Robust stability of C_0–semigroups and an application to stability of delay equations

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Abstract - Let A be a closed linear operator on a complex Banach space X and let $\lambda \in \varrho(A)$ be a fixed element of the resolvent set of A. Let U and Y be Banach spaces and let $D \in \mathcal{L}(U, X)$ and $E \in \mathcal{L}(X, Y)$ be bounded linear operators. We define $r_\lambda(A; D, E)$ by

$$\sup\big\{r \geq 0 : \ \lambda \in \varrho(A + D\Delta E) \text{ for all } \Delta \in \mathcal{L}(Y, U) \text{ with } \|\Delta\| \leq r\big\}$$

and prove that

$$r_\lambda(A; D, E) = \frac{1}{\|ER(\lambda, A)D\|}.$$

We give two applications of this result. The first is an exact formula for the so-called stability radius of the generator of a C_0–semigroup of linear operators on a Hilbert space; it is derived from a precise result about robustness under perturbations of uniform boundedness in the right half-plane of the resolvent of an arbitrary semigroup generator. The second application gives sufficient conditions on the norm of the operators $B_j \in \mathcal{L}(X)$ in order that the classical solutions of the delay equation

$$\dot{u}(t) = Au(t) + \sum_{j=1}^{n} B_j u(t - h_j), \quad t \geq 0,$$

are exponentially stable in $L^p([-h, 0]; X)$.

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0. Introduction

In this paper we investigate robustness of certain properties of a closed linear operator A on a Banach space X under small additive perturbations. Some 'structure' in the perturbation will be allowed, in the following sense: we fix Banach spaces U and Y and two operators $D \in \mathcal{L}(U, X)$ and $E \in \mathcal{L}(X, Y)$ (or even $E \in \mathcal{L}(\mathcal{D}(A), Y)$), and consider perturbations of the form $D\Delta E$, with $\Delta \in \mathcal{L}(Y, U)$. The question we address is the following:

> If A has a certain property (P), what is the supremum of all $r \geq 0$ with the following property: for all bounded linear operators $\Delta \in \mathcal{L}(Y, U)$ with norm $\|\Delta\| \leq r$, the perturbed operator $A + D\Delta E$ has property (P) as well.

Among the properties we consider are the following: containment of a given complex number $\lambda \in \mathbb{C}$ in the resolvent set of the operator, containment of a given set $\Omega \subset \mathbb{C}$ in the resolvent set, and uniform boundedness of the resolvent on Ω. For these properties we give a precise answer to the above question in terms of the so-called *transfer function* $\lambda \mapsto ER(\lambda, A)D$, where $R(\lambda, A) := (\lambda - A)^{-1}$ is the resolvent of A.

In two subsequent sections, we give two applications of the abstract results of Section 1. In Section 2 we prove some new results on robust stability. Among others we obtain an exact formula for the stability radius for generators of Hilbert space semigroups. In Section 3 we study the delay equation

$$\dot{u}(t) = Au(t) + \sum_{j=1}^{n} B_j u(t - h_j), \quad t \geq 0,$$

where A is the generator of a C_0−semigroup on a Banach space X. Regarding the bounded operators B_j as a perturbation of an appropriate Cauchy problem corresponding to the absence of delays, we obtain sufficient conditions on A and B_j for exponential stability of classical solutions.

1. The abstract perturbation results

Throughout this section, X, U, and Y are fixed complex Banach spaces, A is a closed linear operator on X with domain $\mathcal{D}(A)$, and $D \in \mathcal{L}(U, X)$ and $E \in \mathcal{L}(\mathcal{D}(A), Y)$ are bounded linear operators; we regard $\mathcal{D}(A)$ as a Banach space with respect to the graph norm $\|\cdot\|_{\mathcal{D}(A)}$.

Proposition 1.1. *Let A be a closed linear operator on X and suppose $\lambda \in \varrho(A)$. If $\Delta \in \mathcal{L}(Y,U)$ satisfies*

$$\|\Delta\| \leq (1-\delta)\frac{1}{\|ER(\lambda,A)D\|} \tag{1.1}$$

for some $\delta \in (0,1)$, then $\lambda \in \varrho(A+D\Delta E)$, and

$$\|R(\lambda, A+D\Delta E)\| \leq \|R(\lambda,A)\| \left(1+\frac{1}{\delta}\|D\|\,\|\Delta E R(\lambda,A)\|\right).$$

Proof: Fix $\lambda \in \varrho(A)$. From $\|\Delta ER(\lambda,A)D\| \leq 1-\delta$ we see that $I-\Delta ER(\lambda,A)D$ is invertible. Using the Neumann series we estimate

$$\|(I-\Delta ER(\lambda,A)D)^{-1}\| \leq \sum_{n=0}^{\infty}(1-\delta)^n = \frac{1}{\delta}.$$

It follows that $I - D\Delta ER(\lambda,A)$ is invertible as well, and its inverse is given by

$$(I-D\Delta ER(\lambda,A))^{-1} = I + D(I-\Delta ER(\lambda,A)D)^{-1}\Delta ER(\lambda,A).$$

By the above estimate,

$$\|(I-D\Delta ER(\lambda,A))^{-1}\| \leq 1+\frac{1}{\delta}\|D\|\,\|\Delta ER(\lambda,A)\|.$$

