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# Asymptotic behaviour of $C_0-$semigroups with bounded local resolvents

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**Abstract.** Let $\{T(t)\}_{t\geq 0}$ be a $C_0$-semigroup on a Banach space $X$ with generator $A$, and let $H^\infty_{\mathbf{T}}$ be the space of all $x \in X$ such that the local resolvent $\lambda \mapsto R(\lambda, A)x$ has a bounded holomorphic extension to the right half-plane. For the class of integrable functions $\phi$ on $[0, \infty)$ whose Fourier transforms are integrable, we construct a functional calculus $\phi \mapsto T_\phi$, as operators on $H^\infty_{\mathbf{T}}$. We show that each orbit $T(\cdot)T_\phi x$ is bounded and uniformly continuous, and $T(t)T_\phi x \to 0$ weakly as $t \to \infty$, and we give a new proof that $\|T(t)R(\mu, A)x\| = O(t)$. We also show that $\|T(t)T_\phi x\| \to 0$ when $\mathbf{T}$ is sun-reflexive, and that $\|T(t)R(\mu, A)x\| = O(\ln t)$ when $\mathbf{T}$ is a positive semigroup on a normal ordered space $X$ and $x$ is a positive vector in $H^\infty_{\mathbf{T}}$.

## 1. Introduction and preliminaries

This paper is concerned with the asymptotic behaviour of orbits $T(\cdot)Sx$, where $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is a $C_0$-semigroup on a complex Banach space $X$ with generator $A$, $x$ is a vector in $X$ such that the local resolvent $R(\cdot, A)x$ has a bounded holomorphic extension to the right half-plane $\mathbf{C}_+$, and $S$ is an operator in one of various classes associated with $\mathbf{T}$. In [Ne1], it was shown that there is a constant $c$ such that $\|T(t)R(\mu, A)x\| \leq c(1 + t)$, $t \geq 0$, and this gave a proof that

$$\inf\Big\{\omega \in \mathbf{R} : \text{for all } x \in D(A),\ \|T(t)x\| = O(e^{\omega t}) \text{ as } t \to \infty \Big\}$$
$$\leq \inf\Big\{\omega \in \mathbf{R} : R(\lambda, A) \text{ exists whenever } \mathrm{Re}\lambda > \omega \text{ and } \sup_{\mathrm{Re}\lambda>\omega} \|R(\lambda, A)\| < \infty\Big\},$$

an inequality originally established in [WW]. It remains an open question whether such orbits $T(\cdot)R(\mu, A)x$ are bounded in general, but it has been shown in [HN] (see

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also [Ne3, Chapter 4]) that $\lim_{t\to\infty} T(t)R(\mu, A)x = 0$ strongly (in norm) if $X$ is $B$-convex, and that, for $\alpha > 1$, $\lim_{t\to\infty} T(t)R(\mu, A)^\alpha x = 0$ weakly for arbitrary $X$, and strongly if $X$ has the analytic Radon-Nikodým property (ARNP), in particular, if $X$ is reflexive. When $X$ has the ARNP, it was shown more generally in [Ch1] (see also [Ch2]) that $\lim_{t\to\infty} T(t)T_\phi x = 0$ whenever

$$T_\phi x := \int_0^\infty \phi(t)T(t)x\,dt$$

is an absolutely convergent integral. This was deduced as a consequence of general Tauberian theorems originating in work of Ingham [In].

Here, we extend these results in various directions. In Section 2, we show that it is possible to define $T_\phi x$ whenever $\phi \in L^1(\mathbf{R}_+)$ and its Fourier transform $\mathcal{F}\phi \in L^1(\mathbf{R})$ (and $x$ has bounded local resolvent on $\mathbf{C}_+$), thereby creating a functional calculus on the space of such vectors. In Section 3 we give some general estimates for orbits $T(\cdot)T_\phi x$, thereby extending and sharpening results in [Ne1] and [HN]. In Section 4, we show that $\lim_{t\to\infty} T(t)T_\phi x = 0$ strongly if $\mathbf{T}$ is sun-reflexive, and indeed that this is also a case of a Tauberian theorem.

Throughout this paper, we shall let $\mathbf{R}_+ = [0,\infty)$ and $\mathbf{C}_+ = \{z \in \mathbf{C} : \ \mathrm{Re} z > 0\}$. Given a locally integrable function $\phi : \mathbf{R}_+ \to \mathbf{C}$, we shall let $\widehat{\phi}$ be the Laplace transform of $\phi$:

$$\widehat{\phi}(z) = \int_0^\infty e^{-tz}\phi(t)\,dt$$

whenever this integral is absolutely convergent.

When $\phi \in L^1(\mathbf{R}_+)$, we shall let $\mathcal{F}\phi$ be the Fourier transform of $\phi$ (where $\phi$ is regarded as vanishing on $(-\infty, 0)$). For $a \geq 0$, we denote by $L^1_{a,\mathcal{F}}(\mathbf{R}_+)$ the space of all $\phi \in L^1(\mathbf{R}_+)$ such that $\|\phi\|_{a,1} := \int_0^\infty |\phi(t)|e^{at}\,dt < \infty$ and $\mathcal{F}\phi \in L^1(\mathbf{R})$. This is a linear subspace of $L^1(\mathbf{R}_+)$, and a Banach space with respect to the norm

$$\|\phi\|_{L^1_{a,\mathcal{F}}(\mathbf{R}_+)} := \|\phi\|_{a,1} + \|\mathcal{F}\phi\|_1.$$

It is even a commutative Banach algebra with respect to convolution. To see this, note that

$$\begin{aligned}
\|\phi * \psi\|_{L^1_{a,\mathcal{F}}(\mathbf{R}_+)} &= \|\phi * \psi\|_{a,1} + \|\mathcal{F}\phi \cdot \mathcal{F}\psi\|_1 \\
&\leq \|\phi\|_{a,1}\,\|\psi\|_{a,1} + \|\mathcal{F}\phi\|_1 \|\mathcal{F}\psi\|_\infty \\
&\leq \|\phi\|_{a,1}\,\|\psi\|_{a,1} + \|\mathcal{F}\phi\|_1 \|\psi\|_1 \\
&\leq \|\phi\|_{L^1_{a,\mathcal{F}}(\mathbf{R}_+)} \|\psi\|_{a,1} \\
&\leq \|\phi\|_{L^1_{a,\mathcal{F}}(\mathbf{R}_+)} \|\psi\|_{L^1_{a,\mathcal{F}}(\mathbf{R}_+)}.
\end{aligned}$$

When $a = 0$, we shall write $L^1_{\mathcal{F}}(\mathbf{R}_+)$ for $L^1_{0,\mathcal{F}}(\mathbf{R}_+)$. By the Riemann-Lebesgue Lemma (applied to $\mathcal{F}\phi$), every function $\phi \in L^1_{\mathcal{F}}(\mathbf{R}_+)$ is continuous, vanishes at infinity, and satisfies $\phi(0) = 0$.

Throughout, $X$ will be a complex Banach space, and we shall denote by $H^\infty = H^\infty(\mathbf{C}_+; X)$ the Banach space of all bounded holomorphic $X$-valued functions on $\mathbf{C}_+$, with the norm

$$\|F\|_{H^\infty} = \sup_{z\in\mathbf{C}_+} \|F(z)\|.$$

By the Phragmén-Lindelöf Principle,

$$\|F\|_{H^\infty} = \lim_{\alpha\to 0+} \sup_{s\in\mathbf{R}} \|F(\alpha+is)\|.$$

We let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a $C_0$-semigroup with infinitesimal generator $A$ on $X$. We shall denote the resolvent set of $A$ by $\varrho(A)$ and write $R(\mu, A)$ for the resolvent $(\mu I - A)^{-1}$, $\mu \in \varrho(A)$. We shall frequently use the resolvent identity:

$$R(\lambda, A) = R(\mu, A) + (\mu - \lambda)R(\mu, A)R(\lambda, A).$$

We shall denote the growth bound of $\mathbf{T}$ by $\omega_0(\mathbf{T})$:

$$\begin{aligned}\omega_0(\mathbf{T}) &= \lim_{t\to\infty} \frac{\ln\|T(t)\|}{t}\\ &= \inf\left\{\omega\in\mathbf{R} : \text{there exists } M \text{ such that } \|T(t)\| \leq Me^{\omega t},\ t\geq 0\right\}.\end{aligned}$$

We let $H^\infty_{\mathbf{T}}$ be the linear subspace of $X$ consisting of all $x \in X$ whose local resolvent $\lambda \mapsto R(\lambda, A)x$ has a holomorphic extension to an element $F_x \in H^\infty$. The space $H^\infty_{\mathbf{T}}$ is a Banach space with respect to the norm

$$\|x\|_{H^\infty_{\mathbf{T}}} := \|x\| + \|F_x\|_{H^\infty}.$$

Indeed, suppose $(x_n)$ is a Cauchy sequence in $H^\infty_{\mathbf{T}}$. Then $(x_n)$ is Cauchy in $X$, with limit $x$, say. Moreover, $(F_{x_n})$ is Cauchy in $H^\infty$, say with limit $G \in H^\infty$. But for all $\lambda \in \mathbf{C}_+$ with $\mathrm{Re}\lambda > \omega_0(\mathbf{T})$,

$$G(\lambda) = \lim_{n\to\infty} F_{x_n}(\lambda) = \lim_{n\to\infty} R(\lambda, A)x_n = R(\lambda, A)x$$

and therefore $G$ is a bounded holomorphic extension of the local resolvent $\lambda \mapsto R(\lambda, A)x$. It follows that $G = F_x$ and completeness of $H^\infty_{\mathbf{T}}$ is proved.

