# Non-symmetric Ornstein-Uhlenbeck semigroups in Banach spaces

J.M.A.M. van Neerven[1,2]

Department of Mathematics
Delft University of Technology
PO Box 5031, 2600 GA Delft
The Netherlands

J.vanNeerven@twi.tudelft.nl

*Abstract* - Let $E$ be a separable real Banach space and let $Q \in \mathcal{L}(E^*, E)$ be positive and symmetric. Let $\mathbf{S} = \{S(t)\}_{t \geqslant 0}$ be a $C_0$−semigroup on $E$. We study the relations between the reproducing kernel Hilbert spaces associated with the operators $Q_t := \int_0^t S(s)QS^*(s)\,ds$. Under the assumption that these operators are the covariances of centered Gaussian measures $\mu_t$ on $E$, we also study equivalence $\mu_t \sim \mu_s$ for different values of $s$ and $t$, and we calculate their Radon-Nikodym derivatives.

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## 0. Introduction

In this paper we investigate the reproducing kernel Hilbert spaces and Gaussian measures associated with a non-symmetric Ornstein-Uhlenbeck semigroup on a separable real Banach space $E$. This study is usually carried out in a Hilbert space setting, and one of the motivations of this paper was to see to what extent the theory can be extended to the Banach space setting.

The main difference between the Banach space- and the Hilbert space situation is that the covariance operator of a Gaussian measure on a Banach space $E$ is a positive symmetric operator $Q$ (the precise definitions are given in Section 1) from the dual $E^*$ into $E$, rather than an operator on $E$. Thus, in contrast to the Hilbert space situation, it is no longer possible to represent the reproducing kernel Hilbert space $H$ associated with $Q$ as $H = \operatorname{Im} Q^{1/2}$. When working in a Banach space setting, any reference to the operator $Q^{1/2}$ has therefore to be avoided. This turns out to be, at least for the

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questions considered in this paper, more of an advantage than a disadvantage, as we believe that the resulting proofs have gained some transparency.

Another difference from the Hilbert space situation is that no necessary and sufficient conditions on a positive symmetric $Q \in \mathcal{L}(E^*, E)$ seem to be known in order that $Q$ be the covariance operator of a Gaussian measure on $E$. As we will show, for several well-known results on non-symmetric Ornstein-Uhlenbeck semigroups it is not relevant whether or not the positive self-adjoint operators that one is led to, are covariances or not. In the remaining results we get around this difficulty by simply imposing that $Q$ be the covariance of a Gaussian measure; this replaces the usual assumption in the Hilbert space setting that $Q$ should be trace class.

Let us now describe in more detail the contents of this paper. Let $E$ be a real Banach space, let $Q \in \mathcal{L}(E^*, E)$ be positive and symmetric, and let $\mathbf{S} = \{S(t)\}_{t\geqslant 0}$ be a $C_0-$semigroup on $E$. The operators

$$Q_t := \int_0^t S(s)QS^*(s)\, ds$$

are well-defined in the strong sense (cf. Proposition 1.2 below), and positive and symmetric. In case $E$ is a Hilbert space and $Q_t$ is also trace class, $Q_t$ can be identified as the covariance of the distribution $\mu_t$ of the $E$-valued Gaussian random variable

$$X(t) = \int_0^t S(t-s)\, dW_Q(s),$$

where $W_Q$ denotes a cylindrical $Q-$Wiener process and the integral is an Itô type stochastic integral. The importance of this resides in the fact that the (strong Markov) process $(X(t))_{t\geqslant 0}$ is the unique weak solution of the Langevin equation

$$\begin{aligned} dX(t) &= AX(t)\, dt + dW_Q(t), \qquad t \geqslant 0, \\ X(0) &= 0 \quad \text{almost surely}, \end{aligned}$$

where $A$ is the infinitesimal generator of $\mathbf{S}$. Without the trace class assumption similar results hold; this time $W_Q$ has to be interpreted as a cylindrical $Q-$Wiener process. For a comprehensive treatment of these concepts we refer to the book [DZ3].

In Section 1 we undertake a detailed study of the reproducing kernel Hilbert spaces (RKHS's) $H_t$ associated with the operators $Q_t$. We do not assume that $Q_t$ be the covariance operators of Gaussian measures on $E$. We prove that

$$S(s)H_{t_0} \subset H_{t_0+s}$$

for all $s > 0$ and $t_0 > 0$, and that $H_{t_0} = H_{t_0+s}$ (as subsets of $E$) if and only if $S(t_0)$, regarded as an operator from $H_s$ into $H_{t_0+s}$, is a strict contraction. We also show that

$$S(s)H_{t_0} \subset H_{t_0}$$

for given $s > 0$ and $t_0 > 0$ implies

$$H_t = H_{\max\{s,t_0\}} \qquad \text{for all} \quad t \in [\max\{s, t_0\}, \infty),$$

and that this result is the best possible.

In Sections 2 through 5 we assume that $E$ is separable and that each of the $Q_t$ is the covariance operator of a centered Gaussian measures $\mu_t$ on $E$. After some preliminary observations in Section 2, we study equivalence of measures $\mu_t \sim \mu_s$ under various conditions in Section 3. For instance, it is shown that

$$\mu_t \sim \mu_{t_0} \qquad \text{for all} \quad t \in [t_0, \infty)$$

whenever there exist $s \in (0, \infty)$ and $t_1 \in (t_0, \infty)$ such that $S(s)Q = QS^*(s)$ and $\mu_{t_1} \sim \mu_{t_0}$.

In Section 4 we derive an explicit formula for the Radon-Nikodym derivative $d\mu_{t_0}/d\mu_{t_1}$ whenever these measures are equivalent. The approach depends on second quantization, existence of linear $\mu-$measurable extensions, and a classical theorem of Shale concerning absolute continuity of image measures, and may be of some interest in its own right.

In Section 5 we proceed to show that for $t_1$ fixed the Radon-Nikodym derivative $d\mu_{t_0}/d\mu_{t_1}$ depends continuously upon $t_0$.

In Section 6 we return to the cylindrical case and study the RKHS $H_\infty$ associated with the strong limit $Q_\infty = \lim_{t\to\infty} Q_t$ whenever this limit exists. Assuming that $Q_\infty$ is the covariance of an (invariant) centered Gaussian measure $\mu_\infty$ on $E$, we discuss versions for $\mu_\infty$ of some of the results obtained in the previous sections. For Hilbert spaces $E$, the main results of this section have been obtained recently by Chojnowska-Michelak and Goldys [CG1-3], [Go] and Fuhrman [Fu]. In particular this is true for the expression of the Radon-Nikodym derivative $d\mu_t/d\mu_\infty$, which was established by [Fu] in the null controllable case, and was extended to the more general situation considered here by [CG3]. We point out, however, that the approach taken in these references in very different for ours. To the best of our knowledge the principal results of Sections 1 through 5 are new even in the Hilbert space setting. Some of these (Theorems 1.4, 1.7, 3.2, and 4.1) extend in a natural corresponding results about invariant measures to finite $t$, but others have no analogue for invariant measures (Theorems 1.9 and 3.5) or its analogue seems to be new as well (Theorem 5.5).

In the final Section 7 we discuss some extensions of our results to the class of so-called (cylindrical) Gaussian Mehler semigroups recently introduced by Bogachev, Röckner, and Schmuland [BRS].

## 1. The reproducing kernel Hilbert spaces $H_t$

Throughout this section, $E$ is a fixed arbitrary real Banach space. A bounded linear operator $Q \in \mathcal{L}(E^*, E)$ is called *positive* if $\langle Qx^*, x^*\rangle \geqslant 0$ for all $x^* \in E^*$ and *symmetric* if $\langle Qx^*, y^*\rangle = \langle Qy^*, x^*\rangle$ for all $x^*, y^* \in E^*$. If $Q$ is positive and symmetric, then on $\operatorname{Im} Q = \{Qx^* : x^* \in E^*\}$ we may define an inner product $[\cdot,\cdot]$ by the formula $[Qx^*, Qy^*] := \langle Qx^*, y^*\rangle$. The completion $H$ of $\operatorname{Im} Q$ with respect to this inner product is a Hilbert space, and the inclusion $i : \operatorname{Im} Q \subset E$ extends to a continuous injection $i : H \to E$. Moreover, if we regard $Q$ as an operator from $E^*$ to $H$ we have the identity $i^* = Q$. We will refer to $H$ as the *reproducing kernel Hilbert space* (RKHS) associated with $Q$. If $E$ is separable, then $H$ is separable as well. If $E$ is a *Hilbert* space and $Q$ is a positive and symmetric operator on $E$ (identifying $E$ and its dual), then $H = \operatorname{Im} Q^{\frac{1}{2}}$ with identical norms. For more information we refer to [VTC, Chapter III], where the simple proofs can be found.

We recall with the following result, which is proved along the lines of [DZ2, Proposition B.1].

**Proposition 1.1.** *Let $Q, \tilde{Q} \in \mathcal{L}(E^*, E)$ be two positive symmetric operators. Then for the associated RKHS's we have $H \subset \tilde{H}$ (as subsets of $E$) if and only if there exists a constant $K > 0$ such that*

$$\langle Qx^*, x^*\rangle \leqslant K\langle \tilde{Q}x^*, x^*\rangle, \qquad \forall x^* \in E^*.$$

In this situation, the operator $V : \operatorname{Im} \tilde{Q} \to \operatorname{Im} Q$ defined by $V\tilde{Q}x^* := Qx^*$ extends to a bounded operator from $\tilde{H}$ into $H$; we will sometimes use the suggestive notation $V = Q\tilde{Q}^{-1}$. If $H = \tilde{H}$ (as subsets of $E$), $V$ is a (Banach space) isomorphism of $\tilde{H}$ onto $H$, the inverse being given by $V^{-1}Qx^* = \tilde{Q}x^*$.

Suppose $Q \in \mathcal{L}(E^*, E)$ is positive and symmetric, and let $\mathbf{S} = \{S(t)\}_{t\geqslant 0}$ be a $C_0$−semigroup on $E$. Our terminology concerning $C_0$−semigroups is standard; we refer to [Pa] for more details. For each $t > 0$ the operator $Q_t$ defined by

$$Q_t x^* := \int_0^t S(s)QS^*(s)x^*\,ds, \qquad x^* \in E^*,$$

is positive and symmetric. Note that this integral exists as a Bochner integral in $E$; strong measurability of the integrand follows from:

**Proposition 1.2.** *For all $x^* \in E^*$, the function $s \mapsto S(s)QS^*(s)x^*$ is strongly measurable.*

*Proof:* As a map from $E^*$ into $H$, the operator $Q$ is the adjoint of the inclusion map $i : H \subset E$, and as such $Q$ is weak$^*$-to-weakly continuous. Hence the weak$^*$-continuity of $S(\cdot)x^*$ implies weak continuity of $QS^*(\cdot)x^*$.

*Step 1* - First we assume that $E$ is separable. Then $H$ is separable and we may choose a countable orthonormal basis $(h_n) \subset H$. Fix $y^* \in E^*$. Expanding $QS^*(s)x^*$ and $QS^*(s)y^*$ in terms of $(h_n)$ we have

$$\langle S(s)QS^*(s)x^*, y^*\rangle = [QS^*(s)x^*, QS^*(s)y^*] = \sum_{n=1}^{\infty}[QS^*(s)x^*, h_n][QS^*(s)y^*, h_n],$$

so $\langle S(\cdot)QS^*(\cdot)x^*, y^*\rangle$ appears as a countable sum of continuous functions. This proves that $S(\cdot)QS^*(\cdot)x^*$ is weakly measurable. Since it is also separably valued by the separability of $E$, strong measurability now follows by an appeal to Pettis's measurability theorem [DU, Chapter II].

*Step 2* - Now let $E$ be arbitrary. Let $H_0$ be the closed linear span in $H$ of the set $\{QS^*(t)x^* : t \geqslant 0\}$. Since $Q$, as a map from $E^*$ to $H$, is weak$^*$–to–weakly continuous and $t \mapsto S^*(t)x^*$ is weak$^*$–continuous, $H_0$ is weakly separable and therefore separable. Denoting by $E_0$ the smallest closed $\mathbf{S}-$invariant subspace in $E$ containing $H_0$, it follows that $E_0$ is separable in $E$. Let $i_0 : H_0 \subset E_0$ and $j_0 : E_0 \subset E$ denote the inclusion maps. Now define $Q_0 \in \mathcal{L}(E_0^*, E_0)$ by

$$Q_0(j_0^* y^*) := (i_0 \circ P_0 \circ Q)y^*, \qquad y^* \in E^*,$$

where $P_0$ is the orthogonal projection of $H$ onto $H_0$. We check that $Q_0$ is well-defined. If $j_0^* y^* = 0$, then $y^*$ annihilates $E_0$ and therefore, for all $t \geqslant 0$,

$$[QS^*(t)x^*, Qy^*] = \langle QS^*(t)x^*, y^*\rangle = 0.$$

This means that $Qy^* \perp H_0$, so $P_0Qy^* = 0$ and hence $Q_0(j_0^* y^*) = 0$. Next we check that $Q_0$ is positive and symmetric. For all $y^* \in E^*$ and $z^* \in E^*$ we have

$$\langle Q_0 j_0^* y^*, j_0^* z^*\rangle = \langle iP_0Qy^*, z^*\rangle = [P_0Qy^*, Qz^*] = [P_0Qy^*, P_0Qz^*],$$

which is symmetric in $y^*$ and $z^*$ and non-negative if $y^* = z^*$.

Let $\mathbf{S}_0$ denote the restriction of $\mathbf{S}$ to the invariant subspace $E_0$. The lemma follows from the corresponding result for the separable space $E_0$ once we have realized that for all $s \geqslant 0$,

$$S(s)QS^*(s)x^* = S_0(s)i_0P_0QS^*(s)x^* = S_0(s)Q_0(j_0^*S^*(s)x^*) = S_0(s)Q_0S_0^*(s)(j_0^*x^*).$$

■

We will frequently use the following algebraic relation between the operators $Q_t$, which is immediate from their definition: for all $t, s > 0$ we have

$$Q_{t+s} = Q_s + S(s)Q_tS^*(s).$$

The RKHS associated with $Q_t$ will be denoted by $H_t$; the inclusion map $H_t \subset E$ is denoted by $i_t$. As in the case of a Hilbert space $E$, $H_t$ can be interpreted as the space of reachable states of a certain linear control problem in $E$; this point of view will be elaborated elsewhere.

