

Stochastic integration in Banach spaces – a survey

Jan van Neerven, Mark Veraar and Lutz Weis

Abstract. This paper presents a brief survey of the theory of stochastic integration in Banach spaces. Expositions of the stochastic integrals in martingale type 2 spaces and UMD spaces are presented, as well as some applications of the latter to vector-valued Malliavin calculus and the stochastic maximal regularity problem. A new proof of the stochastic maximal regularity theorem is included.

Mathematics Subject Classification (2010). Primary: 60H05, Secondary: 46B09, 46E40, 60H15.

Keywords. Stochastic integration, martingale type, UMD Banach spaces, γ -radonifying operators, Malliavin calculus, R -boundedness, stochastic maximal regularity.

1. Introduction

Stochastic calculus was developed in the 1950s in the fundamental work of Itô. In its simplest form, the construction of the Itô stochastic integral with respect to a Brownian motion $(B_t)_{t \geq 0}$ relies on an L^2 -isometry, which asserts that if $\phi : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ is an adapted simple process, then

$$\mathbb{E} \left| \int_0^\infty \phi_t dB_t \right|^2 = \mathbb{E} \int_0^\infty |\phi_t|^2 dt.$$

The first named author is supported by VICI subsidy 639.033.604 of the Netherlands Organisation for Scientific Research (NWO). The second author is supported by VENI subsidy 639.031.930 of the Netherlands Organisation for Scientific Research (NWO). The third named author is supported by a grant from the Deutsche Forschungsgemeinschaft (We 2847/1-2).

This paper is based on the lectures delivered by the first-named author during the Semester on Stochastic Analysis and Applications at the Centre Interfacultaire Bernoulli, EPF Lausanne. JvN wishes to thank the organisers Robert Dalang, Marco Dozzi, Franco Flandoli, and Francesco Russo for the excellent conditions and the enjoyable atmosphere.

This isometry is used to extend the stochastic integral to arbitrary progressively measurable processes satisfying $\mathbb{E} \int_0^\infty |\phi_t|^2 dt < \infty$. The stochastic integral process $t \mapsto \int_0^t \phi_s dB_s$ defines a continuous L^2 -martingale, and by means of stopping time techniques the integral can be extended to all progressively measurable processes satisfying

$$\int_0^\infty |\phi_t|^2 dt < \infty \text{ almost surely.}$$

It was immediately realised that the above programme generalises *mutatis mutandis* to stochastic integrals of progressively measurable processes with values in a Hilbert space H . Some of the early works in this direction include [3, 17, 25, 28, 72]. More generally, if H' is another Hilbert space one may allow operator-valued integrands with values in the space of Hilbert-Schmidt operators $\mathcal{L}_2(H, H')$ to define an H' -valued stochastic integral with respect to an H -cylindrical Brownian motion. This integral was popularised by Da Prato and Zabczyk, who used it to study stochastic partial differential equations (SPDE) by functional analytic and operator theoretic methods [26, 27]; see also [73].

From the point of view of SPDE the limitation to the Hilbert space framework is rather restricting, and various authors have attempted to extend the theory of stochastic integration to more general classes of Banach spaces. It was realised soon that a stochastic integral for square integrable functions with values in a Banach space X can be defined if X has type 2 [44], whereas a bounded measurable function $f : [0, 1] \rightarrow \ell^p$ may fail to be stochastically integrable for $1 \leq p < 2$ [99]; see [95] for more detailed results and examples along these lines. A systematic theory of stochastic integration in 2-smooth Banach spaces was developed by Neidhardt in his 1978 PhD thesis and, independently, by Belopol'skaya and Dalecky, [2] and Dettweiler [31] independently developed a parallel theory for martingale type 2 spaces. Interestingly, Pisier [90] had already shown in 1975 that a Banach space has an equivalent 2-smooth norm if and only if it has martingale type 2. The stochastic integrals of Neidhardt and Dettweiler were further developed and applied to SPDEs by Brzeźniak [8, 9, 10, 11]. We shall briefly summarize the martingale type 2 approach in Section 4.

Along a different line, the fundamental work of Burkholder [14, 15] showed that many of the deeper inequalities in the theory of martingales extend to a class of Banach spaces in which martingale differences are unconditional, nowadays called the class of UMD Banach spaces. These spaces were characterised by Burkholder [14] and Bourgain [5] as precisely those Banach spaces X for which the Hilbert transform on $L^p(\mathbb{R})$ extends boundedly to $L^p(\mathbb{R}; X)$. As a consequence, UMD spaces provide a natural framework for vector-valued harmonic analysis, and indeed large parts of the theory of singular integrals have by now been extended to UMD spaces [6, 40, 42, 45, 69, 98, 101].

The probabilistic definition of the UMD property in terms of martingale differences suggests the possibility to develop stochastic calculus in UMD

spaces. The first result in this direction is due to Garling [37], who proved a two-sided L^p -estimate for the stochastic integral of an adapted simple process ϕ with values in a UMD space in terms of the stochastic integral of ϕ with respect to an independent Brownian motion. McConnell [70] proved decoupling inequalities for tangent martingale difference sequences and used them to obtain a sufficient condition for stochastic integrability of an UMD-valued process with respect to a Brownian motion in terms of the almost sure stochastic integrability of its trajectories with respect to an independent Brownian motion. The ideas of Garling and McConnell have been streamlined and extended in a systematic way by the present authors [77, 78, 83] and applied to SPDEs [12, 79, 80, 81]. A key idea in obtaining two-sided estimates of Burkholder-Gundy type is to measure the integrand in a norm that is custom-made for the Gaussian setting, rather than in the traditional Lebesgue-Bochner norms. In an operator-theoretic language, these Gaussian norms are given in terms of certain γ -radonifying operators (see Section 3 for the relevant definitions). The main aim of this paper is to provide a coherent presentation of this theory and some of its applications, in particular to the vector-valued Malliavin calculus and the stochastic maximal L^p -regularity problem. In the final section of this paper we discuss some recent L^p -bounds for vector-valued Poisson stochastic integrals.

Let us mention that various different approaches to stochastic integration in Banach spaces exist in the literature, e.g., [7, 33, 32, 74].

We finish the introduction by fixing some notation. All vector spaces are real. Throughout the paper, H and \mathcal{H} are fixed Hilbert spaces. We will always identify Hilbert spaces with their duals via the Riesz representation theorem. All random variables are supposed to be defined on a fixed probability space (Ω, \mathbb{P}) .