We will sometimes regard the mapping $x \mapsto F_x$ as an operator from $H^\infty_{\mathbf{T}}$ into $H^\infty$; as such it is linear and contractive.

## 2. The main estimate

For $x \in X$ and $\phi \in C_c(\mathbf{R}_+)$, the space of continuous functions with compact support in $\mathbf{R}_+ = [0, \infty)$, we define an element $T_\phi x \in X$ by

$$T_\phi x := \int_0^\infty \phi(t)T(t)x\,dt.$$

**Proposition 2.1.** *For $x \in H^\infty_{\mathbf{T}}$ and $\phi \in C_c(\mathbf{R}_+)$,*

$$\|T_\phi x\| \le \liminf_{\alpha\to 0+} \liminf_{r\to\infty} \frac{1}{2\pi} \left\| \int_{\alpha-ir}^{\alpha+ir} F_x(z)\widehat{\phi}(\alpha - z)\,dz \right\|,$$

*where the integral is along any path in $\mathbf{C}_+$ from $\alpha - ir$ to $\alpha + ir$.*

Proof. By Cauchy's Theorem, we may assume that the integral is along the line segment from $\alpha - ir$ to $\alpha + ir$.

Take $\omega > \max(\omega_0(\mathbf{T}), 0)$. The Laplace transform $\widehat{\phi}(z)$ is defined for all $z \in \mathbf{C}$, and $(\mathcal{F}\phi)(s) = \widehat{\phi}(is)$. Take $0 < \alpha < \omega$ and $x^* \in X^*$. The functions $t \mapsto e^{-\omega t}\langle T(t)x, x^*\rangle$ and $t \mapsto e^{(\omega-\alpha)t}\overline{\phi(t)}$ belong to $L^2(\mathbf{R}_+)$, and their respective Fourier transforms are $s \mapsto \langle R(\omega + is, A)x, x^*\rangle$ and $s \mapsto \overline{\widehat{\phi}(\alpha - \omega - is)}$. By Plancherel's Theorem,

$$\begin{aligned}\int_0^\infty e^{-\alpha t}\langle T(t)x, x^*\rangle\phi(t)\,dt &= \int_0^\infty e^{-\omega t}\langle T(t)x, x^*\rangle e^{(\omega-\alpha)t}\phi(t)\,dt \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty \langle R(\omega + is, A)x, x^*\rangle\widehat{\phi}(\alpha - \omega - is)\,ds.\end{aligned} \tag{2.1}$$

Now consider the contour integral

$$\int \langle F_x(z), x^*\rangle\widehat{\phi}(\alpha - z)\,dz$$

around the rectangle with vertices $\alpha \pm ir$, $\omega \pm ir$, where $r > 0$. The integral along the bottom edge is

$$\int_\alpha^\omega \langle F_x(\xi - ir), x^*\rangle\widehat{\phi}(\alpha - \xi + ir)\,d\xi.$$

For $\alpha < \xi < \omega$,

$$\widehat{\phi}(\alpha - \xi + ir) = \int_0^\infty e^{-(\alpha-\xi)t}\phi(t)e^{-irt}\,dt \to 0$$

as $r \to \infty$, by the Riemann-Lebesgue Lemma. Moreover,

$$\begin{aligned}\left|\widehat{\phi}(\alpha - \xi + ir)\right| &\le \int_0^\infty e^{\omega t}|\phi(t)|\,dt \\ |\langle F_x(\xi - ir), x^*\rangle| &\le \|F_x\|_{H^\infty}\|x^*\|,\end{aligned}$$

whenever $r > 0$, $\alpha < \xi < \omega$. By Lebesgue's Dominated Convergence Theorem,

$$\lim_{r\to\infty}\int_\alpha^\omega \langle F_x(\xi - ir), x^*\rangle\widehat{\phi}(\alpha - \xi + ir)\,d\xi = 0.$$

A similar argument shows that the integral along the top edge of the rectangle tends to 0 as $r \to \infty$. By Cauchy's Theorem,

$$\lim_{r\to\infty}\left\{\int_{-r}^r \langle F_x(\omega + is), x^*\rangle\widehat{\phi}(\alpha - \omega - is)\,ds - \int_{-r}^r \langle F_x(\alpha + is), x^*\rangle\widehat{\phi}(-is)\,ds\right\} = 0.$$

By (2.1),

$$\int_0^\infty e^{-\alpha t}\langle T(t)x, x^*\rangle \phi(t)\,dt = \lim_{r\to\infty} \frac{1}{2\pi} \int_{-r}^{r} \langle F_x(\alpha + is), x^*\rangle (\mathcal{F}\phi)(-s)\,ds,$$

so

$$\left\| \int_0^\infty e^{-\alpha t}\phi(t)T(t)x\,dt \right\| \le \liminf_{r\to\infty} \frac{1}{2\pi} \left\| \int_{-r}^{r} F_x(\alpha + is)(\mathcal{F}\phi)(-s)\,ds \right\|.$$

The result now follows by letting $\alpha \to 0+$. $\square$

**Corollary 2.2.** *For all $x \in H^\infty_{\mathbf{T}}$ and $\phi \in C_c(\mathbf{R}_+)$ with $\mathcal{F}\phi \in L^1(\mathbf{R})$ we have*

$$\|T_\phi x\| \le \frac{1}{2\pi}\|F_x\|_{H^\infty}\|\mathcal{F}\phi\|_1.$$

**Lemma 2.3.** *Let $a \ge 0$ and $\phi \in L^1_{a,\mathcal{F}}(\mathbf{R}_+)$. There is a sequence $(\phi_n)$ in $L^1_{a,\mathcal{F}}(\mathbf{R}_+) \cap C_c(\mathbf{R}_+)$ such that $|\phi_n| \le |\phi|$ for all $n$ and $\lim_{n\to\infty} \|\phi_n - \phi\|_{L^1_{a,\mathcal{F}}(\mathbf{R}_+)} = 0$.*

Proof. Let $\psi \in C_c^\infty(\mathbf{R})$ be an arbitrary function satisfying $0 \le \psi \le 1$, $\psi(0) = 1$, and $\int_{-\infty}^\infty \psi(t)\,dt = 1$. Let $\psi_n(t) := \psi(t/n)$, $t \in \mathbf{R}$, $n \ge 1$; $\phi_n := \phi \cdot \psi_n|_{\mathbf{R}_+}$, $n \ge 1$. Then $\phi_n \in C_c(\mathbf{R}_+)$ and $|\phi_n| \le |\phi|$.

Since each $\psi_n$ belongs to the Schwartz space $\mathcal{S}(\mathbf{R})$, so does its Fourier transform $\mathcal{F}\psi_n$. In particular $\mathcal{F}\psi_n \in L^1(\mathbf{R})$, so $\phi_n \in L^1_{a,\mathcal{F}}(\mathbf{R}_+)$. Moreover, $\lim_{n\to\infty} \|\phi_n - \phi\|_{a,1} = 0$, and

$$\lim_{n\to\infty} \mathcal{F}\phi_n = (2\pi)^{-1} \lim_{n\to\infty} \mathcal{F}\phi * \mathcal{F}\psi_n = (2\pi)^{-1} \lim_{n\to\infty} \mathcal{F}\phi * n\mathcal{F}\psi(n\,\cdot) = \mathcal{F}\phi$$

in $L^1(\mathbf{R})$; cf. [Ka, Theorem VI.1.10]. Hence, $\lim_{n\to\infty} \|\phi_n - \phi\|_{L^1_{a,\mathcal{F}}(\mathbf{R}_+)} = 0$. $\square$

The following result now follows easily from Corollary 2.2, Lemma 2.3 and the Dominated Convergence Theorem.

**Proposition 2.4.** *For each $x \in H^\infty_{\mathbf{T}}$ the linear operator $T_x : \phi \mapsto T_\phi x$ from $L^1_{\mathcal{F}}(\mathbf{R}_+) \cap C_c(\mathbf{R}_+)$ to $X$ has a unique extension to a bounded linear operator $T_x$ from $L^1_{\mathcal{F}}(\mathbf{R}_+)$ into $X$. For each $\phi \in L^1_{\mathcal{F}}(\mathbf{R}_+)$ there exists a unique bounded linear operator $T_\phi : H^\infty_{\mathbf{T}} \to X$ given by $T_\phi x = T_x\phi$. Moreover*

$$\|T_\phi x\| \le \frac{1}{2\pi}\|F_x\|_{H^\infty}\|\mathcal{F}\phi\|_1$$

*for all $\phi \in L^1_{\mathcal{F}}(\mathbf{R}_+)$. If $\int_0^\infty |\phi(t)|\,\|T(t)x\|\,dt < \infty$, then $T_\phi x = \int_0^\infty \phi(t)T(t)x\,dt$.*

In Corollary 3.4, we will give a general formula for $T_\phi x$.