The present section is devoted to a systematic study of the relation between the spaces $H_t$ for different values of $t$. The first observation is a direct consequence of Proposition 1.1:

**Proposition 1.3.** *If $0 < t_0 \leqslant t_1$, then $H_{t_0} \subset H_{t_1}$.*

From the identity $S(s)Q_{t_0}S^*(s) = Q_{t_0+s} - Q_s$ combined with Proposition 1.3 we see that $S(s)$ maps the linear subspace $\mathrm{Im}\,(Q_{t_0}S^*(s))$ of $H_{t_0}$ into $H_{t_0+s}$. The next result shows that we actually have $S(s)H_{t_0} \subset H_{t_0+s}$:

**Theorem 1.4.** *For all $s > 0$ and $t_0 > 0$ we have $S(s)H_{t_0} \subset H_{t_0+s}$. Moreover, $\|S(s)\|_{\mathcal{L}(H_{t_0},H_{t_0+s})} \leqslant 1$.*

*Proof:* For all $x^* \in E^*$ we have

$$\begin{aligned}
\|Q_{t_0}S^*(s)x^*\|^2_{H_{t_0}} &= \langle Q_{t_0}S^*(s)x^*, S^*(s)x^*\rangle \\
&= \langle Q_{t_0+s}x^*, x^*\rangle - \langle Q_s x^*, x^*\rangle \\
&\leqslant \langle Q_{t_0+s}x^*, x^*\rangle = \|Q_{t_0+s}x^*\|^2_{H_{t_0+s}}.
\end{aligned} \tag{1.1}$$

Hence,

$$|\langle Q_{t_0}S^*(s)x^*, y^*\rangle| = |[Q_{t_0}S^*(s)x^*, Q_{t_0}y^*]_{H_{t_0}}| \leqslant \|Q_{t_0+s}x^*\|_{H_{t_0+s}}\|Q_{t_0}y^*\|_{H_{t_0}}. \tag{1.2}$$

Define a linear functional $\psi_{s,y^*} : \mathrm{Im}\, Q_{t_0+s} \to \mathbb{R}$ by

$$\psi_{s,y^*}(Q_{t_0+s}x^*) := \langle Q_{t_0}S^*(s)x^*, y^*\rangle.$$

If $Q_{t_0+s}x^* = 0$, then $Q_{t_0}S^*(s)x^* = 0$ by (1.1), so $\psi_{s,y^*}$ is well-defined. By (1.2), $\psi_{s,y^*}$ extends to a bounded linear functional on $H_{t_0+s}$ of norm $\leqslant \|Q_{t_0}y^*\|_{H_t}$. Identifing $\psi_{s,y^*}$ with an element of $H_{t_0+s}$, for all $x^* \in E^*$ we have

$$\langle \psi_{s,y^*}, x^*\rangle = [Q_{t_0+s}x^*, \psi_{s,y^*}]_{H_{t_0+s}} = \langle Q_{t_0}S^*(s)x^*, y^*\rangle = \langle S(s)Q_{t_0}y^*, x^*\rangle.$$

Hence, $S(s)Q_{t_0}y^* = \psi_{s,y^*} \in H_{t_0+s}$ and $\|S(s)Q_{t_0}y^*\|_{H_{t_0+s}} \leqslant \|Q_{t_0}y^*\|_{H_{t_0}}$. ■

Whenever it is convenient, the restriction of $S(s)$ as an operator in $\mathcal{L}(H_t, H_{t+s})$ will be denoted by $S_{t\to t+s}(s)$ and its adjoint $(S_{t\to t+s}(s))^* \in \mathcal{L}(H_{t+s}, H_t)$ by $S^*_{t\to t+s}(s)$.

**Corollary 1.5.** *For all $0 < t_0 < t_1$ the inclusion mapping $H_{t_0} \subset H_{t_1}$ is contractive.*

*Proof:* For all $x^* \in E^*$ we have

$$\begin{aligned}
\|Q_{t_0}x^*\|^2_{H_{t_1}} &= [Q_{t_0}x^*, Q_{t_1}x^* - S(t_0)Q_{t_1-t_0}S^*(t_0)x^*]_{H_{t_1}} \\
&= \langle Q_{t_0}x^*, x^*\rangle - [Q_{t_0}x^*, S(t_0)Q_{t_1-t_0}S^*(t_0)x^*]_{H_{t_1}} \\
&= \|Q_{t_0}x^*\|^2_{H_{t_0}} - [Q_{t_0}x^*, S(t_0)Q_{t_1-t_0}S^*(t_0)x^*]_{H_{t_1}}.
\end{aligned}$$

But $S^*_{t_1-t_0\to t_1}(t_0)Q_{t_1} = Q_{t_1-t_0}S^*(t_0)$. Hence,

$$\begin{aligned}
&[Q_{t_0}x^*, S(t_0)Q_{t_1-t_0}S^*(t_0)x^*]_{H_{t_1}} \\
&\quad = [Q_{t_1}x^*, S(t_0)Q_{t_1-t_0}S^*(t_0)x^*]_{H_{t_1}} - \|S(t_0)Q_{t_1-t_0}S^*(t_0)x^*\|^2_{H_{t_1}} \\
&\quad = \langle Q_{t_1-t_0}S^*(t_0)x^*, S^*(t_0)x^*\rangle - \|S(t_0)Q_{t_1-t_0}S^*(t_0)x^*\|^2_{H_{t_1}} \\
&\quad = \|Q_{t_1-t_0}S^*(t_0)x^*\|^2_{H_{t_1-t_0}} - \|S_{t_1-t_0\to t_0}(t_0)Q_{t_1-t_0}S^*(t_0)x^*\|^2_{H_{t_1}} \\
&\quad \geqslant 0;
\end{aligned}$$

for the inequality we used that $\|S_{t_1-t_0\to t_1}(t_0)\|_{\mathcal{L}(H_{t_1-t_0},H_{t_1})} \leqslant 1$. We conclude that $\|Q_{t_0}x^*\|_{H_{t_1}} \leqslant \|Q_{t_0}x^*\|_{H_{t_0}}$ for all $x^* \in E^*$, and the corollary follows. ■

Next we characterize equality of $H_{t_0}$ and $H_{t_0+s}$ in terms of the restriction $S(t_0) \in \mathcal{L}(H_{t_0}, H_{t_0+s})$. For later reference, we first isolate a simple lemma.

**Lemma 1.6.** *Let $t_1 \geqslant t_0 > 0$, $s > 0$, and assume that $S(s)$ maps $H_{t_0}$ into $H_{t_1}$. Then $S(s) \in \mathcal{L}(H_{t_0}, H_{t_1})$, and for all $x^* \in E^*$ we have*

$$\|Q_{t_0} S^*(s)x^*\|_{H_{t_0}} \leqslant \|S(s)\|_{\mathcal{L}(H_{t_0},H_{t_1})} \cdot \|Q_{t_1} x^*\|_{H_{t_1}}.$$

*Proof:* By the closed graph theorem, $S(s)$ is bounded as an operator from $H_{t_0}$ into $H_{t_1}$. For all $x^* \in E^*$ we have

$$\begin{aligned}
\|Q_{t_0} S^*(s)x^*\|_{H_{t_0}} &= \sup\{[Q_{t_0} S^*(s)x^*, Q_{t_0} y^*]_{H_{t_0}} :\ y^* \in E^*, \|Q_{t_0} y^*\|_{H_{t_0}} \leqslant 1\} \\
&= \sup\{\langle S(s)Q_{t_0} y^*, x^*\rangle :\ y^* \in E^*, \|Q_{t_0} y^*\|_{H_{t_0}} \leqslant 1\} \\
&= \sup\{[S(s)Q_{t_0} y^*, Q_{t_1} x^*]_{H_{t_1}} :\ y^* \in E^*, \|Q_{t_0} y^*\|_{H_{t_0}} \leqslant 1\} \\
&\leqslant \|S(s)\|_{\mathcal{L}(H_{t_0},H_{t_1})} \cdot \|Q_{t_1} x^*\|_{H_{t_1}}.
\end{aligned}$$

■

**Theorem 1.7.** *Let $t_0 > 0$ and $h > 0$ be fixed. Then $H_{t_0} = H_{t_0+h}$ (as subsets of $E$) if and only if $\|S(t_0)\|_{\mathcal{L}(H_h,H_{t_0+h})} < 1$.*

*Proof:* We have already seen that $H_{t_0} \subset H_{t_0+h}$. It remains to prove that $H_{t_0+h} \subset H_{t_0}$ if and only if $\|S(t_0)\|_{\mathcal{L}(H_h,H_{t_0+h})} < 1$.

First assume $\|S(t_0)\|_{\mathcal{L}(H_h,H_{t_0+h})} < 1$. For all $x^* \in E^*$ we have

$$\begin{aligned}
\|Q_{t_0} x^*\|^2_{H_{t_0}} &= \langle Q_{t_0} x^*, x^*\rangle \\
&= \langle Q_{t_0+h} x^*, x^*\rangle - \langle S(t_0)Q_h S^*(t_0)x^*, x^*\rangle \\
&= \|Q_{t_0+h} x^*\|^2_{H_{t_0+h}} - \|Q_h S^*(t_0)x^*\|^2_{H_h}.
\end{aligned}$$

But by Lemma 1.6,

$$\|Q_h S^*(t_0)x^*\|_{H_h} \leqslant \|S(t_0)\|_{\mathcal{L}(H_h,H_{t_0+h})} \cdot \|Q_{t_0+h} x^*\|_{H_{t_0+h}}.$$

Hence,

$$\begin{aligned}
\langle Q_{t_0} x^*, x^*\rangle = \|Q_{t_0} x^*\|^2_{H_{t_0}} &\geqslant \left(1 - \|S(t_0)\|^2_{\mathcal{L}(H_h,H_{t_0+h})}\right) \|Q_{t_0+h} x^*\|^2_{H_{t_0+h}} \\
&= \left(1 - \|S(t_0)\|^2_{\mathcal{L}(H_h,H_{t_0+h})}\right) \langle Q_{t_0+h} x^*, x^*\rangle.
\end{aligned}$$

By Proposition 1.1 this gives the inclusion $H_{t_0+h} \subset H_{t_0}$.

Conversely, assume that $H_{t_0+h} \subset H_{t_0}$. Then there exists $K > 1$ such that

$$\langle Q_{t_0+h} x^*, x^*\rangle \leqslant K\langle Q_{t_0} x^*, x^*\rangle = K\langle Q_{t_0+h} x^*, x^*\rangle - K\langle S(t_0)Q_h S^*(t_0)x^*, x^*\rangle$$

for all $x^* \in E^*$, or equivalently,

$$\|Q_h S^*(t_0)x^*\|^2_{H_h} \leqslant (1 - K^{-1})\|Q_{t_0+h} x^*\|^2_{H_{t_0+h}}$$

for all $x^* \in E^*$. Hence for all $x^*, y^* \in E^*$,

$$\begin{aligned}
|[S(t_0)Q_h y^*, Q_{t_0+h} x^*]_{H_{t_0+h}}| &= |[Q_h y^*, Q_h S^*(t_0)x^*]_{H_h}| \\
&\leqslant \|Q_h y^*\|_{H_h} \|Q_h S^*(t_0)x^*\|_{H_h} \\
&\leqslant \sqrt{1 - K^{-1}} \|Q_h y^*\|_{H_h} \|Q_{t_0+h} x^*\|_{H_{t_0+h}}.
\end{aligned}$$

This shows that $\|S(t_0)\|_{\mathcal{L}(H_h,H_{t_0+h})} \leqslant \sqrt{1 - K^{-1}} < 1$. ■

Throughout the rest of this paper, the notation '$H_{t_1} = H_{t_0}$' means equality of $H_{t_1}$ and $H_{t_0}$ as subsets of $E$; as Hilbert spaces, $H_{t_1}$ and $H_{t_0}$ will usually carry different inner products.

**Corollary 1.8.** *If $0 < t_0 < t_1$ are such that $H_{t_1} = H_{t_0}$, then $H_t = H_{t_0}$ for all $t \in [t_0, \infty)$.*

*Proof:* It is clear that $H_{t_0} = H_t = H_{t_1}$ for all $t \in [t_0, t_1]$. Furthermore, Theorem 1.7 implies that $H_{t_0+\delta} = H_{t_1+\delta}$ for all $\delta \geqslant 0$. These two observations clearly lead to the desired result. ∎

The following theorem relates equality of different spaces $H_t$ to their invariance under $\mathbf{S}$:

**Theorem 1.9.** *If $S(s)H_{t_0} \subset H_{t_0}$ for some $s > 0$, then $H_t = H_{\max\{s,t_0\}}$ for all $t \in [\max\{s, t_0\}, \infty)$.*

*Proof:* In view of the Proposition 1.3 we only need to prove the inclusion $H_t \subset H_{\max\{s,t_0\}}$ for $t \in [\max\{s, t_0\}, \infty)$.
*Step 1* - In this step we prove the following: *If $\sigma \in (0, t_0]$ and $t_1 \in (t_0, 2t_0]$ are such that $S(\sigma)$ maps $H_{t_1-t_0}$ into $H_{t_0}$, then $H_{t_1} \subset H_{2t_0-\sigma}$.* By Lemma 1.5, using that $0 < t_1 - t_0 \leqslant t_0$, for all $x^* \in E^*$ we have

$$\|Q_{t_1-t_0}S^*(\sigma)x^*\|_{H_{t_1-t_0}} \leqslant \|S(\sigma)\|_{\mathcal{L}(H_{t_1-t_0},H_{t_0})}\|Q_{t_0}x^*\|_{H_{t_0}}.$$

It follows that

$$\begin{aligned}\langle Q_{t_1}x^*, x^*\rangle &= \langle Q_{t_0}x^*, x^*\rangle + \langle Q_{t_1-t_0}S^*(t_0)x^*, S^*(t_0)x^*\rangle \\ &= \langle Q_{t_0}x^*, x^*\rangle + \|Q_{t_1-t_0}S^*(\sigma)S^*(t_0-\sigma)x^*\|^2_{H_{t_1-t_0}} \\ &\leqslant \langle Q_{t_0}x^*, x^*\rangle + \|S(\sigma)\|^2_{\mathcal{L}(H_{t_1-t_0},H_{t_0})}\|Q_{t_0}S^*(t_0-\sigma)x^*\|^2_{H_{t_0}}.\end{aligned}$$

Now

$$\begin{aligned}\|Q_{t_0}S^*(t_0-\sigma)x^*\|^2_{H_{t_0}} &= \langle Q_{t_0}S^*(t_0-\sigma)x^*, S^*(t_0-\sigma)x^*\rangle \\ &= \langle Q_{2t_0-\sigma}x^*, x^*\rangle - \langle Q_{t_0-\sigma}x^*, x^*\rangle \\ &\leqslant \langle Q_{2t_0-\sigma}x^*, x^*\rangle.\end{aligned}$$

Putting these estimates together, we obtain

$$\begin{aligned}\langle Q_{t_1}x^*, x^*\rangle &\leqslant \langle Q_{t_0}x^*, x^*\rangle + \|S(\sigma)\|^2_{\mathcal{L}(H_{t_1-t_0},H_{t_0})}\langle Q_{2t_0-\sigma}x^*, x^*\rangle \\ &\leqslant \left(1 + \|S(\sigma)\|^2_{\mathcal{L}(H_{t_1-t_0},H_{t_0})}\right)\langle Q_{2t_0-\sigma}x^*, x^*\rangle\end{aligned}$$

By Proposition 1.1 this implies the inclusion $H_{t_1} \subset H_{2t_0-\sigma}$.
*Step 2* - In this step we prove: *If $s \in (0, t_0]$ is such that $S(s)H_{t_0} \subset H_{t_0}$, then for all $t_1 \in [t_0 + s, 2t_0]$ we have $H_{t_1} \subset H_{t_1-s}$.* Indeed, by Theorem 1.4 we know that $S(2t_0 - t_1)$ maps $H_{t_1-t_0}$ into $H_{t_0}$. Therefore also $S(2t_0 - t_1 + s)H_{t_1-t_0} \subset H_{t_0}$, and from Step 1 we obtain $H_{t_1} \subset H_{2t_0-(2t_0-t_1+s)} = H_{t_1-s}$.