2. Isonormal processes

It is a well-known result in the theory of Gaussian measures that an infinite-dimensional Hilbert space \mathcal{H} does not support a standard Gaussian measure (cf. [4]). By this we mean that there exists no Radon probability measure γ on \mathcal{H} with the property that for all $h \in \mathcal{H}$ of norm one the image measure of γ under the mapping $h : \mathcal{H} \rightarrow \mathbb{R}$ is standard Gaussian. The following definition serves as a substitute.

Definition 2.1. An \mathcal{H} -isonormal process is a bounded linear mapping $W : \mathcal{H} \rightarrow L^2(\Omega)$ with the following properties:

- (i) for all $h \in \mathcal{H}$ the random variable Wh is Gaussian;
- (ii) for all $h_1, h_2 \in \mathcal{H}$ we have $\mathbb{E}(Wh_1 \cdot Wh_2) = [h_1, h_2]$.

It is an easy exercise to check that for any Hilbert space \mathcal{H} , an \mathcal{H} -isonormal process does indeed exist. The random variables Wh , $h \in \mathcal{H}$, are jointly Gaussian, as every linear combination $\sum_{j=1}^k c_j Wh_j = W(\sum_{j=1}^k c_j h_j)$

is Gaussian. In particular this implies that if h_1, \dots, h_k are orthogonal, then Wh_1, \dots, Wh_k are independent. For more details we refer to [85, Chapter 1].

Example. If $(B_t)_{t \geq 0}$ is a standard Brownian motion in \mathbb{R}^d , then the Itô stochastic integral

$$W(f) := \int_0^\infty f(t) dB_t, \quad f \in L^2(\mathbb{R}_+; \mathbb{R}^d),$$

defines an $L^2(\mathbb{R}_+; \mathbb{R}^d)$ -isonormal process W . In the converse direction, if W is an $L^2(\mathbb{R}_+; \mathbb{R}^d)$ -isonormal process, we let $(e_j)_{j=1}^d$ denote the standard unit basis of \mathbb{R}^d and note that

$$B_t^{(j)} := W(\mathbf{1}_{[0,t]} \otimes e_j), \quad t \geq 0,$$

defines a standard Brownian motion for each $1 \leq j \leq d$; these Brownian motions are independent and define the coordinates of a standard Brownian motion in \mathbb{R}^d .

Definition 2.2. An H -cylindrical Brownian motion is an $L^2(\mathbb{R}_+; H)$ -isonormal process.

Definition 2.3. A space-time white noise on a domain $D \subseteq \mathbb{R}^d$ is an $L^2(\mathbb{R}_+ \times D)$ -isonormal process.

Under the natural identification $L^2(\mathbb{R}_+ \times D) = L^2(\mathbb{R}_+; L^2(D))$, a space-time white noise may be identified with an $L^2(D)$ -cylindrical Brownian motion.

3. Radonifying operators

Let $H \otimes X$ denote the linear space of all finite rank operators from H to X . Every element in $H \otimes X$ can be represented in the form $\sum_{n=1}^N h_n \otimes x_n$, where $h_n \otimes x_n$ is the rank one operator mapping the vector $h \in H$ into $[h, h_n]x_n \in X$. By a Gram-Schmidt orthogonalisation argument we may assume that the vectors h_1, \dots, h_N are orthonormal in H .

Let $(\gamma_n)_{n \geq 1}$ be a Gaussian sequence, i.e., a sequence of independent real-valued standard Gaussian random variables.

Definition 3.1. The Banach space $\gamma(H, X)$ is defined as the completion of $H \otimes X$ with respect to the norm

$$\left(\left\| \sum_{n=1}^N h_n \otimes x_n \right\|_{\gamma(H, X)}^2 \right)^{1/2} := \left(\mathbb{E} \left\| \sum_{n=1}^N \gamma_n x_n \right\|^2 \right)^{1/2},$$

where it is assumed that h_1, \dots, h_N are orthonormal in H .

The quantity on the right-hand side is independent of the above representation as long as the vectors in H are taken to be orthonormal; this is an easy consequence of the fact that the distribution of a Gaussian vector in \mathbb{R}^N is invariant under orthogonal transformations. As a result, the norm $\|\cdot\|_{\gamma(H, X)}$ is well defined.

The celebrated Kahane-Khintchine inequality asserts that for all $0 < p, q < \infty$ there exists a constant $\kappa_{q,p} \geq 0$, depending only on p and q , such that

$$\left(\mathbb{E}\left\|\sum_{n=1}^N \gamma_n x_n\right\|^q\right)^{1/q} \leq \kappa_{q,p} \left(\mathbb{E}\left\|\sum_{n=1}^N \gamma_n x_n\right\|^p\right)^{1/p}. \quad (3.1)$$

Proofs can be found in [34, 59, 61]. It was shown in [60] that the optimal constant is given by $\kappa_{q,p} = \max\{\frac{\|\gamma_1\|_q}{\|\gamma_1\|_p}, 1\}$. In particular, $\kappa_{q,p} \leq C_p \sqrt{q}$ for $q \geq 1$.

It follows from (3.1) that for each $p \in [1, \infty)$ we obtain an equivalent norm on $\gamma(H, X)$ if we replace the exponent 2 by p in Definition 3.1. The resulting space will be indicated by $\gamma^p(H, X)$.

The identity mapping on $H \otimes X$ extends to an injective and contractive embedding of $\gamma(H, X)$ into $\mathcal{L}(H, X)$, the space of all bounded linear operators from H into X (for the simple proof see [76, Section 3]). We may thus identify $\gamma(H, X)$ with a linear subspace in $\mathcal{L}(H, X)$. Assuming this identification, we call a bounded operator $T \in \mathcal{L}(H, X)$ γ -radonifying if it belongs to $\gamma(H, X)$.

Example. If X is a Hilbert space, then we have an isometric isomorphism

$$\gamma(H, X) = \mathcal{L}_2(H, X),$$

where $\mathcal{L}_2(H, X)$ is the space of all Hilbert-Schmidt operators from H to X .

Example. For $1 \leq p < \infty$ we have an isometric isomorphism of Banach spaces

$$\gamma^p(H, L^p(\mu; X)) \simeq L^p(\mu; \gamma^p(H; X))$$

which is obtained by associating with $f \in L^p(\mu; \gamma(H; X))$ the mapping $h' \mapsto f(\cdot)h'$ from H to $L^p(\mu; X)$. The proof is an easy application of Fubini's theorem. In particular, upon identifying $\gamma^p(H, \mathbb{R})$ isomorphically with H , we obtain an isomorphism of Banach spaces

$$\gamma^p(H, L^p(\mu)) \simeq L^p(\mu; H). \quad (3.2)$$

4. Stochastic integration in martingale type 2 spaces

In this section we shall give a brief account of the construction of the Itô stochastic integral in martingale type 2 spaces. In order to bring out the analogy with the UMD approach more clearly we will first consider the simpler case of deterministic integrands, for which it suffices to assume that X has type 2.