We will now consider the question whether $T_\phi$ maps $H^\infty_{\mathbf{T}}$ into $H^\infty_{\mathbf{T}}$. The main result will be Proposition 2.7, but we first make a simple observation.

If $S$ is a bounded linear operator on $X$ commuting with $\mathbf{T}$, then $S$ maps $H^\infty_{\mathbf{T}}$ into $H^\infty_{\mathbf{T}}$, with $\|Sx\|_{H^\infty_{\mathbf{T}}} \le \|S\|_{\mathcal{L}(\mathcal{X})}\|x\|_{H^\infty_{\mathbf{T}}}$. In particular, this applies with $S = T_\phi$, where $\int_0^\infty |\phi(t)|\,\|T(t)\|\,dt < \infty$ and

$$T_\phi x = \int_0^\infty \phi(t)T(t)x\,dt.$$

Note that this is consistent with the definition of $T_\phi x$ for $\phi \in L^1_{\mathcal{F}}(\mathbf{R}_+)$, by Proposition 2.4. The following is an interesting example of this situation.

**Example 2.5.** Let $\alpha > 1$ and $\mu > \max(\omega_0(\mathbf{T}), 0)$, and define $\phi(t) := \Gamma(\alpha)^{-1}t^{\alpha-1}e^{-\mu t}$, $t \ge 0$. Then $\mathcal{F}\phi(s) = (\mu + is)^{-\alpha}$, $s \in \mathbf{R}$; thus $\mathcal{F}\phi \in L^1(\mathbf{R})$. Moreover, $T_\phi x = R(\mu, A)^\alpha x$, where $R(\mu, A)^\alpha$ is the negative fractional power of the sectorial operator $\mu I - A$ (see [Ne3, Appendix A1]).

**Proposition 2.6.** *Let $x \in H^\infty_{\mathbf{T}}$ and $\lambda \in \mathbf{C}_+$. Then*

(1) $F_x(\lambda) \in H^\infty_{\mathbf{T}}$;

(2) $F_{F_x(\lambda)}(\mu) = \dfrac{F_x(\lambda) - F_x(\mu)}{\mu - \lambda}, \qquad \mu \in \mathbf{C}_+, \mu \ne \lambda$;

(3) $\left\|F_{F_x(\lambda)}\right\|_{H^\infty} \le \dfrac{4\|F_x\|_{H^\infty}}{\mathrm{Re}\lambda}$;

(4) *The $H^\infty_{\mathbf{T}}$-valued function $\lambda \mapsto F_x(\lambda)$ is holomorphic on $\mathbf{C}_+$.*

Proof. We first show that, for each $\lambda \in \mathbf{C}_+$, the function $\mu \mapsto G_{x,\lambda}(\mu) := R(\mu, A)F_x(\lambda)$ has a holomorphic extension to $\mathbf{C}_+$ and that this extension is bounded.

By the resolvent identity and analytic continuation, for all $\mathrm{Re}\mu > \max(0, \omega_0(\mathbf{T}))$, $\mu \ne \lambda$, we have

$$G_{x,\lambda}(\mu) = \frac{F_x(\lambda) - R(\mu, A)x}{\mu - \lambda}.$$

A holomorphic extension to $\mathbf{C}_+$ is given by

$$\begin{aligned} G_{x,\lambda}(\mu) &= \frac{F_x(\lambda) - F_x(\mu)}{\mu - \lambda}, \quad \mu \ne \lambda, \\ G_{x,\lambda}(\lambda) &= -F_x'(\lambda). \end{aligned}$$

For fixed $\mu \in \mathbf{C}_+$, the function $\lambda \mapsto G_{x,\lambda}(\mu) = G_{x,\mu}(\lambda)$ is holomorphic on $\mathbf{C}_+$, so Cauchy's Integral Formula gives

$$G_{x,\lambda}(\mu) = \frac{1}{2\pi i}\int_\gamma \frac{F_x(z) - F_x(\mu)}{(\mu - z)(z - \lambda)}\,dz \tag{2.2}$$

for any contour $\gamma$ in $\mathbf{C}_+$ around $\lambda$. Taking $\gamma$ to be a circle with centre $\lambda$ and radius $r$ where $0 < r < \mathrm{Re}\lambda$, we obtain

$$\|G_{x,\lambda}(\mu)\| \le \frac{2\|F_x\|_{H^\infty}}{|r - |\lambda-\mu||}.$$

Letting $r \to 0$ if $|\lambda - \mu| \ge \mathrm{Re}\lambda/2$ and $r \to \mathrm{Re}\lambda$ otherwise,

$$\|G_{x,\lambda}(\mu)\| \le \frac{4\|F_x\|_{H^\infty}}{\mathrm{Re}\lambda}.$$

This establishes (1), (2) and (3).

If $|\lambda' - \lambda| < r < \mathrm{Re}\lambda$, then applying (2.2) for $\lambda$ and $\lambda'$, we obtain

$$\begin{aligned} \|G_{x,\lambda'}(\mu) - G_{x,\lambda}(\mu)\| &= \left\| \frac{\lambda'-\lambda}{2\pi i} \int_{|z-\lambda|=r} \frac{F_x(z) - F_x(\mu)}{(\mu - z)(z-\lambda)(z-\lambda')}\, dz \right\| \\ &\le \frac{2|\lambda'-\lambda|\|F_x\|_{H^\infty}}{|r - |\lambda-\mu||\,(r - |\lambda'-\lambda|)}. \end{aligned}$$

Letting $r = \mathrm{Re}\lambda/3$ if $|\lambda - \mu| \ge \mathrm{Re}\lambda/2$ and $r \to \mathrm{Re}\lambda$ otherwise, it follows that

$$\|F_x(\lambda') - F_x(\lambda)\|_{H^\infty_{\mathbf{T}}} \le \|F_x(\lambda') - F_x(\lambda)\| + \frac{72|\lambda'-\lambda|\|F_x\|_{H^\infty}}{(\mathrm{Re}\lambda)^2}$$

whenever $|\lambda' - \lambda| < \mathrm{Re}\lambda/6$. Thus $F_x$ is continuous as an $H^\infty_{\mathbf{T}}$-valued function, and holomorphic as an $X$-valued function. It follows from Cauchy's Theorem and Morera's Theorem that $F_x$ is holomorphic as an $H^\infty_{\mathbf{T}}$-valued function. □

**Proposition 2.7.** *Let $\phi \in L^1_{\mathcal{F}}(\mathbf{R}_+)$ and $x \in H^\infty_{\mathbf{T}}$.*

(1) *$\lambda \mapsto R(\lambda, A)T_\phi x$ has a holomorphic extension to $\mathbf{C}_+$ given by $\lambda \mapsto T_\phi(F_x(\lambda))$.*

(2) *If $R(\cdot, A)^2 x$ has a bounded holomorphic extension $G_x$ to $\mathbf{C}_+$, then $T_\phi x \in H^\infty_{\mathbf{T}}$ and*

$$\|T_\phi x\|_{H^\infty_{\mathbf{T}}} \le \frac{1}{2\pi}\|\mathcal{F}\phi\|_1 \left(\|F_x\|_{H^\infty} + \|G_x\|_{H^\infty}\right).$$

(3) *If $\phi \in L^1_{a,\mathcal{F}}(\mathbf{R}_+)$ for some $0 < a < 1$, then $T_\phi x \in H^\infty_{\mathbf{T}}$ and*

$$\|T_\phi x\|_{H^\infty_{\mathbf{T}}} \le \frac{1}{2\pi}\|F_x\|_{H^\infty} \left(3\|\mathcal{F}\phi\|_1 + (4 - 2\ln a)\|\phi\|_{a,1}\right).$$

Proof. (1). By Proposition 2.6, $F_x$ is a holomorphic map of $\mathbf{C}_+$ into $H^\infty_{\mathbf{T}}$. By Proposition 2.4, $T_\phi$ is a bounded linear map of $H^\infty_{\mathbf{T}}$ into $X$. Hence $T_\phi(F_x(\cdot)) : \mathbf{C}_+ \to X$ is holomorphic. For $\mathrm{Re}\lambda > \max(0, \omega_0(\mathbf{T}))$, the formula

$$T_\phi(F_x(\lambda)) = R(\lambda, A)T_\phi x$$

is valid for $\phi \in C_c(\mathbf{R}_+) \cap L^1_{\mathcal{F}}(\mathbf{R}_+)$ by definition of $T_\phi$, and hence for all $\phi \in L^1_{\mathcal{F}}(\mathbf{R}_+)$ by density (Lemma 2.3) and continuity with respect to $\phi$ (Proposition 2.4).

