A stochastic Datko-Pazy theorem

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Let H be a Hilbert space and E a Banach space. In this note we present a sufficient condition for an operator $R : H \to E$ to be γ–radonifying in terms of Riesz sequences in H. This result is applied to recover a result of Lutz Weis and the second named author on the R-boundedness of resolvents, which is used to obtain a Datko-Pazy type theorem for the stochastic Cauchy problem. We also present some perturbation results.

1 Introduction

The well-known Datko-Pazy theorem states that if $(T(t))_{t\geq 0}$ is a strongly continuous semigroup on a Banach space E such that all orbits $T(\cdot)x$ belong to the space $L^p(\mathbb{R}_+, E)$ for some $p \in [1, \infty)$, then $(T(t))_{t\geq 0}$ is uniformly exponentially stable, or equivalently, there exists an $\varepsilon > 0$ such that all orbits $t \mapsto e^{\varepsilon t}T(t)x$ belong to $L^p(\mathbb{R}_+, E)$. For $p = 2$ and Hilbert spaces E this result is due to Datko [3], and the general case was obtained by Pazy [14].

In this note we prove a stochastic version of the Datko-Pazy theorem for spaces of γ–radonifying operators (cf. Section 2). Let us denote by $\gamma(\mathbb{R}_+, E)$ the space of all strongly measurable functions $\phi : \mathbb{R}_+ \to E$ for which the integral operator

$$f \mapsto \int_0^\infty f(t)\phi(t)\, dt$$

is well-defined and γ-radonifying from $L^2(\mathbb{R}_+)$ to E.

Theorem 1.1a (Stochastic Datko-Pazy Theorem, first version)**.** *Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space E. The following assertions are equivalent:*

(a) *For all $x \in E$, $T(\cdot)x \in \gamma(\mathbb{R}_+, E)$.*

(b) *There exists an $\varepsilon > 0$ such that for all $x \in E$, $t \mapsto e^{\varepsilon t}T(t)x \in \gamma(\mathbb{R}_+, E)$.*

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If E is a Hilbert space, $\gamma(\mathbb{R}_+, E) = L^2(\mathbb{R}_+, E)$ and Theorem 1.1a is equivalent to the Datko's theorem mentioned above.

As explained in [12], γ–radonifying operators play an important role in the study of the following stochastic abstract Cauchy problem on E:

$$(\mathrm{SCP})_{(A,B)} \qquad \begin{cases} dU(t) &= AU(t)\,dt + B\,dW_H(t), \qquad t \geq 0, \\ U(0) &= 0. \end{cases}$$

Here, H is a separable Hilbert space, $B \in \mathcal{B}(H, E)$ is a bounded operator, and W_H is an H-cylindrical Brownian motion. Theorem 1.1a can be reformulated in terms of invariant measures for $(\mathrm{SCP})_{(A,B)}$ as follows.

Theorem 1.1b (Stochastic Datko-Pazy theorem, second version)**.** *With the above notations, the following assertions are equivalent:*

(a) *For all rank one operators $B \in \mathcal{B}(H, E)$, the problem $(\mathrm{SCP})_{(A,B)}$ admits an invariant measure.*

(b) *There exists an $\varepsilon > 0$ such that for all rank one operators $B \in \mathcal{B}(H, E)$, the problem $(\mathrm{SCP})_{(A+\varepsilon,B)}$ admits an invariant measure.*

For unexplained terminology and more information on the stochasic Cauchy problem and invariant measures we refer to [2, 11, 12].

2 Riesz bases and γ-radonifying operators

Let $\mathcal{H}$ be a Hilbert space and E a Banach space. Let $(\gamma_n)_{n\geq 1}$ be a sequence of independent standard Gaussian random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A bounded linear operator $R : \mathcal{H} \to E$ is called *almost summing* if

$$\|R\|_{\gamma_\infty(\mathcal{H},E)} := \sup \Big\| \sum_{n=1}^{N} \gamma_n R h_n \Big\|_{L^2(\Omega,E)} < \infty,$$

where the supremum is taken over all $N \in \mathbb{N}$ and all orthonormal systems $\{h_1, \ldots, h_N\}$ in $\mathcal{H}$. Endowed with this norm, the space $\gamma_\infty(\mathcal{H}, E)$ of all almost summing operators is a Banach space. Moreover, $\gamma_\infty(\mathcal{H}, E)$ is an operator ideal in $\mathcal{B}(\mathcal{H}, E)$. The closure of the finite rank operators in $\gamma_\infty(\mathcal{H}, E)$ will be denoted by $\gamma(\mathcal{H}, E)$. Operators belonging to this space are called *γ-radonifying*. Again $\gamma(\mathcal{H}, E)$ is an operator ideal in $\mathcal{B}(\mathcal{H}, E)$.

