \circ \mathscr{S}(\tfrac{t_k}{n_k})\right]^{n_k} f(x_k)\right| \geqslant \tfrac{1}{2}\varepsilon \tag{3.11}$$

for all k. By passing to a further subsequence we may assume that $\lim_{k\to\infty} t_k = t \in [0,T]$ and $\lim_{k\to\infty} x_k = x \in K$ exist.

Let π_k denote the equipartition of $[0,t_k]$ into k subintervals of equal length, and note that $\lim_{k\to\infty} \operatorname{mesh}(\pi_k) = 0$. Applying Theorem 3.4 to the sequences (t_k), (x_k) and the partitions (π_k), and recalling that $\{\mathscr{P}(t)\}_{t\geqslant 0}$ is τ_c-continuous, we see that

$$\lim_{k\to\infty} \mathscr{P}(t_k)f(x_k) = \mathscr{P}(t)f(x) = \lim_{k\to\infty} \left[\mathscr{T}(\tfrac{t_k}{n_k}) \circ \mathscr{S}(\tfrac{t_k}{n_k})\right]^{n_k} f(x_k).$$

This contradicts (3.11). $\square$

4. The case when $\{S(t)\}_{t\geqslant 0}$ restricts to a C_0-semigroup on H_Q

In this section we will show that condition (3.7) holds whenever the RKHS H_Q associated with Q is $\{S(t)\}_{t\geqslant 0}$-invariant and $\{S(t)\}_{t\geqslant 0}$ restricts to a C_0-semigroup on H_Q.

Let us fix $t \geqslant 0$ and recall that $Q_t \in \mathscr{L}(E^*, E)$ is the positive symmetric operator defined by

$$Q_t x^* := \int_0^t S(s)QS^*(s)x^*\, ds, \qquad x^* \in E^*.$$

The RKHS associated with Q_t will be denoted by H_t and the inclusion operator of $H_t \hookrightarrow E$ by i_t. It is well-known that

$$H_t = \left\{ \int_0^t S(s)i_Q f(s)\, ds : \ f \in L^2((0,t); H_Q) \right\}$$

and that

$$\|h\|_{H_t} = \inf \left\{ \|f\|_{L^2((0,t);H_Q)} : \ h = \int_0^t S(s)i_Q f(s)\, ds \right\}.$$

For Hilbert spaces E this is shown in [7, Appendix B]; the proof carries over to the Banach space case without difficulty.

Given a partition $P = \{t_0, \dots, t_N\}$ of the interval $[0,t]$, we define the positive symmetric operator $Q_t^P \in \mathscr{L}(E^*, E)$ as before by

$$Q_t^P x^* := \sum_{j=1}^N (t_j - t_{j-1})\, S(t_j) Q S^*(t_j) x^*, \qquad x^* \in E^*.$$

Let H_t^P denote the associated RKHS with inclusion mapping $i_t^P : H_t^P \hookrightarrow E$. Define

$$\mathscr{H}_t^P := \left\{ \sum_{j=1}^N \int_{t_{j-1}}^{t_j} S(t_j) i_Q f(s)\, ds : \ f \in L^2((0,t); H_Q) \right\}.$$

Endowed with the norm

$$\|h\|_{\mathscr{H}_t^P} := \inf \left\{ \|f\|_{L^2((0,t);H_Q)} : \ h = \sum_{j=1}^N \int_{t_{j-1}}^{t_j} S(t_j) i_Q f(s)\, ds \right\},$$

it is easy to see that $\mathscr{H}_t^P$ is a separable real Hilbert space.

Lemma 4.1. *For all $x^* \in E^*$ we have $Q_t^P x^* \in \mathscr{H}_t^P$ and*

$$\|Q_t^P x^*\|_{\mathscr{H}_t^P} \leqslant \|Q_t^P x^*\|_{H_t^P}.$$

Proof. Fix an arbitrary $x^* \in E^*$ and define $f_{x^*} \in L^2((0,t); H_Q)$ by

$$f_{x^*}(s) := i_Q^* S^*(t_j) x^*, \qquad s \in (t_{j-1}, t_j),\ j = 1, \dots, N.$$