(2). Since the derivative of $R(\cdot,A)x$ is $-R(\cdot,A)^2x$, $G_x=-F_x'$. Propositions 2.4 and 2.6 show that

$$\|T_\phi(F_x(\lambda))\| \le \frac{1}{2\pi}\|\mathcal{F}\phi\|_1 \left\|F_{F_x(\lambda)}\right\|_{H^\infty} \le \frac{1}{2\pi}\|\mathcal{F}\phi\|_1\|F_x'\|_{H^\infty}.$$

Hence $T_\phi x \in H^\infty_{\mathbf{T}}$ and

$$\|T_\phi x\|_{H^\infty_{\mathbf{T}}} = \|T_\phi x\| + \sup_{\lambda\in\mathbf{C}_+}\|T_\phi(F_x(\lambda))\| \le \frac{1}{2\pi}\|\mathcal{F}\phi\|_1\left(\|F_x\|_{H^\infty}+\|G_x\|_{H^\infty}\right).$$

(3). Suppose that $0<\mathrm{Re}\lambda<a/2$. Consider first the case when $\phi\in C_c(\mathbf{R}_+)\cap L^1_{\mathcal{F}}(\mathbf{R}_+)$. By Proposition 2.6, and Proposition 2.1 with $x$ replaced by $F_x(\lambda)$,

$$\|T_\phi(F_x(\lambda))\| \le \liminf_{\alpha\to0+}\liminf_{r\to\infty}\frac{1}{2\pi}\left\|\int_{\gamma_{\alpha,r}}\left(\frac{F_x(\lambda)-F_x(z)}{z-\lambda}\right)\widehat{\phi}(\alpha-z)\,dz\right\|,$$

where, for $0<\alpha<a$ and $r>|\mathrm{Im}\lambda|+1$, we choose $\gamma_{\alpha,r}$ to be the path consisting of five line segments: $\gamma_1$ from $\alpha-ir$ to $\alpha+i(\mathrm{Im}\lambda-1)$; $\gamma_2$ from $\alpha+i(\mathrm{Im}\lambda-1)$ to $a+i(\mathrm{Im}\lambda-1)$; $\gamma_3$ from $a+i(\mathrm{Im}\lambda-1)$ to $a+i(\mathrm{Im}\lambda+1)$; $\gamma_4$ from $a+i(\mathrm{Im}\lambda+1)$ to $\alpha+i(\mathrm{Im}\lambda+1)$; $\gamma_5$ from $\alpha+i(\mathrm{Im}\lambda+1)$ to $\alpha+ir$. We use the following estimates

$$\begin{array}{rcll}
\|F_x(\lambda)-F_x(z)\| & \le & 2\|F_x\|_{H^\infty}, & z\in\gamma_{\alpha,r},\\
|\widehat{\phi}(\alpha-z)| & \le & \|\phi\|_{a,1}, & z\in\gamma_2\cup\gamma_3\cup\gamma_4,\\
|z-\lambda| & \ge & 1, & z\in\gamma_1\cup\gamma_2\cup\gamma_4\cup\gamma_5,\\
|z-\lambda| & \ge & a-\mathrm{Re}\lambda, & z=a+i(\mathrm{Im}\lambda+s)\in\gamma_3, |s|\le a-\mathrm{Re}\lambda,\\
|z-\lambda| & \ge & |s|, & z=a+i(\mathrm{Im}\lambda+s)\in\gamma_3, a-\mathrm{Re}\lambda<|s|\le 1.
\end{array}$$

These give

$$\begin{aligned}
\|T_\phi(F_x(\lambda))\| &\le \frac{\|F_x\|_{H^\infty}}{\pi}\left\{\int_{-\infty}^{\infty}|\widehat{\phi}(-is)|\,ds+\left(2a+2+2\int_{a-\mathrm{Re}\lambda}^{1}\frac{ds}{s}\right)\|\phi\|_{a,1}\right\}\\
&\le \frac{\|F_x\|_{H^\infty}}{\pi}\left\{\|\mathcal{F}\phi\|_1+(4-2\ln(a-\mathrm{Re}\lambda))\,\|\phi\|_{a,1}\right\}. \qquad (2.3)
\end{aligned}$$

This estimate remains valid for $\phi\in L^1_{a,\mathcal{F}}(\mathbf{R}_+)$ by density (Lemma 2.3) and continuity (Proposition 2.4).

For $\phi\in L^1_{a,\mathcal{F}}(\mathbf{R}_+)$, Propositions 2.4 and 2.6 give

$$\|T_\phi(F_x(\lambda))\| \le \frac{1}{2\pi}\|\mathcal{F}\phi\|_1\left\|F_{F_x(\lambda)}\right\|_{H^\infty} \le \frac{4}{\pi a}\|\mathcal{F}\phi\|_1\|F_x\|_{H^\infty},$$

for $\mathrm{Re}\lambda\ge a/2$. Now (2.3) shows that $T_\phi x\in H^\infty_{\mathbf{T}}$ and

$$\begin{aligned}
\|T_\phi x\|_{H^\infty_{\mathbf{T}}} &= \|T_\phi x\|+\lim_{\alpha\to0+}\sup_{s\in\mathbf{R}}\|T_\phi(F_x(\alpha+is))\|\\
&\le \frac{\|F_x\|_{H^\infty}}{2\pi}\left(3\|\mathcal{F}\phi\|_1+(4-2\ln a)\|\phi\|_{a,1}\right).
\end{aligned}$$

□

**Theorem 2.8.** *For $a > 0$, $\phi \mapsto T_\phi$ is a continuous Banach algebra homomorphism from $L^1_{a,\mathcal{F}}(\mathbf{R}_+)$ into $\mathcal{L}(H^\infty_{\mathbf{T}})$.*

Proof. Proposition 2.7 shows that the map is a continuous linear map of $L^1_{a,\mathcal{F}}(\mathbf{R}_+)$ into $\mathcal{L}(H^\infty_{\mathbf{T}})$. It remains to prove that the map $\phi \mapsto T_\phi$ is an algebra homomorphism, i.e. that $T_{\psi*\phi}x = (T_\psi \circ T_\phi)x$ for all $x \in H^\infty_{\mathbf{T}}$. But this is almost trivial for $\psi, \phi \in C_c(\mathbf{R}_+)$, and the general case follows by density and continuity via Lemma 2.3 and Proposition 2.4. □

## 3. Applications to individual orbits

In this section, we apply the results of Section 2 to obtain information about orbits $T(\cdot)T_\phi x$. This is possible because $T(t)T_\phi x = T_{\phi_t}x$, where

$$\phi_t(s) = \begin{cases} \phi(s-t), & s > t, \\ 0, & 0 \le s < t. \end{cases}$$

For the proofs in this and the following section we want to recall some results concerning vector-valued holomorphic functions. Let

$$H^1_{\mathrm{loc}}(\mathbf{C}_+; X) := \Big\{ F : \mathbf{C}_+ \to X : F \text{ is holomorphic and for all } R > 0 \quad \limsup_{\alpha \to 0} \int_{-R}^{R} \|F(\alpha + is)\|\, ds < \infty \Big\}.$$

Let $Y \subset X^*$ be a norming subspace in the sense that for each $x \in X$ one has $\|x\| = \sup\{|\langle x, x^*\rangle| : x^* \in Y, \|x^*\| \le 1\}$. Note that if $Y \subset X^*$ is norming, then $X$ can be identified in a natural way with a closed subspace of $Y^*$.

Let $F \in H^1_{\mathrm{loc}}(\mathbf{C}_+; X)$. We say that a function $\widetilde{F} : \mathbf{R} \to Y^*$ is a *boundary function* for $F$ if for each $x^* \in Y$

$$\lim_{\alpha \to 0^+} \langle F(\alpha + is), x^*\rangle = \langle x^*, \widetilde{F}(s)\rangle \quad \text{a.e.}(s).$$

If $F \in H^1_{\mathrm{loc}}(\mathbf{C}_+; X)$ has a boundary function $\widetilde{F} : \mathbf{R} \to Y^*$, then for each $x^* \in Y$ the function $\langle \widetilde{F}(\cdot), x^*\rangle$ is the limit in $L^1_{\mathrm{loc}}(\mathbf{R})$ of the functions $\langle F(\alpha + i\cdot), x^*\rangle$ as $\alpha \to 0^+$ (cf. [Du, Sections 2.3, 11.3]).

In the vector-valued case, boundary functions have been studied by Bukhvalov [Bu] (actually, he considered holomorphic functions on the disc, but the generalization to the right half plane by conformal mappings is standard [Ch2, Section 3]). When $X$

has the analytic Radon-Nikodým property (ARNP), every function $F \in H^1_{\mathrm{loc}}(\mathbf{C}_+, X)$ has a boundary function $\widetilde{F} : \mathbf{R} \to X$ such that $\lim_{\alpha\downarrow 0} \|F(\alpha + is) - \widetilde{F}(s)\| = 0$ a.e. For general $X$, the following proposition was proved in [Bu, Theorems 2.3, 2.4]. For completeness, we give here a short direct proof in the case of bounded holomorphic functions, which will suffice for our application in Theorem 4.1.

**Proposition 3.1.** *Let $Y$ be a norming subspace of $X^*$. Then for each $F \in H^1_{\mathrm{loc}}(\mathbf{C}_+; X)$ there exists a boundary function $\widetilde{F} : \mathbf{R} \to Y^*$.*

Proof. We consider the case when $F \in H^\infty(\mathbf{C}_+; X)$ (see [Bu] for the general case). By the boundedness of $K = \{F(\lambda) : \ \lambda \in \mathbf{C}_+\}$ and the Banach-Alaoglu theorem, $K$ is relatively weak*-compact as a subset of $Y^*$. For each $s \in \mathbf{R}$ let $\widetilde{F}(s) \in Y^*$ be a weak*-limit point of the net $\{F(\alpha + is) : \ \alpha \downarrow 0\}$.