*Step 3* - In this step we prove the theorem for the case $s \in (0, t_0]$. First assume $t \in [t_0, 2t_0]$. Write $t = t_0 + ks + \varepsilon$, where $k$ is a nonnegative integer and $\varepsilon \in [0, s)$. If $k = 0$, then by Proposition 1.3 and Step 2,

$$H_t \subset H_{t_0+s} \subset H_{t_0}.$$

If $k \geqslant 1$, then we apply Step 2 $k$ times to see that

$$H_t \subset H_{t-s} \subset H_{t-2s} \subset \ldots \subset H_{t-ks} = H_{t_0+\varepsilon},$$

and therefore by the previous case, $H_t \subset H_{t_0}$.

*Step 4* - In this step we prove the theorem for the case $s \geqslant t_0$.

First observe that by dualizing the identity $i_{t_0} S(s)|_{H_{t_0}} = S(s) i_{t_0}$, where $i_{t_0} : H_{t_0} \subset E$ is the inclusion map, we obtain $(S(s)|_{H_{t_0}})^* Q_{t_0} = Q_{t_0} S^*(s)$. Fix $t_1 \in (s, s+t_0]$ arbitrary. For all $x^* \in E^*$ we have

$$\begin{aligned}
\langle Q_{t_1} x^*, x^* \rangle &= \langle Q_s x^*, x^* \rangle + \langle Q_{t_1-s} S^*(s) x^*, S^*(s) x^* \rangle \\
&\leqslant \langle Q_s x^*, x^* \rangle + \langle Q_{t_0} S^*(s) x^*, S^*(s) x^* \rangle \\
&= \langle Q_s x^*, x^* \rangle + \|Q_{t_0} S^*(s) x^*\|_{H_{t_0}}^2 \\
&= \langle Q_s x^*, x^* \rangle + \|(S(s)|_{H_{t_0}})^* Q_{t_0} x^*\|_{H_{t_0}}^2 \\
&\leqslant \langle Q_s x^*, x^* \rangle + \|S(s)\|_{\mathcal{L}(H_{t_0})}^2 \langle Q_{t_0} x^*, x^* \rangle \\
&\leqslant \left(1 + \|S(s)\|_{\mathcal{L}(H_{t_0})}^2\right) \langle Q_s x^*, x^* \rangle.
\end{aligned}$$

Hence, $H_{t_1} \subset H_s$. But then for any $\tau \in (0, t_1 - s]$, by Theorem 1.4 and Proposition 1.3 we have $S(\tau) H_s \subset H_{s+\tau} \subset H_{t_1} \subset H_s$. Since $0 < \tau \leqslant s$, Step 3 now shows that $H_t = H_s$ for all $t \geqslant s$. ∎

Notice that the case $s = t_0$ already follows from Step 1. In Example 1.14 below we show that the bound $\max\{s, t_0\}$ is the best possible.

Next we study the situation where $H$, the RKHS associated with $Q$, is $\mathbf{S}$−invariant. Then by the closed graph theorem, for each $t > 0$ the restriction $S^H(t) := S(t)|_H$ is a bounded operator on $H$, and it is easy to see that the function $s \mapsto \|S^H(s)\|_{\mathcal{L}(H)}$ is Borel.

**Lemma 1.10.** *Suppose $S(t)H \subset H$ for all $t \geqslant 0$. If there exists $T > 0$ such that*

$$\int_0^T \|S^H(s)\|_{\mathcal{L}(H)}^2 \, ds < \infty,$$

*then $H_t \subset H$ for all $t > 0$.*

*Proof:* By the semigroup property, for all $t > 0$ we have

$$\int_0^t \|S^H(s)\|^2_{\mathcal{L}(H)}\, ds < \infty.$$

Then,

$$\begin{aligned}
\langle Q_t x^*, x^* \rangle &= \int_0^t \langle QS^*(s)x^*, S^*(s)x^* \rangle\, ds \\
&= \int_0^t \|QS^*(s)x^*\|^2_H\, ds \\
&= \int_0^t \|(S^H(s))^* Q x^*\|^2_H\, ds \\
&\leqslant \langle Qx^*, x^* \rangle \int_0^t \|S^H(s)\|^2_{\mathcal{L}(H)}\, ds,
\end{aligned}$$

where we used that $iS^H(s) = S(s)i$ and $i^* = Q$ (recall that $i : H \subset E$ is the inclusion map) imply $(S^H(s))^* Q = QS^*(s)$. From Proposition 1.1 it follows that $H_t \subset H$. ∎

**Theorem 1.11.** *Suppose $S(t)H \subset H$ for all $t \geqslant 0$ and assume there exists $T > 0$ such that*

$$\int_0^T \|S^H(s)\|^2_{\mathcal{L}(H)}\, ds < \infty.$$

*Then for each $t > 0$,*

$$Q_t^H(Qx^*) := Q_t x^*, \qquad x^* \in E^*,$$

*defines a bounded self-adjoint operator $Q_t^H$ on $H$. Denoting the RKHS associated with the operator $Q_t^H$ by $\mathcal{H}_t$, we have $H_t = \mathcal{H}_t$ with identical norms.*

*Proof:* For all $x^* \in E^*$ and $y^* \in E^*$ we have

$$\begin{aligned}
[Q_t^H(Qx^*), Qy^*]_H &= \int_0^t [S^H(s)QS^*(s)x^*, Qy^*]_H\, ds \\
&= \int_0^t [S^H(s)(S^H(s))^* Qx^*, Qy^*]_H\, ds.
\end{aligned}$$

Since by assumption $\|S^H(\cdot)\|_{\mathcal{L}(H)} \in L^2_{loc}[0,\infty)$, Hölder's inequality shows that $Q_t^H$ extends to a bounded operator on $H$. The above identities show that this extension is self-adjoint.

By Lemma 1.10 we have $H_t \subset H$, which implies that for all $x^* \in E^*$ and $y^* \in E^*$,

$$\begin{aligned}
[Q_t^H(Qx^*), Q_t^H(Qy^*)]_{\mathcal{H}_t} &= [Q_t x^*, Q_t^H(Qy^*)]_{\mathcal{H}_t} \\
&= [Q_t x^*, Qy^*]_H \\
&= \langle Q_t x^*, y^* \rangle \\
&= [Q_t x^*, Q_t y^*]_{H_t} \\
&= [Q_t^H(Qx^*), Q_t^H(Qy^*)]_{H_t}.
\end{aligned}$$

Hence the identity map restricted to $\operatorname{Im}(Q_t^H \circ Q) = \operatorname{Im} Q_t$ extends to an inner product preserving isomorphism of $\mathcal{H}_t$ onto $H_t$. ∎

The following examples illustrate the results of this section.

**Example 1.12.** Let $E = L^2[0,1]$ and let $w$ be the Wiener measure on $E$; thus, $w$ is the centered Gaussian measure on $E$ whose covariance operator $Q$ is the integral operator on $E$ defined by

$$(Qf)(s) = \int_0^1 (s \wedge \tau) f(\tau)\, d\tau.$$

The associated RKHS is the Hilbert space $H$ of all absolutely continuous functions $f$ on $[0,1]$ for which $f(0) = 0$ and the a.e. derivative $f'$ belongs to $L^2[0,1]$. The inner product of $H$ is given by $[f,g]_H = [f',g']_E$.

Let $\mathbf{S}$ be the nilpotent right shift semigroup on $E$, i.e.

$$S(t)f(s) = \begin{cases} f(s-t), & t \in [0,s]; \\ 0, & \text{otherwise}, \end{cases} \qquad s \in [0,1],\ t \geqslant 0.$$

We will show that $H_t = H_s$ for all $t > 0$ and $s > 0$. Since $S(t) = 0$ for $t \geqslant 1$ it is clear that $Q_t = Q_1$ and therefore $H_t = H_1$ for all $t \geqslant 1$. For this reason we will only consider $t \in (0,1]$.

Denote by $\mathbf{S}^H$ the restiction of $\mathbf{S}$ to $H$ and note that $\mathbf{S}^H$ is a $C_0-$contraction semigroup on $H$. Therefore by Theorem 1.11, for all $t > 0$ we have $H_t = \mathcal{H}_t$ with identical norms.

We compute the space $H_t$ explicitly. From

$$S^H(s)(S^H(s))^* h(\tau) = \chi_{[s,1]}(\tau) h(\tau), \qquad s \in [0,t],\ \tau \in [0,1],\ h \in H,$$

we have

$$Q_t^H h(\tau) = (t \wedge \tau) h(\tau), \qquad \tau \in [0,1],\ h \in H.$$

Therefore,

$$\begin{aligned} H_t = \mathcal{H}_t &= \operatorname{Im}((Q_t^H)^{1/2}) \\ &= \{h \in H :\ \text{the function } \tau \mapsto (t \wedge \tau)^{-1/2} h(\tau) \text{ belongs to } H\} \\ &= \{h \in H :\ \text{the function } \tau \mapsto \tau^{-1/2} h(\tau) \text{ belongs to } H\}. \end{aligned}$$

Thus, $H_t$ is independent of $t$, and its norm is given by

$$\begin{aligned} \|h\|_{H_t}^2 = \|h\|_{\mathcal{H}_t}^2 &= [Q_t^H h, h]_H \\ &= [\chi_{[0,t]} h + (t \wedge \cdot) h', h']_E \\ &= \int_0^t h(\tau) h'(\tau)\, d\tau + \int_0^1 (t \wedge \tau)(h'(\tau))^2\, d\tau. \end{aligned}$$

$\blacksquare$

From the representation of $H_t$ it is clear that $S(s)H_t \subset H_t$ for all $s \geqslant 0$, so that *a posteriori* Theorem 1.9 applies. On the other hand, by control theoretic methods it is not difficult to show that the assumptions of Theorem 1.11 already imply the inclusion $S^H(t)H \subset \mathcal{H}_t$. Therefore by Lemma 1.10, $S(t)H_t \subset S(t)H \subset \mathcal{H}_t = H_t$ for all $t > 0$.

The next example shows that the inclusion $H_{t_0} \subset H_{t_1}$ may fail to be dense for certain $0 < t_0 < t_1$. The construction is based upon an example shown to the author by Szymon Peszat.

**Example 1.13.** Let $E = C_0[0,1]$ be the Banach space of continuous real-valued functions $f$ on $[0,1]$ with $f(0) = 0$. Let $\mathbf{S}$ be the nilpotent right shift semigroup on $E$. Fix $a \in (0,1)$ arbitrary and let $Q \in \mathcal{L}(E^*, E)$ be the rank one operator defined by $Q\nu := \langle f_0, \nu\rangle f_0$, where $f_0 \in E$ is a function which is strictly positive on the interval $(0,a)$ and vanishes on $[a,1]$. Clearly $Q$ is positive and symmetric. From

$$Q_t\nu = \int_0^t \langle f_0, S^*(s)\nu\rangle S(s)f_0\,ds$$

it follows that for each $t > 0$ the RKHS $H_t$ is contained in the closed linear span $G_t$ of the set $\{S(s)f_0 : s \in [0,t]\}$.

Suppose $0 < t_0 < t_1 \leqslant 1$ are such that $t_1 - t_0 > a$. Then $G_{t_0}$, hence also $H_{t_0}$, is contained in the closed subspace $E_{a+t_0}$ of $E$ consisting of all functions vanishing on $[a + t_0, 1]$. On the other hand,

$$\begin{aligned}
(Q_{t_1}\delta_{t_1})(t_1) &= \left(\int_0^{t_1} S(s)Q\delta_{t_1-s}\,ds\right)(t_1) \\
&= \int_0^{t_1} f_0(t_1 - s)(S(s)f_0)(t_1)\,ds \\
&= \int_0^{t_1} (f_0(t_1 - s))^2\,ds > 0,
\end{aligned}$$

where $\delta_{t_1}$ denotes the Dirac measure at $t_1$. Since $t_1 > t_0 + a$, $Q_{t_1}\delta_{t_1} \notin E_{a+t_0}$. But if $H_{t_0}$ were dense in $H_{t_1}$, we would have $Q_{t_1}\delta_{t_1} \in H_{t_1} = \overline{H_{t_0}}^{H_{t_1}} \subset \overline{H_{t_0}}^{E} \subset E_{a+t_0}$. Therefore the inclusion $H_{t_0} \subset H_{t_1}$ cannot be dense. ∎

This example can be extended to show that Theorem 1.9 is the best possible:

**Example 1.14.** For each $n$ let $E_n := C_0[0,1]$, let $\mathbf{S}_n$ be the nilpotent right shift semigroup on $E_n$, and let $Q_n$ be as in Example 1.13 with $a_n := 1/n$. Let $E$ be the $c_0$−direct sum of the spaces $E_n$, and define $\mathbf{S}$ and $Q$ as direct sums of the $\mathbf{S}_n$ and $Q_n$ in the natural way. Then the inclusion $H_{t_0} \subset H_{t_1}$ fails to be dense for all $0 < t_0 < t_1 \leqslant 1$, this being the case in the $k$th summand whenever $t_1 - t_0 > 1/k$. On the other hand, the fact that $S(t) = 0$ for all $t \geqslant 1$ implies that $H_t = H_1$ for all $t \geqslant 1$.

Trivially, $S(1)H_{t_0} \subset H_{t_0}$ for all $t_0 > 0$. In particular this holds for any $t_0 \in (0,1)$, although $H_t$ is constant only after $t \geqslant 1$. This shows that Theorem 1.9 is the best possible in case $\max\{s, t_0\} = s$.

Similarly, for all $s > 0$ we have $S(s)H_1 \subset H_1$. In particular this holds for any $s \in (0,1)$, although $H_t$ is constant only after $t \geqslant 1$. This shows that Theorem 1.9 is also the best possible in case $\max\{s, t_0\} = t_0$. ∎

## 2. The associated Gaussian measures $\mu_t$

In this section and the next, $E$ will always denote a separable real Banach space, and $\mathbf{S}$ is a fixed $C_0-$semigroup on $E$.

It is not hard to see that for each positive symmetric $Q \in \mathcal{L}(E^*, E)$ there exists a unique finitely additive cylindrical measure $\mu$, defined on the ring of all cylindrical sets of $E$, whose Fourier transform is given by

$$\hat{\mu}(x^*) = \exp\left(-\frac{1}{2}\langle Q\,x^*, x^*\rangle\right), \qquad x^* \in E^*. \tag{2.1}$$

In this section we fix a positive symmetric operator $Q \in \mathcal{L}(E^*, E)$ and make the following

**Assumption 2.1.** *For each $t > 0$ the cylindrical measure $\mu_t$ associated with the positive symmetric operator $Q_t \in \mathcal{L}(E^*, E)$ is countably additive.*

In other words, we assume that the operators $Q_t$ are the covariances of centered Gaussian measures $\mu_t$ on the Borel $\sigma-$algebra of $E$.

