NORM DISCONTINUITY AND SPECTRAL PROPERTIES OF ORNSTEIN-UHLENBECK SEMIGROUPS

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Abstract. Let E be a real Banach space. We study the Ornstein-Uhlenbeck semigroup $P = \{P(t)\}_{t\geqslant 0}$ associated with the Ornstein-Uhlenbeck operator

$$Lf(x) = \tfrac{1}{2}\mathrm{Tr}\, QD^2f(x) + \langle Ax, Df(x)\rangle, \qquad x \in E.$$

Here $Q \in \mathscr{L}(E^*, E)$ is a positive symmetric operator and A is the generator of a C_0-semigroup $S = \{S(t)\}_{t\geqslant 0}$ on E. Under the assumption that P admits an invariant measure μ_∞ we prove that if S is eventually compact and the spectrum of its generator is nonempty, then

$$\|P(t) - P(s)\|_{\mathscr{L}(L^1(E,\mu_\infty))} = 2 \ \text{ for all } t, s \geqslant 0 \text{ with } t \neq s.$$

This result is new even when $E = \mathbb{R}^n$. We also study the behaviour of P in the space $BUC(E)$. We show that if $A \neq 0$ there exists $t_0 > 0$ such that

$$\|P(t) - P(s)\|_{\mathscr{L}(BUC(E))} = 2 \ \text{ for all } 0 \leqslant t, s \leqslant t_0 \text{ with } t \neq s.$$

Moreover, under a nondegeneracy assumption or a strong Feller assumption, the following dichotomy holds: either

$$\|P(t) - P(s)\|_{\mathscr{L}(BUC(E))} = 2 \ \text{ for all } t, s \geqslant 0, \ \ t \neq s,$$

or S is the direct sum of a nilpotent semigroup and a finite-dimensional periodic semigroup. Finally we investigate the spectrum of L in the spaces $L^1(E, \mu_\infty)$ and $BUC(E)$.

1. Introduction and Preliminaries

In this paper we study certain properties of the Ornstein-Uhlenbeck semigroup in spaces of continuous functions and integrable functions. This semigroup is associated with the stochastic linear Cauchy problem

$$\begin{cases} dU(t) = AU(t)\,dt + B\,dW_H(t), \\ U(0) = x. \end{cases} \tag{1.1}$$

Here A is assumed to be the infinitesimal generator of a C_0-semigroup $S = \{S(t)\}_{t\geqslant 0}$ on a real Banach space E, B is a bounded operator from a real Hilbert space H into E, $W_H = \{W_H(t)\}_{t\geqslant 0}$ is an H-cylindrical Brownian motion, and $x \in E$ is an initial value. As is well known, the above problem admits a unique weak solution if and only if for all $t \geqslant 0$ there exists a centred Gaussian Radon measure μ_t on E whose covariance operator $Q_t \in \mathscr{L}(E^*, E)$ is given by

$$\langle Q_t x^*, y^*\rangle = \int_0^t \langle S(s)BB^*S^*(s)x^*, y^*\rangle\, ds, \qquad x^*, y^* \in E^*,$$

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where E^* denotes the topological dual of E. Under this assumption the solution $U = \{U(t,x)\}_{t\geqslant 0}$ of (1.1) is given by the stochastic Itô integral

$$U(t,x) = S(t)x + \int_0^t S(t-s)B\,dW_H(s),$$

see [3, 9, 23]. For more information on Gaussian measures in infinite dimensions we refer to [2, 27].

The *Ornstein-Uhlenbeck semigroup* $P = \{P(t)\}_{t\geqslant 0}$ associated with A and B is defined on the space $C_b(E)$ of bounded real-valued continuous functions on E by

$$P(t)f(x) := \mathbb{E}(f(U(t,x))) = \int_E f(S(t)x+y)\,d\mu_t(y), \qquad x\in E, \quad f\in C_b(E). \tag{1.2}$$

This semigroup leaves $BUC(E)$, the space of bounded real-valued uniformly continuous functions on E, invariant and has been studied by many authors [4, 8, 14, 15, 16, 25, 26]. It is well known that P fails to be strongly continuous with respect to the supremum norm of $BUC(E)$ unless $A = 0$. Therefore it is natural to introduce the closed subspace $BUC^\circ(E)$ consisting of all functions on which P acts in a strongly continuous way. This subspace is invariant under P, and the restriction P° of P is strongly continuous on $BUC^\circ(E)$. It is well known that the behaviour of P° is quite pathological. For instance, in the setting of a Hilbert space E it was shown in [25] that one has

$$\|P^\circ(t) - P^\circ(s)\|_{\mathscr{L}(BUC^\circ(E))} = 2 \tag{1.3}$$

whenever $\mu_t \perp \mu_s$, i.e., the measures μ_t and μ_s are mutually singular. Here $\|\cdot\|_{\mathscr{L}(X)}$ denotes the uniform operator norm of the Banach space $\mathscr{L}(X)$ of all bounded linear operators on X. For the heat semigroup, which corresponds to the case $A = 0$, (1.3) was established earlier in [11]. In Section 2 we extend this result to Banach spaces and complement it by showing that (1.3) also holds whenever $S(t) \neq S(s)$. It follows that if $A \neq 0$, then there exists $t_0 > 0$ such that

$$\|P^\circ(t) - P^\circ(s)\|_{\mathscr{L}(BUC^\circ(E))} = 2 \ \text{ for all } 0 \leqslant s,t \leqslant t_0, \ \ t \neq s. \tag{1.4}$$

In particular, if $A \neq 0$, then P° always fails to be norm continuous on $BUC^\circ(E)$ for $t > 0$. In the converse direction we show that for fixed $t, s \geqslant 0$, (1.3) and $S(t) = S(s)$ imply $\mu_t \perp \mu_s$. These results are used to prove the following dichotomy: either

$$\|P^\circ(t) - P^\circ(s)\|_{\mathscr{L}(BUC^\circ(E))} = 2 \ \text{ for all } t,s \geqslant 0, \ \ t \neq s, \tag{1.5}$$

or S is the direct sum of a nilpotent semigroup and a finite-dimensional periodic semigroup. Note that this result is new even when E is finite-dimensional. The probabilistic interpretation of (1.4) and (1.5) is that $\sup_{x\in E}\ \|\mu_{t,x} - \mu_{s,x}\|_{\rm var} = 2$, for $t,\ s \geqslant 0$ with $t \neq s$, where $\mu_{t,x}$ denotes the law of the process $U(t,x)$ which solves (1.1), and $\|\cdot\|_{\rm var}$ is the total variation norm.

Related to the problem of norm discontinuity is the problem of characterizing the spectrum of the generator L_{P° of P°. For finite-dimensional spaces E, it was shown in [17] that if the operator $Q := B \circ B^*$ is invertible and the spectrum $\sigma(A)$ of A is contained in $\{\lambda \in \mathbb{C} :\ \operatorname{Re}\lambda < 0\}$, then

$$\sigma(L_{P^\circ}) = \mathbb{C}^-, \tag{1.6}$$

where $\mathbb{C}^- := \{\lambda\in\mathbb{C} : \operatorname{Re}\lambda \leqslant 0\}$, and every $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda < 0$ is an eigenvalue. By standard results from semigroup theory, (1.6) already implies that P° cannot be eventually norm continuous in $BUC^\circ(E)$. Below we obtain an extension of (1.6) to the case where S is an eventually compact semigroup on a Banach space E.