Then, using that $i_Q \circ i_Q^* = Q$, we have

$$\sum_{j=1}^N \int_{t_{j-1}}^{t_j} S(t_j) i_Q f_{x^*}(s)\, ds = \sum_{j=1}^N (t_j - t_{j-1})\, S(t_j) Q^* S^*(t_j) x^* = Q_t^P x^*.$$

This shows that $Q_t^P x^* \in \mathscr{H}_t^P$. Furthermore,

$$\begin{aligned} \|Q_t^P x^*\|_{H_t^P}^2 &= \langle Q_t^P x^*, x^* \rangle \\ &= \sum_{j=1}^N (t_j - t_{j-1})\, \langle Q S^*(t_j) x^*, S^*(t_j) x^* \rangle = \sum_{j=1}^N (t_j - t_{j-1})\, \|i_Q^* S^*(t_j) x^*\|_{H_Q}^2. \end{aligned}$$

Hence,

$$\|Q_t^P x^*\|_{\mathscr{H}_t^P}^2 \leqslant \|f_{x^*}\|_{L^2((0,t);H_Q)}^2 = \sum_{j=1}^N (t_j - t_{j-1})\, \|i_Q^* S^*(t_j) x^*\|_{H_Q}^2 = \|Q_t^P x^*\|_{H_t^P}^2.$$

□

As a consequence, we see that the identity mapping $Q_t^P x^* \mapsto Q_t^P x^*$ extends uniquely to a linear contraction mapping $I_t^P : H_t^P \to \mathscr{H}_t^P$. We will see below that I_t^P is injective.

Lemma 4.2. *Suppose that $\{S(t)\}_{t \geqslant 0}$ restricts to a C_0−semigroup $\{S_Q(t)\}_{t \geqslant 0}$ on H_Q. Then $\mathscr{H}_t^P \subseteq H_t$ as subsets of E, and for all $h \in \mathscr{H}_t^P$ we have*

$$\|h\|_{H_t} \leqslant \left(\sup_{s \in [0,t]} \|S_Q(s)\| \right) \|h\|_{\mathscr{H}_t^P}. \tag{4.1}$$

Proof. Let $h \in \mathscr{H}_t^P$ be arbitrary and fixed, and choose $f \in L^2((0,t); H_Q)$ such that $h = \sum_{j=1}^N \int_{t_{j-1}}^{t_j} S(t_j) i_Q f(s)\,ds$. Define $g \in L^2((0,t); H_Q)$ by

$$g(s) := S_Q(t_j - s) f(s), \qquad s \in (t_{j-1}, t_j),\ j = 1, \dots, N.$$

Noting that $S(t_j) \circ i_Q = S(s) \circ S(t_j - s) \circ i_Q = S(s) \circ i_Q \circ S_Q(t_j - s)$ we have

$$h = \sum_{j=1}^N \int_{t_{j-1}}^{t_j} S(t_j) i_Q f(s)\,ds = \sum_{j=1}^N \int_{t_{j-1}}^{t_j} S(s) i_Q g(s)\,ds = \int_0^t S(s) i_Q g(s)\,ds \in H_t.$$

This proves the inclusion $\mathscr{H}_t^P \subseteq H_t$. Moreover,

$$\|h\|_{H_t} \leqslant \|g\|_{L^2((0,t);H_Q)} \leqslant \left(\sup_{s\in[0,t]} \|S_Q(s)\| \right) \|f\|_{L^2((0,t);H_Q)}.$$

Taking the infimum over all function f representing h we obtain (4.1). $\square$

Putting things together we obtain the following commutative diagram:

$$\begin{array}{ccc} H_t^P & \xrightarrow[I_t^P]{} & \mathscr{H}_t^P \\ \downarrow & & \downarrow \\ E & \longleftarrow & H_t \end{array}$$