**Remark 2.2.** We state two sufficient conditions for Assumption 2.1 to be satisfied:

(i) $E$ is a Hilbert space and $Q$ is trace class (i.e. the cylindical measure associated with $Q$ is countably additive);

(ii) The cylindrical measure associated with $Q$ is countably additive, $S(s)H \subset H$ for all $s \geqslant 0$, and

$$\int_0^t \|S(s)\|^2_{\mathcal{L}(H)}\,ds < \infty$$

for all $t \geqslant 0$ [MS].

For the reader's convenience we reproduce the simple proofs; more information about (cylindrical) Gaussian measures can be found in the books [Ku], [VTC], and [DZ3].

(i): If $(e_n)$ is an orthonormal basis in $E$, then by Fubini's theorem

$$\begin{aligned}\sum_{n=1}^{\infty}[Q_t e_n, e_n]_E &= \int_0^t \sum_{n=1}^{\infty}[S(s)QS^*(s)e_n, e_n]_E\,ds \\ &\leqslant \left(\sup_{0\leqslant s\leqslant t}\|S(s)\|^2\right)\cdot t\,\|Q\|_1 < \infty,\end{aligned}$$

where $\|Q\|_1$ is the trace class norm of $Q$; we used the fact that for any bounded $T$, the operator $TQT^*$ is trace class whenever $Q$ is, with $\|TQT^*\|_1 \leqslant \|T\|\,\|Q\|_1\,\|T^*\| = \|T\|^2\,\|Q\|_1$.

(ii): By Lemma 1.10, for each $t > 0$ there is a constant $K_t > 0$ such that

$$\langle Q_t x^*, x^*\rangle \leqslant K_t \langle Q x^*, x^*\rangle, \qquad \forall x^* \in E^*.$$

Countable additivity of $\mu_t$ then follows from [VTC, Corollary VI.3.4.2]. ∎

On $B_b(E)$, the space of bounded, Borel measurable functions on $E$, the formula

$$(P(t)f)(x) := \int_E f(S(t)x + y)\, d\mu_t(y), \qquad x \in E,$$

defines a semigroup $\mathbf{P} = \{P(t)\}_{t\geqslant 0}$ of contractions. This semigroup will be referred to as the (non-symmetric) *Ornstein-Uhlenbeck semigroup* associated with $\mathbf{S}$ and $Q$. In this section we state, without proof, a number of results about $\mathbf{P}$, the analogues of which are well-known in the Hilbert space setting. Their proofs carry over to the Banach space setting without difficulty and are therefore omitted.

**Theorem 2.3.** *Let $x \in E$ and $t_0 > 0$ be fixed. The following assertions are equivalent:*

*(i)* $S(t_0)x \in H_{t_0}$;
*(ii) For all $f \in B_b(E)$, the function $\varepsilon \mapsto P(t_0)f(\varepsilon x)$ is continuous at $\varepsilon = 0$;*
*(iii) For all $f \in B_b(E)$ and $y \in E$, $P(t_0)f$ is smooth at $y$ in the direction of $x$.*

In this situation, the first directional derivative can be computed explicitly. For this purpose we introduce the following notation. If $\mu$ is a centered Gaussian measure on $E$, then $\phi^\mu : H \to L^2(E, \mu)$ denotes the isometric embedding uniquely defined by

$$\phi^\mu(Qx^*) := \langle x^*, \cdot\rangle,$$

where $Q$ is the covariance operator of $\mu$. For $h \in H$, the RHKS associated with $Q$, we will write $\phi_h^\mu$ to denote the function $\phi^\mu(h) \in L^2(E, \mu)$. To see that this map is well-defined, recall that the support of $\mu$ is contained in the closure $E_0$ of $H$ in $E$, whereas $Qx^* = Qy^*$ implies that $x^*|_{E_0} = x^*|_{E_0}$.

With this notation, the partial derivative $\partial P(t_0)f/\partial x$ is given by

$$\frac{\partial P(t_0)f}{\partial x}(y) = \int_E f(S(t_0)y + z)\phi^{\mu_{t_0}}_{S(t_0)x}(z)\, d\mu_{t_0}(z).$$

For the Wiener semigroup these results are due to Gross [Gr]; for $E$ Hilbert they were extended to arbitrary semigroups $\mathbf{S}$ in [CG3].

The semigroup $\mathbf{P}$ is said to be *strongly Feller at time* $t_0 > 0$ if $P(t_0)f(\cdot)$ is a continuous function for all $f \in B_b(E)$. We refer to [DZ3] for more information in the Hilbert space setting.

**Corollary 2.4.** *For $t_0 > 0$ fixed, the following conditions are equivalent:*

*(i)* $\mathbf{P}$ *is strongly Feller at* $t_0$;
*(ii)* $S(t_0)E \subset H_{t_0}$.

## 3. Equivalence of the measures $\mu_t$

Two measures $\mu$, $\nu$ are said to be *equivalent*, notation $\mu \sim \nu$, if they are absolutely continuous with respect to each other, i.e. if $\mu \ll \nu$ and $\nu \ll \mu$. We will study the question under what conditions we have equivalence $\mu_{t_0} \sim \mu_{t_1}$ for certain $t_0$ and $t_1$. Our result is based on the following version of the Feldman-Hajek theorem, due to Tarieladze; see also the review paper [VT].

**Theorem 3.1 [Ta].** *Let $\mu, \nu$ be two centered Gaussian measures on a Banach space $E$, and denote by $Q_\mu$, $Q_\nu \in \mathcal{L}(E^*, E)$, and $H_\mu$, $H_\nu$ their covariance operators and RHKS's, respectively. Then $\mu \sim \nu$ if and only if the following two conditions are satisfied:*

*(i)* $H_\mu = H_\nu$;
*(ii)* $I - j \circ V$ *is Hilbert-Schmidt on* $H_\mu$, *where* $V : H_\mu \to H_\nu$ *and* $j : H_\nu \to H_\mu$ *are defined by*

$$VQ_\mu x^* := Q_\nu x^*, \qquad x^* \in E^*,$$
$$jh := h, \qquad h \in H_\nu.$$

*Otherwise,* $\mu \perp \nu$.

Throughout this section we consider a $C_0-$semigroup $\mathbf{S}$ on $E$ and a positive symmetric operator $Q \in \mathcal{L}(E^*, E)$ verifying Assumption 2.1.

Let $0 < t_0 < t_1 < \infty$. In terms of the operators $S_{t_1 - t_0 \to t_1}(t_0) := S(t_0)|_{H_{t_1 - t_0}} \in \mathcal{L}(H_{t_1 - t_0}, H_{t_1})$ (whose existence follows from Theorem 1.4) we can characterize equivalence of the measures $\mu_{t_0}$ and $\mu_{t_1}$ as follows:

**Theorem 3.2.** *Let $0 < t_0 < t_1 < \infty$. Then $\mu_{t_0} \sim \mu_{t_1}$ if and only if the following two conditions are satisfied:*

*(i)* $\|S_{t_1 - t_0 \to t_1}(t_0)\|_{\mathcal{L}(H_{t_1 - t_0}, H_{t_1})} < 1$;
*(ii) The operator* $S_{t_1 - t_0 \to t_1}(t_0) S^*_{t_1 - t_0 \to t_1}(t_0)$ *is Hilbert-Schmidt on* $H_{t_1}$.

*Proof:* By Theorem 1.7, strict contractivity of $S_{t_1 - t_0 \to t_1}(t_0)$ is equivalent to $H_{t_0} = H_{t_1}$. Next we consider the Hilbert-Schmidt condition. We have

$$Q_{t_1} - Q_{t_0} = S(t_0) Q_{t_1 - t_0} S^*(t_0) = S_{t_1 - t_0 \to t_1}(t_0) S^*_{t_1 - t_0 \to t_1}(t_0) Q_{t_1}.$$

Letting $j : H_{t_0} \to H_{t_1}$ be the identity map, it follows that $I - j \circ Q_{t_0} Q_{t_1}^{-1} : Q_{t_1} x^* \mapsto Q_{t_1} x^* - Q_{t_0} x^*$ is Hilbert-Schmidt on $H_{t_1}$ if and only if $S_{t_1 - t_0 \to t_1}(t_0) S^*_{t_1 - t_0 \to t_1}(t_0)$ is Hilbert-Schmidt on $H_{t_1}$. ∎

**Corollary 3.3.** *Suppose $0 < t_0 < t_1 < \infty$ are such that $\mu_{t_0} \sim \mu_{t_1}$.*

*(i) For all* $\delta \geqslant 0$ *we have* $\mu_{t_0 + \delta} \sim \mu_{t_1 + \delta}$;
*(ii) If* $t_2 \in [t_1, \infty)$ *is such that* $\mu_t \sim \mu_{t_2}$ *for all* $t \in [t_2, \infty)$, *then* $\mu_t \sim \mu_{t_0}$ *for all* $t \in [t_0, \infty)$.

*Proof:* (i): Fix $\delta \geqslant 0$ arbitrary. By Corollary 1.8, $H_{t_0+\delta} = H_{t_1+\delta}$. Hence from the identity

$$\begin{aligned} S_{(t_1+\delta)-(t_0+\delta)\to t_1+\delta}&(t_0+\delta)S^*_{(t_1+\delta)-(t_0+\delta)\to t_1+\delta}(t_0) \\ &= S_{t_1\to t_1+\delta}(\delta)\big(S_{t_1-t_0\to t_1}(t_0)S^*_{t_1-t_0\to t_1}(t_0)\big)S^*_{t_1\to t_1+\delta}(\delta) \end{aligned}$$

and Theorem 3.2 we conclude that $\mu_{t_0+\delta} \sim \mu_{t_1+\delta}$.

(ii): Pick $k \in \mathbb{N}$ such that $t_1 + k(t_1 - t_0) \geqslant t_2$. By (i) we have

$$\mu_{t_0} \sim \mu_{t_1} \sim \mu_{t_1+(t_1-t_0)} \sim \dots \sim \mu_{t_1+k(t_1-t_0)} \sim \mu_{t_2}.$$

Hence, $\mu_t \sim \mu_{t_0}$ for all $t \in [t_2, \infty)$. But then, by another application of (i) we have, for all $t \in [t_0, t_2]$,

$$\mu_t = \mu_{t_0+(t-t_0)} \sim \mu_{t_2+(t-t_0)} \sim \mu_{t_0}.$$

It follows that $\mu_t \sim \mu_{t_0}$ for all $t \in [t_0, \infty)$. ∎

If $Q$ 'commutes' with $S(s)$ for some $s > 0$, in the sense that $S(s)Q = QS^*(s)$, we can prove more. We start with a lemma (which was shown to the author by Ben de Pagter).

**Lemma 3.4.** *Suppose $\mu$ and $\nu$ are centered Gaussian measures on $E$ such that $H_\mu = H_\nu$. Let $V : H_\mu \to H_\nu$ and $j : H_\nu \to H_\mu$ be defined by*

$$\begin{aligned} VQ_\mu x^* &:= Q_\nu x^*, \qquad x^* \in E^*, \\ jh &:= h, \qquad h \in H_\nu. \end{aligned}$$

*Then $V \circ j$ is positive and self-adjoint on $H_\nu$, and $(V \circ j)^{\frac{1}{2}} \circ j^{-1}$ is an inner product preserving isomorphism of $H_\mu$ onto $H_\nu$.*

*Proof:* Clearly, the inner product $[\cdot,\cdot]_{H_\mu}$ defines a bounded symmetric bilinear form on $H_\nu$. Consequently there exists a unique self-adjoint operator $V_1 \in \mathcal{L}(H_\nu)$ such that

$$[jg, jh]_{H_\mu} = [V_1 g, h]_{H_\nu}, \qquad \forall g, h \in H_\nu.$$

Moreover $V_1$ is positive and invertible. Let $(e_n)$ be an orthonormal basis in $H_\mu$. Then,

$$[V_1^{\frac{1}{2}} j^{-1} e_n, V_1^{\frac{1}{2}} j^{-1} e_m]_{H_\nu} = [V_1 j^{-1} e_n, j^{-1} e_m]_{H_\nu} = [e_n, e_m]_{H_\mu} = \delta_{nm}.$$

Hence, $(V_1^{\frac{1}{2}} j^{-1} e_n)$ is an orthonormal basis for $H_\nu$; note that $V_1^{\frac{1}{2}} \in \mathcal{L}(H_\nu)$ is surjective. Since $[V_1^{\frac{1}{2}} g, V_1^{\frac{1}{2}} h]_{H_\nu} = [jg, jh]_{H_\mu}$ for all $g, h \in H_\nu$, it follows that $V_1^{\frac{1}{2}} \circ j^{-1} : H_\mu \to H_\nu$ is an inner product preserving isometric isomorphism onto.

Returning to the map $V$, for all $x^*, y^* \in E^*$ we have

$$[Q_\mu x^*, Q_\mu y^*]_{H_\mu} = \langle Q_\mu y^*, x^* \rangle = [Q_\nu x^*, j^{-1} Q_\mu y^*]_{H_\nu} = [V(Q_\mu x^*), j^{-1} Q_\mu y^*]_{H_\nu}.$$

Hence via density, $[jg, jh]_{H_\mu} = [Vjg, h]_{H_\nu}$ for all $g, h \in H_\nu$. This shows that $V \circ j = V_1$. ∎

**Theorem 3.5.** *Suppose we have $S(s)Q = QS^*(s)$ for some $s > 0$. Let $t_0 > 0$ be fixed. If there exists $t_1 \in (t_0, \infty)$ such that $\mu_{t_1} \sim \mu_{t_0}$, then $\mu_t \sim \mu_{t_0}$ for all $t \in [t_0, \infty)$.*

*Proof: Step 1* - First assume $s = t_0$.

Fix $t_2 \geqslant 2t_0$ and assume for the moment that $\mu_{t_2} \sim \mu_{t_0}$. We will prove that $\mu_t \sim \mu_{t_2}$ for all $t \in [t_2, \infty)$.