Let us next assume that the limit $Q_\infty := \lim_{t\to\infty} Q_t$ exists in the weak operator topology of $\mathscr{L}(E^*, E)$ and that there exists a centred Gaussian Radon measure μ_∞ with covariance operator Q_∞. A sufficient condition for this is that the Gaussian Radon measures μ_t exist and S is uniformly exponentially stable; cf. [9, Chapter 9], [24]. The measure μ_∞ is *invariant* for P, in the sense that for all $f\in BUC(E)$ and $t \geqslant 0$,

$$\int_E P(t)f(x)\,d\mu_\infty(x) = \int_E f(x)\,d\mu_\infty(x).$$

By a standard argument, the semigroup P has a unique extension to a strongly continuous contraction semigroup on $L^p(E,\mu_\infty)$ for all $p\in[1,\infty)$. For $p\in(1,\infty)$, the behaviour of this semigroup is well understood. We refer to [6, 7, 18, 19, 22], where the domain of the generator, its spectrum, and analyticity properties are characterized.

The behaviour of P in $L^1(E,\mu_\infty)$ is much less well understood. For finite-dimensional spaces E it is shown in [18] that the $L^1(E,\mu_\infty)$-spectrum of its generator L_P equals $\mathbb{C}^-$. To the best of our knowledge, it is an open problem whether this result extends to infinite dimensions. Furthermore no L^1-analogue of (1.3) seems to be known. In Section 3 we will first show, for finite-dimensional spaces E, that

$$\|P(t)-P(s)\|_{\mathscr{L}(L^1(E,\mu_\infty))}=2$$

whenever $t>s\geqslant 0$. Then we extend this result to infinite dimensions in the setting of eventually compact semigroups S, and, extending a result for $E=\mathbb{R}^d$ in [18], we prove that the spectrum of L_P equals $\mathbb{C}^-$.

Our approach is based on a technique introduced by Davies and Simon [10] which may be described as follows. If B_1 and B_2 generate C_0-semigroups of contractions T_1 and T_2 on a Banach space X, then B_1 *belongs to the limit class* of B_2 if there exists a sequence of invertible isometries $V_n : X\to X$ such that

$$R(\lambda,B_1)x=\lim_{n\to\infty}V_n^{-1}R(\lambda,B_2)V_nx,\qquad x\in E.$$

Here $R(\lambda,B_k)=(\lambda-B_k)^{-1}$, $k=1,2$. This is equivalent to require that, for each $t>0$,

$$T_1x=\lim_{n\to\infty}V_n^{-1}T_2V_nx,\qquad x\in E.$$

In this situation one has

$$\|T_2(t)-T_2(s)\|_{\mathscr{L}(X)}\geqslant\|T_1(t)-T_1(s)\|_{\mathscr{L}(X)},\qquad t,s\geqslant 0,$$

and

$$\|R(\lambda,B_2)\|_{\mathscr{L}(X)}\geqslant\|R(\lambda,B_1)\|_{\mathscr{L}(X)},\qquad \lambda\in\varrho_\infty(B_1)\cap\varrho_\infty(B_2),$$

where $\varrho_\infty(B_k)$ denotes the connected component of the resolvent set $\varrho(B_k)$ containing $+\infty$, $k=1,2$. This technique is applied in the situation where B_2 is a suitable realization of the generator of P and B_1 is a realization of the generator of the *drift semigroup* R associated with A. This semigroup is defined on $C_b(E)$ by

$$R(t)f(x):=f(S(t)x),\qquad x\in E,\quad f\in C_b(E). \tag{1.7}$$

Throughout this paper, a *Gaussian measure* is a centred Gaussian Radon measure.

2. The Ornstein-Uhlenbeck semigroup in spaces of continuous functions

In this section we study various properties of the Ornstein-Uhlenbeck semigroup P and the drift semigroup R in the spaces $C_b(E)$ and $BUC(E)$. We denote by $\|\cdot\|$ the supremum norm.

As semigroups on $C_b(E)$, both P and R are strongly continuous with respect to the mixed topology. This topology is defined as the finest locally convex topology in $C_b(E)$ which agrees on every norm bounded set with the topology of uniform convergence on compact sets; see [28, 29] for a detailed investigation of its properties. This topology is complete and may be used to define the infinitesimal generators L_P and L_R of P and R by taking, for $T=P$ or R,

$$\mathscr{D}(L_T):=\Big\{f\in C_b(E):\ \lim_{t\downarrow 0}\frac{1}{t}(T(t)f-f)\ \text{ exists }\Big\},$$

$$L_Tf:=\lim_{t\downarrow 0}\frac{1}{t}(T(t)f-f),\qquad f\in\mathscr{D}(L_T),$$

where the limits are taken with respect to the mixed topology. We have $f\in\mathscr{D}(L_T)$ if and only if the following two conditions hold:

(i) $\limsup_{t\downarrow 0} \frac{1}{t}\|T(t)f - f\| < \infty$;

(ii) there exists a function $g \in C_b(E)$ such that for all $x \in E$,

$$\lim_{t\downarrow 0} \frac{1}{t}(T(t)f(x) - f(x)) = g(x).$$

In this situation, $L_T f = g$.

On a suitable core of smooth cylindrical functions, L_P and L_R are given by

$$\begin{aligned} L_P f(x) &= \tfrac{1}{2}\mathrm{Tr}\, QD^2 f(x) + \langle Ax, Df(x)\rangle, \\ L_R f(x) &= \langle Ax, Df(x)\rangle, \end{aligned}$$

where 'Tr' denotes the trace and $Q := BB^*$. We refer to [14, 15] for proofs and more details. Alternative approaches to diffusion semigroups in spaces of continuous functions may be found in [4, 16, 26].

Both P and R leave the closed subspace $BUC(E)$ of $C_b(E)$ invariant, but even on this smaller space both semigroups fail to be strongly continuous with respect to the supremum norm, unless $A = 0$. It is easy to see, cf. [8, Lemma 3.2], that the closed subspaces of $BUC(E)$ on which P and R act in a strongly continuous way with respect to the supremum norm coincide. This common subspace will be denoted by $BUC^\circ(E)$. Thus,

$$\begin{aligned} BUC^\circ(E) &= \big\{f \in BUC(E) : \lim_{t\downarrow 0} \|P(t)f - f\| = 0\big\} \\ &= \big\{f \in BUC(E) : \lim_{t\downarrow 0} \|R(t)f - f\| = 0\big\}. \end{aligned}$$

The restrictions of P and R to $BUC^\circ(E)$, denoted by P° and R° respectively, are strongly continuous with respect to the supremum norm. Their generators L_{P° and L_{R° are characterized as follows; see [8, Proposition 3.5] for a related result.

Proposition 2.1. *We have*

$$\begin{aligned} \mathscr{D}(L_{P^\circ}) &= \big\{f \in \mathscr{D}(L_P) \cap BUC^\circ(E) : \ L_P f \in BUC^\circ(E)\big\}, \\ \mathscr{D}(L_{R^\circ}) &= \big\{f \in \mathscr{D}(L_R) \cap BUC^\circ(E) : \ L_R f \in BUC^\circ(E)\big\}. \end{aligned}$$

Proof. Let $T = P$ or R.