Let us now assume that $\mathcal{H}$ is a separable Hilbert space. Under this assumption one has $R \in \gamma_\infty(\mathcal{H}, E)$ if and only if for some (every) orthonormal basis $(h_n)_{n\geq 1}$ for $\mathcal{H}$,

$$M := \sup_{N\geq 1} \Big\| \sum_{n=1}^{N} \gamma_n R h_n \Big\|_{L^2(\Omega,E)} < \infty.$$

In that case, $\|R\|_{\gamma_\infty(\mathcal{H},E)} = M$. Furthermore, one has $R \in \gamma(\mathcal{H}, E)$ if and only if for some (every) orthonormal basis $(h_n)_{n\geq 1}$ for $\mathcal{H}$, $\sum_{n\geq 1} \gamma_n R h_n$ converges in $L^2(\Omega, E)$. In that case,

$$\|R\|_{\gamma(\mathcal{H},E)} = \Big\| \sum_{n\geq 1} \gamma_n R h_n \Big\|_{L^2(\Omega,E)}.$$

If E does not contain a closed subspace isomorphic to c_0, then by a result of Hoffmann-Jørgensen and Kwapień [10, Theorem 9.29], $\gamma(\mathcal{H},E) = \gamma_\infty(\mathcal{H},E)$.

We will apply the above notions to the space $\mathcal{H} = L^2(\mathbb{R}_+,H)$ where H is a separable Hilbert space. For an operator-valued function $\phi : \mathbb{R}_+ \to \mathcal{B}(H,E)$ which is *H-strongly measurable* in the sense that $t \mapsto \phi(t)h$ is strongly measurable for all $h \in H$, and *weakly square integrable* in the sense that $t \mapsto \phi^*(t)x^*$ is square Bochner integrable for all $x^* \in E^*$, let $R_\phi \in \mathcal{B}(L^2(\mathbb{R}_+,H),E)$ be defined as the Pettis integral operator

$$R_\phi(f) := \int_{\mathbb{R}_+} \phi(t)f(t)\,dt.$$

We say that $\phi \in \gamma(\mathbb{R}_+,H,E)$ if $R_\phi \in \gamma(L^2(\mathbb{R}_+,H),E)$ and write

$$\|\phi\|_{\gamma(\mathbb{R}_+,H,E)} := \|R_\phi\|_{\gamma(L^2(\mathbb{R}_+,H),E)}.$$

If $H = \mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is the underlying scalar field, we write $\gamma(\mathbb{R}_+,E)$ for $\gamma(\mathbb{R}_+,H,E)$. For almost summing operators we use an analogous notation.

For more information we refer to [4, 8, 11, 12].

Hilbert and Bessel sequences. Let $\mathcal{H}$ be a Hilbert space and $I \subseteq \mathbb{Z}$ an index set. A sequence $(h_i)_{i\in I}$ in $\mathcal{H}$ is said to be a *Hilbert sequence* if there exists a constant $C > 0$ such that for all scalars $(\alpha_i)_{i\in I}$,

$$\Big(\Big\|\sum_{i\in I}\alpha_i h_i\Big\|^2\Big)^{1/2} \le C\Big(\sum_{i\in I}|\alpha_i|^2\Big)^{1/2}.$$

The infimum of all admissible constants $C > 0$ will be denoted by $C_H(\{h_i : i \in I\})$. A Hilbert sequence that is a Schauder basis is called a *Hilbert basis* (cf. [17, Section 1.8]).