In this diagram, the lower three arrows denote inclusion mappings. Since they are injective, it follows that also I_t^P is injective and we obtain a (contractive) inclusion mapping $I_t^P : H_t^P \hookrightarrow \mathscr{H}_t^P$. Composing this mapping with the inclusion $\mathscr{H}_t^P \hookrightarrow H_t$ we obtain an inclusion mapping $J_t^P : H_t^P \hookrightarrow H_t$, which by Lemma 4.2 has norm

$$\|J_t^P\| \leqslant \sup_{s\in[0,t]} \|S_Q(s)\|. \tag{4.2}$$

Theorem 4.3. *Suppose $\{S(t)\}_{t\geqslant 0}$ restricts to a C_0-semigroup on H_Q. Let (t_n) be a sequence of strictly positive real numbers with $\lim_{n\to\infty} t_n = t$. For each n let P_n be a partition of $[0, t_n]$, and assume that $\lim_{n\to\infty} \mathrm{mesh}\,(P_n) = 0$. Then,*

$$\lim_{n\to\infty} \mu_{t_n}^{P_n} = \mu_t \quad \textit{weakly.} \tag{4.3}$$

Proof. Choose $T \geqslant 0$ so large that $0 \leqslant t_n \leqslant T$ for all n. Combination of Lemma 2.4, (3.1), and (4.2), shows that for all n we have

$$\langle Q_{t_n}^{P_n} x^*, x^* \rangle \leqslant \left(\sup_{s\in[0,t_n]} \|S_Q(s)\| \right)^2 \langle Q_{t_n} x^*, x^* \rangle \leqslant \left(\sup_{s\in[0,T]} \|S_Q(s)\| \right)^2 \langle Q_T x^*, x^* \rangle.$$

Hence, by Proposition 2.1, the sequence $(\mu_{t_n}^{P_n})$ is tight.

By Lemma 3.3 we have

$$\lim_{n\to\infty} \langle Q_{t_n}^{P_n} x^*, y^* \rangle = \langle Q_t x^*, y^* \rangle,$$

so from (2.2) we conclude that μ_t is the only possible weak limit point of the tight sequence $(\mu_{t_n}^{P_n})$. A standard argument now gives (4.3). $\square$

Upon combining this result with Theorem 3.6, we obtain:

Theorem 4.4. *If $\{S(t)\}_{t\geqslant 0}$ restricts to a C_0-semigroup on H_Q, then for all $f \in C_b(E)$ and all $t \geqslant 0$ and $x \in E$ we have*

$$\mathscr{P}(t)f(x) = \lim_{n\to\infty} \left[\mathscr{T}(\tfrac{t}{n}) \circ \mathscr{S}(\tfrac{t}{n})\right]^n f(x),$$

the convergence being uniform on finite time intervals $[0,T]$ and compact subsets $K \subseteq E$.

5. The case when E is a Hilbert space

In this section we will show that condition (3.7) always holds if E is a separable real *Hilbert* space.

In the following lemma, E is still allowed to be a separable real Banach space. Recall the standing assumption that ν is a centred Gaussian measure on E with covariance operator Q. For $t \geqslant 0$ let ρ_t denote the image measure of ν under the operator $S(t)$; this is a centred Gaussian measure on E with covariance operator $S(t)QS^*(t)$.

Lemma 5.1. *The function $t \mapsto \int_E \|x\|^2\, d\rho_t(x)$ is continuous on $[0,\infty)$.*

Proof. We start by showing that for all $t \geqslant 0$, the family $\{\rho_s : s \in [0,t]\}$ is tight. Fix $\varepsilon > 0$ and choose a compact subset K of E with $\nu(K) \geqslant 1-\varepsilon$. Define $L = \{S(s)x : s \in [0,t],\, x \in K\}$. Being the image of the compact set $[0,t]\times K$ under the continuous mapping $(s,x) \mapsto S(s)x$, L is compact. For all $s \in [0,t]$ we now have

$$\rho_s(L) \geqslant \rho_s(S(s)K) = \nu\{y \in E : S(s)y \in S(s)K\} \geqslant \nu(K) \geqslant 1-\varepsilon.$$

This proves the asserted tightness.