The inclusion '$\subseteq$' is clear. To prove the inclusion '$\supseteq$' let $f \in \mathscr{D}(L_T) \cap BUC^\circ(E)$ be such that $L_T f \in BUC^\circ(E)$. Then,

$$\begin{aligned} &\lim_{t\downarrow 0} \sup_{x\in E} \Big|\frac{1}{t}\big(T(t)f(x) - f(x)\big) - L_T f(x)\Big| \\ &\quad = \lim_{t\downarrow 0} \sup_{x\in E} \Big|\frac{1}{t}\int_0^t T(s)L_T f(x)\, ds - L_T f(x)\Big| = \lim_{t\downarrow 0} \Big\|\frac{1}{t}\int_0^t T^\circ(s)L_T f - L_T f\, ds\Big\| = 0, \end{aligned}$$

where the first identity is a consequence of the fact that T is strongly continuous with respect to the mixed topology. This proves that $f \in \mathscr{D}(L_{T^\circ})$. ☐

We do not know whether $\mathscr{D}(L_{P^\circ})$ is always contained in $\mathscr{D}(L_{R^\circ})$.

The following simple observation, cf. the proof of [25, Lemma 2.3], will be useful.

Lemma 2.2. *Let $T = P$ or R. For $f \in BUC(E)$ and $\delta > 0$ define*

$$f_\delta(x) := \frac{1}{\delta}\int_0^\delta T(t)f(x)\, dt, \quad x \in E.$$

Then $f_\delta \in BUC^\circ(E)$. Moreover, $\lim_{\delta\downarrow 0} f_\delta = f$ in the mixed topology inherited from $C_b(E)$.

Proof. First note that $t \mapsto T(t)f(x)$ is continuous for all $x \in E$, and therefore the function f_δ is well defined. It is clear that $f_\delta \in BUC(E)$ and $\|f_\delta\| \leqslant 1$. For all $x \in E$ and $t \in (0, \delta)$ we have

$$|T(t)f_\delta(x) - f_\delta(x)| = \frac{1}{\delta}\Big|\int_t^{\delta+t} T(s)f(x)\,ds - \int_0^\delta T(s)f(x)\,ds\Big| \leqslant \frac{2t}{\delta}\|f\|.$$

Thus $\|T(t)f_\delta - f_\delta\| \leqslant 2\delta^{-1}t\|f\|$, which shows that $f_\delta \in BUC^\circ(E)$. The final statement is obvious. ☐

Obviously, if $S(t) = S(s)$ for certain $t, s \geqslant 0$, then $R(t) = R(s)$. The following lemma describes what happens if $S(t) \neq S(s)$.

Lemma 2.3. *For all $t, s \geqslant 0$ such that $S(t) \neq S(s)$ we have $\|R^\circ(t) - R^\circ(s)\|_{\mathscr{L}(BUC^\circ(E))} = 2$.*

Proof. Fix $t, s \geqslant 0$ such that $S(t) \neq S(s)$. We may assume that $t > s \geqslant 0$. Choose $x_0^* \in E^*$ such that $S^*(t)x_0^* \neq S^*(s)x_0^*$. Noting that $S^*(s)x_0^* \neq 0$ we pick $x_0 \in E$ such that $\langle x_0, S^*(t)x_0^*\rangle = 0$ and $\langle x_0, S^*(s)x_0^*\rangle = \pi$. The function $f(x) := \cos\langle x, x_0^*\rangle$ defines an element of $BUC(E)$ and we have

$$\|R(t)f - R(s)f\| \geqslant \big|R(t)f(x_0) - R(s)f(x_0)\big| = 2.$$

Given $\varepsilon > 0$ we choose $\delta > 0$ small enough such that

$$\big|R(t)f_\delta(x_0) - R(s)f_\delta(x_0)\big| = \big|(R(t)f)_\delta(x_0) - (R(s)f)_\delta(x_0)\big| \geqslant 2 - \varepsilon,$$

where f_δ is defined as in the previous lemma. Since $f_\delta \in BUC^\circ(E)$, $\|f_\delta\| \leqslant 1$, and $\|R^\circ(t)\| \leqslant 1$, $\|R^\circ(s)\| \leqslant 1$, the lemma follows. ☐

In combination with the technique described in Introduction we obtain a similar result for the Ornstein-Uhlenbeck semigroup:

Proposition 2.4. *For all $t, s \geqslant 0$ such that $S(t) \neq S(s)$ we have $\|P^\circ(t) - P^\circ(s)\|_{\mathscr{L}(BUC^\circ(E))} = 2$.*

Proof. Define the invertible isometries $V_n : BUC(E) \to BUC(E)$ by

$$V_n f(x) = f(n^{-1}x), \qquad x \in E, \quad f \in BUC(E).$$

We will show that L_{R° belongs to the limit class of L_{P°. To this end, for any $f \in BUC(E)$ and $x \in E$, one has

$$|V_n^{-1}P(t)V_n f(x) - R(t)f(x)| \leqslant \int_E \Big|f(S(t)x + n^{-1}y) - f(S(t)x)\Big|\,d\mu_t(y) \leqslant \int_E \omega_f(n^{-1}y)\,d\mu_t(y),$$

where ω_f denotes the modulus of continuity of f. Letting $n \to \infty$, the last term tends to 0 by the dominated convergence theorem. Hence, for any $f \in BUC(E)$,

$$\lim_{n\to\infty} \|V_n^{-1}P(t)V_n f - R(t)f\| = 0.$$

The result now follows from Lemma 2.3. ☐

Corollary 2.5. *If $A \neq 0$, then there exists $t_0 > 0$ such that $\|P^\circ(t) - P^\circ(s)\|_{\mathscr{L}(BUC^\circ(E))} = 2$ for all $0 \leqslant t, s \leqslant t_0$, $t \neq s$.*

Proof. If such t_0 does not exist, there exist sequences $s_n \downarrow 0$ and $t_n \downarrow 0$ with $s_n \leqslant t_n$ such that $\|P^\circ(t_n) - P^\circ(s_n)\|_{\mathscr{L}(BUC^\circ(E))} < 2$ for all n. By Proposition 2.4, $S(s_n) = S(t_n)$ for all n. Fixing an element $x \in D(A)$, for all n we obtain

$$\int_{s_n}^{t_n} S(r)Ax\,dr = S(t_n)x - S(s_n)x = 0.$$

Upon dividing both sides by $t_n - s_n$ and passing to the limit $n \to \infty$ we obtain $Ax = 0$. This being true for all $x \in D(A)$ we conclude that $A = 0$. ☐

By a result of [25] the same conclusion holds for $A = 0$ if the range of Q is infinite-dimensional; see also [11] where the special case of a Hilbert space E was considered.

We proceed with a different sufficient condition for norm discontinuity which, for the case of a Hilbert spaces E, is implicitly contained in [25]. Two probability measures μ and ν are called *disjoint*, notation $\mu \perp \nu$, if there exist disjoint measurable sets A and B such that $\mu(A) = \nu(B) = 1$. The measures μ and ν are called *equivalent*, notation $\mu \sim \nu$, if they are mutually absolutely continuous, i.e., $\mu \ll \nu$ and $\nu \ll \mu$.