The sequence $(h_i)_{i\in I}$ is said to be *Bessel sequence* if there exists a constant $c > 0$ such that for all scalars $(\alpha_i)_{i\in I}$,

$$c\Big(\sum_{i\in I}|\alpha_i|^2\Big)^{1/2} \le \Big(\Big\|\sum_{i\in I}\alpha_i h_i\Big\|^2\Big)^{1/2}.$$

The supremum of all admissible constants $c > 0$ will be denoted by $C_B(\{h_i : i \in I\})$. Notice that every Bessel sequence is linearly independent. A Bessel sequence that is a Schauder basis is called a *Bessel basis*. A sequence $(h_i)_{i\in I}$ that is a Bessel sequence and a Hilbert sequence is said to be a *Riesz sequence*. A sequence $(h_i)_{i\in I}$ that is a Bessel basis and a Hilbert basis is said to be a *Riesz basis* (cf. [17, Section 1.8]).

In the above situation if it is clear which sequence in $\mathcal{H}$ we refer to, we use the short-hand notation C_H and C_B for $C_H(\{h_i : i \in I\})$ and $C_B(\{h_i : i \in I\})$.

In the next results we study the relation between γ–radonifying operators and Hilbert and Bessel sequences.

Proposition 2.1. *Let $(f_n)_{n\ge1}$ be a Hilbert sequence in $\mathcal{H}$.*

(a) *If $R \in \gamma_\infty(\mathcal{H},E)$, then*

$$\sup_{N\ge1}\Big\|\sum_{n=1}^N \gamma_n R f_n\Big\|_{L^2(\Omega,E)} \le C_H\,\|R\|_{\gamma_\infty(\mathcal{H},E)}. \tag{1}$$

(b) *If $R \in \gamma(\mathcal{H}, E)$, then $\sum_{n\geq 1} \gamma_n R f_n$ converges in $L^2(\Omega, E)$ and*

$$\Big\| \sum_{n\geq 1} \gamma_n R f_n \Big\|_{L^2(\Omega,E)} \leq C_H \; \|R\|_{\gamma(\mathcal{H},E)}. \tag{2}$$

Proof. (a): Fix $N \geq 1$ and let $\{h_1, \dots, h_N\}$ be an orthonormal system in $\mathcal{H}$. Since $(f_n)_{n\geq 1}$ is a Hilbert sequence there is a unique $T \in \mathcal{B}(\mathcal{H})$ such that $Th_n = f_n$ for $n = 1, \dots, N$ and $Tx = 0$ for all $x \in \{h_1, \dots, h_N\}^{\perp}$. Moreover, $\|T\| \leq C_H$. By the right ideal property we have $R \circ T \in \gamma_\infty(\mathcal{H}, E)$ and, for all $N \geq 1$,

$$\Big\| \sum_{n=1}^{N} \gamma_n R f_n \Big\|_{L^2(\Omega,E)} = \Big\| \sum_{n=1}^{N} \gamma_n R T h_n \Big\|_{L^2(\Omega,E)} \leq \|R \circ T\|_{\gamma_\infty(\mathcal{H},E)} \leq C_H \; \|R\|_{\gamma_\infty(\mathcal{H},E)}.$$

(b): This is proved in a similar way. $\square$

Proposition 2.2. *Let $(f_n)_{n\geq 1}$ be a Bessel sequence in $\mathcal{H}$ and let $\mathcal{H}_f$ denote its closed linear span.*

(a) *If $\sup_{N\geq 1} \Big\| \sum_{n=1}^{N} \gamma_n R f_n \Big\|_{L^2(\Omega,E)} < \infty$, then $R \in \gamma_\infty(\mathcal{H}_f, E)$ and*

$$\|R\|_{\gamma_\infty(\mathcal{H}_f,E)} \leq C_B^{-1} \; \sup_{N\geq 1} \Big\| \sum_{n=1}^{N} \gamma_n R f_n \Big\|_{L^2(\Omega,E)}. \tag{3}$$

(b) *If $\sum_{n\geq 1} \gamma_n R f_n$ converges in $L^2(\Omega, E)$, then $R \in \gamma(\mathcal{H}_f, E)$ and*

$$\|R\|_{\gamma(\mathcal{H}_f,E)} \leq C_B^{-1} \; \Big\| \sum_{n\geq 1} \gamma_n R f_n \Big\|_{L^2(\Omega,E)}. \tag{4}$$