Clearly, $H_{t_2} = H_{t_0}$. Fix $t \geqslant t_2$. In view of $t_2 - t_0 \geqslant t_0$, by Corollary 1.8 we also have $H_{t-t_0} = H_{t_2-t_0} = H_{t_0}$ and $H_t = H_{t_0}$. Define $V : H_{t_2-t_0} \to H_{t-t_0}$ by

$$V : Q_{t_2-t_0}x^* \mapsto Q_{t-t_0}x^*, \qquad x^* \in E^*.$$

From $S(t_0)Q = QS^*(t_0)$ we have $S(t_0)Q_\tau = Q_\tau S^*(t_0)$ for all $\tau > 0$ and hence

$$\begin{aligned} S(t_0)VQ_{t_2-t_0}x^* &= S(t_0)Q_{t-t_0}x^* = Q_{t-t_0}S^*(t_0)x^* \\ &= VQ_{t_2-t_0}S^*(t_0)x^* = VS(t_0)Q_{t_2-t_0}x^*, \qquad x^* \in E^*. \end{aligned}$$

Letting $j : H_{t-t_0} \to H_{t_2-t_0}$ be the identity operator, it follows that on $H_{t-t_0}$ we have

$$S_{t-t_0\to t-t_0}(t_0) \circ (V \circ j) = (V \circ j) \circ S_{t-t_0\to t-t_0}(t_0).$$

By Lemma 3.4, $V \circ j$ is positive on $H_{t-t_0}$, and $U := (V \circ j)^{\frac{1}{2}} \circ j^{-1}$ is an inner product preserving isomorphism of $H_{t_2-t_0}$ onto $H_{t-t_0}$. Moreover, by functional calculus we have

$$S_{t-t_0\to t-t_0}(t_0) \circ (V \circ j)^{\frac{1}{2}} = (V \circ j)^{\frac{1}{2}} \circ S_{t-t_0\to t-t_0}(t_0).$$

Multiplying on the right with $j^{-1}$ gives

$$S_{t-t_0\to t-t_0}(t_0) = (V \circ j)^{\frac{1}{2}} \circ S_{t-t_0\to t-t_0}(t_0) \circ j^{-1} \circ U^* = U \circ S_{t_2-t_0\to t_2-t_0}(t_0) \circ U^*.$$

Therefore,

$$S_{t-t_0\to t}(t_0) = j_{t-t_0\to t} \circ U \circ j_{t_2\to t_2-t_0} \circ S_{t_2-t_0\to t_2}(t_0) \circ U^*,$$

where $j_{t-t_0\to t}$ and $j_{t_2\to t_2-t_0}$ are the identity maps from $H_{t-t_0}$ to $H_t$ and from $H_{t_2}$ to $H_{t_2-t_0}$, respectively. Using the equivalence $\mu_{t_2} \sim \mu_{t_0}$ and Theorem 3.2, we conclude that

$$\begin{aligned} &S_{t-t_0\to t}(t_0)S^*_{t-t_0\to t}(t_0) \\ &\quad = (j_{t-t_0\to t} \circ U \circ j_{t_2\to t_2-t_0} \circ S_{t_2-t_0\to t_2}(t_0) \circ U^*) \\ &\qquad\qquad \circ \big(U \circ S^*_{t_2-t_0\to t_2}(t_0) \circ j^*_{t_2\to t_2-t_0} \circ U^* \circ j^*_{t-t_0\to t}\big) \\ &\quad = j_{t-t_0\to t} \circ U \circ j_{t_2\to t_2-t_0} \circ \big(S_{t_2-t_0\to t_2}(t_0)S^*_{t_2-t_0\to t_2}(t_0)\big) \circ j^*_{t_2\to t_2-t_0} \circ U^* \circ j^*_{t-t_0\to t} \end{aligned}$$

is Hilbert-Schmidt on $H_t$. By another application of Theorem 3.2 it follows that $\mu_t \sim \mu_{t_0}$.

*Step 2* - Using Step 1, we now prove the theorem for the case $s = t_0$.

Let $k \in \mathbb{N}$ be any integer such that $t_1 + k(t_1 - t_0) \geqslant 2t_0$. By Corollary 3.3 (i) we have

$$\mu_{t_0} \sim \mu_{t_1} \sim \mu_{t_1+(t_1-t_0)} \sim \dots \sim \mu_{t_1+k(t_1-t_0)}.$$

We may apply Step 1 to $t_2 := t_1 + k(t_1 - t_0)$. It follows that $\mu_t \sim \mu_{t_2}$ for all $t \geqslant t_2$. But then Corollary 3.3 (ii) shows that $\mu_t \sim \mu_{t_0}$ for all $t \in [t_0, \infty)$.

*Step 3* - We now prove the general case. Choose an integer $m$ such that $ms \geqslant t_1$. Then $S(ms)Q = QS^*(ms)$; further $\mu_{t_0} \sim \mu_{t_1}$ implies $\mu_{ms} \sim \mu_{t_1+ms-t_0}$ by Corollary 3.3 (i). Hence we may apply Step 2 to $\tau_0 := ms$ and $\tau_1 := t_1 + ms - t_0$. It follows that $\mu_t \sim \mu_{ms}$ for all $t \in [ms, \infty)$. But then we apply Corrollary 3.3 (ii) to $t_0$, $t_1$, and $t_2 := ms$ to see that $\mu_t \sim \mu_{t_0}$ for all $t \in [t_0, \infty)$. ∎

In view of this result and by the analogy to Corollary 1.8 we conjecture that $\mu_{t_1} \sim \mu_{t_0}$ *always* implies $\mu_t \sim \mu_{t_0}$ for all $t \in [t_0, \infty)$.

The following corollary gives necessary and sufficient conditions for the situation described by Theorem 3.5:

**Corollary 3.6.** *Suppose we have $S(s)Q = QS^*(s)$ for some $s > 0$. Then the following assertions are equivalent:*

*(i) $\mu_t \sim \mu_{t_0}$ for all $t \in [t_0, \infty)$;*
*(ii) $S(t_0)H_{t_0} \subset H_{t_0}$ and $S(t_0)|_{H_{t_0}}(S(t_0)|_{H_{t_0}})^*$ is Hilbert-Schmidt on $H_{t_0}$.*

*Proof:* Assume (i). Then in particular $\mu_{2t_0} \sim \mu_{t_0}$, and Theorem 3.2 shows that $S_{t_0\to 2t_0}(t_0)(S_{t_0\to 2t_0}(t_0))^*$ is Hilbert-Schmidt on $H_{2t_0}$. But then

$$S_{t_0\to t_0}(t_0)(S_{t_0\to t_0}(t_0))^* = (j_{2t_0\to t_0} \circ S_{t_0\to 2t_0}(t_0)) \circ (S^*_{t_0\to t_0}(t_0) \circ j^*_{2t_0\to t_0})$$

is Hilbert-Schmidt on $H_{t_0}$. This gives (ii).

Conversely, if (ii) holds, then

$$\begin{aligned} S_{t_0\to 2t_0}(t_0)(S_{t_0\to 2t_0}(t_0))^* &= (j_{t_0\to 2t_0} \circ S_{t_0\to t_0}(t_0)) \circ (S^*_{t_0\to t_0}(t_0) \circ j^*_{t_0\to 2t_0}) \\ &= j_{t_0\to 2t_0} \circ S_{t_0\to t_0}(2t_0) \circ j^*_{t_0\to 2t_0} \end{aligned}$$

Hilbert-Schmidt on $H_{2t_0}$. Theorem 3.2 shows that $\mu_{2t_0} \sim \mu_{t_0}$, and therefore (i) holds by Theorem 3.5. ∎

Under the assumptions that $E$ is Hilbert and that for all $s > 0$ the operator $S(s)$ is self-adjoint on $E$ and commutes with $Q$, this result is essentially equivalent to [NZ, Theorem 3.1].

**Remark 3.7.** If $S(t_0)Q = QS^*(t_0)$, then (ii) may be replaced by
(ii)′ *$S(t_0)H_{t_0} \subset H_{t_0}$ and the restriction $S(2t_0)|_{H_{t_0}}$ is Hilbert-Schmidt on $H_{t_0}$.*

This follows from (ii) once we show that the restriction $S(t_0)|_{H_{t_0}}$ is self-adjoint on $H_{t_0}$:

$$\begin{aligned} [(S(t_0)|_{H_{t_0}})^* Q_{t_0}x^*, Q_{t_0}y^*]_{H_{t_0}} &= [Q_{t_0}x^*, S(t_0)|_{H_{t_0}} Q_{t_0}y^*]_{H_{t_0}} \\ &= [Q_{t_0}x^*, Q_{t_0}S^*(t_0)y^*]_{H_{t_0}} \\ &= [Q_{t_0}x^*, (S(t_0)|_{H_{t_0}})^* Q_{t_0}y^*]_{H_{t_0}} \\ &= [S(t_0)|_{H_{t_0}} Q_{t_0}x^*, Q_{t_0}y^*]_{H_{t_0}} \end{aligned}$$

for all $x^*, y^* \in E^*$. ∎

As the following two examples show, it may happen that $H_t = H_{t_0}$ for all $t \in [t_0, \infty)$ although $\mu_t \perp \mu_s$ for all $t \neq s \in [t_0, \infty)$. A third example is given in Section 6 below.

**Example 3.8.** Let $E$ be an infinite-dimensional Hilbert space and let $\mu$ be a non-degenerate centered Gaussian measure on $E$ with covariance operator $Q$. Let $\mathbf{S}$ be a periodic $C_0-$semigroup on $E$, with period 1. Then Assumption 2.1 holds, we have $Q_k = kQ_1$ for all $k = 1, 2, ...$, and consequently $H_k = H_1$ for all such $k$. Hence, $H_t = H_1$ for all $t \in [1, \infty)$. On the other hand, let us suppose that $\mu_t \sim \mu_s$ for certain $t, s \in [1, \infty)$ with $t < s$, then for any integer $k \geqslant t$ we also have $\mu_k \sim \mu_{s+k-t}$ by Corollary 3.3. But $S(k) = I$ commutes with $Q$, and therefore Theorem 3.5 implies $\mu_\tau \sim \mu_k$ for all $\tau \in [k, \infty)$; in particular, $\mu_k \sim \mu_l$ for all integers $l > k$. But these measures have covariances $kQ$ and $lQ$, respectively, and therefore they are singular by the Feldman-Hajek theorem; a contradiction. ∎

**Example 3.9.** We continue Example 1.12. By Remark 2.2 (i), each of the operators $Q_t$ is the covariance of a centered Gaussian measure $\mu_t$ on $E = L^2[0, 1]$. We will show that $\mu_t \perp \mu_s$ if $t \in (0, 1)$ and $s \neq t$, whereas it is trivial that $\mu_t = \mu_s$ whenever $t \geqslant 1$ and $s \geqslant 1$.

Fix $t \in (0, 1)$ and $s > 0$, $s \neq t$. Since $\mu_s = \mu_1$ if $s \geqslant 1$ we may assume that $s \in (0, 1]$. By interchanging the roles of $t$ and $s$ if necessary, we may also assume that $t < s$, say $s = t + h$ for some $h \in (0, 1 - t]$. Let $F$ denote the closed subspace of $H_{t+h}$ consisting of all functions with support in $[t, 1]$. For all $f \in F$, $S_{h\to t+h}(t)S^*_{h\to t+h}(t)f = f$, so $S^H(t)(S^H(t))^*|_F = I_F$, the identity operator on $F$. Since $\dim F = \infty$ it follows that $S_{h\to t+h}(t)S^*_{h\to t+h}(t)$ is not compact on $H_{t+h}$ and therefore not Hilbert-Schmidt. This shows that $\mu_t \perp \mu_{t+h} = \mu_s$. ∎

## 4. Computation of the Radon-Nikodym derivative

It is possible to give an explicit expression for the Radon-Nikodym density $d\mu_{t_1}/d\mu_{t_0}$ whenever we have $\mu_{t_0} \sim \mu_{t_1}$. This will occupy us in the present section.

We start by recalling some notation and results concerning second quantization. For more details we refer to [Ne] and the book [Si]. Fix a centered Gaussian measure $\mu$ on $E$ with covariance operator $Q \in \mathcal{L}(E^*, E)$, and let $H$ denote the associated RKHS. Let $\phi^\mu : H \mapsto L^2(E, \mu)$ be the isometric embedding from $H$ into $L^2(E, \mu)$ defined by $\phi^\mu(Qx^*) = \langle x^*, \cdot\rangle$ as in Section 2. Whenever the measure $\mu$ is understood, we omit it from the notation and write $\phi_h$ to denote the function $\phi(h) = \phi^\mu(h) \in L^2(E, \mu)$.

Let $(H_n)_{n\in\mathbb{N}}$ be the sequence of Hermite polynomials and denote by $\mathcal{H}_n$ the closure in $L^2(E, \mu)$ of the linear span of the set $\{H_n(\phi_h) : \|h\|_H = 1\}$. Note that $\mathcal{H}_0$ is the one-dimensional subspace spanned by the constant one function and that $\mathcal{H}_1$ is precisely the image of $H$ under the isometry $\phi$. One has the orthogonal *Wiener-Itô decomposition*,

$$L^2(E, \mu) = \oplus_{n\in\mathbb{N}}\, \mathcal{H}_n.$$

The orthogonal projection onto $\mathcal{H}_n$ will be denoted by $I_n$.

For all $h \in H$, the functions

$$K_h(x) := \exp\left(\phi_h(x) - \frac{1}{2}\|h\|_H^2\right),$$

belong to $L^2(E,\mu)$, their linear span is dense in $L^2(E,\mu)$, and we have the identity

$$K_h = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(\phi_h^n), \qquad h \in H.$$

Now assume we have two pairs $(E_0,\mu_0)$ and $(E_1,\mu_1)$, and let $T \in \mathcal{L}(H_0,H_1)$ be a contraction. The *second quantization* of $T$ is the contraction $\Gamma(T) \in \mathcal{L}(L^2(E_0,\mu_0), L^2(E_1,\mu_1))$ defined by

$$\Gamma(T)\big(I_n(\phi_{h_1}^{k_1} \cdot \ldots \cdot \phi_{h_j}^{k_j})\big) := I_n(\phi_{Th_1}^{k_1} \cdot \ldots \cdot \phi_{Th_j}^{k_j}),$$

where it is assumed that $k_1 + \ldots + k_j = n$.

Now let a positive symmetric operator $Q \in \mathcal{L}(E^*,E)$ and a $C_0-$semigroup $\mathbf{S}$ on $E$ be given such that Assumption 2.1 holds. Let $\mathbf{P}$ be the Ornstein-Uhlenbeck semigroup on $B_b(E)$ associated with $\mathbf{S}$ and $Q$. We are going to apply second quantization to $E_0 = E_1 = E$, $\mu_0 := \mu_{t_0+h}$, $\mu_1 := \mu_h$, and the adjoint $S^*_{h\to t_0+h}(t_0) \in \mathcal{L}(H_{t_0+h}, H_h)$ of the Hilbert space contraction $S_{h\to t_0+h}(t_0) \in \mathcal{L}(H_h, H_{t_0+h})$.