Proposition 2.6. *For all $t, s \geqslant 0$ such that $\mu_t \perp \mu_s$ we have $\|P^\circ(t) - P^\circ(s)\|_{\mathscr{L}(BUC^\circ(E))} = 2$.*

Proof. By assumption we have $\|\mu_t - \mu_s\|_{\rm var} = 2$, where $\|\cdot\|_{\rm var}$ denotes the total variation norm of a finite signed Radon measure. Identifying μ_t and μ_s with elements from the dual of $BUC^\circ(E)$, it will be enough to show that $\|\mu_t - \mu_s\|_{(BUC^\circ(E))^*} = 2$. Indeed, once we know this, given $\varepsilon > 0$ we choose $g \in BUC^\circ(E)$ with $\|g\| = 1$ such that $|\langle g, \mu_t - \mu_s\rangle| \geqslant 2 - \varepsilon$ and observe that

$$\|P^\circ(t) - P^\circ(s)\|_{\mathscr{L}(BUC^\circ(E))} \geqslant |P^\circ(t)g(0) - P^\circ(s)g(0)| = |\langle g, \mu_t - \mu_s\rangle| \geqslant 2 - \varepsilon.$$

Suppose ν is a finite signed Radon measure on E. Generalizing [25, Lemma 2.3], the proof will be finished by showing that

$$\|\nu\|_{(BUC^\circ(E))^*} = \|\nu\|_{\rm var}. \tag{2.1}$$

The inequality '$\leqslant$' is clear. To check the inequality '$\geqslant$', by the Jordan-Hahn decomposition it is enough to prove the assertion when ν is a Radon probability measure on E. By [1, Section 1.1], for any given $\varepsilon > 0$ there exists $f \in BUC(E)$ with $\|f\| \leqslant 1$ such that $\langle f, \nu\rangle \geqslant 1 - \varepsilon$. For $\delta > 0$ define $f_\delta \in BUC^\circ(E)$ as in Lemma 2.2. By inner regularity of ν, the supremum of $\nu(K)$ with K ranging over all compact subsets of E equals 1. Hence to prove (2.1) it is enough to observe that by Lemma 2.2 we have $\lim_{\delta\downarrow 0} f_\delta = f$ uniformly on compact sets. ☐

In the converse direction we have the following result.

Proposition 2.7. *If $t, s \geqslant 0$ are such that $S(t) = S(s)$ and $\|P^\circ(t) - P^\circ(s)\|_{\mathscr{L}(BUC^\circ(E))} = 2$, then $\mu_t \perp \mu_s$.*

Proof. Given $\varepsilon > 0$, there exist $f \in BUC^\circ(E)$ and $x \in E$ such that $|P^\circ(t)f(x) - P^\circ(s)f(x)| \geqslant 2 - \varepsilon$. Defining $g \in BUC^\circ(E)$ by $g(y) = f(S(s)x + y)$, this may be restated as

$$\begin{aligned}
\Big| \int_E g(y)\, d\mu_t(y) - \int_E g(y)\, d\mu_s(y)\Big| \\
&= \Big| \int_E f(S(s)x + y)\, d\mu_t(y) - \int_E f(S(s)x + y)\, d\mu_s(y)\Big| \\
&= \Big| \int_E f(S(t)x + y)\, d\mu_t(y) - \int_E f(S(s)x + y)\, d\mu_s(y)\Big| \\
&= |P^\circ(t)f(x) - P^\circ(s)f(x)| \geqslant 2 - \varepsilon.
\end{aligned}$$

This shows that $\|\mu_t - \mu_s\|_{(BUC^\circ(E))^*} \geqslant 2 - \varepsilon$. Since the choice of $\varepsilon > 0$ is arbitrary we obtain that

$$2 \leqslant \|\mu_t - \mu_s\|_{(BUC^\circ(E))^*} \leqslant \|\mu_t - \mu_s\|_{\rm var} \leqslant 2,$$

the second and third of these inequalities being obvious. Hence $\|\mu_t - \mu_s\|_{\rm var} = 2$, which implies that $\mu_t \perp \mu_s$. ☐

By putting these results together we have proved:

Theorem 2.8. *For all $t, s \geqslant 0$ the following assertions are equivalent:*

1. $\|P^\circ(t) - P^\circ(s)\|_{\mathscr{L}(BUC^\circ(E))} = 2$;
2. $S(t) \neq S(s)$ *or* $\mu_t \perp \mu_s$.

It should be observed that neither $S(t) \neq S(s)$ implies $\mu_t \perp \mu_s$ or conversely. An example of a periodic semigroup with period 1 such that $\mu_t \perp \mu_s$ for all $t, s \geqslant 1$ is given in [21, Example 3.8]. On the other hand, if $\dim E < \infty$, then for any choice of S and B the measures μ_t and μ_s are mutually absolutely continuous for all $t, s \geqslant t_0$.

We continue with two examples which show that $\|P(t) - P(s)\|_{\mathscr{L}(BUC^\circ(E))} < 2$ may occur for certain values of $t \neq s$.

Example 2.9 (Nilpotent S). Let $E = L^2(0,1)$ and let S be the nilpotent shift semigroup on $L^2(0,1)$, see for instance [12, page 120]. Then $S(t) = S(s) = 0$ and $\mu_t = \mu_s = \mu_1$ for all $t, s \geqslant 1$. Hence, $P(t) = P(s)$ for all $t \geqslant s \geqslant 1$.

Example 2.10 (Periodic S in finite dimensions). Let $H = E = \mathbb{R}^2$ and let S be the rotation group on $\mathbb{R}^2$. Let $B := I$, the identity operator on $\mathbb{R}^2$. Since $S^*(t) = S(-t)$ for all $t \geqslant 0$, the covariance operator of μ_t is given by $Q_t = tI$. Hence μ_t is the Gaussian measure on $\mathbb{R}^2$ with variance t. For $k = 0, 1, 2, \ldots$ and $f \in BUC^\circ(\mathbb{R}^2)$,

$$P^\circ(2k\pi)f(x) = \int_E f(S(2k\pi)x + y)\, d\mu_{2k\pi}(y) = \int_E f(x + y)\, d\mu_{2k\pi}(y).$$

For $j \geqslant 1$, $k \geqslant 1$, $j \neq k$, we have $S(2j\pi) = S(2k\pi)$ and $\mu_{2j\pi} \sim \mu_{2k\pi}$. Theorem 2.8 shows that $\|P^\circ(2j\pi) - P^\circ(2k\pi)\| < 2$.

We will show next that the above two examples are in some sense the only possible ones.

Recall that a Gaussian measure ν on E is called *nondegenerate* if there exists no proper closed subspace E_0 of E with $\nu(E_0) = 1$. It is easy to see that ν is nondegenerate if and only if its covariance operator has dense range.

For $t > 0$ fixed, P is said to be *strongly Feller at time* t if $P(t)f \in C_b(E)$ for all $f \in B_b(E)$. Here $B_b(E)$ denotes the space of real-valued bounded Borel functions on E. As is well known, P is strongly Feller at time t if and only if we have $S(t)E \subseteq H_{Q_t}$, where H_{Q_t} is the reproducing kernel Hilbert space associated with Q_t; cf. [9, 21].

Theorem 2.11. *Suppose $t > s \geqslant 0$ are such that $\|P^\circ(t) - P^\circ(s)\|_{\mathscr{L}(BUC^\circ(E))} < 2$. Assume in addition that one of the following two assumptions is satisfied:*

(i) *μ_{t-s} is nondegenerate;*
(ii) *P is strongly Feller at time $t - s$.*

Then there exists a direct sum decomposition into S-invariant subspaces $E = E_0 \oplus E_1$, with $\dim E_1 < \infty$, such that S is nilpotent on E_0 and periodic on E_1 with period $t - s$.

Proof. By Theorem 2.8, the assumption of the theorem implies that $S(t) = S(s)$ and $\mu_t \not\perp \mu_s$. By the Feldman-Hajek theorem [2, Theorem 2.7.2], $\mu_t \sim \mu_s$.