Proof. Let $(h_n)_{n\geq 1}$ an orthonormal basis for $\mathcal{H}_f$. Since $(f_n)_{n\geq 1}$ is a Bessel sequence there is a unique $T \in \mathcal{B}(\mathcal{H}, E)$ such that $Tf_n = h_n$ and $Tx = 0$ for $x \in \mathcal{H}_f^{\perp}$. Notice that $\|T\| \leq C_B^{-1}$. On the linear span $\mathcal{H}_0$ of the sequence $(f_n)_{n\geq 1}$ we define an inner product by $[x, y]_T := [Tx, Ty]_{\mathcal{H}}$. Note that this is well defined by the linear independence of the sequence $(f_n)_{n\geq 1}$. Let $\mathcal{H}_T$ denote the Hilbert space completion of $\mathcal{H}_0$ with respect to $[\cdot, \cdot]_T$. The identity mapping on $\mathcal{H}_f$ extends to a bounded operator $j : \mathcal{H}_f \hookrightarrow \mathcal{H}_T$ with norm $\|j\| \leq C_B^{-1}$. Clearly, $(jf_n)_{n\geq 1}$ is an orthonormal sequence in $\mathcal{H}_T$ with dense span, and therefore it is an orthonormal basis for $\mathcal{H}_T$. It is elementary to verify that the assumption on R may now be translated as saying that R extends in a unique way to an almost summing operator (in part (a)), respectively a γ-radonifying operator (in part (b)), denoted by R_T, from $\mathcal{H}_T$ to E. We estimate

$$\Big\| \sum_{n\geq 1} \alpha_n j h_n \Big\|_{\mathcal{H}_T} = \Big\| \sum_{n\geq 1} \alpha_n T h_n \Big\|_{\mathcal{H}} \; \leq \; C_B^{-1} \; \Big\| \sum_{n\geq 1} \alpha_n h_n \Big\|_{\mathcal{H}} = C_B^{-1} \Big(\sum_{n\geq 1} |\alpha_n|^2 \Big)^{1/2}.$$

From this we deduce that $(jh_n)_{n\geq 1}$ is a Hilbert sequence in $\mathcal{H}_T$ with constant $\leq C_B^{-1}$. Hence we may apply Proposition 2.1 to the operator $R_T : \mathcal{H}_T \to E$ and the Hilbert sequence $(jh_n)_{n\geq 1}$ in $\mathcal{H}_T$ to obtain the result. $\square$

As a consequence of the above results we obtain:

Theorem 2.3. *Let $(f_n)_{n\geq 1}$ be a Riesz basis in the Hilbert space $\mathcal{H}$.*

(a) *One has $R \in \gamma_\infty(\mathcal{H}, E)$ if and only if $\sup_{N\geq 1}\Big\|\sum_{n=1}^{N} \gamma_n R f_n\Big\|_{L^2(\Omega,E)} < \infty$. In that case (1) and (3) hold.*

(b) *One has $R \in \gamma(\mathcal{H}, E)$ if and only if $\sum_{n\geq 1} \gamma_n R f_n$ converges in $L^2(\Omega, E)$. In that case (2) and (4) hold.*

The following well-known lemma identifies a class of Riesz sequences in $L^2(\mathbb{R})$. For convenience we include the short proof from [1, Theorem 2.1]. Let $\mathbb{T}$ be the unit circle in $\mathbb{C}$.

Lemma 2.4. *Let $f \in L^2(\mathbb{R})$ and define the sequence $(f_n)_{n\in\mathbb{Z}}$ in $L^2(\mathbb{R})$ by $f_n(t) = e^{2\pi nit} f(t)$. Define $F : \mathbb{T} \to \overline{\mathbb{R}}$ as*

$$F(e^{2\pi it}) := \sum_{k\in\mathbb{Z}} |f(t + k)|^2$$

(a) *The sequence $(f_n)_{n\in\mathbb{Z}}$ is a Bessel sequence in $L^2(\mathbb{R})$ if and only if there exists a constant $A > 0$ such that $A \leq F(e^{2\pi it})$ for almost all $t \in [0, 1]$.*

(b) *The sequence $(f_n)_{n\in\mathbb{Z}}$ is a Hilbert sequence in $L^2(\mathbb{R})$ if and only if there exists a constant $B > 0$ such that $F(e^{2\pi it}) \leq B$ for almost all $t \in [0, 1]$.*

In these cases, $C_B^2 = \operatorname{ess\,inf} F$ and $C_H^2 = \operatorname{ess\,sup} F$ respectively.