**Theorem 4.1.** *For all $t_0 > 0$ and $h > 0$, the operator $P(t_0)$ extends to a contraction from $L^2(E,\mu_{t_0+h})$ into $L^2(E,\mu_h)$. This extension is realized as the second quantization of $S^*_{h\to t_0+h}(t_0)$:*

$$P(t_0) = \Gamma(S^*_{h\to t_0+h}(t_0)).$$

*Proof:* We denote the image measure of $\mu_t$ with respect to an element $x^* \in E^*$ by $\langle x^*, \mu_t\rangle$. For all $x^* \in E^*$ we then have

$$\begin{aligned}
P(t_0)K_{Q_{t_0+h}x^*}(x) &= \int_E \exp\left(\langle x^*, S(t_0)x + y\rangle - \frac{1}{2}\|Q_{t_0+h}x^*\|^2_{H_{t_0+h}}\right) d\mu_{t_0}(y) \\
&= K_{Q_{t_0+h}x^*}(S(t_0)x) \int_E \exp\left(\langle x^*, y\rangle\right) d\mu_{t_0}(y) \\
&= K_{Q_{t_0+h}x^*}(S(t_0)x) \int_{\mathbb{R}} \exp(s)\, d\langle x^*, \mu_{t_0}\rangle(s) \\
&= K_{Q_{t_0+h}x^*}(S(t_0)x) \exp\left(\frac{1}{2}\|Q_{t_0}x^*\|^2_{H_{t_0}}\right) \\
&= \exp\left(\langle x^*, S(t_0)x\rangle - \frac{1}{2}\left(\|Q_{t_0+h}x^*\|^2_{H_{t_0+h}} - \|Q_{t_0}x^*\|^2_{H_{t_0}}\right)\right) \\
&= \exp\left(\langle x^*, S(t_0)x\rangle - \frac{1}{2}\|Q_h S^*(t_0)x^*\|^2_{H_h}\right) \\
&= K_{Q_h S^*(t_0)x^*}(x) \\
&= K_{S^*_{h\to t_0+h}(t_0)Q_{t_0+h}x^*}(x).
\end{aligned}$$

Hence the identity

$$K_g = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(\phi_g^n), \qquad g \in H_{t_0+h},$$

implies

$$\begin{aligned} P(t_0) K_{Q_{t_0+h}x^*} &= K_{S^*_{h\to t_0+h}(h) Q_{t_0+h}x^*} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} I_n(\phi^n_{S^*_{h\to t_0+h}(t_0) Q_{t_0+h}x^*}) \\ &= \Gamma(S^*_{h\to t_0+h}(t_0)) K_{Q_{t_0+h}x^*}. \end{aligned}$$

By a density argument, it follows that

$$P(t_0) K_g = \Gamma(S^*_{h\to t_0+h}(t_0)) K_g, \qquad \forall g \in H_{t_0+h}.$$

Since the linear span of the functions $K_g$, $g \in H_{t_0+h}$, is dense in $L^2(E, \mu_{t_0+h})$, this proves the theorem. ∎

The next aim is to apply the so-called *Mehler formula* for second quantized operators to the above situation.

To this end, we consider the situation of two pairs $(E_0, \mu_0)$ and $(E_1, \mu_1)$, with $\mu_k$ a centered Gaussian measure on $E_k$ with RHKS $H_k$; $k = 0, 1$. The following result, due to Feyel and La Pradelle, shows that every bounded operator in $\mathcal{L}(H_0, H_1)$ has an extension to a linear $\mu_0-$measurable extension from $E_0$ into $E_1$. Recall that a mapping $f : E_0 \to E_1$ is $\mu_0-$*measurable* if $f^{-1}(B)$ belongs to the $\mu_0-$completion of the Borel $\sigma-$algebra of $E_0$, for all Borel sets $B \subset E_1$.

**Proposition 4.2 [F-LP, Théorème 5].** *Let $T \in \mathcal{L}(H_0, H_1)$. Then there exists a $\mu_0-$measurable linear operator $\overline{T}$ from $E_0$ into $E_1$ which extends $T$. This extension is $\mu_0-$essentially unique in the sense that any two such $\mu_0-$measurable linear extensions agree $\mu_0-$a.e. Moreover, for all $h \in H_1$ we have $\phi_h^{\mu_1}(\overline{T}(x)) = \phi^{\mu_0}_{T^*h}(x)$ for $\mu_0-$almost all $x \in E_0$.*

The uniqueness part implies that for a bounded operator $T \in \mathcal{L}(E_0, E_1)$ which maps $H_0$ into $H_1$ we have $T = \overline{T|_{H_0}}$ $\mu_0-$a.e.

In terms of these extensions, one has the following *Mehler formula* for the second quantization of a Hilbert space contraction:

**Proposition 4.3 [F-LP, Théorème 10].** *Let $T$ be a contraction in $\mathcal{L}(H_0, H_1)$. Then for all $f \in L^2(E_0, \mu_0)$ and $\mu_1-$almost all $x \in E_1$ we have*

$$\Gamma(T) f(x) = \int_{E_0} f\left(\overline{T^*}(x) + \overline{\sqrt{I - T^*T}}(y)\right) d\mu_0(y).$$

This result motivates the consideration of the image measure of $\mu_0$ under the $\mu_0-$measurable transformation $\overline{\sqrt{I - T^*T}}$. Let us denote this measure by $\mu_0^T$. One has the following extension of a result of Shale [Sh]:

**Proposition 4.4 [F-LP, Proposition 12].** *Let $T \in \mathcal{L}(H_0, H_1)$ be a strict contraction such that $T^*T$ is Hilbert-Schmidt on $H_0$. Then $\mu_0^T \ll \mu_0$, with Radon-Nikodym derivative $\mu_0-$a.e. given by*

$$\frac{d\mu_0^T}{d\mu_0}(x) = \frac{1}{\sqrt{\det(I - T^*T)}} \exp\left( -\frac{1}{2} \left\| \overline{\left(\sqrt{I - T^*T}\right)^{-1} (T^*T)^{\frac{1}{2}}}(x) \right\|^2 \right).$$

If $\mu_{t_0} \sim \mu_{t_0+h}$ for some $t_0 > 0$ and $h > 0$, then $S_{h\to t_0+h}(t_0)S^*_{h\to t_0+h}(t_0)$ is Hilbert-Schmidt on $H_{t_0+h}$. If we assume that $S_{h\to t_0+h}(t_0)$ itself is Hilbert-Schmidt as an operator from $H_h$ into $H_{t_0+h}$ we can prove more:

**Theorem 4.5.** *Suppose we have $\mu_{t_0} \sim \mu_{t_0+h}$ for some $t_0 > 0$ and $h > 0$. If $S_{h\to t_0+h}(t_0)$ is Hilbert-Schmidt from $H_h$ into $H_{t_0+h}$, then the Radon-Nikodym derivative $d\mu_{t_0}/d\mu_{t_0+h}(x)$ is $\mu_{t_0+h}-$a.e. given by*

$$\frac{1}{\sqrt{\det(I - T)}} \exp\left( -\frac{1}{2} \left\| \overline{\left(\sqrt{I - T}\right)^{-1} T^{\frac{1}{2}}}(x) \right\|^2 \right),$$

*where $T := S_{h\to t_0+h}(t_0)S^*_{h\to t_0+h}(t_0)$.*

*Proof:* We note that for all $0 \leqslant f \in B_b(E)$ we have

$$P(t_0)f(x) = \int_E f(S(t_0)x + y)\, d\mu_{t_0}(y) = \int_E f(S(t_0)x + y) \frac{d\mu_{t_0}}{d\mu_{t_0+h}}(y)\, d\mu_{t_0+h}(y).$$

Combining this with Theorem 4.1 and Proposition 4.3, we see that

$$\begin{aligned} P(t_0)f(x) &= \Gamma(S^*_{h\to t_0+h}(t_0))f(x) \\ &= \int_E f\left( \overline{S_{h\to t_0+h}(t_0)}(x) + \overline{\sqrt{I - T}}(y) \right) d\mu_{t_0+h}(y) \\ &= \int_E f\left( S(t_0)x + \overline{\sqrt{I - T}}(y) \right) d\mu_{t_0+h}(y) \end{aligned}$$

By Proposition 4.4, the image measure of $\mu_{t_0+h}$ under the $\mu_{t_0+h}-$measurable transformation $\overline{\sqrt{I - T}}$ is absolutely continuous with respect to $\mu_{t_0+h}$, with Radon-Nikodym derivative given, for $\mu_{t_0+h}-$a.a. $y \in E$, by

$$\frac{1}{\sqrt{\det(I - T)}} \exp\left( -\frac{1}{2} \left\| \overline{\left(\sqrt{I - T}\right)^{-1} T^{\frac{1}{2}}}(y) \right\|^2 \right).$$

Hence,

$$\begin{aligned} &P(t_0)f(x) = \\ &= \frac{1}{\sqrt{\det(I - T)}} \int_E f(S(t_0)x + y) \exp\left( -\frac{1}{2} \left\| \overline{\left(\sqrt{I - T}\right)^{-1} T^{\frac{1}{2}}}(y) \right\|^2 \right) d\mu_{t_0+h}(y), \end{aligned}$$

and the desired result follows by comparing the two identities for $P(t_0)f(x)$. ∎

## 5. Continuous dependence of the Radon-Nikodym derivative

Thoughout this section $E$ is a separable real Banach space, $Q \in \mathcal{L}(E^*, E)$ is positive and symmetric, and $\mathbf{S}$ is a $C_0-$semigroup on $E$ such that Assumption 2.1 is verified. We will show that for $t_1$ fixed the Radon-Nikodym derivative $d\mu_{t_0}/d\mu_{t_1}$ depends continuously upon $t_0$.

Our first aim is to establish a result concerning continuity of determinants.

**Lemma 5.1.** *Fix $\tau > 0$. For all $g \in H_\tau$ we have*

$$\lim_{h \downarrow 0} \|S_{\tau-h\to\tau}(h)S^*_{\tau-h\to\tau}(h)g - g\|_{H_\tau} = 0.$$

*Proof:* Fix $x^* \in E^*$ and $h \in [0, \tau)$. Writing $T_h := S_{\tau-h\to\tau}(h)$, for all $y^* \in E^*$ we have

$$\begin{aligned}
[T_h^* Q_\tau x^*, T_h^* Q_\tau y^*]_{H_\tau} &= [Q_{\tau-h} S^*(h)x^*, Q_{\tau-h} S^*(h)y^*]_{H_{\tau-h}}, \\
&= \langle S(h)Q_{\tau-h}S^*(h)x^*, y^*\rangle \\
&= \langle Q_\tau x^* - Q_h x^*, y^*\rangle \\
&= [Q_\tau x^* - Q_h x^*, Q_\tau y^*]_{H_\tau}.
\end{aligned}$$

Hence,

$$[T_h T_h^* Q_\tau x^* - Q_\tau x^*, Q_\tau y^*]_{H_\tau} = -[Q_h x^*, Q_\tau y^*]_{H_\tau}.$$

Taking the supremum with respect to all $Q_\tau y^*$ of norm $\leqslant 1$, it follows that

$$\|T_h T_h^* Q_\tau x^* - Q_\tau x^*\|_{H_\tau} = \|Q_h x^*\|_{H_\tau} \leqslant \|Q_h x^*\|_{H_h},$$

the inequality being a consequence of Corollary 1.5. As $h \downarrow 0$ the right hand side tends to 0. Since $\|T_h\| \leqslant 1$ for all $h$ by Theorem 1.4, the lemma now follows by a density argument. ∎

**Lemma 5.2.** *Let $H_0$ and $H_1$ be separable Hilbert spaces, let $S \in \mathcal{L}(H_0, H_1)$ be Hilbert-Schmidt and let $(T_n) \subset \mathcal{L}(H_0)$ be a sequence of operators converging to $I$ strongly. Then*

$$\lim_{n\to\infty} ST_nS^* = SS^*$$

*in the space $\mathcal{L}_1(H_1)$ of trace class operators on $H_1$.*

*Proof:* The lemma is obvious if $S$ is a rank one operator. By taking linear combinations, it also holds for finite rank operators $S$. The general case then follows from a $3\varepsilon-$argument, approximating $S$ in the Hilbert-Schmidt norm by finite rank operators. ∎

The preceding two lemmas combined with the fact [GGK, p. 119] that the mapping $T \mapsto \det(I - T)$ is continuous with respect to the trace class norm lead to the following result:

**Lemma 5.3.** *Let $0 < t_0 < t_1$ be fixed and assume that the operator $S_{t_1-t_0\to t_1}(t_0) \in \mathcal{L}(H_{t_1-t_0}, H_{t_1})$ is Hilbert-Schmidt. Then the function*

$$h \mapsto \det(I - S_{t_1-t_0-h\to t_1}(t_0+h)S^*_{t_1-t_0-h\to t_1}(t_0+h)), \qquad h \in [0, t_1-t_0),$$

*is continuous.*

In the following lemma, $C_b(\Omega)$ denotes the space of bounded real-valued continuous functions on a topological space $\Omega$.

**Lemma 5.4.** *Suppose $\tilde{f} \in C_b(\mathbb{R}^n)$ and $x_1^*, ..., x_n^* \in E^*$ are given, and define $f \in C_b(E)$ by*

$$f(x) := \tilde{f}(\langle x_1^*, x\rangle, ..., \langle x_n^*, x\rangle), \qquad x \in E.$$

*Then for all $t_0 \geqslant 0$ and $x \in E$ we have we have*

$$\lim_{h\downarrow 0} P(t_0+h)f(x) - P(t_0)f(x) = 0.$$

*Proof:* We have

$$\begin{aligned} P(t_0+h)f(x) &= \int_E \tilde{f}(\langle x_1^*, S(t_0+h)x+y\rangle, ..., \langle x_n^*, S(t_0+h)x+y\rangle)\, d\mu_{t_0+h}(y) \\ &= \int_E \tilde{f}(\langle x_1^*, z\rangle, ..., \langle x_n^*, z\rangle)\, d\mu_{t_0+h}^{(S(t_0+h)x)}(z) \\ &= \int_{\mathbb{R}^n} \tilde{f}(\tau_1, ..., \tau_n)\, d\nu_{t_0+h}^{(S(t_0+h)x)}(\tau), \end{aligned}$$

where $\mu_{t_0+h}^{(S(t_0+h)x)}$ is the translation of $\mu_{t_0+h}$ along $S(t_0+h)x$, and $\nu_{t_0+h}^{(S(t_0+h)x)}$ is the image measure on $\mathbb{R}^n$ of $\mu_{t_0+h}^{(S(t_0+h)x)}$ under the map $T : E \to \mathbb{R}^n$ given by $Tz := (\langle x_1^*, z\rangle, ..., \langle x_n^*, z\rangle)$. Thus, the Gaussian measure $\nu_{t_0+h}^{(S(t_0+h)x)}$ has mean $(\langle x_1^*, S(t_0+h)x\rangle, ..., \langle x_n^*, S(t_0+h)x\rangle)$ and covariance $TQ_{t_0+h}T^*$. By Lévy's theorem,

$$\lim_{h\downarrow 0} \nu_{t_0+h}^{(S(t_0+h)x)} = \nu_{t_0}^{(S(t_0)x)} \quad \text{weakly.}$$

But then

$$\begin{aligned} \lim_{h\downarrow 0} P(t_0+h)f(x) &= \lim_{h\downarrow 0} \int_{\mathbb{R}^n} \tilde{f}(\tau_1, ..., \tau_n)\, d\nu_{t_0+h}^{(S(t_0+h)x)}(\tau) \\ &= \int_{\mathbb{R}^n} \tilde{f}(\tau_1, ..., \tau_n)\, d\nu_{t_0}^{(S(t_0)x)}(\tau) \\ &= P(t_0)f(x). \end{aligned}$$

■

It is well-known that the space of all cylindrical $C_b(E)-$functions as considered in Lemma 5.4 are dense in $L^2(E, \mu)$, for any Gaussian measure $\mu$ defined on the Borel $\sigma-$algebra of $E$. This will be used in the following theorem, which is the main result of this section.

**Theorem 5.5.** *Assume that $\mu_t \sim \mu_{t_0}$ for all $t \in [t_0, \infty)$ and that for all $h > 0$ the operator $S_{h \to t_0+h}(t_0)$ is Hilbert-Schmidt from $H_h$ to $H_{t_0+h}$. Fix $t_1 > t_0$, and for $\tau \in [t_0, t_1]$ let $g_\tau := d\mu_\tau / d\mu_{t_1}$ denote the Radon-Nikodym derivative. Then*

$$\lim_{h \downarrow 0} \|g_{t_0+h} - g_{t_0}\|_{L^2(E,\mu_{t_1})} = 0.$$

*Proof:* The proof is divided into two steps.