Let H_Q be the reproducing kernel Hilbert space associated with $Q = BB^*$ and let E_s denote the closure of the range of $S(s)$. Define $j : H_Q \to E_s$ by $j := S(s)B$ and $R \in \mathscr{L}(E_s^*, E_s)$ by $R := jj^* = S_s(s)QS_s^*(s)$, where $S_s(s)$ is the operator $S(s)$ regarded as an operator from E to E_s. For $\tau > 0$ introduce the operators $R_\tau \in \mathscr{L}(E_s^*, E_s)$ by

$$R_\tau y^* := \int_0^\tau S(u)RS^*(u)y^*\, du, \qquad y^* \in E_s^*,$$

where by abuse of notation we think of S as a semigroup on E_s. Then R_τ is the covariance operator of the image measure $\nu_\tau = S_s(s)\mu_\tau$ on E_s, i.e., $R_\tau = S_s(s)Q_\tau S_s^*(s)$. Moreover,

$$\nu_s = S_s(s)\mu_s \sim S_s(s)\mu_t = \nu_t. \tag{2.2}$$

Clearly,

$$S(t-s)|_{E_s} = I|_{E_s}. \tag{2.3}$$

By (2.2) and [21, Corollary 3.3], for $\tilde{k} \in \mathbb{N}$ such that $\tilde{k}(t-s) \geqslant s$ we obtain

$$\nu_{\tilde{k}(t-s)} = \nu_{s+(\tilde{k}(t-s)-s)} \sim \nu_{t+(\tilde{k}(t-s)-s)} = \nu_{(\tilde{k}+1)(t-s)}.$$

But by (2.3) we have $R_{(\tilde{k}+1)(t-s)} = (\tilde{k}+1)R_{t-s}$, and therefore the Feldman-Hajek theorem implies that the reproducing kernel Hilbert space $H_{R_{t-s}}$ associated with R_{t-s} is finite-dimensional; cf. [2, Example 2.7.4].

We will show below that each of the conditions (i) and (ii) implies that the measure ν_{t-s} is nondegenerate. Once we know this, the proof can be finished as follows. Since ν_{t-s} is nondegenerate, the reproducing kernel Hilbert space $H_{R_{t-s}}$ is dense in E_s. It follows that $H_{R_{t-s}} = E_s$, which means that E_s is finite-dimensional. Hence E_s equals the range of $S(s) = S(t)$. By the semigroup property, E_s equals also the range of $S(k(t-s))$, where the integer $k \geqslant 1$ is such that $s \leqslant k(t-s) < t$. In combination with (2.3) it follows that $S(k(t-s))$ is a projection in E. This proves the theorem, with $E_0 := \ker S(k(t-s))$ and $E_1 := E_s = \operatorname{range} S(k(t-s))$.

To finish the proof we show that both (i) and (ii) imply the nondegeneracy of the measure ν_{t-s}.

First assume (i). It is immediate from the definition that the image of a nondegenerate Gaussian measure under a bounded operator with dense range is nondegenerate. Thus the nondegeneracy assumption on μ_{t-s} implies that ν_{t-s} is nondegenerate.

Next assume (ii). Write $H_{t-s} := H_{Q_{t-s}}$ for brevity and let $i_{t-s} : H_{t-s} \hookrightarrow E$ be the inclusion mapping. Recalling that $Q_{t-s} = i_{t-s} \circ i_{t-s}^*$, for all $u^* \in E_s^*$ and $x^* \in E^*$ such that $x^*|_{E_s} = u^*$ we have

$$\langle R_{t-s}u^*, u^* \rangle = \langle S_s(s)Q_{t-s}S_s^*(s)u^*, u^* \rangle = \langle S(s)Q_{t-s}S^*(s)x^*, x^* \rangle = \|i_{t-s}^*S^*(s)x^*\|_{H_{t-s}}^2.$$

By the strong Feller property and a closed graph argument, $S(t-s)$ is bounded as an operator from E to H_{t-s}. Denoting this operator by $T(t-s)$ we have $S(t-s) = i_{t-s} \circ T(t-s)$. Let $I : E_s \to E$ be the inclusion mapping. On E_s we have $S(t) \circ I = I \circ S(t)$, where as before we abuse of notation by writing S for the restriction of S to E_s. Then, for all $x^* \in E^*$,

$$\begin{aligned}\|S^*(t)I^*x^*\| &= \|I^*S^*(t-s)S^*(s)x^*\| \\ &= \|I^*T^*(t-s)i_{t-s}^*S^*(s)x^*\| \leqslant \|T(t-s)I\|_{\mathscr{L}(E_s,H_{t-s})}\,\|i_{t-s}^*S^*(s)x^*\|_{H_{t-s}}.\end{aligned}$$

Combining these things we obtain

$$\|T(t-s)I\|_{\mathscr{L}(E_s,H_{t-s})}^2\,\langle R_{t-s}I^*x^*, I^*x^* \rangle \geqslant \|S^*(t)I^*x^*\|^2 \geqslant c_t^2\|I^*x^*\|^2, \qquad x^* \in E^*,$$

where the last estimate follows from the fact that S is periodic on E_s. Since I^* is a surjection from E^* onto E_s^*, this gives that either R_{t-s} is nondegenerate or $T(t-s)I = 0$. In the first case the proof is complete. If $T(t-s)I = 0$, then $S(t-s)I = 0$ as well, which means that $S(t-s) = 0$ on E_s. By periodicity this implies that $E_s = \{0\}$. This in turn implies that $S(s) = 0$, i.e., S is nilpotent on E. ∎

The nondegeneracy assumption on μ_{t-s} in (i) is fulfilled if Q has dense range; this is proved in the same way as [13, Lemma 5.2].

Corollary 2.12. *Let* $\dim E = \infty$, *and assume that* S *is analytic and condition* (i) *or* (ii) *is satisfied. Then for all* $t > s \geqslant 0$ *we have* $\|P^\circ(t) - P^\circ(s)\|_{BUC^\circ(E)} = 2$.

Proof. An analytic C_0-semigroup on a nonzero Banach space cannot be nilpotent. Hence, Theorem 2.11 shows that if there exist $t > s \geqslant 0$ such that $\|P^\circ(t) - P^\circ(s)\|_{BUC^\circ(E)} < 2$, then $\dim E < \infty$. ∎

Next we consider the case $A = 0$. In this situation one has $Q_t = tQ = tBB^*$, and since by our standing assumption these operators are Gaussian covariances, it follows that Q is a Gaussian covariance. We denote the Gaussian measure on E with covariance operator Q by ν. The semigroup P is then the heat semigroup given by

$$P(t)f(x) = \int_E f(x+y)\,d\mu_t(y) = \int_E f(x+\sqrt{t}\,y)\,d\nu(y), \qquad f \in C_b(E).$$

The restriction of P to $BUC(E)$ is strongly continuous with respect to the supremum norm. The infinitesimal generator L_P of P is given, on a suitable core of cylindrical functions, by

$$L_P f(x) = \tfrac{1}{2}\operatorname{Tr} QD^2 f(x).$$

The following result was proved in [20] for the special case of an infinite-dimensional Hilbert space E. Our proof is essentially the same, the main difference being that the coordinate-free presentation simplifies matters somewhat. The spectrum and approximate point spectrum of L_P in $BUC(E)$ are denoted by $\sigma(L_P)$ and $\sigma_{\mathrm{a}}(L_P)$, respectively.