From the identity $\lambda - A - D\Delta E = (I-D\Delta ER(\lambda,A))(\lambda-A)$ we see that $\lambda-A-D\Delta E$ is closed, being the composition of a closed operator and a bounded invertible operator. It also shows that $\lambda - A - D\Delta E$ maps $\mathcal{D}(A)$ injectively onto X. Hence, the inverse mapping $(\lambda-A-D\Delta E)^{-1}$ is well defined on X, and being the inverse of a closed operator, it is closed. Hence by the closed graph theorem, $(\lambda - A - D\Delta E)^{-1}$ is bounded, which means that $\lambda \in \varrho(A+D\Delta E)$. By the previous estimate, we obtain

$$\begin{aligned}\|R(\lambda, A+D\Delta E)\| &= \|R(\lambda,A)(I-D\Delta ER(\lambda,A))^{-1}\| \\ &\leq \|R(\lambda,A)\| \left(1+\frac{1}{\delta}\|D\|\,\|\Delta ER(\lambda,A)\|\right).\end{aligned}$$

∎

This result shows that the property '$\lambda \in \varrho(A)$' is stable under small perturbations. Next we show that the bound (1.1) is actually the best possible. To this end, for $\lambda \in \varrho(A)$ we introduce the quantity

$$r_\lambda(A;D,E) := \sup\{r \geq 0 :\ \lambda \in \varrho(A+D\Delta E) \text{ for all } \Delta \in \mathcal{L}(Y,U) \text{ with } \|\Delta\| \leq r\}.$$

Theorem 1.2. *Let A be a closed linear operator on X. Then for all $\lambda \in \varrho(A)$ we have*

$$r_\lambda(A; D, E) = \frac{1}{\|ER(\lambda, A)D\|}.$$

Proof: If $0 \le r < \|ER(\lambda, A)D\|^{-1}$ and $\|\Delta\| \le r$, then $\lambda \in \varrho(A + D\Delta E)$ by Proposition 1.1. Hence, $r_\lambda(A; D, E) \ge \|ER(\lambda, A)D\|^{-1}$. In order to prove the converse inequality, let us fix $\varepsilon > 0$. Choose $u \in U$, $\|u\| = 1$, such that

$$\frac{1}{\|ER(\lambda, A)Du\|} \le \frac{1}{\|ER(\lambda, A)D\|} + \varepsilon.$$

By the Hahn-Banach theorem we may choose $y^* \in Y^*$, $\|y^*\| = 1$, such that

$$\left\langle \frac{ER(\lambda, A)Du}{\|ER(\lambda, A)Du\|}, y^* \right\rangle = 1.$$

Define $\Delta \in \mathcal{L}(Y, U)$ by

$$\Delta y := \frac{\langle y, y^* \rangle u}{\|ER(\lambda, A)Du\|}, \qquad y \in Y.$$

Then $\Delta ER(\lambda, A)Du = u$ and

$$\|\Delta\| \le \frac{1}{\|ER(\lambda, A)D\|} + \varepsilon.$$

Set $v := R(\lambda, A)Du$. Then $\Delta Ev = u \ne 0$, so $v \ne 0$, and

$$(\lambda - A - D\Delta E)v = Du - D\Delta ER(\lambda, A)Du = Du - Du = 0.$$

This shows that $\lambda - A - D\Delta E$ is not injective, which implies $\lambda \in \sigma(A + D\Delta E)$. ∎

We remark that the proofs of Proposition 1.1 and Theorem 1.2 are entirely based on techniques in a paper of Latushkin, Montgomery-Smith, and Randolph [13], where they are used to obtain the two-sided bounds (2.4) below for robust stability.

For a subset $\Omega \subset \varrho(A)$ we define

$$r_\Omega(A; D, E) := \sup \{ r \ge 0 : \ \Omega \subset \varrho(A + D\Delta E) \text{ for all } \Delta \in \mathcal{L}(Y, U) \text{ with } \|\Delta\| \le r \}.$$

We then have the following straightforward generalization of Theorem 1.2:

Corollary 1.3. *Let A be a closed linear operator on X. If $\Omega \subset \varrho(A)$, then*

$$r_\Omega(A; D, E) = \inf_{\lambda \in \Omega} \frac{1}{\|ER(\lambda, A)D\|}.$$

We may also impose uniform boundedness of the resolvent on the set Ω by defining, for a subset $\Omega \subset \varrho(A)$ such that $\sup_{\lambda\in\Omega} \|R(\lambda, A)\| < \infty$,

$$r_\Omega^\infty(A; D, E) := \sup\Big\{r \geq 0 : \ \Omega \subset \varrho(A + D\Delta E) \text{ and } \sup_{\lambda\in\Omega} \|R(\lambda, A + D\Delta E)\| < \infty \\ \text{for all } \Delta \in \mathcal{L}(Y, U) \text{ with } \|\Delta\| \leq r\Big\}.$$

Corollary 1.4. *Let A be a closed linear operator on X and assume that E extends to a bounded operator from X into Y. If $\Omega \subset \varrho(A)$ with $\sup_{\lambda\in\Omega} \|R(\lambda, A)\| < \infty$, then*

$$r_\Omega^\infty(A; D, E) = \frac{1}{\sup_{\lambda\in\Omega} \|ER(\lambda, A)D\|}.$$

Proof: It is clear from the definition that $r_\Omega^\infty(A; D, E) \leq r_\Omega(A; D, E)$. Hence by Corollary 1.3 we only need to prove the inequality $r_\Omega^\infty(A; D, E) \geq \inf_{\lambda\in\Omega} \|ER(\lambda, A)D\|^{-1}$. But this inequality follows immediately from Proposition 1.1, since $\|\Delta ER(\lambda, A)\| \leq \|\Delta E\| \, \|R(\lambda, A)\|$ and $\sup_{\lambda\in\Omega} \|R(\lambda, A)\| < \infty$. ∎

2. Application to robust stability of C_0-semigroups

Throughout this section we fix complex Banach space X, U, and Y, and bounded linear operators $D \in \mathcal{L}(U, X)$ and $E \in \mathcal{L}(X, Y)$. We further consider a C_0-semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ of bounded linear operators on X, and denote by A its generator. Our terminology concerning semigroups is standard; for more information we refer to the books [16] and [18].