*Step 1* - We first prove that

$$\lim_{h \downarrow 0} \|g_{t_0+h}\|_{L^2(E,\mu_{t_1})} = \|g_{t_0}\|_{L^2(E,\mu_{t_1})}.$$

For $\tau \in [t_0, t_1]$, we define $T_\tau \in \mathcal{L}(H_{t_1})$ by

$$T_\tau := S_{t_1-\tau \to t_1}(\tau) S^*_{t_1-\tau \to t_1}(\tau).$$

Then,

$$\begin{aligned}
\|g_{t_0+h}\|^2_{L^2(E,\mu_{t_1})} &= \\
&= \frac{1}{\det(I - T_{t_0+h})} \int_E \exp\left( - \left\| \overline{\left(\sqrt{I - T_{t_0+h}}\right)^{-1} \sqrt{T_{t_0+h}}(x)} \right\|^2 \right) d\mu_{t_1}(x) \\
&\leqslant \frac{1}{\det(I - T_{t_0+h})} \int_E \exp\left( -\frac{1}{2} \left\| \overline{\left(\sqrt{I - T_{t_0+h}}\right)^{-1} \sqrt{T_{t_0+h}}(x)} \right\|^2 \right) d\mu_{t_1}(x) \\
&= \frac{1}{\sqrt{\det(I - T_{t_0+h})}} \|g_{t_0+h}\|_{L^1(E,\mu_{t_1})} \\
&= \frac{1}{\sqrt{\det(I - T_{t_0+h})}}.
\end{aligned}$$

Therefore Step 1 is a consequence of Lemma 5.3.

*Step 2* - The cylindrical functions as described in Lemma 5.4 are dense in $L^2(E, \mu_{t_1})$, and for each such $f$ we have

$$\lim_{h \downarrow 0} \int_E f(x)(g_{t_0+h}(x) - g_{t_0}(x))\, d\mu_{t_1}(x) = \lim_{h \downarrow 0} P(t_0 + h) f(0) - P(t_0) f(0) = 0.$$

Since by Step 1 the norms $\|g_{t_0+h}\|_{L^2(E,\mu_{t_1})}$ remain bounded as $h \downarrow 0$, it follows that $\lim_{h \downarrow 0} g_{t_0+h} = g_{t_0}$ weakly in $L^2(E, \mu_{t_1})$. Together with Step 1 this implies that $\lim_{h \downarrow 0} g_{t_0+h} = g_{t_0}$ strongly in $L^2(E, \mu_{t_1})$. ∎

We will apply this result to show that under certain conditions the Ornstein-Uhlenbeck semigroup $\mathbf{P}$ associated with $\mathbf{S}$ and $Q$ is pointwise continuous for $t \geqslant t_0$, uniformly on bounded sets in $E$, in the space $BUC(E)$ of bounded real-valued uniformly continuous functions on $E$. Before doing so we make the following simple observation.

**Proposition 5.6.** *Let $f \in C_b(E)$ and $t_0 > 0$ be fixed. If $S(t_0)$ is compact on $E$ and*

$$\lim_{h\downarrow 0}\left(\sup_{x\in K}|P(h)f(x) - f(x)|\right) = 0 \tag{5.1}$$

*for all compact sets $K \subset E$, then for all bounded sets $B \subset E$ we have*

$$\lim_{h\downarrow 0}\left(\sup_{x\in B}|P(t_0+h)f(x) - P(t_0)f(x)|\right) = 0.$$

*Proof:* Given $\varepsilon > 0$ and a bounded set $B \subset E$, let $K_0 := \overline{S(t_0)B}$ and let $K_1 \subset E$ be a compact set such that $\mu_{t_0}(K_1) > 1 - \varepsilon$. Writing $g_h := P(h)f - f$, we have $\lim_{h\downarrow 0} g_h = 0$ uniformly on the compact set $\{y_0 + y_1 : y_0 \in K_0,\, y_1 \in K_1\}$, and hence

$$\begin{aligned}
\lim_{h\downarrow 0}&\left(\sup_{x\in B}|P(t_0+h)f(x) - P(t_0)f(x)|\right)\\
&= \lim_{h\downarrow 0}\left(\sup_{x\in B}\left|\int_E g_h(S(t_0)x+y)\,d\mu_{t_0}(y)\right|\right)\\
&\leqslant 2\varepsilon\,\|f\| + \lim_{h\downarrow 0}\left(\sup_{x\in B}\int_{K_1} g_h(S(t_0)x+y)\,d\mu_{t_0}(y)\right)\\
&= 2\varepsilon\,\|f\|.
\end{aligned}$$

∎

For Hilbert spaces $E$ it is known that (5.1) holds for all $f \in BUC(E)$; semigroups on $BUC(E)$ satisfying (5.1) have been studied from an abstract point of view in [Ce] and [CG]. In our more general setting we do not know whether (5.1) holds without additional assumptions. For this reason we will impose stronger assumptions on $\mathbf{S}$ and $Q$.

Let $t_0 > 0$ be fixed. The pair $(\mathbf{S}, Q)$ is said to be *null controllable at* $t_0$ if $S(t_0)E \subset H_{t_0}$. This condition arises in control theory in a natural way; for its interpretation and further discussion we refer to [DZ3]. If the domain $D(A)$ of the generator $A$ of a differentiable semigroup $\mathbf{S}$ is contained in the RKHS $H$ associated with $Q$, then $(\mathbf{S}, Q)$ is null controllable at all $t > 0$; this follows from [Nv, Lemma 2.2]. Under a null controllability condition, the results of Section 4 and Theorem 5.5 are applicable. This is the content of the following proposition.

**Proposition 5.7.** *If $(\mathbf{S}, Q)$ is null controllable at $t_0$, then:*

*(i) For all $t \geqslant t_0$ we have $\mu_t \sim \mu_{t_0}$;*
*(ii) For all $h > 0$ the operator $S_{h\to t_0+h}(t_0)$ is Hilbert-Schmidt from $H_h$ into $H_{t_0+h}$;*
*(iii) For all $t \geqslant t_0$ the operator $S(t)$ is compact in $E$.*

*Proof:* First notice that the null controllability condition implies $S(t_0)H_{t_0} \subset H_{t_0}$, so that $H_t = H_{t_0}$ for all $t \in [t_0, \infty)$, and for each $h > 0$, $S_{h\to t_0+h}(t_0)$ is a strict contraction.

If we regard $S(t_0)$ as an element of $\mathcal{L}(E, H_{t_0})$, then $S_{h\to t_0+h}(t_0)$ admits the factorization $S_{h\to t_0+h}(t_0) = j_{t_0\to t_0+h} \circ S(t_0) \circ i_h$, where $i_h : H_h \subset E$ and $j_{t_0\to t_0+h} : H_{t_0} \subset H_{t_0+h}$ are the inclusion maps. By a result of Kwapień and Szymański [KS], there exists an orthonormal basis $(g_n)$ of $H_h$ such that $\sum_{n=1}^{\infty} \|i_h g_n\|_E^2 < \infty$. But then also

$$\sum_{n=1}^{\infty} \|S_{h\to t_0+h}(t_0) g_n\|^2_{H_{t_0+h}} \leqslant \|j_{t_0\to t_0+h} \circ S(t_0)\|_{\mathcal{L}(E,H_{t_0+h})} \sum_{n=1}^{\infty} \|i_h g_n\|_E^2 < \infty,$$

proving that $S_{h\to t_0+h}(t_0)$ is Hilbert-Schmidt.

The last assertion follows from the fact that by assumption $S(t_0)$ factors through $H_{t_0}$ and the general fact from the theory of abstract Wiener spaces (cf. [Ku, Section 1.4]) that the inclusion map $i_{t_0} : H_{t_0} \subset E$ is compact. ∎

**Corollary 5.8.** *Let $t_0 > 0$ be fixed and suppose the pair $(\mathbf{S}, Q)$ is null controllable at $t_0$. Then for all bounded sets $B \subset E$ and all $f \in BUC(E)$ we have*

$$\lim_{h\downarrow 0} \Big( \sup_{x\in B} |P(t_0+h)f(x) - P(t_0)f(x)| \Big) = 0.$$

*Proof:* The null controllability assumption $S(t_0)E \subset H_{t_0}$ implies that $\mu_t \sim \mu_{t_0}$ for $t \in [t_0, \infty)$, and that for all $h > 0$ the operator $S_{h\to t_0+h}(t_0)$ is Hilbert-Schmidt.

Fix $f \in BUC(E)$, $x \in E$, and $t_1 > t_0$ arbitrary. Then for $h \in [0, t_1 - t_0]$,

$$\begin{aligned} P(t_0+h)f(x) - P(t_0)f(x) &= \int_E f(S(t_0+h)x + y)(g_{t_0+h}(y) - g_{t_0}(y))\, d\mu_{t_1}(y) \\ &\quad + \int_E f(S(t_0+h)x + y) - f(S(t_0)x + y)\, d\mu_{t_0}(y). \end{aligned}$$

As $h \downarrow 0$, by Theorem 5.5 the first integral tends to 0, uniformly in $x$. In order to estimate the second integral, we note that $S(t_0)$ is compact by Proposition 5.7 (iii). If $B \subset E$ is a given bounded set, it then follows from the uniform continuity of $f$ and the strong continuity of $\mathbf{S}$ that

$$\lim_{h\downarrow 0} \Big( \sup_{x\in B} \Big| \int_E f(S(t_0+h)x + y) - f(S(t_0)x + y)\, d\mu_{t_0}(y) \Big| \Big) = 0.$$

This shows that $\lim_{h\downarrow 0} P(t_0+h)f(x) - P(t_0)f(x) = 0$, uniformly for $x \in B$. ∎

The following example shows that the convergence is generally not uniformly on $E$, even if $E$ is one-dimensional.

**Example 5.9.** Let $E = \mathbb{R}$, $Q = I$, and $S(t) = e^{-t}$. Then

$$\int_{\mathbb{R}} \exp(-i(e^{-t}s + \tau))\, d\mu_t(\tau) = \exp(-i(e^{-t}s))\hat{\mu}_t(1) = (1 - e^{-2t}) \exp(-i(e^{-t}s)).$$

Hence, for $f(s) := \cos s$ we have

$$P(t)f(s) = (1 - e^{-2t}) \cos(e^{-t}s),$$

from which we deduce that $\|P(t_0+h)f - P(t_0)f\| = 2$ for all $t_0 > 0$ and $h > 0$. ∎

**Remark 5.10.** Strong continuity in $BUC(E)$ with $E$ a Hilbert space was investigated in [DL], where it was shown that for a given $f \in BUC(E)$ we have $\lim_{h\downarrow 0} \|P(h)f - f\| = 0$ if and only if

$$\lim_{h\downarrow 0} \left( \sup_{x\in E} |f(S(h)x) - f(x)| \right) = 0.$$

## 6. The reproducing kernel Hilbert space $H_\infty$

In this section we will discuss some versions of the previous results assuming that an invariant measure $\mu_\infty$ exists.

We return to the cylindrical setting in an arbitrary real Banach space $E$, i.e. Assumption 2.1 is *not* adopted and $E$ need not be separable. Instead, will make the following

**Assumption 6.1.** *The strong limit (in $E$)*

$$Q_\infty x^* := \lim_{t\to\infty} Q_t x^*$$

*exists for all $x^* \in E^*$ and defines a bounded linear operator $Q_\infty \in \mathcal{L}(E^*, E)$.*

It is clear that the operator $Q_\infty$ defined in this way is positive symmetric; its RKHS is denoted by $H_\infty$, and the inclusion map $H_\infty \subset E$ is denoted by $i_\infty$. The proof of Proposition 1.3 extends to show that $H_t \subset H_\infty$ for all $t > 0$.

**Theorem 6.2.** *For all $s > 0$ we have $S(s)H_\infty \subset H_\infty$, and $\mathbf{S}$ restricts to a $C_0-$contraction semigroup $\mathbf{S}_\infty$ on $H_\infty$.*

*Proof:* The invariance of $H_\infty$ is proved by repeating the proof of Theorem 1.4 with $t$ replaced by $\infty$; this also gives contractivity. It remains to prove strong continuity of $\mathbf{S}_\infty$ on $H_\infty$.

For all $h \in H_\infty$ and $x^* \in E^*$ we have

$$\lim_{t\downarrow 0}[S_\infty(t)h, Q_\infty x^*]_{H_\infty} = \lim_{t\downarrow 0}\langle S(t)h, x^*\rangle = \langle h, x^*\rangle = [h, Qx^*]_{H_\infty}.$$

But $\mathbf{S}_\infty$ being uniformly bounded on $H_\infty$, the linear subspace $H_\infty^0$ of all $g \in H_\infty$ such that $\lim_{t\downarrow 0}[S_\infty(t)h, g]_{H_\infty} = [h, g]_{H_\infty}$ is closed. Therefore, $H_\infty^0 = H_\infty$ and $\mathbf{S}_\infty$ is weakly continuous. By a standard result from semigroup theory [Pa, Theorem 2.1.4], this implies that $\mathbf{S}_\infty$ is strongly continuous. ∎

Under the assumption that $E$ is a Hilbert space and $Q_\infty$ is trace class, this result is due to Chojnowska-Michalik and Goldys [CG3, Proposition 1] (see also [CG2, Lemma 4]). Our proof is a modification of the proof of [CG2]. In fact, an analysis of this proof led us to the discovery of Theorem 1.4.

**Theorem 6.3.** *Let $t_0 > 0$. Then $H_{t_0} = H_\infty$ if and only if $\|S_\infty(t_0)\|_{H_\infty} < 1$. In this case, $H_{t_0} = H_t = H_\infty$ for all $t \in (t_0, \infty)$.*

*Proof:* We only need to prove that $H_\infty \subset H_{t_0}$ if and only if $\|S_\infty(t_0)\|_{H_\infty} < 1$.

We note that

$$S_\infty^*(t)Q_\infty = (i_\infty S_\infty(t))^* = (S(t)i_\infty)^* = Q_\infty S^*(t); \tag{6.1}$$

here $i_\infty : H_\infty \to E$ is the inclusion map. First assume $\|S_\infty(t_0)\|_{H_\infty} < 1$. Using (6.1), for all $x^* \in E^*$ we have

$$\begin{aligned}
\|Q_{t_0}x^*\|_{H_{t_0}}^2 &= \langle Q_\infty x^*, x^* \rangle - \langle S(t_0)Q_\infty S^*(t_0)x^*, x^* \rangle \\
&= \|Q_\infty x^*\|_{H_\infty}^2 - \|Q_\infty S^*(t_0)x^*\|_{H_\infty}^2 \\
&= \|Q_\infty x^*\|_{H_\infty}^2 - \|S_\infty^*(t_0)Q_\infty x^*\|_{H_\infty}^2 \\
&\geqslant \left(1 - \|S_\infty(t_0)\|_{H_\infty}^2\right) \|Q_\infty x^*\|_{H_\infty}^2
\end{aligned}$$

This gives the inclusion $H_\infty \subset H_{t_0}$.