Proposition 2.13. *If $A = 0$ and Q is not of finite rank, then $\sigma(L_P) = \sigma_{\mathrm{a}}(L_P) = \mathbb{C}^-$.*

Proof. Fix a sequence (x_n^*) in E^* such that $(B^* x_n^*)$ is an orthonormal sequence in H. Such a sequence exists since the range of B^* is not finite-dimensional in H. For each $n \geqslant 1$ we consider the map $T_n : E \to \mathbb{R}^n$ defined by

$$T_n x := (\langle x, x_1^*\rangle, \dots, \langle x, x_n^*\rangle).$$

The image measure of ν under T_n equals γ_n, the standard Gaussian measure on $\mathbb{R}^n$.

Let Δ_n be the Laplace operator acting in $BUC(\mathbb{R}^n)$. Denoting the heat semigroup on $BUC(\mathbb{R}^n)$ generated by $\frac{1}{2}\Delta_n$ by $\{P_n(t)\}_{t\geqslant 0}$, for all $f \in BUC(\mathbb{R}^n)$ and $x \in E$ we have

$$P(t)f(T_n x) = \int_E f(T_n(x + \sqrt{t}\,y))\,d\nu(y) = \int_{\mathbb{R}^n} f(T_n x + \sqrt{t}\,\eta))\,d\gamma_n(\eta) = P_n(t)f(T_n(x)).$$

From this it is immediate that $f \circ T_n \in D(L_P)$ whenever $f \in D(\Delta_n)$ and in this case,

$$L_P(f \circ T_n) = (\tfrac{1}{2}\Delta_n f) \circ T_n.$$

Fix $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda < 0$ and consider the functions $f_{n,\lambda}, g_{n,\lambda} \in BUC(\mathbb{R}^n)$ defined by

$$f_{n,\lambda}(\xi) = \exp\Big(\frac{\lambda}{n}|\xi|^2\Big) \quad \text{and} \quad g_{n,\lambda}(\xi) = \frac{-2\lambda^2|\xi|^2}{n^2} f_{n,\lambda}(\xi), \qquad \xi \in \mathbb{R}^n.$$

We have $f_{n,\lambda} \in D(\Delta_n)$ and

$$(\lambda - \tfrac{1}{2}\Delta_n) f_{n,\lambda} = g_{n,\lambda}.$$

Hence $f_{n,\lambda} \circ T_n \in D(L_P)$ and

$$(\lambda - L_P)(f_{n,\lambda} \circ T_n) = g_{n,\lambda} \circ T_n.$$

Moreover,

$$\|f_{n,\lambda} \circ T_n\|_{BUC(E)} = \|f_{n,\lambda}\|_{BUC(\mathbb{R}^n)} = 1$$

and we compute

$$\|g_{n,\lambda} \circ T_n\|_{BUC(E)} = \|g_{n,\lambda}\|_{BUC(\mathbb{R}^n)} = \frac{2|\lambda|^2}{ne\,|\operatorname{Re}\lambda|}.$$

This proves that the sequence $(f_{n,\lambda} \circ T_n)$ is an approximate eigenvector for L_P, with approximate eigenvalue λ. It follows that $\{\operatorname{Re}\lambda < 0\} \subseteq \sigma_{\mathrm{a}}(L_P)$. On the other hand, since $\{P(t)\}_{t\geqslant 0}$ is a contraction semigroup on $BUC(E)$, we have $\{\operatorname{Re}\lambda > 0\} \subseteq \varrho(L_P)$, where $\varrho(L_P)$ denotes the resolvent set of L_P. Combining this, we see that $\sigma(L_P) = \mathbb{C}^-$. Moreover, $i\mathbb{R} = \partial\sigma(L_P) \subseteq \sigma_{\mathrm{a}}(L_P)$ by the general theory of semigroups, and therefore $\sigma(L_P) = \sigma_{\mathrm{a}}(L_P) = \mathbb{C}^-$. ☐

If $A = 0$ and $E = \mathbb{R}^d$, then P is analytic and therefore $\sigma(L_P)$ is contained in a strict subsector in $\mathbb{C}^-$. For $A \neq 0$, Q invertible, and $E = \mathbb{R}^d$, it was shown in [17] that $\sigma(L_{P^\circ}) = \mathbb{C}^-$ if $\sigma(A) \subseteq \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\}$ and that $\sigma(L_{P^\circ}) \supseteq \mathbb{C}^-$ if $\sigma(A) \subseteq \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$, and that in both cases every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda < 0$ is an eigenvalue. We have the following extension of this result to infinite dimensions:

Theorem 2.14. *Assume that the operator Q has dense range. Assume also that S is eventually compact and that $\sigma(A)$ is nonempty.*

1. *If $\sigma(A) \subseteq \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\}$, then $\sigma(L_{P^\circ}) = \mathbb{C}^-$ and every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda < 0$ is an eigenvalue.*
2. *If $\sigma(A) \subseteq \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$, then $\sigma(L_{P^\circ}) \supseteq \mathbb{C}^-$ and every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda < 0$ is an eigenvalue.*

The proof is based on the same Riesz projection argument as Theorem 3.7 below and is left to the reader.

3. The Ornstein-Uhlenbeck semigroup in spaces of integrable functions

Our approach to proving norm discontinuity of Ornstein-Uhlenbeck semigroups in L^1-spaces is based on the following observation.

Lemma 3.1. *For all $t > s \geqslant 0$ with $S(t-s) \neq I$ there exist $x_0 \in E$ and $r > 0$ such that*

$$\{x \in E : \ \|S(t)x - x_0\| < r\} \cap \{x \in E : \ \|S(s)x - x_0\| < r\} = \varnothing.$$

Proof. Choose $x_0 \in E$ such that $S(t-s)x_0 \neq x_0$. Let $M := \max\{1, \|S(t-s)\|_{\mathscr{L}(E)}\}$ and put

$$r := \frac{1}{2M}\|S(t-s)x_0 - x_0\|.$$

Suppose $x \in E$ is such that $\|S(s)x - x_0\| < r$. We will prove that $\|S(t)x - x_0\| \geqslant r$. By assumption there exists a vector $x_1 \in E$ with $\|x_1\| < r$ such that $S(s)x = x_0 + x_1$. Then,

$$\begin{aligned}\|S(t)x - x_0\| &= \|S(t-s)(x_0 + x_1) - x_0\| \\ &\geqslant \|S(t-s)x_0 - x_0\| - \|S(t-s)x_1\| \geqslant 2Mr - Mr = Mr \geqslant r.\end{aligned}$$

□

Until further notice we now specialize to the case where $E = \mathbb{R}^d$ and assume that A is an $(d \times d)$-matrix with real coefficients. We write $S(t) = e^{tA}$. As before, R indicates the drift semigroup given by (1.7). Let $C_c(\mathbb{R}^d)$ denote the space of continuous compactly supported functions on $\mathbb{R}^d$.