In this section and the next we will be concerned with the behaviour under small perturbations of the following four quantities (see [17, Chapter 1]):

- The *spectral bound* $s(A) = \sup\{\operatorname{Re}\lambda : \ \lambda \in \sigma(A)\}$;
- The *abscissa of uniform boundedness* $s_0(A)$ of the resolvent of A,

$$s_0(A) := \inf\Big\{\omega \in \mathbb{R} : \ \{\operatorname{Re}\lambda > \omega\} \subset \varrho(A) \text{ and } \sup_{\operatorname{Re}\lambda > \omega} \|R(\lambda, A)\| < \infty\Big\};$$

- The *growth bound* $\omega_1(A)$,

$$\omega_1(A) := \inf\Big\{\omega \in \mathbb{R} : \quad \text{there exists } M > 0 \text{ such that } \|T(t)x\| \leq M e^{\omega t} \|x\|_{\mathcal{D}(A)} \\ \text{for all } x \in \mathcal{D}(A) \text{ and } t \geq 0\Big\};$$

- The *uniform growth bound* $\omega_0(A)$,

$$\omega_0(A) := \inf\Big\{\omega \in \mathbb{R} : \quad \text{there exists } M > 0 \text{ such that } \|T(t)\| \leq M e^{\omega t} \\ \text{for all } t \geq 0\Big\}.$$

It is well-known [17, Sections 1.2, 4.1] that

$$-\infty \leq s(A) \leq \omega_1(A) \leq s_0(A) \leq \omega_0(A) < \infty. \tag{2.1}$$

If $\omega_0(A) < 0$ (resp. $\omega_1(A) < 0$), then $\mathbf{T}$ is said to be *uniformly exponentially stable* (resp. *exponentially stable*). Below we will use the following simple fact concerning $s_0(A)$: if $\{\operatorname{Re}\lambda > 0\} \subset \varrho(A)$ and $\sup_{\operatorname{Re}\lambda>0} \|R(\lambda, A)\| < \infty$, then $s_0(A) < 0$; see [17, Lemma 2.3.4].

We start by studying the behaviour of abscissa of uniform boundedness under small additive perturbations. To this end, for a semigroup with $s_0(A) < 0$ we define

$$r_{s_0}(A; D, E) := \sup\{r \geq 0 : \; s_0(A + D\Delta E) < 0 \text{ for all } \Delta \in \mathcal{L}(Y, U) \text{ with } \|\Delta\| \leq r\}.$$

Recalling that the suprema along vertical lines $\operatorname{Re}\lambda = c$ of a bounded holomorphic $X-$valued function on $\{\operatorname{Re}\lambda > 0\}$ decrease as c increases, an application of Corollary 1.4 to $\Omega = \{\operatorname{Re}\lambda > 0\}$ shows the following:

Theorem 2.1. *Suppose A is the generator of a C_0-semigroup on X. If $s_0(A) < 0$, then*

$$r_{s_0}(A; D, E) = \frac{1}{\sup_{\omega\in\mathbb{R}} \|ER(i\omega, A)D\|}.$$

For a uniformly exponentially stable C_0-semigroup we now define

$$r_{\omega_0}(A; D, E) := \sup\{r \geq 0 : \; \omega_0(A + D\Delta E) < 0 \text{ for all } \Delta \in \mathcal{L}(Y, U) \text{ with } \|\Delta\| \leq r\}.$$

It is a well-known theorem of Gearhart [4], cf. [17, Corollary 2.2.5], that for C_0-semigroups on a *Hilbert* space, the abscissa of uniform boundedness of the resolvent and the uniform growth bound always coincide. Hence if X is isomorphic to a Hilbert space, Theorem 2.1 assumes the following form:

Corollary 2.2. *Suppose A is the generator of a C_0-semigroup on X. If X is isomorphic to a Hilbert space, and if $\omega_0(A) < 0$, then*

$$r_{\omega_0}(A; D, E) = \frac{1}{\sup_{\omega \in \mathbb{R}} \|ER(i\omega, A)D\|}.$$

Remark: It is not assumed that U and Y are isomorphic to Hilbert spaces.

The quantity $r_{\omega_0}(A; D, E)$ is called the *stability radius* of A with respect to the 'perturbation structure' (D, E) and was introduced, in the finite-dimensional setting, by Hinrichsen and Pritchard [5]; see also their survey paper [6]. In order to state some known results about the stability radius, for $p \in [1, \infty)$ we define the *input-output operator* $\mathbb{L}_p(A; D, E) \in \mathcal{L}(L^p(\mathbb{R}_+; U), L^p(\mathbb{R}_+; Y))$ by

$$\mathbb{L}_p(A; D, E)f(s) := E \int_0^s T(s-t)Df(t)\,dt \qquad s \geq 0, \; f \in L^p(\mathbb{R}_+; U).$$

This operator is easily seen to be bounded if $\omega_0(A) < 0$; conversely, if $U = Y = X$, then boundedness of $\mathbb{L}_p(A; I, I)$ implies $\omega_0(A) < 0$ [17, Theorem 3.3.1]. The following results are well-known:

- If X, U, and Y are finite dimensional, then

$$r_{\omega_0}(A; D, E) = \frac{1}{\sup_{\omega \in \mathbb{R}} \|ER(i\omega, A)D\|}. \tag{2.2}$$