Fix a nonnegative convergent sequence (t_n) with limit t. Consider an arbitrary subsequence (t_{n_k}). The lemma will be proved if we find a further subsequence with the property that

$$\lim_{j\to\infty} \int_E \|x\|^2\, d\rho_{t_{n_{k_j}}}(x) = \int_E \|x\|^2\, d\rho_t(x). \tag{5.1}$$

By the above, the sequence $(\rho_{t_{n_k}})$ is tight. Consequently, there is a subsequence $(\rho_{t_{n_{k_j}}})$ converging weakly to some probability measure $\tilde{\rho}_t$. Since the weak limit of a sequence of centred Gaussian measures is a centred Gaussian measure and since

$$\lim_{j\to\infty} \langle S(t_{n_{k_j}})QS^*(t_{n_{k_j}})x^*, y^*\rangle = \langle S(t)QS^*(t)x^*, y^*\rangle$$

for all $x^*, y^* \in E^*$, it follows that $\tilde{\rho}_t = \rho_t$. Hence, (5.1) follows from Proposition 2.3, part (i). □

Suppose now that E is a separable real Hilbert space. Then, we may identify Q with a positive selfadjoint operator on E. Since, by assumption, Q is a Gaussian covariance, Q may be identified with a trace class operator on E and by (2.5), Lemma 5.1 may be reformulated as saying that the function $t \mapsto \operatorname{tr} S(t)QS^*(t)$ is continuous on $[0,\infty)$. Only this fact will be needed below, and it is worthwile to point out that this can be proved more directly as follows. Let (e_j) be an orthonormal basis of E and suppose that $t_n \to t$ in $[0,\infty)$. Then,

$$\begin{aligned}
\lim_{n\to\infty} \operatorname{tr} S(t_n)QS^*(t_n) &= \lim_{n\to\infty} \sum_j \|Q^{\frac12}S^*(t_n)e_j\|^2 \\
&= \lim_{n\to\infty} \sum_j \sum_k [Q^{\frac12}S^*(t_n)e_j, e_k]_E^2 = \lim_{n\to\infty} \sum_k \|S(t_n)Q^{\frac12}e_k\|^2 \\
&= \sum_k \|S(t)Q^{\frac12}e_k\|^2 = \sum_k \sum_j [S(t)Q^{\frac12}e_k, e_j]_E^2 \\
&= \sum_j \|Q^{\frac12}S^*(t)e_j\|^2 = \operatorname{tr} S(t)QS^*(t),
\end{aligned}$$

the convergence of the series being justified by dominated convergence, since we have, for some constant $C \geqslant 0$, $\|S(t_n)Q^{\frac12}e_j\|^2 \leqslant C\|Q^{\frac12}e_j\|^2$, and the latter is a summable sequence:

$$\sum_j \|Q^{\frac12}e_j\|^2 = \operatorname{tr} Q.$$

Theorem 5.2. *Let E be a separable real Hilbert space. Let (t_n) be a sequence of strictly positive real numbers with $\lim_{n\to\infty} t_n = t$. For each n let P_n be a partition of $[0,t_n]$, and assume that $\lim_{n\to\infty} \operatorname{mesh}(P_n) = 0$. Then,*

$$\lim_{n\to\infty} \mu_{t_n}^{P_n} = \mu_t \quad \textit{weakly.}$$

Proof. By Lemma 3.3, for all $x,y \in E$ we have

$$\lim_{n\to\infty} [Q_{t_n}^{P_n}x, y]_E = [Q_t x, y]_E.$$

Hence, by part (ii) of Proposition 2.3, it remains to check that

$$\lim_{n\to\infty} \int_E \|x\|^2 \, d\mu_{t_n}^{P_n}(x) = \int_E \|x\|^2 \, d\mu_t(x).$$

This is equivalent to the condition

$$\lim_{n\to\infty} \operatorname{tr} Q_{t_n}^{P_n} = \operatorname{tr} Q_t.$$

Choose an orthonormal basis $\{e_k\}_{k=1}^\infty$ for E. Then,

$$\begin{aligned}
\operatorname{tr} Q_{t_n}^{P_n} &= \sum_{k=1}^\infty \sum_{j=1}^{N_n} (t_{j,n} - t_{j-1,n}) \left[S(t_{j,n})QS^*(t_{j,n})e_k, e_k\right]_E \\
&= \sum_{j=1}^{N_n} (t_{j,n} - t_{j-1,n}) \sum_{k=1}^\infty \left[S(t_{j,n})QS^*(t_{j,n})e_k, e_k\right]_E \qquad (5.2)\\
&= \sum_{j=1}^{N_n} (t_{j,n} - t_{j-1,n}) \operatorname{tr} S(t_{j,n})QS^*(t_{j,n}),
\end{aligned}$$

where the change in the order of summation is justified by the fact that each term $\left[S(t_{j,n})QS^*(t_{j,n})e_k, e_k\right]_E = \left[QS^*(t_{j,n})e_k, S^*(t_{j,n})e_k\right]_E$ is nonnegative.