4.1. Deterministic integrands

Let $(r_n)_{n \geq 1}$ be a Rademacher sequence, i.e., a sequence of independent random variables taking the values ± 1 with probability $\frac{1}{2}$.

Definition 4.1. Let $p \in [1, 2]$. A Banach space X has *type p* if there exists a constant $\tau \geq 0$ such that for all finite sequences $(x_n)_{n=1}^N$ in X we have

$$\mathbb{E} \left\| \sum_{n=1}^N r_n x_n \right\|^p \leq \tau^p \sum_{n=1}^N \|x_n\|^p.$$

The least admissible constant is denoted by $\tau_{p,X}$. In the next subsection we will give some examples of spaces with type p ; in fact these examples have the stronger property of martingale type p .

In the proof of the next proposition we shall use the following randomisation identity. If $(\xi_n)_{n \geq 1}$ is a sequence of independent symmetric random variables in $L^p(\Omega; X)$, and if $(\tilde{r}_n)_{n \geq 1}$ is an independent Rademacher sequence defined on another probability space $(\tilde{\Omega}, \tilde{\mathbb{P}})$, then for all $N \geq 1$ we have

$$\mathbb{E} \left\| \sum_{n=1}^N \xi_n \right\|^p = \mathbb{E} \tilde{\mathbb{E}} \left\| \sum_{n=1}^N \tilde{r}_n \xi_n \right\|^p. \quad (4.1)$$

This follows readily from Fubini's theorem, noting that for each $\tilde{\omega} \in \tilde{\Omega}$ the sequences $(\xi_n)_{n \geq 1}$ and $(\tilde{r}_n(\tilde{\omega})\xi_n)_{n \geq 1}$ are identically distributed.

Suppose now that W is an H -cylindrical Brownian motion (i.e. an $L^2(\mathbb{R}_+; H)$ -isonormal process). A function $\phi : \mathbb{R}_+ \rightarrow H \otimes X$ is called an *elementary function* if it is a linear combination of functions of the form $\mathbf{1}_{(s,t]} \otimes (h \otimes x)$ with $0 \leq s < t < \infty$, $h \in H$ and $x \in X$. The stochastic integral with respect to W of such a function is defined by putting

$$\int_0^\infty \mathbf{1}_{(s,t]} \otimes (h \otimes x) dW := W(\mathbf{1}_{(s,t]} \otimes h) \otimes x$$

and extending this definition by linearity.

Proposition 4.2. *Suppose that X has type 2 and let $\phi : \mathbb{R}_+ \rightarrow H \otimes X$ be elementary. Then*

$$\mathbb{E} \left\| \int_0^\infty \phi dW \right\|^2 \leq \tau_{2,X}^2 \int_0^\infty \|\phi(t)\|_{\gamma(H,X)}^2 dt.$$

Proof. We may write

$$\phi = \sum_{n=1}^N \mathbf{1}_{(t_{n-1}, t_n]} \otimes \sum_{j=1}^k h_j \otimes x_{jn}$$

for some fixed orthonormal system $(h_j)_{j=1}^k$ in X and suitable $0 \leq t_0 < \dots < t_N < \infty$ and $x_{jn} \in X$.

Since the functions $(\mathbf{1}_{(t_{n-1}, t_n)} \otimes h_j) / (t_n - t_{n-1})^{1/2}$ are orthonormal in $L^2(\mathbb{R}_+; H)$, their images under W , denoted by γ_{jn} , form a Gaussian sequence. Hence, by (4.1) and the type 2 property,

$$\mathbb{E} \left\| \int_0^\infty \phi dW \right\|^2 = \mathbb{E} \left\| \sum_{n=1}^N \sum_{j=1}^k \gamma_{jn} \otimes [(t_n - t_{n-1})^{1/2} x_{jn}] \right\|^2$$

$$\begin{aligned}
&= \mathbb{E} \tilde{\mathbb{E}} \left\| \sum_{n=1}^N \tilde{r}_n \sum_{j=1}^k \gamma_{jn} \otimes [(t_n - t_{n-1})^{1/2} x_{jn}] \right\|^2 \\
&\leq \tau_{2,X}^2 \sum_{n=1}^N \mathbb{E} \left\| \sum_{j=1}^k \gamma_{jn} \otimes [(t_n - t_{n-1})^{1/2} x_{jn}] \right\|^2 \\
&= \tau_{2,X}^2 \sum_{n=1}^N (t_n - t_{n-1}) \mathbb{E} \left\| \sum_{j=1}^k \gamma_{jn} \otimes x_{jn} \right\|^2 \\
&= \tau_{2,X}^2 \sum_{n=1}^N (t_n - t_{n-1}) \left\| \sum_{j=1}^k h_j \otimes x_{jn} \right\|_{\gamma(H,X)}^2 \\
&= \tau_{2,X}^2 \int_0^\infty \|\phi(t)\|_{\gamma(H,X)}^2 dt.
\end{aligned}$$

□

The following proposition shows that there is no hope of extending Proposition 4.2 beyond the type 2 case, even in the case $H = \mathbb{R}$ (in which case W can be identified with a standard Brownian motion B and $\gamma(H, X)$ with X).