- If X is a Banach space, and U and Y are Hilbert spaces, then

$$\frac{1}{\|\mathbb{L}_2(A; D, E)\|} = r_{\omega_0}(A; D, E) = \frac{1}{\sup_{\omega \in \mathbb{R}} \|ER(i\omega, A)D\|}. \tag{2.3}$$

- If X, U, and Y are arbitrary Banach spaces, then for all $p \in [1, \infty)$,

$$\frac{1}{\|\mathbb{L}_p(A; D, E)\|} \leq r_{\omega_0}(A; D, E) \leq \frac{1}{\sup_{\omega \in \mathbb{R}} \|ER(i\omega, A)D\|}. \tag{2.4}$$

The identities (2.2) and (2.3) are due to Hinrichsen and Pritchard [6] and Pritchard and Townley [19] (where a more general setup is considered), respectively. Notice that in some sense our Corollary 2.2 complements the second identity in (2.3).

The inequalities (2.4) were obtained by Latushkin, Montgomery-Smith and Randolph [13] by using the theory of evolutionary semigroups; this further enabled them to extend certain results on time-varying systems due to Hinrichsen and Pritchard [7]. They also showed that the inequality between the first and the third term in (2.4) may be strict. More results on the time-varying case may be found in [2].

In the case of positive semigroups, Theorem 2.1 and Corollary 2.2 simplify somewhat:

Corollary 2.3. *If X, U, and Y are Banach lattices, $D \in \mathcal{L}(U, X)$ and $E \in \mathcal{L}(X, Y)$ are positive, and A is the generator of a positive C_0-semigroup on X with $s_0(A) < 0$, then*

$$r_{s_0}(A; D, E) = \frac{1}{\|EA^{-1}D\|}.$$

If in addition X is isomorphic to a Hilbert space, then the same result holds for the uniform growth bound.

Proof: From

$$|ER(i\omega, A)Du| \leq E|R(i\omega, A)|D|u| \leq ER(0, A)D|u|$$

[16, Corollary C-III-1.3] it follows that $\|ER(i\omega, A)D\| \leq \|ER(0, A)D\| = \|EA^{-1}D\|$ for all $\omega \in \mathbb{R}$. Accordingly, the supremum in the expressions in Theorem 2.1 and Corollary 2.2 is taken for $\omega = 0$. ∎

For a detailed treatment of the theory of positive semigroups we refer to the book [16].

The next application is concerned with semigroups which are uniformly continuous for $t > 0$. First we recall that if A is the generator of a C_0-semigroup which is uniformly continuous for $t > t_0$ for some $t_0 \geq 0$, then the spectral mapping theorem

$$\sigma(T(t))\backslash\{0\} = \exp(t\sigma(A))\backslash\{0\}$$

holds for all $t \geq 0$ [16, Theorem A-III-6.6], [17, Theorem 2.3.2]. In particular, this implies that $s(A) = s_0(A) = \omega_0(A)$. We will combine Theorem 2.1 with the following simple observation [16, Theorem A-II-1.30], a proof of which is included for the reader's convenience.

Lemma 2.4. *If A is the generator of a C_0-semigroup $\mathbf{T}$ on X which is uniformly continuous for $t > 0$, and if B is a bounded linear operator on X, then the semigroup generated by $A + B$ is uniformly continuous for $t > 0$.*

Proof: Let $\mathbf{S} = \{S(t)\}_{t\geq 0}$ denote the semigroup generated by $A+B$. Put $V_0(t) := T(t)$, $t \geq 0$, and define the operators $V_n(t)$ inductively by

$$V_{n+1}(t)x := \int_0^t T(t-s)BV_n(s)x\,ds, \qquad n \in \mathbb{N},\ x \in X,\ t \geq 0.$$

As is well-known [18, Section 3.1], if $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$, then

$$\|V_n(t)\| \leq Me^{\omega t}\frac{M^n\|B\|^n t^n}{n!}, \qquad n \in \mathbb{N},\ t \geq 0,$$

and

$$S(t) = \sum_{n=0}^{\infty} V_n(t), \qquad t \geq 0,$$

the convergence being uniform on compact subsets of $[0,\infty)$. Fix $n \geq 0$ and positive real numbers $0 < \varepsilon < \delta_0 < \delta_1 < \infty$. For $\delta_0 \leq t \leq t' \leq \delta_1$, from

$$V_{n+1}(t')x - V_{n+1}(t)x = \int_t^{t'} T(t'-s)BV_n(s)x\,ds + \int_0^t (T(t'-s) - T(t-s))BV_n(s)x\,ds$$

we obtain, by splitting the second integral as $\int_0^t = \int_{t-\varepsilon}^t + \int_0^{t-\varepsilon}$,

$$\|V_{n+1}(t') - V_{n+1}(t)\| \leq C_n \left((t'-t) + \varepsilon + \sup_{s\in[0,t-\varepsilon]} \|T(t'-s) - T(t-s)\| \right),$$

where C_n is a finite constant depending on M, ω, $\|B\|$, δ_0, δ_1, and n only. It follows that

$$\limsup_{t'\downarrow t} \|V_{n+1}(t') - V_{n+1}(t)\| \leq C_n\varepsilon,$$

and since ε can be taken arbitrarily small, we see that $V_{n+1}(\cdot)$ is uniformly continuous for $t > 0$. Therefore the same is true for $S(\cdot)$. ∎

Corollary 2.5. *Suppose A is the generator of a uniformly exponentially stable C_0-semigroup on X which is uniformly continuous for $t > 0$. Then*

$$r_{\omega_0}(A; D, E) = \frac{1}{\sup_{\omega\in\mathbb{R}} \|ER(i\omega, A)D\|}.$$

This result applies to compact semigroups, differentiable semigroups, and analytic semigroups, since each of these is uniformly continuous for $t > 0$.