The right hand side of (5.2) is a Riemann sum of the integral

$$\int_0^{t_n} \operatorname{tr} S(s)QS(s)\, ds.$$

As we noted, by Lemma 5.1 the function $s \mapsto \operatorname{tr} S(s)QS(s)$ is continuous on $[0,t]$. Arguing as in the proof of Lemma 3.3, this implies

$$\limsup_{n\to\infty} \left|\operatorname{tr} Q_{t_n}^{P_n} - \int_0^t \operatorname{tr} S(s)QS(s)\,ds\right| < \varepsilon$$

for all $\varepsilon > 0$. Hence,

$$\lim_{n\to\infty} \operatorname{tr} Q_{t_n}^{P_n} = \int_0^t \operatorname{tr} S(s)QS(s)\,ds = \operatorname{tr} Q_t.$$

□

Upon combining this result with Theorem 3.6, we obtain:

Corollary 5.3. *If E is a separable real Hilbert space, then for all $f \in C_b(E)$ and all $t \geqslant 0$ and $x \in E$ we have*

$$\mathscr{P}(t)f(x) = \lim_{n\to\infty} \left[\mathscr{T}(\tfrac{t}{n}) \circ \mathscr{S}(\tfrac{t}{n})\right]^n f(x),$$

the convergence being uniform on finite time intervals $[0,T]$ and compact subsets $K \subseteq E$.

For separable real Hilbert spaces E, a Lie-Trotter product formula for a class of transition semigroups on $C_b(E)$ associated with nonlinear stochastic equations of the form

$$\begin{aligned} dX(t) &= F(X(t))\,dt + B(X(t))\,dW(t), \qquad t \geqslant 0, \\ X(0) &= x_0, \end{aligned} \tag{5.3}$$

has been obtained recently by G. Tessitore and J. Zabczyk. Here, $\{W(t)\}_{t\geqslant 0}$ is a Brownian motion with values in E, and $F : E \to E$ and $B : E \to \mathscr{L}(E)$ are Lipschitz functions. In the linear case there is a small overlap with our Corollary 5.3. To make this explicit we make two special choices of F and B in (5.3). First, we let $\mathscr{T}$ be the transition semigroup on $C_b(E)$ obtained by taking $F \equiv A$, with A a given bounded operator on E, and $B \equiv 0$ in (5.3); thus,

$$\mathscr{T}(t)f(x) = f(e^{tA}x), \qquad t \geqslant 0,\ x \in E.$$

Second, we let $\mathscr{S}$ be the transition semigroup on $C_b(E)$ obtained by taking $F \equiv 0 \in \mathscr{L}(E)$ and $B \equiv I$ in (5.3); thus,

$$\mathscr{S}(t)f(x) = \int_E f(x+y)\,d\mu_t(y), \qquad t \geqslant 0,\ x \in E,$$

where μ_t is the distribution of $W(t)$. This puts us into the setting considered in Corollary 5.3. From [17, Proposition 3.5] (the special case for uniformly bounded F and B of the main result, [17, Theorem 3.4]) we now see the following. Let Y denote the closure with respect to the supremum norm in $C_b(E)$ of the space of all functions which are bounded and uniformly continuous along with their first and second Fréchet derivatives. Then for all $f \in Y$, $t \geqslant 0$, and $x \in E$,

$$\mathscr{P}(t)f(x) = \lim_{n\to\infty} \left[\mathscr{T}(\tfrac{t}{n}) \circ \mathscr{S}(\tfrac{t}{n})\right]^n f(x),$$

the convergence being uniform on finite time intervals and *bounded* subsets $B \subseteq E$.

6. Appendix: Proof of Proposition 2.3, part (ii)

Although part (ii) of Proposition 2.3 may be well-known to specialists, we could not find an explicit reference for it, and for the convenience of the reader we include a proof here.

Let E be a separable real Hilbert space and let $(\mu_n)_{n=1}^{\infty}$ and μ satisfy the conditions (1) and (2) in Proposition 2.3. We choose an orthonormal basis $\{e_j\}_{j=1}^{\infty}$ of E and denote by P_j the orthogonal projection onto the linear span of $\{e_1, \dots, e_j\}$.