As a preparation for the proof we recall the Kahane contraction principle, which asserts that if $(\xi_n)_{n=1}^N$ is a sequence of independent and symmetric random variables, then for all scalar sequences $(a_n)_{n=1}^N$ we have

$$\mathbb{E} \left\| \sum_{n=1}^N a_n \xi_n \right\|^p \leq \max_{1 \leq n \leq N} |a_n|^p \mathbb{E} \left\| \sum_{n=1}^N \xi_n \right\|^p$$

for all $1 \leq p < \infty$. Using this result together with the observation that $(\xi_n)_{n=1}^N$ and $(r_n |\xi_n|)_{n=1}^N$ are identically distributed we see that if $\inf_{1 \leq n \leq N} \mathbb{E} |\xi_n| \geq \delta$, then

$$\begin{aligned}
\tilde{\mathbb{E}} \left\| \sum_{n=1}^N \tilde{r}_n x_n \right\|^p &= \tilde{\mathbb{E}} \left\| \mathbb{E} \sum_{n=1}^N \frac{\tilde{r}_n |\xi_n|}{\mathbb{E} |\xi_n|} x_n \right\|^p \\
&\leq \tilde{\mathbb{E}} \mathbb{E} \left\| \sum_{n=1}^N \frac{\tilde{r}_n |\xi_n|}{\mathbb{E} |\xi_n|} x_n \right\|^p \leq \frac{1}{\delta^p} \mathbb{E} \left\| \sum_{n=1}^N \xi_n x_n \right\|^p.
\end{aligned} \tag{4.2}$$

In the case of standard Gaussian variables, note that

$$\mathbb{E} |\gamma| = \sqrt{2/\pi} \tag{4.3}$$

Proposition 4.3. *If there exists a constant $C \geq 0$ such that for all elementary functions $\phi : \mathbb{R}_+ \rightarrow X$ we have*

$$\mathbb{E} \left\| \int_0^\infty \phi dB \right\|^2 \leq C^2 \int_0^\infty \|\phi(t)\|^2 dt,$$

then X has type 2.

Proof. Fix $x_1, \dots, x_N \in X$ and consider the function $\phi = \sum_{n=1}^N \mathbf{1}_{(n-1, n]} \otimes x_n$. If a constant $C \geq 0$ with the above property exists, then, using that the increments $B_n - B_{n-1}$ are standard Gaussian and independent,

$$\begin{aligned} \mathbb{E} \left\| \sum_{n=1}^N \gamma_n x_n \right\|^2 &= \mathbb{E} \left\| \sum_{n=1}^N (B_n - B_{n-1}) x_n \right\|^2 \\ &= \mathbb{E} \left\| \int_0^\infty \phi \, dB \right\|^2 \leq C^2 \int_0^\infty \|\phi(t)\|^2 \, dt = C^2 \sum_{n=1}^N \|x_n\|^2. \end{aligned}$$

This proves that X has *Gaussian type 2*, with Gaussian type 2 constant $\tau_{2,X}^\gamma \leq C$. By (4.2) and (4.3), this implies that X has type 2 with $\tau_{2,X} \leq C\sqrt{\pi/2}$. \square

Further examples may be found in [95, 99].

4.2. Random integrands

If one tries to extend the above proof to the case of a random integrand, one sees that the type 2 property does not suffice. Indeed, the coupling between the integrand and W destroys the Gaussianity. However, the martingale structure is retained, and this can be exploited to make a variation of the argument work under a slightly stronger assumption on the Banach space X , viz. that it has martingale type 2.

Definition 4.4. Let $p \in [1, 2]$. A Banach space X has *martingale type p* if there exists a constant $\mu \geq 0$ such that for all finite X -valued martingale difference sequences $(d_n)_{n=1}^N$ we have

$$\mathbb{E} \left\| \sum_{n=1}^N d_n \right\|^p \leq \mu^p \sum_{n=1}^N \mathbb{E} \|d_n\|^p.$$

The least admissible constant in this definition is denoted by $\mu_{p,X}$.

Example. Here are some examples.

- Every Banach space has martingale type 1.
- Every Hilbert space has martingale type 2.
- Every $L^p(\mu)$ space, $1 \leq p < \infty$, has martingale type $p \wedge 2$.

Since every Gaussian sequence is a martingale difference sequence, we see immediately that every Banach space with martingale type p has type p , with constant $\tau_{p,X} \leq \mu_{p,X}$.

Suppose now that an H -cylindrical Brownian motion W is given. We shall denote by $(\mathcal{F}_t)_{t \geq 0}$ the filtration induced by W , i.e., \mathcal{F}_t is the σ -algebra generated by all random variables $W(f)$ with $f \in L^2(0, t; H)$. The following lemma is proved by a standard monotone class argument.

Lemma 4.5. *If the functions $f_1, \dots, f_k \in L^2(\mathbb{R}_+; H)$ have support in $[t, \infty)$, then $(W(f_1), \dots, W(f_k))$ is independent of \mathcal{F}_t .*

More generally, we could consider any filtration with the property stated in the lemma.

Let $\phi : \mathbb{R}_+ \times \Omega \rightarrow H \otimes X$ be an *adapted elementary process*. By this we mean that ϕ is a linear combination of processes of the form

$$\mathbf{1}_{(s,t] \times F} \otimes (h \otimes x)$$

with $0 \leq s < t$, $F \in \mathcal{F}_s$, $h \in H$, and $x \in X$. The stochastic integral of ϕ with respect to W is then defined by putting

$$\int_0^\infty \mathbf{1}_{(s,t] \times F}(h \otimes x) dW := \mathbf{1}_F W(\mathbf{1}_{(s,t]} \otimes h) \otimes x$$

and extending this definition by linearity.

Theorem 4.6. *Suppose that the Banach space X has martingale type 2 and let $\phi : \mathbb{R}_+ \times \Omega \rightarrow H \otimes X$ be an adapted elementary process. Then*

$$\mathbb{E} \left\| \int_0^\infty \phi dW \right\|^2 \leq \mu_{2,X}^2 \mathbb{E} \int_0^\infty \|\phi_t\|_{\gamma(H,X)}^2 dt.$$

Proof. By assumption we may represent ϕ as

$$\phi = \sum_{n=1}^N \mathbf{1}_{(t_{n-1}, t_n]} \sum_{m=1}^M \mathbf{1}_{F_{mn}} \otimes \sum_{j=1}^k h_j \otimes x_{jmn}. \quad (4.4)$$

Here, $(h_j)_{j=1}^k$ is an orthonormal system in H , for each $1 \leq n \leq N$ the sets F_{mn} , $1 \leq m \leq M$, are disjoint and belong to $\mathcal{F}_{t_{n-1}}$, and the vectors x_{jmn} are taken from X . Then

$$\int_0^\infty \phi dW = \sum_{n=1}^N \sum_{m=1}^M \sum_{j=1}^k \mathbf{1}_{F_{mn}} W(\mathbf{1}_{(t_{n-1}, t_n]} \otimes h_j) \otimes x_{jmn}.$$