3. Delay equations in $L^p([-h,0];X)$

Throughout this section, we fix a C_0-semigroup $\mathbf{T}$ with generator A on a complex Banach space X. We also fix $p \in [1,\infty)$ and non-negative real numbers $0 \leq h_1 < ... < h_n =: h$.

Given bounded linear operators $B_1, ..., B_n$ on X, we will study the delay equation

$$(DE_{B_1,...,B_n}) \qquad \begin{aligned} &\dot{u}(t) = Au(t) + \sum_{j=1}^n B_j u(t-h_j), \qquad t \geq 0,\\ &u(0) = x,\\ &u(t) = f(t), \qquad t \in [-h, 0). \end{aligned}$$

Here, $x \in X$ is the initial value and $f \in L^p([-h,0];X)$ is the 'history' function. This equation has been investigated by Nakagiri [14, 15]; see also [1, 3, 8, 9, 12, 21] for related studies.

A *mild solution* of $(\mathrm{DE}_{B_1,\dots,B_n})$ is a function $u(\cdot\,;x,f) \in L^p_{loc}([-h,\infty);X)$ satisfying

$$u(t;x,f) = \begin{cases} T(t)x + \int_0^t T(t-s)\sum_{j=1}^n B_j u(s-h_j;x,f)\,ds, & t \geq 0, \\ f(t), & t \in [-h,0). \end{cases}$$

It follows from [14, Theorem 2.1] that for all $x \in X$ and $f \in L^p([-h,0];X)$ a unique mild solution $u(\cdot\,;x,f)$ exists; this solution is continuous on $[0,\infty)$ and exponentially bounded. In order to study the asymptotic behaviour of these solutions by semigroup methods, we introduce the product space $\mathcal{X} := X \times L^p([-h,0];X)$ and define bounded linear operators $\mathcal{T}_{B_1,\dots,B_n}(t)$ on $\mathcal{X}$ as follows. Given a function $u \in L^p_{loc}([-h,\infty);X)$, for each $t \geq 0$ we define $u_t \in L^p([-h,0];X)$ by $u_t(s) := u(t+s)$, $s \in [-h,0]$. Denoting the unique mild solution of $(\mathrm{DE}_{B_1,\dots,B_n})$ by $u(\cdot\,;x,f)$, we now define

$$\mathcal{T}_{B_1,\dots,B_n}(t)(x,f) := (u(t;x,f), u_t(\cdot\,;x,f)), \qquad t \geq 0.$$

By [15, Proposition 3.1] we have:

Proposition 3.1. *The family $\mathcal{T}_{B_1,\dots,B_n} = \{\mathcal{T}_{B_1,\dots,B_n}(t)\}_{t\geq 0}$ defines a C_0-semigroup of linear operators on $\mathcal{X}$. Its generator $\mathcal{A}_{B_1,\dots,B_n}$ is given by*

$$\mathcal{D}(\mathcal{A}_{B_1,\dots,B_n}) = \left\{(x,f) \in \mathcal{X} : \ f \in W^{1,p}([-h,0];X),\ f(0) = x \in \mathcal{D}(A)\right\},$$

$$\mathcal{A}_{B_1,\dots,B_n}(x,f) = \Big(Ax + \sum_{j=1}^n B_j f(-h_j), f'\Big), \qquad (x,f) \in \mathcal{D}(\mathcal{A}_{B_1,\dots,B_n}).$$

Here $W^{1,p}([-h,0];X)$ is the space of absolutely continuous $X-$valued functions f on $[-h,0]$ which are strongly differentiable a.e. with derivative $f' \in L^p([-h,0];X)$.

Whenever the operators $B_1,\dots,B_n$ are understood, we will drop them from the notation and simply write $\mathcal{T}$ and $\mathcal{A}$.

The spectrum and resolvent of $\mathcal{A}$ are described by [15, Theorem 6.1]:

Proposition 3.2. *We have $\lambda \in \varrho(\mathcal{A})$ if and only if $\lambda \in \varrho(A + \sum_{j=1}^n e^{-\lambda h_j} B_j)$. In this case the resolvent of $\mathcal{A}$ is given by*

$$R(\lambda,\mathcal{A}) = E_\lambda R\Big(\lambda, A + \sum_{j=1}^n e^{-\lambda h_j} B_j\Big) H_\lambda F + T_\lambda,$$

where $E_\lambda \in \mathcal{L}(X, \mathcal{X})$, $H_\lambda \in \mathcal{L}(\mathcal{X}, X)$, $F \in \mathcal{L}(\mathcal{X}, \mathcal{X})$, *and* $T_\lambda \in \mathcal{L}(\mathcal{X}, \mathcal{X})$ *are defined by*

$$\begin{aligned} E_\lambda x &:= (x, e^{\lambda\,\cdot}\, x); \\ H_\lambda(x, f) &:= x + \int_{-h}^{0} e^{\lambda s} f(s)\, ds; \\ F(x, f) &:= \Big(x, \sum_{j=1}^{n} \chi_{[-h_j,0]}(\cdot) B_j f(-h_j - \cdot\,)\Big); \\ T_\lambda(x, f) &:= \Big(0, \int_{\cdot}^{0} e^{\lambda(\,\cdot\,-\xi)} f(\xi)\, d\xi\Big). \end{aligned}$$

Our first result relates the abscissa $s_0(\mathcal{A})$ to $s_0(A)$:

Theorem 3.3. *Assume that* $s_0(A) < 0$. *If*

$$\sup_{\omega\in\mathbb{R}} \Big\| \sum_{j=1}^{n} e^{i\omega h_j} B_j \Big\| < \frac{1}{\sup_{\omega\in\mathbb{R}} \|R(i\omega, A)\|},$$

then $s_0(\mathcal{A}) < 0$.