Proof. Both assertions are obtained by observing that for $I \subseteq \mathbb{Z}$ and $(a_n)_{n\in I}$ in $\mathbb{C}$ we may write

$$\begin{aligned}\Big\|\sum_{n\in I} a_n f_n\Big\|^2_{L^2(\mathbb{R})} &= \sum_{k\in\mathbb{Z}} \int_k^{(k+1)} \Big|\sum_{n\in I} a_n e^{2\pi nit} f(t)\Big|^2 dt \\ &= \sum_{k\in\mathbb{Z}} \int_0^1 \Big|\sum_{n\in I} a_n e^{2\pi nit} f(t+k)\Big|^2 dt = \int_0^1 \Big|\sum_{n\in I} a_n e^{2\pi nit}\Big|^2 F(e^{2\pi it})\, dt.\end{aligned}$$

□

The following application of Lemma 2.4 will be used below.

Example 2.5. Let $\rho \in [0, 1)$ and $a > 0$. For $n \in \mathbb{Z}$ let

$$f_n(t) = e^{-at+2\pi(n+\rho)it} \mathbb{1}_{[0,\infty)}(t).$$

Then $(f_n)_{n\in\mathbb{Z}}$ is a Riesz sequence in $L^2(\mathbb{R})$ with constants $C_B^2 = \frac{e^{-2a}}{e^{2a}-1}$ and $C_H^2 = \frac{e^{2a}}{e^{2a}-1}$. Indeed, let $f(t) := e^{-at+2\pi\rho it} \mathbb{1}_{[0,\infty)}(t)$. For all $t \in [0, 1)$,

$$F(e^{2\pi it}) = \sum_{k\in\mathbb{Z}} |f(t + k)|^2 = \sum_{k=0}^{\infty} e^{-2a(t+k)} = \frac{e^{2a(1-t)}}{e^{2a} - 1}.$$

Now Lemma 2.4 implies the result.

Remark 2.6. Necessary and sufficient conditions on the complex coefficients c_n and λ_n with $\mathrm{Re}\lambda_n > 0$ in order that the functions $z \mapsto c_n \exp(-\lambda_n z)$ form a Riesz sequence can be found in [13, Section 10.3] and [7].

3 Main results

In this section we use Proposition 2.1 to obtain an alternative proof of [12, Theorem 3.4] on the R–boundedness of certain Laplace transforms. This result is applied to strongly continuous semigroups to obtain estimates for the abscissa of R–boundedness of the resolvent. From this we deduce Theorem 1.1a as well as bounded perturbation results for the existence of solutions and invariant measures for the problem $(\mathrm{SCP})_{(A,B)}$.

Let $(r_n)_{n\geq 1}$ be a Rademacher sequence on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. A family of operators $\mathscr{T} \subseteq \mathcal{B}(E)$ is called *R-bounded* if there exists a constant $C > 0$ such that for all $N \geq 1$ and all sequences $(T_n)_{n=1}^N \subseteq \mathcal{T}$ and $(x_n)_{n=1}^N \subseteq E$ we have

$$\mathbb{E}\Big\|\sum_{n=1}^N r_n T_n x_n\Big\|^2 \leq C^2 \mathbb{E}\Big\|\sum_{n=1}^N r_n x_n\Big\|^2.$$

The least possible constant C is called the *R-bound* of $\mathscr{T}$, notation $\mathscr{R}(\mathscr{T})$. Clearly, every R-bounded family $\mathscr{T}$ is uniformly bounded and $\sup_{T\in\mathscr{T}} \|T\| \leq \mathscr{R}(\mathscr{T})$.

Following [12], for an operator $T \in \mathcal{B}(L^2(\mathbb{R}_+), E)$ we define the *Laplace transform* $\widehat{T} : \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\} \to E$ as

$$\widehat{T}(\lambda) := T e_\lambda.$$

Here $e_\lambda \in L^2(\mathbb{R}_+)$ is given by $e_\lambda(t) = e^{-\lambda t}$. For a Banach space F and a bounded operator $\Theta : F \to \mathcal{B}(L^2(\mathbb{R}_+), E)$ we define the *Laplace transform* $\widehat{\Theta} : \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\} \to \mathcal{B}(F, E)$ as

$$\widehat{\Theta}(\lambda) y := \widehat{\Theta y}(\lambda) \qquad \operatorname{Re}\lambda > 0,\ y \in F.$$

The following result is a slight refinement of [12, Theorem 3.4]. The main novelty is the simple proof of the estimate (5).