Fix $x^* \in Y$. By the theory of (scalar) Hardy spaces (cf. [Du, Section 11.3]), we know that for almost all $s \in \mathbf{R}$ the scalar limit $\lim_{\alpha\to 0^+} \langle F(\alpha + is), x^*\rangle$ exists. By construction of $\widetilde{F}$ we have

$$\lim_{\alpha\to 0^+} \langle F(\alpha + is), x^*\rangle = \langle x^*, \widetilde{F}(s)\rangle \quad \text{a.e.}(s).$$

Thus $\widetilde{F}$ is a boundary function of $F$. □

The following Tauberian theorem is originally due to Ingham [In, Theorem I] with a somewhat stronger form of convergence to the boundary function. Actually, Ingham considered only the scalar-valued case which is all that we shall need in this section, but we will use the vector-valued case in Section 4. Our version of the theorem is proved in [Ch2, Prop. 1.3, Rem 1.4]. Note that in this theorem the boundary function of $\widehat{f}$ is assumed to be strongly (Bochner) measurable.

**Theorem 3.2.** *Let $f : \mathbf{R}_+ \to X$ be uniformly continuous, and suppose that the Laplace transform $\widehat{f}$ of $f$ has a boundary function in $L^1_{\mathrm{loc}}(\mathbf{R}, X)$. Then $\lim_{t\to\infty} \|f(t)\| = 0$.*

Our next result is a generalisation of [HN, Theorem 0.3].

**Theorem 3.3.** *Let $x \in H^\infty_{\mathbf{T}}$ and $\phi \in L^1_{\mathcal{F}}(\mathbf{R}_+)$. Then the orbit $T(\cdot)T_\phi x$ is bounded and uniformly continuous, and*

$$\lim_{t\to\infty} T(t)T_\phi x = 0 \quad \textit{weakly.}$$

Proof. For $t \geq 0$, let

$$\phi_t(s) = \begin{cases} \phi(s-t), & s \geq t, \\ 0, & 0 < s < t. \end{cases}$$

Then $(\mathcal{F}\phi_t)(s) = e^{ist}(\mathcal{F}\phi)(s)$ and $T_{\phi_t}x = T(t)T_\phi x$. By Proposition 2.4,

$$\|T(t)T_\phi x\| \le \frac{1}{2\pi}\|F_x\|_{H^\infty}\|\mathcal{F}\phi_t\|_1 = \frac{1}{2\pi}\|F_x\|_{H^\infty}\|\mathcal{F}\phi\|_1,$$

and

$$\begin{aligned}\|T(t+h)T_\phi x - T(t)T_\phi x\| &\le \frac{1}{2\pi}\|F_x\|_{H^\infty}\left\|\mathcal{F}\left(\phi_{t+h}-\phi_t\right)\right\|_1 \\ &= \frac{1}{2\pi}\|F_x\|_{H^\infty}\int_{-\infty}^{\infty}\left|\left(e^{ish}-1\right)(\mathcal{F}\phi)(s)\right|\,ds \\ &\to 0\end{aligned}$$

as $h \to 0$, uniformly in $t$, by the Dominated Convergence Theorem.

For the final statement, we first consider $\psi \in L^1_{\mathcal{F}}(\mathbf{R}_+) \cap C_c(\mathbf{R}_+)$. Let $x^* \in X^*$ and let $f(t) = \langle T(t)T_\psi x, x^*\rangle$. Then $f$ is bounded and uniformly continuous, and $\widehat{f}(\lambda) = \langle R(\lambda, A)T_\psi x, x^*\rangle$ for $\mathrm{Re}\lambda > \max(0, \omega_0(\mathbf{T}))$. Since $T_\psi x \in H^\infty_{\mathbf{T}}$, it follows that $\widehat{f}$ is bounded on $\mathbf{C}_+$, and therefore has a boundary function in $L^\infty(\mathbf{R})$ by Proposition 3.1. By Theorem 3.2, $\lim_{t\to\infty} f(t) = 0$, so $\lim_{t\to\infty} T(t)T_\psi x = 0$ weakly.

Since

$$\|T(t)T_\phi x - T(t)T_\psi x\| \le \frac{1}{2\pi}\|F_x\|_{H^\infty}\|\mathcal{F}(\phi_t - \psi_t)\|_1 = \frac{1}{2\pi}\|F_x\|_{H^\infty}\|\mathcal{F}(\phi-\psi)\|_1,$$

it follows from Lemma 2.3 that $T(t)T_\phi x \to 0$ weakly. □

There is an alternative proof of Theorem 3.3 which uses [HN, Theorem 0.3] and an argument similar to the proof of [HN, Corollary 1.6]. On the other hand, we show now that Theorem 3.3 includes [HN, Theorem 0.3] as a special case, and this enables us to give a general formula for $T_\phi x$.

**Corollary 3.4.** *Let $x \in H^\infty_{\mathbf{T}}$, $\mu > \max(\omega_0(\mathbf{T}), 0)$, and $\alpha > 1$. Then*

(1) $\lim_{t\to\infty} T(t)R(\mu, A)^\alpha x = 0$ *weakly,*

(2) $\sup_{t\ge 0}\|T(t)R(\mu, A)^\alpha x\| \le c_{\alpha,\mu}\|F_x\|_{H^\infty}$, *where*

$$c_{\alpha,\mu} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{ds}{(\mu^2+s^2)^{\alpha/2}} = \frac{\Gamma\left(\frac{\alpha-1}{2}\right)}{2\sqrt{\pi}\,\Gamma\left(\frac{\alpha}{2}\right)\mu^{\alpha-1}}.$$

(3) *For $\phi \in L^1_{\mathcal{F}}(\mathbf{R}_+)$, $T_\phi x = (\mu I - A)^\alpha\left(\int_0^\infty \phi(t)T(t)R(\mu, A)^\alpha x\,dt\right)$.*

Proof. Parts (1) and (2) follow from Example 2.5 and Theorem 3.3 (see also [HN, Theorem 0.3]). The equality

$$R(\mu, A)^\alpha T_\phi x = \int_0^\infty \phi(t)T(t)R(\mu, A)^\alpha x\,dt$$

holds for $\phi \in C_c(\mathbf{R}_+) \cap L^1_{\mathcal{F}}(\mathbf{R}_+)$, and hence for $\phi \in L^1_{\mathcal{F}}(\mathbf{R}_+)$ by density (Lemma 2.3) and continuity (using Proposition 2.4 and (2)). Now (3) follows. □

While Theorem 3.3 shows that the orbit $T(\cdot)T_\phi x$ is bounded in $X$ for any $\phi \in L^1_{\mathcal{F}}(\mathbf{R}_+)$, we can show that $\|T(t)T_\phi x\|_{H^\infty_{\mathbf{T}}}$ grows at most logarithmically if we make a slightly stronger assumption on $\phi$.

**Proposition 3.5.** *Let $x \in H^\infty_{\mathbf{T}}$ and $\phi \in L^1_{\mathcal{F}}(\mathbf{R}_+)$.*

(1) *If $\phi \in L^1_{a,\mathcal{F}}(\mathbf{R}_+)$ for some $0 < a < 1$, then*
$$\|T(t)T_\phi x\|_{H^\infty_{\mathbf{T}}} \le 2(1+\ln t)\|F_x\|_{H^\infty}\|\phi\|_{L^1_{a,\mathcal{F}}(\mathbf{R}_+)}, \quad t \ge 1/a.$$

(2) *If $R(\cdot, A)^2 x$ has a bounded holomorphic extension $G_x$ to $\mathbf{C}_+$, then*
$$\|T(t)T_\phi x\|_{H^\infty_{\mathbf{T}}} \le \frac{1}{2\pi}\|\mathcal{F}\phi\|_1 \left(\|F_x\|_{H^\infty} + \|G_x\|_{H^\infty}\right), \quad t \ge 0.$$

Proof. Suppose that $\phi \in L^1_{a,\mathcal{F}}(\mathbf{R}_+)$. For $0 < \alpha \le a$, $\|\phi_t\|_{\alpha,1} \le e^{\alpha t}\|\phi\|_{a,1}$. By Proposition 2.7 (3),
$$\|T(t)T_\phi x\|_{H^\infty_{\mathbf{T}}} = \|T_{\phi_t} x\|_{H^\infty_{\mathbf{T}}} \le \frac{1}{2\pi}\|F_x\|_{H^\infty}\left(3\|\mathcal{F}\phi\|_1 + (4 - 2\ln\alpha)e^{\alpha t}\|\phi\|_{a,1}\right).$$
For $t \ge 1/a$, we may put $\alpha = 1/t$, so
$$\|T(t)T_\phi x\|_{H^\infty_{\mathbf{T}}} \le \frac{1}{2\pi}\|F_x\|_{H^\infty}\left(3\|\mathcal{F}\phi\|_1 + (4 + 2\ln t)e\|\phi\|_{a,1}\right).$$
This suffices to give (1).