Proof: Choose $\delta \in (0, 1)$ such that

$$\sup_{\omega\in\mathbb{R}} \Big\| \sum_{j=1}^{n} e^{i\omega h_j} B_j \Big\| \le (1-\delta) \frac{1}{\sup_{\omega\in\mathbb{R}} \|R(i\omega, A)\|}.$$

Recalling that the suprema along vertical lines $\operatorname{Re}\lambda = c$ of bounded analytic functions decrease as c increases, for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$ we have

$$\Big\| \sum_{j=1}^{n} e^{-\lambda h_j} B_j \Big\| \le \sup_{\omega\in\mathbb{R}} \Big\| \sum_{j=1}^{n} e^{i\omega h_j} B_j \Big\| \le (1-\delta) \frac{1}{\sup_{\omega\in\mathbb{R}} \|R(i\omega, A)\|} \le (1-\delta) \frac{1}{\|R(\lambda, A)\|}.$$

Therefore by Proposition 1.1, $\{\operatorname{Re}\lambda > 0\} \subset \varrho(A + \sum_{j=1}^{n} e^{-\lambda h_j} B_j)$, and for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$ we have

$$\begin{aligned} \Big\| R\Big(\lambda, A + \sum_{j=1}^{n} e^{-\lambda h_j} B_j\Big) \Big\| &\le \|R(\lambda, A)\| \left(1 + \frac{1}{\delta} \Big\| \sum_{j=1}^{n} e^{-\lambda h_j} B_j R(\lambda, A) \Big\| \right) \\ &\le \|R(\lambda, A)\| \left(1 + \frac{1-\delta}{\delta}\right) = \frac{1}{\delta} \|R(\lambda, A)\|. \end{aligned}$$

Hence by Proposition 3.2, $\{\operatorname{Re}\lambda > 0\} \subset \varrho(\mathcal{A})$ and

$$\begin{aligned} \|R(\lambda, \mathcal{A})\| &= \Big\| E_\lambda R\Big(\lambda, A + \sum_{j=1}^{n} e^{-\lambda h_j} B_j\Big) H_\lambda F + T_\lambda \Big\| \\ &\le \frac{1}{\delta} \|R(\lambda, A)\| \,(1 + h^{\frac{1}{p}})\, (1 + h^{\frac{1}{q}}) \Big(1 + \sum_{j=1}^{n} \|B_j\|\Big) + \|T_0\| \end{aligned}$$

for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$; $1/p + 1/q = 1$. Therefore, $\sup_{\operatorname{Re}\lambda > 0} \|R(\lambda, \mathcal{A})\| < \infty$, which implies $s_0(\mathcal{A}) < 0$. ∎

Note that by (2.1) in particular we have $\omega_1(\mathcal{A}) < 0$, which means that the semigroup $\mathcal{T}$ is exponentially stable. This, in turn, implies that there exist $M > 0$ and $\omega > 0$ such that for all $(x, f) \in \mathcal{D}(\mathcal{A}) = \{(x, f) \in X \times W^{1,p}([-h, 0]; X) : \ f(0) = x \in \mathcal{D}(A)\}$ we have

$$\|u(t; x, f)\| \leq Me^{-\omega t}\|(x, f)\|_{\mathcal{D}(\mathcal{A})}.$$

Corollary 3.4. *Suppose $p = 2$ and X is isomorphic to a Hilbert space. If $\omega_0(A) < 0$ and*

$$\sup_{\omega\in\mathbb{R}} \Big\|\sum_{j=1}^{n} e^{i\omega h_j} B_j\Big\| < \frac{1}{\sup_{\omega\in\mathbb{R}} \|R(i\omega, A)\|},$$

then $\omega_0(\mathcal{A}) < 0$.

Another situation in which information about $\omega_0(\mathcal{A})$ may be obtained from $s_0(\mathcal{A})$ is described in the following proposition.

Proposition 3.5. *If the semigroup generated by A is uniformly continuous for $t > 0$, then the semigroup generated by $\mathcal{A}$ is uniformly continuous for $t > h$.*

Proof: We proceed as in the proof of [15, Proposition 3.1]. For $(x, f) \in \mathcal{X}$ we define

$$k(s; x, f) := \sum_{j=1}^{n} B_j u(s - h_j; x, f), \qquad s \geq 0,$$

where $u(\cdot\,; x, f)$ is the unique mild solution of $(\mathrm{DE}_{B_1,\ldots,B_n})$ with initial value (x, f). For $t > 0$ set

$$Q_t(x, f) := \int_0^t T(t - s)k(s; x, f)\, ds, \qquad (x, f) \in \mathcal{X},$$

and for $\varepsilon \in (0, t]$ set

$$Q_{t,\varepsilon}(x, f) := \int_0^{t-\varepsilon} T(t - s)k(s; x, f)\, ds, \qquad (x, f) \in \mathcal{X}.$$

The argument in [15] shows that there exist constants $M_t > 0$ and $N_t > 0$, both increasing with t, such that

$$\|k(\cdot\,; x, f)\|_{L^p([0,t];X)} \leq M_t\|(x, f)\|_{\mathcal{X}}$$

and

$$\|Q_{t,\varepsilon}(x, f) - Q_t(x, f)\| \leq \varepsilon^{\frac{1}{q}} N_t\|(x, f)\|_{\mathcal{X}}, \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