For all $f \in C_c(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} |R(t)f(x)|\,dx = \frac{1}{|\det(S(t))|}\int_{\mathbb{R}^d} |f(S(t)x)|\,|\det(S(t))|\,dx = e^{-t\operatorname{Tr}A}\int_{\mathbb{R}^d} |f(y)|\,dy. \tag{3.1}$$

It follows that the restrictions of $R(t)$ to $C_c(\mathbb{R}^d)$ extend to bounded operators on $L^1(\mathbb{R}^d)$ of norm $\|R(t)\|_{L^1(\mathbb{R}^d)} = e^{-t\operatorname{Tr}A}$. Since also $\lim_{t\downarrow 0}\|R(t)f - f\|_{L^1(\mathbb{R}^d)} = 0$ for all $f \in C_c(\mathbb{R}^d)$ it follows that R has a unique extension to a C_0-semigroup on $L^1(\mathbb{R}^d)$. The space $C_c^1(\mathbb{R}^d)$ is a core for its generator L_R and we have

$$L_R f(x) = \langle Ax, Df(x)\rangle, \qquad x \in \mathbb{R}^d, \ f \in C_c^1(\mathbb{R}^d).$$

Proposition 3.2. *For all $t > s \geqslant 0$ with $S(t) \neq S(s)$ we have $\big\|e^{t\operatorname{Tr}A}R(t) - e^{s\operatorname{Tr}A}R(s)\big\|_{\mathscr{L}(L^1(\mathbb{R}^d))} = 2$.*

Proof. Let $x_0 \in \mathbb{R}^d$ and $r > 0$ be as in Lemma 3.1. By Lemma 3.1,

$$\begin{aligned}&\big\|e^{t\operatorname{Tr}A}R(t)\mathbf{1}_{\{\|x-x_0\|<r\}} - e^{s\operatorname{Tr}A}R(s)\mathbf{1}_{\{\|x-x_0\|<r\}}\big\| \\ &\quad= \big\|e^{t\operatorname{Tr}A}\mathbf{1}_{\{\|S(t)x-x_0\|<r\}} - e^{s\operatorname{Tr}A}\mathbf{1}_{\{\|S(s)x-x_0\|<r\}}\big\| \\ &\quad= e^{t\operatorname{Tr}A}\|\mathbf{1}_{\{\|S(t)x-x_0\|<r\}}\| + e^{s\operatorname{Tr}A}\|\mathbf{1}_{\{\|S(s)x-x_0\|<r\}}\| \\ &\quad= e^{t\operatorname{Tr}A}\|R(t)\mathbf{1}_{\{\|x-x_0\|<r\}}\| + e^{s\operatorname{Tr}A}\|R(s)\mathbf{1}_{\{\|x-x_0\|<r\}}\| = 2\|\mathbf{1}_{\{\|x-x_0\|<r\}}\|,\end{aligned}$$

where in the last step we used (3.1). It follows that $\big\|e^{t\operatorname{Tr}A}R(t) - e^{s\operatorname{Tr}A}R(s)\big\| \geqslant 2$. Since by (3.1) we also have $e^{\tau\operatorname{Tr}A}\|R(\tau)\| \leqslant 1$ for all $\tau \geqslant 0$, the proposition is proved. □

Our next aim is to extend the Ornstein-Uhlenbeck semigroup P to $L^1(\mathbb{R}^d)$ as well. For all $f \in C_c(\mathbb{R}^d)$ we have

$$\begin{aligned}\int_{\mathbb{R}^d} |P(t)f(x)|\,dx &\leqslant \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(S(t)x+y)|\,dx\,d\mu_t(y) \\ &= e^{-t\operatorname{Tr}A}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(\xi)|\,d\xi\,d\mu_t(y) = e^{-t\operatorname{Tr}A}\int_{\mathbb{R}^d} |f(\xi)|\,d\xi\end{aligned} \tag{3.2}$$

with equality for nonnegative functiones f. It follows that the restrictions of the operators $P(t)$ to $C_c(\mathbb{R}^d)$ extend to bounded operators on $L^1(\mathbb{R}^d)$ of norm $\|P(t)\|_{\mathscr{L}(L^1(\mathbb{R}^d))} = e^{-t\operatorname{Tr}A}$. Since also $\lim_{t\downarrow 0}\|P(t)f - f\|_{L^1(\mathbb{R}^d)} = 0$ for all $f \in C_c(\mathbb{R}^d)$ it follows that the restriction of P to $C_c(\mathbb{R}^d)$ has a unique extension to a C_0-semigroup on $L^1(\mathbb{R}^d)$, which is still given by formula (1.2). The space $C_c^2(\mathbb{R}^d)$ is a core for its generator L_P and we have

$$Lf(x) = \tfrac{1}{2}\mathrm{Tr}\, QD^2f(x) + \langle Ax, Df(x)\rangle, \qquad x \in \mathbb{R}^d,\ f \in C_c^2(\mathbb{R}^d).$$

Theorem 3.3. *For all $t > s \geqslant 0$ with $S(t) \neq S(s)$ we have $\left\|e^{-t\mathrm{Tr}\, A}P(t) - e^{-s\mathrm{Tr}\, A}P(s)\right\|_{\mathscr{L}(L^1(\mathbb{R}^d))} = 2$.*

Proof. For $n = 1, 2, \ldots$ let $V_n : L^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ denote the invertible isometry

$$V_nf(x) = n^{-d}f(n^{-1}x), \qquad x \in \mathbb{R}^d,\ f \in L^1(\mathbb{R}^d).$$

As in the proof of Proposition 2.4 we see that L_R belongs to the limit class of L_P. Hence by Proposition 3.2 and [10, Proposition 12], applied to the operators $L_P - \mathrm{Tr}\, A$ and $L_R - \mathrm{Tr}\, A$,

$$\left\|e^{-t\mathrm{Tr}\, A}P(t) - e^{-s\mathrm{Tr}\, A}P(s)\right\|_{\mathscr{L}(L^1(\mathbb{R}^d))} \geqslant \left\|e^{-t\mathrm{Tr}\, A}R(t) - e^{-s\mathrm{Tr}\, A}R(s)\right\|_{\mathscr{L}(L^1(\mathbb{R}^d))} \geqslant 2.$$

Since by (3.2) we also have $e^{\tau\mathrm{Tr}\, A}\|P(\tau)\| \leqslant 1$ for all $\tau \geqslant 0$, the theorem is proved. □

Alternatively this theorem may be derived from Proposition 2.4 via the duality argument of [17, Lemma 3.6].

After these preparations we come to the main results of this section, which give conditions for norm discontinuity of P in the space $L^1(E, \mu_\infty)$, where μ_∞ is the invariant measure for P discussed in Section 1. Note that in finite dimensions, its existence is guaranteed under the mere assumption that the limit $Q_\infty := \lim_{t\to\infty} Q_t$ exists in $\mathscr{L}(\mathbb{R}^d)$. This will be assumed in the next result, in which P denotes Ornstein-Uhlenbeck semigroup on $L^1(\mathbb{R}^d, \mu_\infty)$ and L_P its generator. Since we are dealing with the finite-dimensional case, a sufficient condition for the existence of Q_∞ is that $\sigma(A) \subseteq \{\mathrm{Re}\,\lambda < 0\}$.

Theorem 3.4. *Assume that the limit $Q_\infty := \lim_{t\to\infty} Q_t$ exists in $\mathscr{L}(\mathbb{R}^d)$ and let μ_∞ be the Gaussian measure on $\mathbb{R}^d$ with covariance matrix Q_∞. Then for all $t, s \geqslant 0$ with $t \neq s$ we have*

$$\|P(t) - P(s)\|_{\mathscr{L}(L^1(\mathbb{R}^d,\mu_\infty))} = 2.$$

Proof. As is well known [6, Proposition 1], the range of Q_∞ is invariant under the action of S and therefore we may assume without loss of generality that μ_∞ is nondegenerate. Moreover, the existence of μ_∞ implies that $S(t) \neq S(s)$ for all $t, s \geqslant 0$ with $t \neq s$, since otherwise the improper integral defining Q_∞ will diverge.