As before, the images under W of the functions $(\mathbf{1}_{(t_{n-1}, t_n]} \otimes h_j) / (t_n - t_{n-1})^{1/2}$, which we denote by γ_{jn} , form a Gaussian sequence. The random variables

$$d_n := (t_n - t_{n-1})^{1/2} \sum_{m=1}^M \sum_{j=1}^k \mathbf{1}_{F_{mn}} \gamma_{jn} \otimes x_{jmn}$$

form a martingale difference sequence $(d_n)_{n=1}^N$ with respect to $(\mathcal{F}_{t_n})_{n=0}^N$. To see this, note that d_n is \mathcal{F}_{t_n} -measurable and $F_{nm} \in \mathcal{F}_{t_{n-1}}$, and therefore

$$\mathbb{E}(\mathbf{1}_{F_{mn}} \gamma_{jn} | \mathcal{F}_{t_{n-1}}) = \mathbf{1}_{F_{mn}} \mathbb{E} \gamma_{jn} = 0$$

since γ_{jn} is independent of $\mathcal{F}_{t_{n-1}}$. Using the martingale type 2 property of X , the lemma, and the disjointness of the sets F_{1n}, \dots, F_{Mn} , we may now estimate

$$\begin{aligned} \mathbb{E} \left\| \int_0^\infty \phi dW \right\|^2 &= \mathbb{E} \left\| \sum_{n=1}^N (t_n - t_{n-1})^{1/2} \sum_{m=1}^M \sum_{j=1}^k \mathbf{1}_{F_{mn}} \gamma_{jn} \otimes x_{jmn} \right\|^2 \\ &= \mathbb{E} \left\| \sum_{n=1}^N d_n \right\|^2 \leq \mu_{2,X}^2 \sum_{n=1}^N \mathbb{E} \|d_n\|^2 \end{aligned}$$

$$\begin{aligned}
&= \mu_{2,X}^2 \sum_{n=1}^N (t_n - t_{n-1}) \sum_{m=1}^M \mathbb{E} \mathbf{1}_{F_{mn}} \mathbb{E} \left\| \sum_{j=1}^k \gamma_{jn} x_{jmn} \right\|^2 \\
&= \mu_{2,X}^2 \sum_{n=1}^N (t_n - t_{n-1}) \sum_{m=1}^M \mathbb{E} \mathbf{1}_{F_{mn}} \left\| \sum_{j=1}^k h_j \otimes x_{jmn} \right\|_{\gamma(H,X)}^2 \\
&= \mu_{2,X}^2 \mathbb{E} \int_0^\infty \|\phi_t\|_{\gamma(H,X)}^2 dt.
\end{aligned}$$

□

By Doob's inequality, this improves to the maximal inequality

$$\mathbb{E} \sup_{t \geq 0} \left\| \int_0^t \phi dW \right\|^2 \leq 4\mu_{2,X}^2 \mathbb{E} \int_0^\infty \|\phi_t\|_{\gamma(H,X)}^2 dt.$$

From here, it is a routine density argument to extend the stochastic integral to arbitrary progressively measurable processes $\phi : \mathbb{R}_+ \times \Omega \rightarrow \gamma(H, X)$ that satisfy $\mathbb{E} \int_0^\infty \|\phi_t\|_{\gamma(H,X)}^2 dt < \infty$; the process $t \mapsto \int_0^t \phi dW$ is then a continuous martingale. Then, the usual stopping time techniques apply to extend the integral to progressively measurable processes satisfying $\int_0^\infty \|\phi_t\|_{\gamma(H,X)}^2 dt < \infty$ almost surely.

The following version of Burkholder's inequality holds:

Theorem 4.7. *Let X have martingale type 2. Then for any strongly measurable adapted process $\phi : \mathbb{R}_+ \times \Omega \rightarrow \gamma(H, X)$ and $0 < p < \infty$,*

$$\mathbb{E} \sup_{t \geq 0} \left\| \int_0^t \phi dW \right\|^p \leq C_{p,X}^p \|\phi\|_{L^p(\Omega; L^2(\mathbb{R}_+; \gamma(H,X)))}^p.$$

For $p \geq 2$ this result is due to Dettweiler [31] who gave a proof based on a martingale version of Rosenthal's inequality. A particularly simple proof, based on a good- λ inequality, was obtained by Ondreját [86]. Both proofs produce non-optimal constants $C_{p,X}$ as $p \rightarrow \infty$. A proof with the optimal constant

$$C_{p,X} \leq C_X \sqrt{p}, \quad p \geq 2,$$

was obtained by Seidler [97] using square function techniques in combination with a maximal inequality for discrete-time martingales due to Pinelis [89].

The drawback of the martingale type 2 theory is not so much the fact that the class of spaces to which it applies is rather limited (e.g., it applies to L^p -spaces only for $p \in [2, \infty)$) but rather the fact that the inequalities of Theorems 4.6 and 4.7 are not sharp. In applications to parabolic SPDE, this lack of sharpness prevents one from proving the sharp endpoint inequalities needed for maximal regularity of mild solutions. As we will outline next, the theory of stochastic integration in UMD spaces does produce the sharp estimates that are needed for this purpose.

5. Stochastic integration in UMD spaces

5.1. Deterministic integrands

Let X be an arbitrary Banach space and W be an H -cylindrical Brownian motion. For an elementary function $\phi : \mathbb{R}_+ \rightarrow H \otimes X$ we define the stochastic integral $\int_0^\infty \phi dW$ as before. The following proposition provides a *two-sided* estimate for the L^p -norms of this integral. As a preliminary observation we note that ϕ , being an elementary function, defines an element in the algebraic tensor product $L^2(\mathbb{R}_+) \otimes (H \otimes X)$. In view of the linear isomorphism of vector spaces

$$L^2(\mathbb{R}_+) \otimes (H \otimes X) \simeq (L^2(\mathbb{R}_+) \otimes H) \otimes X \quad (5.1)$$

we may view ϕ as an element of $(L^2(\mathbb{R}_+) \otimes H) \otimes X$. Identifying $L^2(\mathbb{R}_+) \otimes H$ with a dense subspace of $L^2(\mathbb{R}_+; H)$, we may view ϕ as an element in $\gamma(L^2(\mathbb{R}_+; H), X)$.