The statement (2) is proved by applying Proposition 2.7 (2) with $\phi$ replaced by $\phi_t$. □

Now we will consider orbits $t \mapsto T(t)R(\mu, A)x$ for $x \in H^\infty_{\mathbf{T}}$. It was shown in [Ne1] that $\|T(t)R(\mu, A)x\|$ grows at most linearly, and it is an open problem whether these orbits are always bounded. The difficulty here is the fact that the Fourier transform of the function $\phi_\mu(t) = e^{-\mu t}$ does not belong to $L^1(\mathbf{R})$, so Theorem 3.3 and Proposition 3.5 are not applicable. Nevertheless we will show that the result of [Ne1] can be derived from the results of Section 2, and indeed strengthened to show that $\|T(t)R(\mu, A)x\|_{H^\infty_{\mathbf{T}}}$ grows at most linearly.

**Theorem 3.6.** *Let $\mu \in \varrho(A)$. There exists a constant $c_\mu > 0$ such that*
$$\|T(t)R(\mu, A)x\|_{H^\infty_{\mathbf{T}}} \le c_\mu(1+t)\|x\|_{H^\infty_{\mathbf{T}}}, \qquad x \in H^\infty_{\mathbf{T}}, t \ge 0.$$

Proof. Let $\omega > \max(1, \omega_0(\mathbf{T}))$, $0 < \alpha < 1 < t$, and
$$\phi(s) = \begin{cases} 0, & 0 \le s \le t - \alpha, \\ 1 + \frac{s-t}{\alpha}, & t - \alpha < s \le t, \\ e^{-\omega(s-t)}, & s \ge t. \end{cases}$$

Then

$$(\mathcal{F}\phi)(s) = e^{-ist}\left(\frac{1}{\omega+is} + \frac{i}{s} + \frac{1-e^{i\alpha s}}{\alpha s^2}\right),$$

so

$$\begin{aligned}
\|\mathcal{F}\phi\|_1 &\le 2\|\phi\|_1 + \int_{|s|\ge 1}\left|\frac{1}{\omega+is} + \frac{i}{s} + \frac{1-e^{i\alpha s}}{\alpha s^2}\right| ds \\
&\le \frac{2}{\omega} + \alpha + \int_{1\le|s|\le\alpha^{-1}}\left(\left|\frac{1}{\omega+is}\right| + \left|\frac{i}{s} + \frac{1-e^{i\alpha s}}{\alpha s^2}\right|\right) ds \\
&\qquad\qquad + \int_{|s|>\alpha^{-1}}\left(\left|\frac{1}{\omega+is} + \frac{i}{s}\right| + \frac{|1-e^{i\alpha s}|}{\alpha s^2}\right) ds \\
&\le \frac{2}{\omega} + \alpha - 2\log\alpha + \int_{-1}^{1}\left|\frac{i}{r} + \frac{1-e^{ir}}{r^2}\right| dr + 2\alpha\omega + 2\int_1^\infty \frac{|1-e^{ir}|}{r^2}\, dr \\
&\le \frac{2}{\omega} + (1+2\omega)\alpha - 2\log\alpha + 5.
\end{aligned}$$

For $0 < a < 1$,

$$\|\phi\|_{a,1} = \int_{t-\alpha}^{t}\left(1 + \frac{s-t}{\alpha}\right)e^{as}\, ds + \int_t^\infty e^{-\omega(s-t)}e^{as}\, ds \le e^{at}\left(\alpha + \frac{1}{\omega-a}\right).$$

Let $M_\omega = \sup_{s\ge 0} e^{-\omega s}\|T(s)\| < \infty$. By Proposition 2.7,

$$\begin{aligned}
&\|T(t)R(\omega,A)x\|_{H^\infty_{\mathbf{T}}} \\
=\ &\left\|\int_0^\infty \phi(s)T(s)x\, ds - \int_{t-\alpha}^{t}\left(1+\frac{s-t}{\alpha}\right)T(s)x\, ds\right\|_{H^\infty_{\mathbf{T}}} \\
\le\ &\frac{1}{2\pi}\|F_x\|_{H^\infty}\left(3\|\mathcal{F}\phi\|_1 + (4-2\ln a)\|\phi\|_{a,1}\right) + \int_{t-\alpha}^{t} M_\omega e^{\omega s}\|x\|_{H^\infty_{\mathbf{T}}}\, ds \\
\le\ &\kappa_\omega(a,\alpha,t)\|x\|_{H^\infty_{\mathbf{T}}},
\end{aligned}$$

where

$$\begin{aligned}
&\kappa_\omega(a,\alpha,t) = \\
&\frac{1}{2\pi}\left(\frac{6}{\omega} + 3\alpha + 6\alpha\omega - 6\log\alpha + 15 + (4-2\ln a)e^{at}\left(\alpha + \frac{1}{\omega-a}\right)\right) + \alpha M_\omega e^{\omega t}.
\end{aligned}$$

For $t \ge 1$, we may take $\alpha = e^{-\omega t}$ and $a = 1/t$, and we obtain

$$\|T(t)R(\omega,A)x\|_{H^\infty_{\mathbf{T}}} \le \widetilde{\kappa}_\omega(t)\|x\|_{H^\infty_{\mathbf{T}}},$$

where

$$\widetilde{\kappa}_\omega(t) = \frac{1}{2\pi}\left(\frac{6}{\omega} + (3+6\omega)e^{-\omega t} + 6\omega t + 15 + (4+2\ln t)e\left(e^{-\omega t} + \frac{t}{\omega t - 1}\right)\right) + M_\omega.$$

Since $\sup_{0\le t\le 1}\|T(t)R(\omega,A)\|_{\mathcal{L}(\mathcal{H}^\infty_{\mathbf{T}})} \le \sup_{0\le t\le 1}\|T(t)R(\omega,A)\|_{\mathcal{L}(\mathcal{X})} < \infty$, the result follows in the case when $\mu = \omega$.

For an arbitrary $\mu \in \varrho(A)$, the result now follows via the resolvent identity. $\square$

Alboth [Al] has used the result of [Ne1] to study rapidly decaying orbits of $\mathbf{T}$. In that context, it is important to know the asymptotic behaviour of the constant $c_\omega$ in Theorem 3.6 for large $\omega$. Both the estimates in [Ne1] and those given above show that $c_\omega$ cannot grow faster than linearly.

Our next result improves the bound $c_\mu(1+t)$ for $\|T(t)R(\mu,A)x\|$ for positive semigroups and positive $x$. Recall that an ordered Banach space $X$ is said to be *normal* if there is a constant $\kappa$ such that $\|x\| \le \kappa \max(\|y_1\|, \|y_2\|)$ whenever $y_1 \le x \le y_2$.

**Theorem 3.7.** *Suppose* $\mathbf{T}$ *is a positive semigroup on a normal ordered Banach space* $X$*. For all* $\mu \in \varrho(A)$ *there exists a constant* $c_\mu > 0$ *such that*

$$\|T(t)R(\mu,A)x\| \le c_\mu(1+\ln t)\|x\|_{H^\infty_{\mathbf{T}}},$$

*whenever* $t \ge 1$ *and* $0 \le x \in H^\infty_{\mathbf{T}}$.

Proof. By the resolvent identity, without loss of generality we may assume that $\mu > \omega_0(\mathbf{T})$. Then $0 \le R(\mu,A)x \in H^\infty_{\mathbf{T}}$.

Fix $t \ge 1$, and let

$$\phi(s) = \left(\chi_{(0,1)} * \chi_{(0,t)}\right)(s) = \begin{cases} s, & 0 \le s \le 1, \\ 1, & 1 \le s \le t, \\ t+1-s, & t \le s \le t+1, \\ 0, & t+1 \le s. \end{cases}$$

Then

$$(\mathcal{F}\phi)(s) = -\left(\frac{1-e^{-is}}{s}\right)\left(\frac{1-e^{-ist}}{s}\right).$$

Hence

$$\begin{aligned} \|\mathcal{F}\phi\|_1 &\le 2\int_2^\infty \frac{4}{s^2}\,ds + \int_{-2}^2 \left|\frac{1-e^{-ist}}{s}\right| ds \\ &\le 4 + 4\int_0^2 \frac{|\sin(st/2)|}{s}\,ds \\ &\le 4 + 4\int_0^t \frac{|\sin s|}{s}\,ds \le 8 + 4\ln t. \end{aligned}$$

For $0 \le y \in H^\infty_{\mathbf{T}}$,

$$0 \le \int_1^t T(s)y\,ds \le T_\phi y,$$

so

$$\left\| \int_1^t T(s)y\,ds \right\| \leq \kappa \|T_\phi y\| \leq \frac{\kappa}{2\pi} \|F_y\|_{H^\infty} \, \|\mathcal{F}\phi\|_1 \leq \frac{\kappa}{\pi} \|F_y\|_{H^\infty} (4 + 2\ln t).$$

Hence

$$\left\| \int_0^t T(s)y\,ds \right\| \leq c(1 + \ln t) \|y\|_{H^\infty_{\mathbf{T}}},$$

where $c = 4\kappa/\pi + \int_0^1 \|T(s)\|\,ds$. Applying this to $y = x$ and $y = \mu R(\mu, A)x$ and using Proposition 2.6 and the identity $AR(\mu, A)x = \mu R(\mu, A)x - x$, it follows that there exists a constant $c'_\mu > 0$ such that

$$\left\| \int_0^t T(s)AR(\mu, A)x\,ds \right\| \leq c'_\mu (1 + \ln t) \|x\|_{H^\infty_{\mathbf{T}}}, \qquad t \geq 1,$$

whenever $0 \leq x \in H^\infty_{\mathbf{T}}$. Hence, the identity

$$T(t)R(\mu, A)x = R(\mu, A)x + \int_0^t T(s)AR(\mu, A)x\,ds$$

implies

$$\|T(t)R(\mu, A)x\| \leq \|R(\mu, A)x\| + c'_\mu (1 + \ln t) \|x\|_{H^\infty_{\mathbf{T}}}, \qquad t \geq 1.$$

□

In the proof above, we could choose $\phi$ to be any non-negative function in $L^1_{\mathcal{F}}(\mathbf{R}_+)$ such that $\phi \geq 1$ on $[1, t]$. We would conclude that $\|T(t)R(\mu, A)x\|$ grows no faster than a constant multiple of

$$\inf \left\{ \|\mathcal{F}\phi\|_1 : 0 \leq \phi \in L^1_{\mathcal{F}}(\mathbf{R}_+), \phi \geq 1 \text{ on } [1, t] \right\}.$$

We do not know whether this quantity grows logarithmically in $t$.