Theorem 3.1. *Let F be a Banach space. Let $\Theta : F \to \gamma_\infty(L^2(\mathbb{R}_+), E)$ be a bounded operator and let $\delta > 0$. Then $\widehat{\Theta}$ is R–bounded on the half-plane $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \delta\}$ and there exists a universal constant C such that*

$$\mathcal{R}\left(\{\widehat{\Theta}(\lambda) : \operatorname{Re}\lambda \geq \delta\}\right) \leq \|\Theta\| \frac{C}{\sqrt{\delta}}.$$

Proof. Let $\delta > 0$. Consider the set $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda = \delta\}$. Fix $\sigma \in [\delta/2, 3/2\delta]$ and $\rho \in [0, 1)$. For $n \in \mathbb{Z}$ let $g_n : \mathbb{R}_+ \to \mathbb{C}$ be given by

$$g_n(t) = e^{-\sigma t + (n+\rho)\delta i t}.$$

By a substitution, this reduces to Example 2.5, whence $(g_n)_{n\geq 1}$ is a Riesz sequence in $L^2(\mathbb{R}_+)$ with constant $0 < C_H \leq \left(\frac{C}{\delta}\right)^{1/2}$ where $C := 2\pi\frac{e^{2\pi}}{e^{2\pi}-1}$. For $y \in F$, we may apply Proposition 2.1 to obtain

$$\begin{aligned}\Big\|\sum_{n=-N}^N \gamma_n \widehat{\Theta}(\sigma - (n+\rho)\delta i) y\Big\|_{L^2(\Omega,E)} &= \Big\|\sum_{n=-N}^N \gamma_n (\Theta y) g_n\Big\|_{L^2(\Omega,E)} \\ &\leq C_H \|\Theta y\|_{\gamma_\infty(\Omega,E)} \leq \left(\frac{C}{\delta}\right)^{1/2} \|\Theta\|\,\|y\|.\end{aligned} \tag{5}$$

The rest of the proof follows the lines in [12]. ∎

In what follows we let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on E with generator A. We recall from [11, 12] that the problem $(\mathrm{SCP})_{(A,B)}$ admits a (unique) solution if and only if $T(\cdot)B$ belongs to $\gamma([0,T],H,E)$ for some (all) $T>0$. Furthermore, an invariant measure exists if and only if $T(\cdot)B$ belongs to $\gamma(\mathbb{R}_+,H,E)$.

The next theorem improves [12, Theorem 1.3], where the bound $s_R(A)\leq 0$ was obtained.

Theorem 3.2. *Assume that for all $x\in E$, $T(\cdot)x\in\gamma_\infty(\mathbb{R}_+,E)$. Then $s_R(A)<0$, i.e., there exists an $\varepsilon>0$ such that $\{R(\lambda,A):\ \mathrm{Re}\lambda\geq-\varepsilon\}$ is R–bounded.*

Proof. By the closed graph theorem there exists an $M>0$ such that $\|T(\cdot)x\|_{\gamma_\infty(\mathbb{R}_+,E)}\leq M\|x\|$. By Theorem 3.1, $\{\lambda\in\mathbb{C}:\ \mathrm{Re}\lambda>0\}\subseteq\varrho(A)$ and

$$\mathcal{R}\left(\{R(\lambda,A):\ \mathrm{Re}\lambda\geq\delta\}\right)\leq\frac{c}{\sqrt{\delta}} \tag{6}$$

for all $\delta>0$, where $c:=CM$ with C the universal constant of Theorem 3.1. The following standard argument shows that this implies the bound

$$s(A)\leq-\frac{1}{4c^2}. \tag{7}$$

Choose $\delta>0$ and let $\mu\in\sigma(A)$ be such that $\mathrm{Re}\mu>s(A)-\delta$. With $\lambda=\frac{1}{4c^2}+i\,\mathrm{Im}\mu$ it follows that

$$\frac{1}{4c^2}-s(A)+\delta\geq\mathrm{dist}(\lambda,\sigma(A))\geq\frac{1}{\|R(\lambda,A)\|}\geq\frac{\sqrt{\mathrm{Re}\lambda}}{c}=\frac{1}{2c^2}.$$

Thus $s(A)\leq-\frac{1}{4c^2}+\delta$. Since $\delta>0$ was arbitrary, this gives (7).