Lemma 6.1. *Let $\varepsilon > 0$ be arbitrary and fixed. For all $k \geqslant 1$ there exists an index J_k with the following property: for all $j \geqslant J_k$ and all $n \geqslant 1$ we have*

$$\mu_n \left\{ x \in E : \ \|x - P_j x\|^2 > \frac{1}{k} \right\} \leqslant \frac{\varepsilon}{2^{k+1}}.$$

Proof. The proof is inspired by an argument in [1].

Denote $\mu_{n,j} := P_j \mu_n$ and $\mu_j := P_j \mu$, and let $Q_{n,j}$ and Q_j denote their covariance operators. By condition (1), for all $x, y \in E$ we have

$$\begin{aligned} \lim_{n\to\infty} [Q_{n,j}x, y]_E = \lim_{n\to\infty} [P_j Q_n P_j x, y]_E = \lim_{n\to\infty} [Q_n P_j x, P_j y]_E \\ = [Q P_j x, P_j y]_E = [P_j Q_j P_j x, y] = [Q_j x, y]_E. \end{aligned}$$

Hence by (2.1),

$$\begin{aligned} \lim_{n\to\infty} \int_E \|P_j x\|^2 \, d\mu_n(x) &= \lim_{n\to\infty} \int_{P_j E} \|y\|^2 \, d\mu_{n,j}(y) \\ &= \lim_{n\to\infty} \sum_{k=1}^{j} \int_{P_j E} [y, e_k]_E^2 \, d\mu_{n,j}(y) = \lim_{n\to\infty} \sum_{k=1}^{j} [Q_{n,j} e_k, e_k]_E \\ &= \sum_{k=1}^{j} [Q_j e_k, e_k]_E = \sum_{k=1}^{j} \int_{P_j E} [y, e_k]_E^2 \, d\mu_j(y) \\ &= \int_{P_j E} \|y\|^2 \, d\mu_j(y) = \int_E \|P_j x\|^2 \, d\mu(x). \end{aligned} \tag{6.1}$$

By the absolute continuity of the measure $\|x\|^2 \, d\mu(x)$ with respect to $d\mu(x)$, for every integer $k \geqslant 1$ we can pick $\delta_k > 0$ such that

$$\int_A \|x\|^2 \, d\mu(x) \leqslant \frac{\varepsilon}{k 2^{k+3}}$$

for all Borel sets $A \subseteq E$ with $\mu(A) \leqslant \delta_k$. Define

$$A_{j,k} := \left\{ x \in E : \ \|x - P_j x\|^2 > \frac{\varepsilon}{k 2^{k+3}} \right\}.$$

By dominated convergence we have $\lim_{j\to\infty} \mu(A_{j,k}) = 0$ for all $k \geqslant 1$. It follows that there exists an index $J(k)$ such that $\mu(A_{j,k}) \leqslant \delta_k$ for all $j \geqslant J(k)$. Then, for all $j \geqslant J(k)$ we have

$$\begin{aligned} \int_E \|x\|^2 - \|P_j x\|^2 \, d\mu(x) &= \int_E \|x - P_j x\|^2 \, d\mu(x) \\ &\leqslant \int_{A_{j,k}} \|x\|^2 \, d\mu(x) + \int_{E \setminus A_{j,k}} \|x - P_j x\|^2 \, d\mu(x) \\ &\leqslant \frac{\varepsilon}{k 2^{k+3}} + \frac{\varepsilon}{k 2^{k+3}} = \frac{\varepsilon}{k 2^{k+2}}. \end{aligned}$$

Next, choose an index N_k so large that for all $n \geqslant N_k$ we have

$$\left| \int_E \|x\|^2 \, d\mu_n(x) - \int_E \|x\|^2 \, d\mu(x) \right| \leqslant \frac{\varepsilon}{k2^{k+3}}$$

and

$$\left| \int_E \|P_{J(k)}x\|^2 \, d\mu(x) - \int_E \|P_{J(k)}x\|^2 \, d\mu_n(x) \right| \leqslant \frac{\varepsilon}{k2^{k+3}}.$$