## 4. The $\odot$−reflexive case

For a given $C_0$-semigroup $\mathbf{T}$ on a Banach space $X$ let $X^\odot$ be the maximal $\mathbf{T}^*$-invariant subspace of $X^*$ such that the restriction $\mathbf{T}^\odot$ of $\mathbf{T}^*$ to $X^\odot$ is a $C_0$-semigroup. Then $X^\odot$ is the norm-closure of $D(A^*)$ in $X^*$. Replacing the norm on $X$ by an equivalent norm, for example

$$\|x\|' = \sup_{t \geq 0} e^{-\omega t} \|T(t)x\|$$

for some $\omega > \omega_0(\mathbf{T})$, we may assume that $\limsup_{t \to 0+} \|T(t)\| = 1$. Then $X^\odot$ is a norming subspace of $X^*$. Using the $C_0$-semigroup $\mathbf{T}^\odot$ on $X^\odot$, one may form the space $X^{\odot\odot}$, and identify $X$ in a natural way with a closed subspace of $X^{\odot\odot}$. Recall that $\mathbf{T}$

is said to be $\odot-$*reflexive* if $X^{\odot\odot} = X$. We refer the reader to [Ne2, Section 1.3] for all these concepts and properties.

In this section we will present some individual stability results for $\odot-$reflexive semigroups. More generally, we will assume that the quotient space $X^{\odot\odot}/X$ is separable. For arbitrary semigroups on spaces with the ARNP, similar results have been obtained in [HN, Theorem 0.2] and [Ch2, Theorem 3.3].

**Theorem 4.1.** *Let* $\mathbf{T}$ *be a* $C_0$*-semigroup on* $X$ *and assume that* $X^{\odot\odot}/X$ *is separable. Let* $x \in H^\infty_{\mathbf{T}}$*. If the orbit* $T(\cdot)x$ *is bounded and uniformly continuous, then*

$$\lim_{t\to\infty} \|T(t)x\| = 0.$$

Proof. If $Y$ is a closed $\mathbf{T}$-invariant subspace of $X$, then $Y^{\odot\odot}/Y$ is canonically isomorphic to a closed subspace of $X^{\odot\odot}/X$ [Ne2, Lemma 6.1.7]. By passing to the closed linear span of the $\mathbf{T}$-orbit of $x$, without loss of generality we may assume that $X$ is separable. The separability of $X^{\odot\odot}/X$ then implies that also $X^{\odot\odot}$ is separable.

Since $F_x \in H^1_{\mathrm{loc}}(\mathbf{C}_+; X)$, we may apply Proposition 3.1 with $Y = X^\odot$, and we find a boundary function $\widetilde{F_x} : \mathbf{R} \to X^{\odot *}$ such that for all $x^\odot \in X^\odot$

$$\lim_{\alpha\to 0} \langle F_x(\alpha + is), x^\odot\rangle = \langle x^\odot, \widetilde{F_x}(s)\rangle \quad \text{a.e.}(s).$$

Fix $\mu \in \varrho(A)$. The resolvent identity and analytic continuation give

$$F_x(\lambda) = R(\mu, A)x + (\mu - \lambda)R(\mu, A)F_x(\lambda), \quad \lambda \in \mathbf{C}_+,$$

so

$$\begin{aligned}
\langle x^\odot, \widetilde{F_x}(s)\rangle &= \lim_{\alpha\to 0+} \langle F_x(\alpha + is), x^\odot\rangle \\
&= \langle R(\mu, A)x, x^\odot\rangle + \lim_{\alpha\to 0+} (\mu - (\alpha + is))\langle F_x(\alpha + is), R(\mu, A^\odot)x^\odot\rangle \\
&= \langle R(\mu, A)x, x^\odot\rangle + (\mu - is)\langle x^\odot, R(\mu, A^{\odot *})\widetilde{F_x}(s)\rangle \quad \text{a.e.}(s).
\end{aligned}$$

Thus

$$\widetilde{F_x}(s) = R(\mu, A)x + (\mu - is)R(\mu, A^{\odot *})\widetilde{F_x}(s) \in D(A^{\odot *}) \subseteq X^{\odot\odot} \quad \text{a.e.}$$

Thus $\widetilde{F_x}$ is separably valued and weak*-measurable, hence strongly measurable by the weak*-version of Pettis's Theorem [DU, Corollary 4, p.42]. Therefore $T(\cdot)x$ is bounded and uniformly continuous and its Laplace transform admits a bounded, strongly measurable boundary function (we may regard these functions as having values in $X^{\odot\odot}$, or alternatively we may deduce from [Bu, Theorem 2.5] that the values of the boundary function lie in $X$). From Theorem 3.2, we obtain

$$\lim_{t\to\infty} \|T(t)x\| = 0.$$

□

There is an alternative proof of Theorem 4.1 using arguments adapted from [HN]. Given $\omega > \max(0, \omega_0(\mathbf{T}))$, let

$$H(s) = (\omega + is)^{-1} R(\mu, A)\widetilde{F_x}(-is).$$

Then $H \in L^1(\mathbf{R}, X^{\odot\odot})$, so $\mathcal{F}H \in C_0(\mathbf{R}, X)$ by the Riemann-Lebesgue Lemma. A calculation similar to one following Theorem 1.2 in [HN] shows that, for $t \geq 0$,

$$(\mathcal{F}H)(t) = 2\pi T(t)R(\omega, A)R(\mu, A)x.$$

Thus, $\lim_{t\to\infty} \|T(t)R(\omega, A)R(\mu, A)x\| = 0$. The assumption that $T(\cdot)x$ is bounded and uniformly continuous can now be used as in the proof of [HN, Corollary 1.6] to obtain the result.

The separability assumption on $X^{\odot\odot}/X$ cannot be omitted from Theorem 4.1, as [HN, Example 1.7] shows. There, $X = C_0(\mathbf{R})$, $\mathbf{T}$ is the semigroup of left translations, and $X^{\odot\odot} = \mathrm{BUC}(\mathbf{R})$ [Ne2, Example 1.3.9]. However, the theorem is true without the assumption if $X$ has the ARNP. Then the bounded holomorphic function $F_x$ has a strongly measurable boundary function with values in $X$ [Bu], and the result follows directly from Theorem 3.2 (see also [HN, Theorem 2.2], [Ne3, Theorem 4.4.2]).

**Corollary 4.2.** *Let $X^{\odot\odot}/X$ be separable and let $x \in H^\infty_{\mathbf{T}}$. Then for all $\phi \in L^1_{\mathcal{F}}(\mathbf{R}_+)$ we have*

$$\lim_{t\to\infty} \|T(t)T_\phi x\| = 0.$$

Proof. Consider first $\psi \in C_c(\mathbf{R}_+) \cap L^1_{\mathcal{F}}(\mathbf{R}_+)$. Then $T_\psi x \in H^\infty_{\mathbf{T}}$, and it follows from Theorem 3.3 and Theorem 4.1 that $\lim_{t\to\infty} \|T(t)T_\psi x\| = 0$. Now the result follows by approximation, as in the proof of Theorem 3.3. □

Combining this with Example 2.5 we obtain:

**Corollary 4.3.** *Let $X^{\odot\odot}/X$ be separable and let $x \in H^\infty_{\mathbf{T}}$. Then for all $\alpha > 1$ and $\mu > \omega_0(\mathbf{T})$ we have*

$$\lim_{t\to\infty} \|T(t)R(\mu, A)^\alpha x\| = 0.$$

Concluding this section we would like to remark that the $\odot$-reflexive case in Theorem 4.1 actually follows from a much more general result in Tauberian theory. If $\mathbf{T}$ is a $\odot$-reflexive semigroup, then $R(\mu, A)$ is weakly compact [Pa, Theorem 3.5], [Ne2, Theorem 2.5.2]. For $x \in H^\infty_{\mathbf{T}}$, the resolvent identity

$$F_x(z) = R(\mu, A)x + (\mu - z)R(\mu, A)F_x(z)$$

yields that $\{F_x(z) : z \in U\} \cap B(0, R)$ is relatively weakly compact in $X$ for all bounded sets $U \subset \mathbf{C}_+$ and all $R > 0$.