Now let $\varepsilon_0:=\frac{1}{4c^2}$. For λ with $-\varepsilon_0<\mathrm{Re}\lambda<3\varepsilon_0$ we may write

$$R(\lambda,A)=\sum_{n\geq 0}(\varepsilon_0-\mathrm{Re}\lambda)^n R(\varepsilon_0+i\mathrm{Im}\lambda,A)^{n+1}.$$

Fix $0<\varepsilon<\varepsilon_0$. We claim that $\{R(\lambda,A):\ \mathrm{Re}\lambda=-\varepsilon\}$ is R–bounded. To see this let $(r_k)_{k=1}^K$ be a Rademacher sequence on $(\Omega,\mathscr{F},\mathbb{P})$, let $(\lambda_k)_{k=1}^K$ be such that $\mathrm{Re}\lambda_k=-\varepsilon$, and let $(x_k)_{k=1}^K$ be a sequence in E. We may estimate

$$\begin{aligned}
\Big\|\sum_{k=1}^K r_k R(\lambda_k,A)x_k\Big\|_{L^2(\Omega,E)} &= \Big\|\sum_{n\geq 0}\sum_{k=1}^K r_k(\varepsilon_0+\varepsilon)^n R(\varepsilon_0+i\mathrm{Im}\lambda_k,A)^{n+1}x_k\Big\|_{L^2(\Omega,E)}\\
&\leq \sum_{n\geq 0}(\varepsilon_0+\varepsilon)^n\Big\|\sum_{k=1}^K r_k R(\varepsilon_0+i\mathrm{Im}\lambda_k,A)^{n+1}x_k\Big\|_{L^2(\Omega,E)}\\
&\leq \sum_{n\geq 0}(\varepsilon_0+\varepsilon)^n\left(\frac{c}{\sqrt{\varepsilon_0}}\right)^{n+1}\Big\|\sum_{k=1}^K r_k x_k\Big\|_{L^2(\Omega,E)}\\
&= \frac{1}{\varepsilon_0-\varepsilon}\Big\|\sum_{k=1}^K r_k x_k\Big\|_{L^2(\Omega,E)},
\end{aligned}$$

where we used that $\varepsilon_0={}^{1}\!/_{4c^2}$. This proves the claim. Now the result is obtained via [16, Proposition 2.8]. $\square$

As an application of Theorem 3.2 we have the following bounded perturbation result for the existence of a solution for the perturbed problem.

Theorem 3.3. *Let $P \in \mathcal{B}(E)$ and $B \in \mathcal{B}(H,E)$. If* $(\mathrm{SCP})_{(A,B)}$ *has a solution, then* $(\mathrm{SCP})_{(A+P,B)}$ *has a solution as well.*

Proof. For $\omega \in \mathbb{R}$ denote $A_\omega = A - \omega$ and $T_\omega(\cdot) := e^{-\omega\cdot}T(\cdot)$. It follows from [12, Proposition 4.5] that for all $\omega > \omega_0(A)$, $T_\omega(\cdot)B \in \gamma(\mathbb{R}_+, H, E)$. From [9, Corollary 2.17] it follows that for all $\omega > \omega_0(A) + 1$,

$$\mathcal{R}\left(\{R(\lambda, A_\omega) :\ \mathrm{Re}\lambda \geq 0\}\right) \leq \frac{c}{\omega - \omega_0(A) - 1},$$

where c is a constant depending only on T. Choose $\omega_1 > \omega_0(A)+1$ so large that $\frac{c}{\omega_1 - \omega_0(A) - 1}\|P\| < 1$. By [12, Lemma 5.1], $R(i\cdot, A_{\omega_1})B \in \gamma(\mathbb{R}_+, H, E)$.