The converse follows from an obvious modification of the proof of Theorem 1.7. ∎

Under the assumption that $E$ is Hilbert and $Q_\infty$ is trace class, this result was obtained in the second part of [CG2, Lemma 4], with a similar proof. In fact, this motivated our Theorem 1.7.

The following result gives a criterion for equality $H_{t_0} = H_\infty$ in terms of mapping properties of $\mathbf{S}$.

**Theorem 6.4.** *If $S(t_0)H_\infty \subset H_{t_0}$, then $H_{t_0} = H_t = H_\infty$ for all $t \in [t_0, \infty]$.*

*Proof:* We always have $H_{t_0} \subset H_t \subset H_\infty$, so we only need prove the inclusion $H_\infty \subset H_{t_0}$.

First note that for all $x^* \in E^*$,

$$Q_\infty x^* = Q_{t_0}x^* + S(t_0)(Q_\infty S^*(t_0)x^*) \in H_{t_0}.$$

Next fix $h \in H_\infty$ arbitrary. Let $(x_n^*) \subset E^*$ be a sequence such that $\lim_{n\to\infty} Q_\infty x_n^* = h$ in $H_\infty$. Then

$$\lim_{n\to\infty} S(t_0)Q_\infty S^*(t_0)x_n^* = \lim_{n\to\infty} S(t_0)S_\infty^*(t_0)Q_\infty x_n^* = S(t_0)S_\infty^*(t_0)h =: g$$

in $H_\infty$. Note that $g \in H_{t_0}$ by the assumption on $S(t_0)$. Moreover, in $H_\infty$ we have

$$\lim_{n\to\infty} Q_{t_0}x_n^* = \lim_{n\to\infty} (Q_\infty x_n^* - S(t_0)Q_\infty S^*(t_0)x_n^*) = h - g.$$

On the other hand, from $\|Q_{t_0}x_n^*\|_{H_{t_0}} \leqslant \|Q_\infty x_n^*\|_{H_\infty}$ we see that the sequence $(Q_{t_0}x_n^*)$ is bounded in $H_{t_0}$. Let $y$ be a weak limit point of $(Q_{t_0}x_n^*)$ in $H_{t_0}$. By the continuity of the inclusion $H_{t_0} \subset H_\infty$, $y$ is also a weak limit point of $(Q_{t_0}x_n^*)$ in $H_\infty$. Therefore we must have $y = h - g$. In particular, $h - g \in H_{t_0}$. But then $h = y + g \in H_{t_0}$. This proves that $H_\infty \subset H_{t_0}$. ∎

For $E$ Hilbert and $Q_\infty$ trace class, this is proved in [CG3, Proposition 3] by control theoretic methods.

The following example, taken from [Go], shows that it may happen that $H_t = H_s$ for all $t, s \in (0,\infty)$, although the inclusions $H_t \subset H_\infty$ are strict. In [Go] these facts are checked by explicit calculations; here, we derive them as consequences of our abstract results and as such the example serves as an interesting illustration of them.

**Example 6.5.** Let $E = l^2$ and denote by $(e_n)$ the standard unit basis of $E$. Define $Q \in \mathcal{L}(E)$ by $Qe_n := e_n/n^3$. Then $Q$ is a non-negative self-adjoint trace class operator and hence the covariance of a Gaussian measure $\mu$ on $E$. Define the operator $A$ by $Ae_n := -e_n/n$. Then $A$ is bounded on $E$ and $S(t) := e^{tA}$ defines a uniformly continuous semigroup of self-adjoint operators on $E$ satisfying $\|S(t)\| = 1$ for all $t \geqslant 0$.

Fix $t > 0$. It is easy to check that

$$\begin{aligned} Q_t &= \frac{A^2}{2}(1 - S(2t)), \\ Q_\infty &= \frac{A^2}{2}. \end{aligned}$$

Since $A^2$ and $S(t)$ commute, so do $Q_t$ and $S(t)$ and we see that $S(t)$ maps $\operatorname{Im} Q_t$ into itself. We check that $S(t)$ extends to a bounded operator on $H_t$. For all $h \in E$ of the form $h = \sum_{k=1}^n a_k e_k$ we have

$$\begin{aligned} \|S(t)Q_t h\|_{H_t}^2 &= \|Q_t S(t) h\|_{H_t}^2 \\ &= [Q_t S(t)h, S(t)h]_E \\ &= [Q_t h, S(2t)h]_E \\ &= \sum_{k=1}^n a_k^2 \cdot e^{-2t/k} \cdot \frac{1}{2k^2}(1 - e^{-2t/k}) \\ &\leqslant \sum_{k=1}^n a_k^2 \cdot \frac{1}{2k^2}(1 - e^{-2t/k}) \\ &= [h, Q_t h]_E = \|Q_t h\|_{H_t}^2. \end{aligned}$$

Since the set of all $Q_t h$, with $h$ of the above form, is dense in $H_t$, this shows that the restriction of $S(t)$ to $\operatorname{Im} Q_t$ extends to a contraction on $H_t$. Theorem 1.9 now shows that $H_t = H_s$ for all $t, s \in (0,\infty)$. On the other hand, $S(t)$ also commutes with $Q_\infty$ and for $t > 0$ fixed we have

$$\begin{aligned} \|S_\infty(t) Q_\infty e_n\|_{H_\infty}^2 &= \|Q_\infty S(t) e_n\|_{H_\infty}^2 \\ &= [Q_\infty S(t)e_n, S(t)e_n]_E \\ &= e^{-2t/n}[Q_\infty e_n, e_n]_E \\ &= e^{-2t/n}\|Q_\infty e_n\|_{H_\infty}^2. \end{aligned}$$

Hence, $\|S_\infty(t)\|_{H_\infty} \geqslant e^{-t/n}$ for all $n$, so $\|S_\infty(t)\|_{H_\infty} = 1$. Hence by Theorem 6.3, the inclusion $H_t \subset H_\infty$ is strict.

Finally a simple computation shows that for all $t_0 > 0$, the restriction of $S(t_0)$ to $H_{t_0}$ fails to be Hilbert-Schmidt. Hence, $\mu_t \perp \mu_s$ for all $t \neq s \in (0,\infty)$ by Corollary 3.6. ∎

For the rest of this section, $E$ is assumed to be separable and we will assume the following simultaneous strengthening of Assumptions 2.1 and 6.1:

**Assumption 6.6.** *Assumption 6.1 holds and the cylindrical measure $\mu_\infty$ associated with $Q_\infty$ is countably additive.*

In other words, we assume that the operator $Q_\infty$ is the covariance of a centered Gaussian measure $\mu_\infty$ on the Borel $\sigma-$algebra of $E$.

**Remark 6.7.** The following conditions are sufficient for Assumption 6.6 to hold:

(i) $E$ is a Hilbert space and $\sup_{t>0} \operatorname{Trace} Q_t < \infty$ [DZ3, Chapter 11];
(ii) $E$ is a Hilbert space, $Q$ is trace class, and $\mathbf{S}$ is uniformly exponentially stable;
(iii) The cylindrical measure associated with $Q$ is countably additive, $\mathbf{S}$ is uniformly exponentially stable, $S(s)H \subset H$ for all $s \geqslant 0$, and

$$\int_0^\infty \|S(s)\|^2_{\mathcal{L}(H)}\, ds < \infty.$$

(iv) Assumption 2.1 holds, $\mathbf{S}$ is uniformly exponentially stable, and the pair $(\mathbf{S}, Q)$ is null controllable at some $t_0 > 0$.

We will investigate the question under what conditions we have equivalence $\mu_{t_0} \sim \mu_\infty$ holds for a given $t_0 \in (0,\infty)$.

**Theorem 6.8.** *For a fixed $t_0 > 0$, the measures $\mu_{t_0}$ and $\mu_\infty$ are equivalent if and only if the following two conditions are satisfied:*

*(i) $\|S_\infty(t_0)\|_{H_\infty} < 1$;*
*(ii) The operator $S_\infty(t_0)S^*_\infty(t_0)$ is Hilbert-Schmidt on $H_\infty$.*

For Hilbert spaces $E$, this was proved in [CG3, Theorem 2]. By the semigroup property, this result implies:

**Corollary 6.9.** *If $\mu_{t_0} \sim \mu_\infty$ for some $t_0 > 0$, then $\mu_t \sim \mu_\infty$ for all $t \in [t_0, \infty]$.*

It is possible to give an explicit expression for the Radon-Nikodym density $d\mu_{t_0}/d\mu_\infty$. If $\mu_{t_0} \sim \mu_\infty$ for some $t_0 > 0$, then $S_\infty(t_0)S^*_\infty(t_0)$ is Hilbert-Schmidt on $H_\infty$. If we assume that $S_\infty(t_0)$ itself is Hilbert-Schmidt we can prove more:

**Theorem 6.10.** *Suppose we have $\mu_{t_0} \sim \mu_\infty$ for some $t_0 > 0$. If $S_\infty(t_0)$ is Hilbert-Schmidt on $H_\infty$, then the Radon-Nikodym derivative $g_{t_0} := d\mu_{t_0}/d\mu_\infty$ is $\mu_\infty-$a.e. given by*

$$g_{t_0}(x) = \Big(\det \sqrt{I - S_\infty(t_0)S^*_\infty(t_0)}\Big)^{-1} \quad \times$$
$$\times \quad \exp\left(-\frac{1}{2}\overline{\|\Big(\sqrt{I - S_\infty(t_0)S^*_\infty(t_0)}\Big)^{-1}(S_\infty(t_0)S^*_\infty(t_0))^{\frac{1}{2}}x\|^2}\right).$$

Concerning continuous dependence, we have:

**Theorem 6.11.** *Under the above assumptions, the Radon-Nikodym derivative* $g_t := d\mu_t/d\mu_\infty$ *exists for all* $t \geqslant t_0$ *and belongs to* $L^2(E, \mu_\infty)$. *The function* $t \mapsto g_t$ *is continuous from* $[t_0, \infty)$ *into* $L^2(E, \mu_\infty)$.

Analogously to the situation encountered in Section 5, the assumptions of the theorem are automatically satisfied under the null controllability assumption $S(t_0)E \subset H_{t_0}$. The proofs of Theorems 6.10 and 6.11 proceed as in Sections 4 and 5, respectively. The main ingredient of Theorem 6.10 is the following version of Theorem 4.1:

**Theorem 6.12.** *The semigroup* $\mathbf{P}$ *extends to a* $C_0-$*semigroup on* $L^2(E, \mu_\infty)$ *and for all* $t \geqslant 0$ *we have*

$$P(t) = \Gamma(S^*_\infty(t)).$$

For Hilbert spaces $E$, Theorems 6.10 and 6.12 are due to Chojnowska-Michalik and Goldys [CG2], [CG3]. Their version of Theorem 6.10 is based on a very general formula for Radon-Nikodym derivatives of Gaussian measures on Hilbert spaces due to Fuhrman [Fu], who obtained the Hilbert space case of Theorem 6.10 under the null controllability assumption $S(t)E \subset H_t$ for all $t > 0$.

## 7. Extension to Gaussian Mehler semigroups

In [BRS], Bogachev, Röckner, and Schmuland introduced the concept of a generalized Mehler semigroup. Under Assumption 2.1, the Ornstein-Uhlenbeck semigroups $\mathbf{P}$ belong to this class. In this final section we will discuss briefly some extensions of our results to this more general framwork.

Let $E$ be a separable real Banach space, let $\mathbf{S}$ be a $C_0-$semigroup on $E$, and let $\{\mu_t\}_{t\geqslant 0}$ be a one-parameter family of probability measures defined on the Borel $\sigma-$algebra of $E$. The pair $(\mathbf{S}, \{\mu_t\}_{t\geqslant 0})$ is called a *Mehler semigroup* on $E$ if

$$\mu_{t+s} = (T(s)\mu_t) * \mu_s, \qquad t, s \geqslant 0, \tag{7.1}$$

where $T(s)\mu_t$ denotes the image measure of $\mu_t$ under $T(s)$. This terminology is explained by the observation [BRS, Proposition 2.2] that $(\mathbf{S}, \{\mu_t\}_{t\geqslant 0})$ is a Mehler semigroup if and only if

$$P(t)f(x) := \int_E f(S(t)x - y)\, d\mu_t(y), \qquad t \geqslant 0,\, x \in E,$$

defines a semigroup on the space $B_b(E)$ of bounded Borel functions on $E$. More generally, a pair $(\mathbf{S}, \{\mu_t\}_{t\geqslant 0})$, where $\mathbf{S}$ is a $C_0-$semigroup on $E$ and $\{\mu_t\}_{t\geqslant 0}$ is a one-parameter family of cylindrical probability measures on the ring of cylindrical sets in $E$, is called a *cylindrical Mehler semigroup* on $E$ if (7.1) holds.

If $Q \in \mathcal{L}(E^*, E)$ is a positive and symmetric operator and $\mathbf{S}$ is a $C_0-$semigroup on $E$, then the pair $(\mathbf{S}, \{\mu_t\}_{t\geqslant 0})$, where $\mu_t$ is the unique cylindrical measure whose Fourier transform is given by

$$\hat{\mu_t}(x^*) = \exp\left(-\frac{1}{2}\langle Q_t x^*, x^*\rangle\right), \qquad x^* \in E^* \tag{7.2}$$

is easily seen to be a cylindrical Mehler semigroup; it is a Mehler semigroup if Assumption 2.1 holds.

Motivated by this example, we say that $(\mathbf{S}, \{\mu_t\}_{t\geqslant 0})$ is *Gaussian* if for each $t > 0$ there exists a positive symmetric operator $Q_t \in \mathcal{L}(E^*, E)$, the *covariance* of $\mu_t$, such that the Fourier transform of $\mu_t$ is given by (7.2). In this situation we denote by $H_t$ the RKHS associated with the covariance operator $Q_t$ of $\mu_t$. By considering the Fourier transform of (7.1) we have the identity [BRS, Proposition 2.2]

$$Q_{t+s} = Q_s + S(s)Q_tS^*(s), \qquad t, s \geqslant 0. \tag{7.3}$$

In particular, $\langle Q_{t+s}x^*, x^*\rangle = \langle Q_s x^*, x^*\rangle + \langle Q_t S^*(s)x^*, S^*(s)x^*\rangle$ for all $t, s \geqslant 0$ and $x^* \in E^*$. By positivity, this shows that the functions $t \mapsto \langle Q_t x^*, x^*\rangle$ are increasing. Hence,

$$H_{t_0} \subset H_{t_1} \qquad \text{whenever} \quad 0 < t_0 < t_1 < \infty. \tag{7.4}$$

Inspection of the proofs shows that (7.3) and (7.4) are all that is needed for most of the results in this paper. These therefore extend to Gaussian (cylindrical) Mehler semigroups without change.

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*Note added in proof* – After this paper had been accepted for publication, the author realized that without any compactness assumption, (5.1) always holds if $f \in BUC(E)$. In fact, it turns out that one always has $\lim_{t\downarrow 0} \mu_t = \delta_0$ weakly; this is a consequence of Anderson's inequality and easily implies the assertion just made. As a consequence, in Corollary 5.8 the null controllability assumption can be omitted, and the characterization of strong continuity in $BUC(E)$ mentioned in Remark 5.10 extends to Banach spaces $E$. The details will be presented elsewhere.