Proposition 5.1 (Itô isometry). *Let X be a Banach space and let $p \in [1, \infty)$. For all elementary functions $\phi : \mathbb{R}_+ \rightarrow H \otimes X$ we have*

$$\mathbb{E} \left\| \int_0^\infty \phi dW \right\|^p = \|\phi\|_{\gamma^p(L^2(\mathbb{R}_+; H), X)}^p.$$

Proof. Representing ϕ as in the proof of Proposition 4.2 and using the notations introduced there, we have

$$\begin{aligned} \mathbb{E} \left\| \int_0^\infty \phi dW \right\|^p &= \mathbb{E} \left\| \sum_{n=1}^N \sum_{j=1}^k \gamma_{jn} \otimes (t_n - t_{n-1})^{1/2} x_{jn} \right\|^p \\ &= \left\| \sum_{n=1}^N \sum_{j=1}^k f_{jn} \otimes (t_n - t_{n-1})^{1/2} x_{jn} \right\|_{\gamma^p(L^2(\mathbb{R}_+; H), X)}^p \\ &= \|\phi\|_{\gamma^p(L^2(\mathbb{R}_+; H), X)}^p, \end{aligned}$$

where we used that the functions $f_{jn} := (\mathbf{1}_{(t_{n-1}, t_n]} \otimes h_j) / (t_n - t_{n-1})^{1/2}$ are orthonormal in $L^2(\mathbb{R}_+; H)$ and satisfy $\sum_{n=1}^N \sum_{j=1}^k f_{jn} \otimes (t_n - t_{n-1})^{1/2} x_{jn} = \phi$. \square

By a density argument, the mapping $\phi \mapsto \int_0^\infty \phi dW$ extends to an isometry from $\gamma^p(L^2(\mathbb{R}_+; H), X)$ into $L^p(\Omega; X)$.

Combining the estimates of Propositions 4.2 and 5.1 under the assumption that X have type 2, we obtain the inequality¹

$$\|\phi\|_{\gamma(L^2(\mathbb{R}_+; H), X)} \leq \tau_{2, X} \|\phi\|_{L^2(\mathbb{R}_+; \gamma(H, X))}$$

for elementary functions ϕ . This implies that if X has type 2, then the natural identification made in (5.1) extends to a bounded inclusion

$$L^2(\mathbb{R}_+; \gamma(H, X)) \hookrightarrow \gamma(L^2(\mathbb{R}_+; H), X)$$

of norm at most $\tau_{2, X}$. For further results along this line we refer the reader to [46, 84, 95].

¹This corrects a misprint in the published version

5.2. UMD spaces

Next we show that it is possible to extend Proposition 5.1 to random integrands if X is a UMD space. We start with a brief introduction of this class of Banach spaces.

Definition 5.2. A Banach space X is called a *UMD space* if for some $p \in (1, \infty)$ (equivalently, for all $p \in (1, \infty)$) there is a constant $\beta \geq 0$ such that for all X -valued L^p -martingale difference sequences $(d_n)_{n \geq 1}$ and all signs $(\epsilon_n)_{n \geq 1}$ one has

$$\mathbb{E} \left\| \sum_{n=1}^N \epsilon_n d_n \right\|^p \leq \beta^p \mathbb{E} \left\| \sum_{n=1}^N d_n \right\|^p, \quad \forall N \geq 1. \quad (5.2)$$

The least admissible constant in this definition is called the *UMD $_p$ -constant* of X and is denoted by $\beta_{p,X}$. It is by no means obvious that once the UMD property holds for one $p \in (1, \infty)$, then it holds for all $p \in (1, \infty)$; this seems to have been first observed by Pisier, whose proof was outlined in [68]. A more systematic proof based on martingale decompositions can be found in [14] and the survey paper [16].

Example. Let us provide some examples of UMD spaces. Fix $p, p' \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{p'} = 1$.

- Every Hilbert space H is a UMD space (with $\tau_{p,H} = \max\{p, p'\}$).
- The spaces $L^p(\mu)$, $1 < p < \infty$, are UMD spaces (with $\beta_{p,L^p(\mu)} = \max\{p, p'\}$). More generally, if X is a UMD space, then $L^p(\mu; X)$, $1 < p < \infty$, is a UMD space (with $\beta_{p,L^p(\mu;X)} = \beta_{p,X}$).
- X is a UMD space if and only if X^* is a UMD space (with $\beta_{p,X} = \beta_{p',X^*}$).
- Every Banach space which is isomorphic to a closed subspace or a quotient of a UMD space is a UMD space.

By applying (5.2) to the martingale difference sequence $(\epsilon_n d_n)_{n \geq 1}$ one obtains the reverse estimate

$$\mathbb{E} \left\| \sum_{n=1}^N d_n \right\|^p \leq \beta_{p,X}^p \mathbb{E} \left\| \sum_{n=1}^N \epsilon_n d_n \right\|^p, \quad \forall N \geq 1. \quad (5.3)$$

If $(r_n)_{n \geq 1}$ is a Rademacher sequence which is independent of $(d_n)_{n \geq 1}$, then (5.2) and (5.3) easily imply the two-sided randomised inequality

$$\frac{1}{\beta_{p,X}^p} \mathbb{E} \left\| \sum_{n=1}^N d_n \right\|^p \leq \mathbb{E} \left\| \sum_{n=1}^N r_n d_n \right\|^p \leq \beta_{p,X}^p \mathbb{E} \left\| \sum_{n=1}^N d_n \right\|^p, \quad \forall N \geq 1, \quad (5.4)$$

In [38] the lower and upper estimates in (5.4) were studied for their own sake (see also Remark 5.9).

We include the simple observation that within the class of UMD spaces, the notions of type and martingale type are equivalent (see [8]).

Proposition 5.3. *Let $p \in [1, 2]$. If X is a UMD space with type p , then X has martingale type p and $\mu_{p,X} \leq \beta_{p,X} \tau_{p,X}$.*

Proof. Let $(\tilde{r}_n)_{n \geq 1}$ be a Rademacher sequence on another probability space $(\tilde{\Omega}, \tilde{\mathbb{P}})$. By (5.4) and Fubini's theorem,

$$\mathbb{E} \left\| \sum_{n=1}^N d_n \right\|^p \leq \beta_{p,X}^p \mathbb{E} \tilde{\mathbb{E}} \left\| \sum_{n=1}^N \tilde{r}_n d_n \right\|^p \leq \beta_{p,X}^p \tau_{p,X}^p \mathbb{E} \sum_{n=1}^N \|d_n\|^p.$$

□

5.3. Decoupling

The extension of Proposition 5.1 to adapted elementary processes will be achieved by means of a decoupling technique, which allows us to replace the cylindrical Brownian motion W by an independent copy \tilde{W} on a second probability space $\tilde{\Omega}$. With respect to \tilde{W} , we may estimate the L^p -norms path-by-path with respect to Ω . The UMD property will provide the relevant estimates for the decoupled integral in terms of the original integral and vice versa.