Let b be the density of μ_∞ with respect to the Lebesgue measure; this density exists since μ_∞ is assumed to be nondegenerate. Proceeding as in [18] we consider the invertible isometry $V : L^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d, \mu_\infty)$ given by $f \mapsto b^{-1}f$ and define $\tilde{P}(t) : L^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ by

$$\tilde{P}(t) = V^{-1} \circ P(t) \circ V, \qquad t \geqslant 0.$$

Then $\tilde{P} = \{\tilde{P}(t)\}_{t\geqslant 0}$ is a C_0-semigroup on $L^1(\mathbb{R}^d)$ and by the computations in [18, Theorem 5.1] its generator $\tilde{L}$ is given by

$$\tilde{L}f(x) = \tfrac{1}{2}\mathrm{Tr}\, QD^2f(x) + \langle \tilde{A}x, Df(x)\rangle + kf(x), \qquad x \in \mathbb{R}^d, \quad f \in C_c^2(\mathbb{R}^d), \tag{3.3}$$

where

$$\tilde{A} = -Q_\infty A^* Q_\infty^{-1}, \quad k = -\mathrm{Tr}\, A = -\mathrm{Tr}\, \tilde{A}.$$

The result now follows from Theorem 3.3 applied to $\tilde{L} - k$. □

Returning to the setting of an arbitrary real Banach space E, we have the following extension of Theorem 3.4.

Theorem 3.5. *Assume that the weak operator limit $Q_\infty := \lim_{t\to\infty} Q_t$ exists in $\mathscr{L}(E^*, E)$ and that it is the covariance operator of a Gaussian measure μ_∞ on E. Let S be an eventually compact C_0-semigroup on E, and assume that its generator A has nonempty spectrum. Then for all $t, s \geqslant 0$ with $t \neq s$ we have*

$$\|P(t) - P(s)\|_{\mathscr{L}(L^1(E,\mu_\infty))} = 2. \tag{3.4}$$

Proof. Replacing E by the closure of the reproducing kernel space associated with Q_∞, which is invariant under S by [6, Proposition 1], see also [21], we may assume without loss in generality that μ_∞ is nondegenerate.

Since $\sigma(A) \neq \varnothing$ we may fix some $\lambda_0 \in \sigma(A)$. Note that λ_0 is an isolated point in $\sigma(A)$. Let $\pi_0 : E \to E$ be the Riesz projection onto E_0, the finite dimensional subspace of E generated by all generalized eigenvectors associated to λ_0, cf. [12, Corollary 3.2, page 330]. The projection π_0 commutes with the operators $S(t)$. Let S_0 denote the restriction of S to E_0, with generator $A_0 \in \mathscr{L}(E_0)$, and define $Q_0 \in \mathscr{L}(E_0^*, E_0)$ by $Q_0 := \pi_0 Q \pi_0^*$. Here we think of π_0 as an operator from E onto E_0. For $0 \leqslant t \leqslant \infty$ the covariance operator $Q_{0,t}$ associated with the image measure $\mu_{0,t} = \pi_0 \mu_t$ on E_0 is given by

$$Q_{0,t} x_0^* = \int_0^t S_0(s) Q_0 S_0^*(s) x_0^* \, ds, \qquad x_0^* \in E_0^*.$$

Since $Q_{0,\infty}$ is nondegenerate and $\sigma(A_0) = \{\lambda_0\}$ we have $\operatorname{Re} \lambda_0 < 0$.

For all $f \in L^1(E_0, \mu_{0,\infty})$, the function

$$f_0(x) := f(\pi_0 x), \qquad x \in E,$$

belongs to $L^1(E, \mu_\infty)$ and we have

$$\int_E |f_0(x)| \, d\mu_\infty(x) = \int_{E_0} |f(\xi)| \, d\mu_{0,\infty}(\xi). \tag{3.5}$$

Let P_0 be the corresponding Ornstein-Uhlenbeck semigroup on E_0, i.e.,

$$P_0(t) f(x_0) = \int_{E_0} f(S_0(t) x_0 + \xi) \, d\mu_{0,t}(\xi), \qquad t \geqslant 0, \quad x_0 \in E_0, \quad f \in L^1(E_0, \mu_{0,\infty}).$$

With these notations,

$$(P_0(t) f)(\pi_0 x) = P(t) f_0(x). \tag{3.6}$$

Now let $t > s \geqslant 0$ be such that (3.4) holds. Then, by virtue of (3.5) and (3.6),

$$\begin{aligned} \|P(t) - P(s)\|_{\mathscr{L}(L^1(E,\mu_\infty))} &\geqslant \sup_{\|f\|_{L^1(E_0,\mu_{0,\infty})} \leqslant 1} \|P(t) f_0 - P(s) f_0\|_{L^1(E,\mu_\infty)} \\ &= \sup_{\|f\|_{L^1(E_0,\mu_{0,\infty})} \leqslant 1} \|P_0(t) f - P_0(s) f\|_{L^1(E_0,\mu_{0,\infty})} = 2, \end{aligned}$$

where the last step follows from the previous theorem. Since P is a contraction semigroup in $L^1(E, \mu_\infty)$, the equality (3.4) follows. □

Our final result concerns the spectrum of L_P. The following description of $\sigma(L_P)$ in $L^1(\mathbb{R}^d, \mu_\infty)$ was shown in [18], where it was derived from the characterization of $\sigma(L_P)$ for $L^1(\mathbb{R}^d)$, see [17].

Theorem 3.6. *Assume that the limit $Q_\infty := \lim_{t\to\infty} Q_t$ exists in $\mathscr{L}(\mathbb{R}^d)$ and let μ_∞ be the Gaussian measure on $\mathbb{R}^d$ with covariance matrix Q_∞. The spectrum of L_P in $L^1(\mathbb{R}^d, \mu_\infty)$ equals $\mathbb{C}^-$, and every $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda < 0$ is an eigenvalue of L_P.*

In setting of a real Banach space E we obtain the following extension:

Theorem 3.7. *Assume that the weak operator limit $Q_\infty := \lim_{t\to\infty} Q_t$ exists in $\mathscr{L}(E^*, E)$ and that it is the covariance operator of a Gaussian measure μ_∞ on E. If S is eventually compact and $\sigma(A) \neq \varnothing$, the spectrum of L_P in $L^1(E, \mu_\infty)$ equals $\mathbb{C}^-$, and every $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda < 0$ is an eigenvalue of L_P.*

Proof. We may assume that μ_∞ is nondegenerate. Fix $\lambda_0 \in \sigma(A)$. Using the notations of the proof of Theorem 3.5, let L_{P_0} denote the generator of the semigroup P_0 on $L^1(E_0, \mu_{0,\infty})$. Theorem 3.6 implies that $\sigma(L_{P_0}) = \mathbb{C}^-$ and that every $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda < 0$ is an eigenvalue of L_{P_0}. Let $f \in L^1(E_0, \mu_{0,\infty})$ be an associated eigenvector. Then $f_0(x) := f(\pi_0 x)$ defines a function $f \in L^1(E, \mu_\infty)$ satisfying

$$P(t)f_0(x) = P_0(t)f(\pi_0 x) = e^{\lambda t} f(\pi_0 x) = e^{\lambda t} f_0(x).$$

Hence, $P(t)f_0 = e^{\lambda t} f_0$, and f_0 is an eigenvector for L_P with eigenvalue λ. □

After the completion of this paper, the authors received the preprint [5] by Chojnowska-Michalik. She proves a related extension of Theorem 3.6: if the part of A in the reproducing kernel Hilbert space of μ_∞ has an eigenvalue λ with $\mathrm{Re}\,\lambda < 0$, then $\sigma(L_P) = \mathbb{C}^-$.

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