Define $K_h := \sup_{\sigma\in[0,h]} \|T(\sigma)\|$. Fix $h+\varepsilon \le t \le t' \le 2h+\varepsilon$. Then, for all $s \in [-h, 0]$,

$$
\begin{aligned}
&\|u(t'+s;x,f) - u(t+s;x,f)\| \\
&\quad \le \|T(t'+s)x - T(t+s)x\| \\
&\qquad + \left\| \int_{t+s}^{t'+s} T(t'+s-\sigma)k(\sigma;x,f)\,d\sigma \right\| \\
&\qquad + \left\| \int_{t+s-\varepsilon}^{t+s} (T(t'+s-\sigma) - T(t+s-\sigma))k(\sigma;x,f)\,d\sigma \right\| \\
&\qquad + \left\| \int_{0}^{t+s-\varepsilon} (T(t'+s-\sigma) - T(t+s-\sigma))k(\sigma;x,f)\,d\sigma \right\| \\
&\quad \le \|T(t'+s) - T(t+s)\| \, \|(x,f)\|_{\mathcal{X}} \\
&\qquad + (t'-t)^{\frac{1}{q}} N_{2h+\varepsilon} \|(x,f)\|_{\mathcal{X}} \\
&\qquad + (K_h+1)\varepsilon^{\frac{1}{q}} N_{2h+\varepsilon} \|(x,f)\|_{\mathcal{X}} \\
&\qquad + (2h)^{\frac{1}{q}} M_{2h} \|(x,f)\|_{\mathcal{X}} \sup_{\sigma\in[0,t+s-\varepsilon]} \|T(t'+s-\sigma) - T(t+s-\sigma)\|.
\end{aligned}
$$

It follows that

$$
\limsup_{t'\downarrow t} \left(\sup_{\|(x,f)\|_{\mathcal{X}}\le 1} \sup_{s\in[-h,0]} \|u(t'+s;x,f) - u(t+s;x,f)\| \right) \le (K_h+1)\varepsilon^{\frac{1}{q}} N_{2h+\varepsilon},
$$

and therefore

$$
\limsup_{t'\downarrow t} \left(\sup_{\|(x,f)\|_{\mathcal{X}}\le 1} \|u_{t'}(x,f) - u_t(x,f)\|_{L^p([-h,0];X)} \right) \le (K_h+1)\varepsilon^{\frac{1}{q}} h^{\frac{1}{p}} N_{2h+\varepsilon}.
$$

It follows that

$$
\limsup_{t'\downarrow t} \|\mathcal{T}(t') - \mathcal{T}(t)\| \le (K_h+1)\varepsilon^{\frac{1}{q}} (1 + h^{\frac{1}{p}}) N_{2h+\varepsilon}.
$$

Since the choice of $\varepsilon > 0$ was arbitrary and $N_{2h+\varepsilon}$ decreases with ε, this proves that $\lim_{t'\downarrow t} \|\mathcal{T}(t') - \mathcal{T}(t)\| = 0$. ∎

Corollary 3.6. *Assume that A generates a uniformly exponentially stable C_0-semigroup which is uniformly continuous for $t > 0$. If*

$$
\sup_{\omega\in\mathbb{R}} \left\| \sum_{j=1}^{n} e^{i\omega h_j} B_j \right\| < \frac{1}{\sup_{\omega\in\mathbb{R}} \|R(i\omega, A)\|},
$$

then $\omega_0(\mathcal{A}) < 0$, i.e. $\mathcal{T}$ is uniformly exponentially stable.

We now consider the case $n = 1$ in more detail and return to the notation $\mathcal{T}_B$ and $\mathcal{A}_B$ to denote the semigroup on $\mathcal{X}$ and its generator governing the solutions of the problem

$$\begin{aligned} \dot{u}(t) &= Au(t) + Bu(t-h), \qquad t \geq 0, \\ u(0) &= x, \\ u(t) &= f(t), \qquad t \in [-h, 0). \end{aligned}$$

Assuming that $s_0(\mathcal{A}_0) < 0$ ($\mathcal{A}_0$ being the generator $\mathcal{A}_B$ corresponding to the zero operator $B = 0$), we define

$$r_{s_0}(\mathcal{A}_0) := \sup \{ r \geq 0 : \ s_0(\mathcal{A}_B) < 0 \text{ for all } B \in \mathcal{L}(X) \text{ with } \|B\| \leq r \} .$$

With this notation, the case $n = 1$ of Theorem 3.3 says that

$$r_{s_0}(\mathcal{A}_0) \geq \frac{1}{\sup_{\omega \in \mathbb{R}} \|R(i\omega, A)\|}.$$

In fact, we have the following more precise result.