Denote by $(S(t))_{t\geq 0}$ the semigroup generated by $A+P$ (cf. [5, Section III.1] or [15, Chapter III]) and let $S_{\omega_1}(t) := e^{-\omega_1 t}S(t)$, $t \geq 0$. Since

$$\mathcal{R}\left(\{R(is, A_{\omega_1})P :\ s \in \mathbb{R}\}\right) \leq \mathcal{R}\left(\{R(is, A_{\omega_1}) :\ s \in \mathbb{R}\}\right)\|P\| =: C < 1,$$

it follows from $i\mathbb{R} \subseteq \varrho(A_{\omega_1})$ that $i\mathbb{R} \subseteq \varrho(A_{\omega_1} + P)$ and

$$R(is, A_{\omega_1}{+}P)B = \sum_{n=0}^{\infty}\big(R(is, A_{\omega_1})P\big)^n R(is, A_{\omega_1})B =: R_{A,P,\omega_1}(s)R(is, A_{\omega_1})B.$$

Moreover, as in Theorem 3.2, and using the fact that $C < 1$, $\{R_{A,P,\omega_1}(s) :\ s \in \mathbb{R}\}$ is R–bounded with constant $\frac{1}{1-C}$. From [8, Proposition 4.11] we deduce that

$$\|R(i\cdot, A_{\omega_1}{+}P)B\|_{\gamma(\mathbb{R},H,E)} \leq \tfrac{1}{1-C}\|R(i\cdot, A_{\omega_1})B\|_{\gamma(\mathbb{R},H,E)}.$$

Now [12, Lemma 5.1] shows that $S_{\omega_1}(\cdot)B \in \gamma(\mathbb{R}_+, H, E)$. It follows from the right ideal property that for all $t > 0$,

$$\|S(\cdot)B\|_{\gamma(0,t,H,E)} \leq e^{t\omega_1}\|S_{\omega_1}(\cdot)B\|_{\gamma(0,t,H,E)}$$

and the result can be obtained via [11, Theorem 7.1]. ∎

Concerning existence and uniqueness of invariant measures we obtain:

Theorem 3.4. *Assume that $s(A) < 0$ and that $\{R(is, A) :\ s \in \mathbb{R}\}$ is R–bounded. Let $B \in \mathcal{B}(H,E)$ such that* $(\mathrm{SCP})_{(A,B)}$ *admits an invariant measure. Then there exists a $\delta > 0$ such that for all $P \in \mathcal{B}(E)$ with $\|P\| < \delta$,* $(\mathrm{SCP})_{(A+P,B)}$ *admits a unique invariant measure.*

Proof. Let $\delta > 0$ such that $\mathcal{R}\left(\{R(is, A) :\ s \in \mathbb{R}\}\right) \leq 1/\delta$. Then, if $\|P\| < \delta$,

$$\mathcal{R}\left(\{R(is, A)P :\ s \in \mathbb{R}\}\right) \leq \mathcal{R}\left(\{R(is, A) :\ s \in \mathbb{R}\}\right)\|P\| =: C < 1.$$

As in Theorem 3.3 it can be deduced that

$$\|R(i\cdot, A{+}P)B\|_{\gamma(\mathbb{R},H,E)} \leq \tfrac{1}{1-C}\|R(i\cdot, A)B\|_{\gamma(\mathbb{R},H,E)}.$$

The existence of an invariant measure now follows from [12, Proposition 4.4 and Lemma 5.1].

By [12, Corollary 4.3], for uniqueness it suffices to note that $R(\lambda, A + P)$ is uniformly bounded for $\mathrm{Re}\lambda > 0$. ∎

In particular, the R-boundedness of $\{R(is,A): s \in \mathbb{R}\}$ implies that an invariant measure for $(\mathrm{SCP})_{(A,B)}$, if one exists, is unique. On the other hand, if $i\mathbb{R} \subseteq \varrho(A)$ but $\{R(is,A): s \in \mathbb{R}\}$ fails to be R-bounded, then Theorem 3.2 shows that there exists a rank one operator $B' \in \mathcal{B}(H,E)$ such that the problem $(\mathrm{SCP})_{(A,B')}$ fails to have an invariant measure. As a result we obtain that if $(\mathrm{SCP})_{(A,B)}$ fails to have a unique invariant measure, then there exists a rank one operator $B' \in \mathcal{B}(H,E)$ such that the problem $(\mathrm{SCP})_{(A,B')}$ fails to have an invariant measure. A related result can be found in [6].

Proof of Theorems 1.1a and 1.1b. If $T(\cdot)x \in \gamma(\mathbb{R}_+,E)$ for all $x \in E$, then by Theorem 3.2 $s(A) < 0$ and $\{R(is,A): s \in \mathbb{R}\}$ is R–bounded. Thus, Theorem 3.4 applies to the bounded perturbation $P = \delta \cdot I_E$. □

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