We begin with a decoupling inequality for martingale transforms due to McConnell [71]. The setting is as follows. We are given a probability space (Ω, \mathbb{P}) with filtration \mathcal{F} , and independent copies $(\tilde{\Omega}, \tilde{\mathbb{P}})$ and $\tilde{\mathcal{F}}$. We identify \mathcal{F} and $\tilde{\mathcal{F}}$ with the filtrations on $\Omega \times \tilde{\Omega}$ given by $\mathcal{F} \times \{\emptyset, \tilde{\Omega}\}$ and $\{\emptyset, \Omega\} \times \tilde{\mathcal{F}}$, respectively. In a similar way, random variables ξ and $\tilde{\xi}$ on Ω and $\tilde{\Omega}$ are identified with the random variables $\xi(\omega, \tilde{\omega}) := \xi(\omega)$ and $\tilde{\xi}(\omega, \tilde{\omega}) := \tilde{\xi}(\tilde{\omega})$ on $\Omega \times \tilde{\Omega}$, respectively.

Theorem 5.4. *Let X be a UMD space and let $p \in (1, \infty)$. Let $(\eta_n)_{n \geq 1}$ be an \mathcal{F} -adapted sequence of centered random variables in $L^p(\Omega)$ such that for each $n \geq 1$, η_n is independent of \mathcal{F}_{n-1} . Let $(\tilde{\eta}_n)_{n \geq 1}$ be an independent $\tilde{\mathcal{F}}$ -adapted copy of this sequence in $L^p(\tilde{\Omega}; X)$. Finally, let $(v_n)_{n \geq 1}$ be an \mathcal{F} -predictable sequence in $L^\infty(\Omega; X)$. Then, for all $N \geq 1$,*

$$\frac{1}{\beta_{p,X}^p} \mathbb{E} \tilde{\mathbb{E}} \left\| \sum_{n=1}^N v_n \tilde{\eta}_n \right\|^p \leq \mathbb{E} \tilde{\mathbb{E}} \left\| \sum_{n=1}^N v_n \eta_n \right\|^p \leq \beta_{p,X}^p \mathbb{E} \tilde{\mathbb{E}} \left\| \sum_{n=1}^N v_n \tilde{\eta}_n \right\|^p.$$

This decoupling inequality was further extended in [71] to more general martingale difference sequences.

Proof. The functions $\eta_n : \Omega \rightarrow X$ and $\tilde{\eta}_n : \tilde{\Omega} \rightarrow X$ will be interpreted as functions on $\Omega \times \tilde{\Omega}$ by considering $(\omega, \tilde{\omega}) \mapsto \eta_n(\omega)$ and $(\omega, \tilde{\omega}) \mapsto \tilde{\eta}_n(\tilde{\omega})$, respectively.

For $n = 1, \dots, N$ define

$$d_{2n-1} := \frac{1}{2} v_n (\eta_n + \tilde{\eta}_n) \quad \text{and} \quad d_{2n} := \frac{1}{2} v_n (\eta_n - \tilde{\eta}_n).$$

We claim that $(d_j)_{j=1}^{2N}$ is an L^p -martingale difference sequence with respect to the filtration $(\mathcal{G}_j)_{j=1}^{2N}$, where for $n \geq 1$,

$$\mathcal{G}_{2n-1} = \sigma(\mathcal{F}_{n-1} \times \tilde{\mathcal{F}}_{n-1}, \eta_n + \tilde{\eta}_n) \quad \text{and} \quad \mathcal{G}_{2n} = \mathcal{F}_n \times \tilde{\mathcal{F}}_n,$$

with $\mathcal{F}_n \times \widetilde{\mathcal{F}}_n$ denoting the product σ -algebra. Clearly, $(d_n)_{n=1}^{2N}$ is $(\mathcal{G}_n)_{n=1}^{2N}$ -adapted. For $n = 1, \dots, N$,

$$\mathbb{E}(d_{2n+1} | \mathcal{G}_{2n}) = \frac{1}{2} v_{n+1} \mathbb{E}(\eta_{n+1} + \widetilde{\eta}_{n+1} | \mathcal{G}_{2n}) = \frac{1}{2} v_{n+1} (\mathbb{E}\eta_{n+1} + \widetilde{\mathbb{E}}\widetilde{\eta}_{n+1}) = 0,$$

since η_{n+1} and $\widetilde{\eta}_{n+1}$ are independent of \mathcal{G}_{2n} and centered. For $n = 1, \dots, N$,

$$\mathbb{E}(d_{2n} | \mathcal{G}_{2n-1}) = \frac{1}{2} v_n \mathbb{E}(\eta_n - \widetilde{\eta}_n | \mathcal{G}_{2n-1}) \stackrel{(i)}{=} \frac{1}{2} v_n \mathbb{E}(\eta_n - \widetilde{\eta}_n | \eta_n + \widetilde{\eta}_n) \stackrel{(ii)}{=} 0.$$

Here (i) follows from the independence of $\sigma(\eta_n, \widetilde{\eta}_n)$ and $\mathcal{F}_{n-1} \times \widetilde{\mathcal{F}}_{n-1}$. For the identity (ii) let $B \subseteq X$ be a Borel set. Let ν and $\widetilde{\nu}$ denote the image measure of η_n and $\widetilde{\eta}_n$ on $\mathcal{B}(X)$, respectively. Then $\nu = \widetilde{\nu}$ and therefore

$$\begin{aligned} \mathbb{E}\widetilde{\mathbb{E}}\mathbf{1}_{\{\eta_n + \widetilde{\eta}_n \in B\}} \eta_n &= \int_X \int_X \mathbf{1}_{\{x+y \in B\}} x \, d\nu(x) d\nu(y) \\ &= \int_X \int_X \mathbf{1}_{\{x+y \in B\}} y \, d\nu(y) d\nu(x) = \mathbb{E}\widetilde{\mathbb{E}}\mathbf{1}_{\{\eta_n + \widetilde{\eta}_n \in B\}} \widetilde{\eta}_n, \end{aligned}$$

which gives (ii) and also finishes the proof of the claim.

Now since

$$\sum_{n=1}^N v_n \eta_n = \sum_{j=1}^{2N} d_j \quad \text{and} \quad \sum_{n=1}^N v_n \widetilde{\eta}_n = \sum_{j=1}^{2N} (-1)^{j+1} d_j,$$

the result follows from the UMD property applied to the sequences $(d_j)_{j=1}^{2N}$ and $((-1)^{j+1} d_j)_{j=1}^{2N}$. \square

5.4. Random integrands

We are now in a position to prove sharp estimates for the stochastic integrals of adapted elementary processes. Similar to what we did in the case of elementary functions, we will identify an adapted elementary process with an element of

$$(L^2(\mathbb{R}_+) \otimes L^p(\Omega)) \otimes (H \otimes X) \simeq L^p(\Omega) \otimes ((L^2(\mathbb{R}_+) \otimes H) \otimes X).$$

In the next theorem we identify the right-hand side with a dense subspace of $L^p(\Omega; \gamma(L^2(\mathbb{R}_+; H), X))$.