**Theorem 4.4.** *Let $f \in \mathrm{BUC}(\mathbf{R}_+;X)$ be such that $\widehat{f} \in H^1_{\mathrm{loc}}(\mathbf{C}_+;X)$. If the set $\{\widehat{f}(z) : z \in U\} \cap B(0,R)$ is relatively weakly compact in $X$ for all bounded sets $U \subset \mathbf{C}_+$ and all $R > 0$, then $\lim_{t\to\infty} \|f(t)\| = 0$.*

Proof. Restricting to the closed linear span of $\{f(t) : t \in \mathbf{R}_+\}$ we may assume without loss of generality that the space $X$ is separable. Let $Y$ be a separable and norming subspace of $X^*$ and let $(x_n^*)_{n\in\mathbf{N}} \subset Y$ be a dense sequence. By Proposition 3.1 we find a boundary function $\widetilde{f} : \mathbf{R} \to Y^*$ for the Laplace transform $\widehat{f}$ and a set $N \subset \mathbf{R}$ of Lebesgue measure 0 such that for all $n \in \mathbf{N}$ and all $s \in \mathbf{R}\backslash N$

$$\lim_{\alpha\to 0} \langle \widehat{f}(\alpha + is), x_n^* \rangle = \langle x_n^*, \widetilde{f}(s) \rangle.$$

This and the assumption of weak compactness yield that $\widetilde{f}(s) \in X$ for all $s \in \mathbf{R}\backslash N$. Since $X$ is separable, the function $\widetilde{f}$ is strongly measurable with values in $X$ by Pettis's Theorem.

The claim follows from Theorem 3.2. □

## 5. Some extensions and questions

### 5.1. Extensions

The following three remarks describe some extensions of the results given above.

**Remark 5.1.** The functional calculus of Section 2 can be extended slightly. Let

$$L^1_*(\mathbf{R}) = \left\{ \psi \in L^1(\mathbf{R}) : \ \mathcal{F}\psi = 0 \text{ on } \mathbf{R}_+ \right\}.$$

It is easy to see that $L^1_*(\mathbf{R})$ is the closure in $L^1(\mathbf{R})$ of $\left\{ \mathcal{F}\phi : \phi \in L^1_{\mathcal{F}}(\mathbf{R}_+) \right\}$. It follows from Corollary 2.2 that there is a unique continuous bilinear map:

$$(\psi, x) \in L^1_*(\mathbf{R}) \times H^\infty_{\mathbf{T}} \mapsto \psi(iA)x \in X$$

satisfying

$$\begin{aligned} (\mathcal{F}\phi)(iA)x &= T_\phi x, && \phi \in L^1_{\mathcal{F}}(\mathbf{R}_+), \\ \|\psi(iA)\| &\le \frac{1}{2\pi} \|F_x\|_{H^\infty} \|\psi\|_1, && \psi \in L^1_*(\mathbf{R}), x \in H^\infty_{\mathbf{T}}. \end{aligned}$$

Since $T(t)\psi(iA)x = \psi^t(iA)x$, where $\psi^t(s) = e^{-ist}\psi(s)$, Theorem 3.3 and Corollary 4.2 remain valid when $T_\phi x$ is replaced by $\psi(iA)x$.

**Remark 5.2.** Another extension of the functional calculus is the following. Let $H^\infty_{\mathbf{T},k}$ be the space of all $x \in X$ such that the local resolvent $R(\cdot, A)x$ has a holomorphic extension $F_x$ to $\mathbf{C}_+$ satisfying

$$\sup_{\lambda\in\mathbf{C}_+} \frac{\|F_x(\lambda)\|}{|(1+\lambda)^k|} =: \|F_x\|_{\infty,k} < \infty.$$

Equipped with the norm $\|x\|_{H^\infty_{\mathbf{T},k}} := \|x\| + \|F_x\|_{\infty,k}$, this space is a Banach space.

Next, define the space

$$L^1_{a,\mathcal{F},k}(\mathbf{R}_+) := \{\phi \in L^1_{a,\mathcal{F}}(\mathbf{R}_+) : \mathcal{F}\phi \in L^1(\mathbf{R}; (1+|t|)^k \, dt)\}.$$

Using the techniques developed in Section 2, it is not difficult to introduce a functional calculus from $L^1_{a,\mathcal{F},k}(\mathbf{R}_+)$ into $H^\infty_{\mathbf{T},k}$. With this functional calculus it is possible to prove similar results on individual stability as in Sections 3 and 4. For example, if $x \in H^\infty_{\mathbf{T},k}$ and $\phi \in L^1_{a,\mathcal{F},k}(\mathbf{R}_+)$, then the orbit $T(\cdot)T_\phi x$ is bounded and uniformly continuous, and weakly asymptotically stable (compare with Theorem 3.3).

Results of this type have been obtained in [HN], and we will not go into details.

**Remark 5.3.** For $1 < p < \infty$, let $H^p_{\mathbf{T}}$ be the space of all $x \in X$ for which $R(\cdot, A)x$ has a holomorphic extension $F_x : \mathbf{C}_+ \to X$ such that

$$\|F_x\|_{H^p} := \sup_{\alpha > 0} \left( \int_{\mathbf{R}} \|F_x(\alpha + is)\|^p \, ds \right)^{1/p} < \infty.$$

It follows from a variant of Proposition 2.1 and Hölder's inequality that

$$\|T_\phi x\| \le \frac{1}{2\pi} \|F_x\|_{H^p} \|\mathcal{F}\phi\|_{p'}$$

whenever $x \in H^p_{\mathbf{T}}$, $\phi \in C^\infty_c(\mathbf{R}_+)$ and $\mathcal{F}\phi \in L^{p'}(\mathbf{R})$ where $\frac{1}{p} + \frac{1}{p'} = 1$. As in Section 2, it is possible to define a functional calculus $(\phi, x) \mapsto T_\phi x$ when $x \in H^p_{\mathbf{T}}$, $\phi \in L^p(\mathbf{R}_+)$ and $\mathcal{F}\phi \in L^{p'}(\mathbf{R})$ (the Hausdorff-Young inequality shows that $T_\phi x$ is defined for all $\phi \in L^p(\mathbf{R}_+)$ when $1 < p \le 2$). As in Remark 1, one can extend this to a functional calculus $(\psi, x) \mapsto \psi(iA)x$ when $x \in H^p_{\mathbf{T}}$, $\psi \in L^{p'}(\mathbf{R})$ and $\mathcal{F}\psi = 0$ on $\mathbf{R}_+$ (the two calculi coincide when $2 \le p < \infty$). As in Theorem 3.3, the corresponding orbits of $\mathbf{T}$ are bounded, uniformly continuous, and converge to 0 weakly (in norm if $X^{\odot\odot}/X$ is separable, as in Corollary 4.2). In particular, Example 2.5 provides the following counterpart to Corollary 3.4, (1) and (2).

**Proposition 5.4.** *Let $x \in H^p_{\mathbf{T}}$, $\mu > \max(\omega_0(\mathbf{T}), 0)$ and $\alpha > 1/p'$. Then*

(1) $\lim_{t\to\infty} T(t)R(\mu, A)^\alpha x = 0$ *weakly,*

(2) $$\sup_{t \ge 0} \|T(t)R(\mu, A)^\alpha x\| \le \frac{\|F_x\|_{H^p}}{2\pi} \left( \int_{-\infty}^{\infty} \frac{ds}{(\mu^2 + s^2)^{\alpha p'/2}} \right)^{1/p'}.$$

This result can be compared with [HN, Theorem 0.1], where it is shown that $\lim_{t\to\infty} \|T(t)R(\mu, A)^\alpha x\| = 0$ if $x \in H^\infty_{\mathbf{T}}$ and $X$ has Fourier type $p'$.

If $\mathbf{T}$ is a holomorphic semigroup, then $H^\infty_{\mathbf{T}} \subset H^p_{\mathbf{T}}$ for all $p > 1$, and it follows that $T(\cdot)R(\mu, A)^\alpha x$ is bounded whenever $x \in H^\infty_{\mathbf{T}}$ and $\alpha > 0$. Mark Blake [Bl] has shown that this is also true when $\alpha = 0$ for some classes of semigroups, including holomorphic semigroups.

### 5.2. Open questions

The following questions remain open:

**Problem 5.5.** *Let $x \in H^\infty_{\mathbf{T}}$ and $\phi \in L^1_{\mathcal{F}}(\mathbf{R}_+)$. Is it always true that $T_\phi x \in H^\infty_{\mathbf{T}}$? See Proposition* 2.7.

**Problem 5.6.** *Let $x \in H^\infty_{\mathbf{T}}$. Is it always true that $T(\cdot)R(\mu, A)x$ is bounded in $X$ (or better in $H^\infty_{\mathbf{T}}$)? See Theorem* 3.6 *and Remark* 5.3 *above. If not, is this true in the context of positive semigroups, as in Theorem* 3.7*?*

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