Theorem 3.7. *If $s_0(A) < 0$, then*

$$r_{s_0}(\mathcal{A}_0) = r_{s_0}(A; I, I) = \frac{1}{\sup_{\omega \in \mathbb{R}} \|R(i\omega, A)\|}.$$

Proof: It only remains to prove the inequality

$$r_{s_0}(\mathcal{A}_0) \leq \frac{1}{\sup_{\omega \in \mathbb{R}} \|R(i\omega, A)\|}.$$

Fix $\varepsilon > 0$ and choose $\omega_0 \in \mathbb{R}$ such that

$$\frac{1}{\|R(i\omega_0, A)\|} \leq \frac{1}{\sup_{\omega \in \mathbb{R}} \|R(i\omega, A)\|} + \varepsilon.$$

Define operators $E \in \mathcal{L}(\mathcal{D}(\mathcal{A}_0), X)$ and $D \in \mathcal{L}(X, \mathcal{X})$ by

$$E(x, f) := f(-h), \qquad Dx := (x, 0).$$

Using Proposition 3.2 it is easily verified that

$$ER(i\omega_0, \mathcal{A}_0)Dx = e^{-i\omega_0 h} R(i\omega_0, A)x, \qquad x \in X,$$

and therefore $\|ER(i\omega_0, \mathcal{A}_0)D\| = \|R(i\omega_0, A)\|$. By Theorem 1.2 there exists $B_0 \in \mathcal{L}(X)$ such that

$$\|B_0\| \leq \frac{1}{\|ER(i\omega_0, \mathcal{A}_0)D\|} + \varepsilon$$

and

$$i\omega_0 \in \sigma(\mathcal{A}_0 + DB_0E).$$

Noting that $\mathcal{A}_{B_0} = \mathcal{A}_0 + DB_0E$, this means that $i\omega_0 \in \sigma(\mathcal{A}_{B_0})$. Therefore $\mathcal{A}_{B_0}$ cannot have a uniformly bounded resolvent on $\{\operatorname{Re}\lambda > 0\}$. The estimate

$$\|B_0\| \leq \frac{1}{\|ER(i\omega_0, \mathcal{A}_0)D\|} + \varepsilon = \frac{1}{\|R(i\omega_0, A)\|} + \varepsilon \leq \frac{1}{\sup_{\omega\in\mathbb{R}} \|R(i\omega, A)\|} + 2\varepsilon$$

then shows that

$$r_{s_0}(\mathcal{A}_0) \leq \frac{1}{\sup_{\omega\in\mathbb{R}} \|R(i\omega, A)\|} + 2\varepsilon.$$

■

In particular, if $p = 2$ and X is isomorphic to a Hilbert space, or if $\mathbf{T}$ is uniformly continuous for $t > 0$, it follows that

$$r_{\omega_0}(\mathcal{A}_0) = r_{\omega_0}(A; I, I) = \frac{1}{\sup_{\omega\in\mathbb{R}} \|R(i\omega, A)\|},$$

where $r_{\omega_0}(\mathcal{A}_0)$ is defined in the obvious way.

For the generator A of a positive semigroup on a Banach lattice X we have $s(A) = \omega_1(A) = s_0(A)$; moreover, $s(A) \in \sigma(A)$ whevener $s(A) > -\infty$ [16, Theorem C-III-1.1]. This will be used to prove the following versions of Theorems 3.3 and 3.7:

Theorem 3.3′. *Let A generate a positive C_0-semigroup on a Banach lattice X, and assume that the operators B_j are positive, $j = 1, .., n$. Then the semigroup $\mathcal{T}_{B_1,\dots,B_n}$ is positive. If $s_0(A) < 0$ and*

$$\Big\|\sum_{j=1}^{n} B_j\Big\| < \frac{1}{\|A^{-1}\|},$$

then $s_0(\mathcal{A}_{B_1,\dots,B_n}) < 0$.

Proof: It is an easy consequence of Proposition 3.2 that $R(\lambda, \mathcal{A}_{B_1,\dots,B_n}) \geq 0$ for sufficiently large real λ. Then $\mathcal{T}_{B_1,\dots,B_n}(t) \geq 0$ for all $t \geq 0$ by the exponential formula [18, Theorem 1.8.3].

Since $\|R(\lambda, A)\| \leq \|R(\operatorname{Re}\lambda, A)\|$ for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > s(A)$, for some $\delta \in (0, 1)$ and all $\lambda \geq 0$ we have

$$\Big\|\sum_{j=1}^{n} e^{-\lambda h_j} B_j\Big\| \leq \Big\|\sum_{j=1}^{n} B_j\Big\| \leq (1-\delta)\frac{1}{\|R(0, A)\|} \leq (1-\delta)\frac{1}{\|R(\lambda, A)\|}.$$

Hence, $[0, \infty) \subset \varrho(\mathcal{A}_{B_1,\dots,B_n})$ by Propositions 1.1 and 3.2. Since $\mathcal{A}_{B_1,\dots,B_n}$ generates a positive semigroup, this implies that $s_0(\mathcal{A}_{B_1,\dots,B_n}) = s(\mathcal{A}_{B_1,\dots,B_n}) < 0$. ■

If $n = 1$ we have:

Theorem 3.7′. *Let A generate a positive C_0-semigroup on a Banach lattice X with $s_0(A) < 0$. Then*

$$r_{s_0}(\mathcal{A}_0) = r_{s_0}(A; I, I) = \frac{1}{\|A^{-1}\|}.$$

The identities $r_{s_0}(\mathcal{A}_0) = r_{s_0}(A; I, I)$ in Theorems 3.7 and 3.7′ can be interpreted as saying that the stability radius for boundedness of the resolvent for the delay problem is independent of the delay h (and equals the stability radius for boundedness of the resolvent for the undelay problem). In the situation of Theorem 3.7′, if in addition B is assumed to be positive, then we further have $s_0(A+B) = \omega_1(A+B)$ and $s_0(\mathcal{A}_B) = \omega_1(\mathcal{A}_B)$, so that we can reformulate this observation in terms of exponential stability of the semigroups involved. In the state space $C([-h, 0]; X)$ this is a well-known phenomenon; cf. [16, Corollary B-IV-3.10], where different methods are used. For further results on stability of delay equations in $C([-h, 0]; X)$ the reader might consult [10, 11, 20].

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