The second condition can be met in view of (6.1). Then, for all $n \geqslant N_k$ and $j \geqslant J(k)$,

$$\begin{aligned}
\int_E \|x - P_j x\|^2 \, d\mu_n(x) &\leqslant \int_E \|x - P_{J(k)}x\|^2 \, d\mu_n(x) = \int_E \|x\|^2 - \|P_{J(k)}x\|^2 \, d\mu_n(x) \\
&\leqslant \left| \int_E \|x\|^2 \, d\mu_n(x) - \int_E \|x\|^2 \, d\mu(x) \right| + \int_E \|x\|^2 - \|P_{J(k)}x\|^2 \, d\mu(x) \\
&\qquad + \left| \int_E \|P_{J(k)}x\|^2 \, d\mu_n(x) - \int_E \|P_{J(k)}x\|^2 \, d\mu(x) \right| \\
&\leqslant \frac{\varepsilon}{k2^{k+3}} + \frac{\varepsilon}{k2^{k+2}} + \frac{\varepsilon}{k2^{k+3}} = \frac{\varepsilon}{k2^{k+1}}.
\end{aligned}$$

It follows that for all $n \geqslant N_k$ and all $j \geqslant J(k)$ we have

$$\mu_n \left\{ x \in E : \ \|x - P_j x\|^2 > \frac{1}{k} \right\} \leqslant k \int_E \|x - P_j x\|^2 \, d\mu_n(x) \leqslant \frac{\varepsilon}{2^{k+1}}.$$

By dominated convergence, for every $k \geqslant 1$ we can find an index $J_k \geqslant J(k)$ such that for all $n = 1, \ldots, N_k - 1$ and all $j \geqslant J_k$ we have

$$\mu_n \left\{ x \in E : \ \|x - P_j x\|^2 > \frac{1}{k} \right\} \leqslant \frac{\varepsilon}{2^{k+1}}.$$

This J_k has the desired properties. $\square$

Proof of Proposition 2.3, part (ii). We follow the argument of [18, Theorem I.3.7]. Define

$$V_{j,k} := \left\{ x \in E : \ \|x - P_j x\|^2 > \frac{1}{k} \right\}.$$

Fix $\varepsilon > 0$ arbitrary. By Lemma 6.1, for every $k \geqslant 1$ we can find an index j_k such that for all $n \geqslant 1$ we have

$$\mu_n(V_{j_k,k}) \leqslant \frac{\varepsilon}{2^{k+1}}.$$

For all $n \geqslant 1$ and $r > 0$ we have

$$\mu_n\{x \in E : \ \|x\| > r\} \leqslant \frac{1}{r^2} \int_E \|x\|^2 \, d\mu_n(x) \leqslant \frac{1}{r^2} \cdot \sup_{m \geqslant 1} \int_E \|x\|^2 \, d\mu_m(x).$$

Hence, we may choose r_0 so large that for all $n \geqslant 1$ we have

$$\mu_n\{x \in E : \ \|x\| > r_0\} \leqslant \frac{\varepsilon}{2}.$$

Set

$$F := \left(\bigcap_{k \geqslant 1} E \backslash V_{j_k,k} \right) \bigcap \big\{ x \in E : \ \|x\| \leqslant r_0 \big\}.$$

Then, F is bounded and closed, and for all $k \geqslant 1$ we have

$$F \subseteq \left\{ x \in E : \ \|x - P_{j_k} x\|^2 \leqslant \frac{1}{k} \right\}.$$

Since every P_{j_k} has finite-dimensional range, an elementary argument implies that F is compact. Moreover, for all $n \geqslant 1$ we have

$$\mu_n(E \backslash F) \leqslant \left(\sum_{k=1}^{\infty} \frac{\varepsilon}{2^{k+1}} \right) + \frac{\varepsilon}{2} \leqslant \varepsilon.$$

We have shown that for every $\varepsilon > 0$ there exists a compact set $F \subseteq E$ with $\mu_n(F) \geqslant 1 - \varepsilon$ for $n \geqslant 1$. This proves that the sequence (μ_n) is tight.

By condition (1) and (2.2), μ is the only possible weak limit point of (μ_n). A standard argument now gives the weak convergence $\lim_{n\to\infty} \mu_n = \mu$. □

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