Theorem 5.5 (Itô isomorphism). *Let X be a UMD space and let $p \in (1, \infty)$. For all adapted elementary processes $\phi : \mathbb{R}_+ \times \Omega \rightarrow H \otimes X$ we have*

$$\begin{aligned} \frac{1}{\beta_{p,X}} \|\phi\|_{L^p(\Omega; \gamma^p(L^2(\mathbb{R}_+; H), X))} &\leq \left\| \int_0^\infty \phi \, dW \right\|_{L^p(\Omega; X)} \\ &\leq \beta_{p,X} \|\phi\|_{L^p(\Omega; \gamma^p(L^2(\mathbb{R}_+; H), X))}. \end{aligned}$$

Proof. Let \widetilde{W} be an H -cylindrical Brownian motion on a probability space $\widetilde{\Omega}$. As before we may view W and \widetilde{W} as independent H -cylindrical Brownian motions on $\Omega \times \widetilde{\Omega}$.

We may represent ϕ as in (4.4), i.e.,

$$\phi = \sum_{n=1}^N \mathbf{1}_{(t_{n-1}, t_n]} \sum_{m=1}^M \mathbf{1}_{F_{mn}} \otimes \sum_{j=1}^k h_j \otimes x_{jmn},$$

where $(h_j)_{j=1}^k$ is orthonormal in H , for each $1 \leq n \leq N$ the sets F_{mn} , $1 \leq m \leq M$, are disjoint and belong to $\mathcal{F}_{t_{n-1}}$, and the vectors x_{jmn} are taken from X . We view ϕ as being defined on $\Omega \times \tilde{\Omega}$.

Define, for $1 \leq j \leq k$ and $1 \leq n \leq N$,

$$\eta_{jn} := W(\mathbf{1}_{(t_{n-1}, t_n]} \otimes h_j), \quad \tilde{\eta}_{jn} := \tilde{W}(\mathbf{1}_{(t_{n-1}, t_n]} \otimes h_j),$$

and

$$v_{jn} := \sum_{m=1}^M \mathbf{1}_{F_{mn}} \otimes x_{jmn}.$$

With these notations,

$$\int_0^T \phi dW = \sum_{n=1}^N \sum_{j=1}^k v_{jn} \eta_{jn}, \quad \int_0^T \phi d\tilde{W} = \sum_{n=1}^N \sum_{j=1}^k v_{jn} \tilde{\eta}_{jn}.$$

We consider the filtration (\mathcal{F}_{jn}) , where

$$\mathcal{F}_{jn} = \sigma(\mathcal{F}_{t_{n-1}}, \eta_{1n}, \dots, \eta_{jn});$$

the indices (jn) are ordered lexicographically by the rule $(j', n') \leq (j, n) \iff n' < n$ or $[n' = n \ \& \ j' \leq j]$. By Theorem 5.4,

$$\begin{aligned} \frac{1}{\beta_{p,X}} \left\| \sum_{n=1}^N \sum_{j=1}^k v_{jn} \tilde{\eta}_{jn} \right\|_{L^p(\Omega \times \tilde{\Omega}; X)} \\ \leq \left\| \sum_{n=1}^N \sum_{j=1}^k v_{jn} \eta_{jn} \right\|_{L^p(\Omega; X)} \leq \beta_{p,X} \left\| \sum_{n=1}^N \sum_{j=1}^k v_{jn} \tilde{\eta}_{jn} \right\|_{L^p(\Omega \times \tilde{\Omega}; X)}. \end{aligned}$$

On the other hand, by Proposition 5.1, for each $\omega \in \Omega$ we have

$$\left\| \sum_{n=1}^N v_n(\omega) \tilde{\eta}_n \right\|_{L^p(\tilde{\Omega}; X)} = \|\phi(\omega)\|_{\gamma^p(L^2(\mathbb{R}_+; H), X)}.$$

Therefore, by Fubini's theorem,

$$\left\| \sum_{n=1}^N \sum_{j=1}^k v_{jn} \eta_{jn} \right\|_{L^p(\Omega \times \tilde{\Omega}; X)} = \|\phi\|_{L^p(\Omega; \gamma^p(L^2(\mathbb{R}_+; H), X))}.$$

□

By an application of Doob's inequality, for $1 < p < \infty$ we obtain the equivalence of norms

$$\begin{aligned} \frac{1}{\beta_{p,X}} \|\phi\|_{L^p(\Omega; \gamma^p(L^2(\mathbb{R}_+; H), X))} \\ \leq \left(\mathbb{E} \sup_{t \geq 0} \left\| \int_0^t \phi dW \right\|^p \right)^{1/p} \leq \frac{p}{p-1} \beta_{p,X} \|\phi\|_{L^p(\Omega; \gamma^p(L^2(\mathbb{R}_+; H), X))}. \end{aligned} \tag{5.5}$$

By a standard application of Lenglar's inequality [62], this equivalence extends to all exponents $0 < p < \infty$ with different constants (see Remark 5.7

below). This yields the UMD analogue of the Burkholder inequality of Theorem 4.7. It is interesting to observe that no additional argument is needed to pass from the case $p = 2$ to the case $1 < p < \infty$; the result for $1 < p < \infty$ is obtained right away from the decoupling inequalities.

By Theorem 5.5, the stochastic integral can be extended to the closure in $L^p(\Omega; \gamma(L^2(\mathbb{R}_+; H), X))$ of all adapted elementary processes. We shall denote this closure by $L^p_{\mathcal{F}}(\Omega; \gamma(L^2(\mathbb{R}_+; H), X))$. In this way the stochastic integral defines an isomorphic embedding

$$I : L^p_{\mathcal{F}}(\Omega; \gamma(L^2(\mathbb{R}_+; H), X)) \rightarrow L^p(\Omega, \mathcal{F}_\infty; X). \quad (5.6)$$

Moreover, by (5.5), the indefinite stochastic integral defines an isomorphic embedding of $L^p_{\mathcal{F}}(\Omega; \gamma(L^2(\mathbb{R}_+; H), X))$ into $L^p(\Omega; C_b(\mathbb{R}_+; X))$.

In the special case of the augmented filtration \mathcal{F}^W generated by W , the isomorphic embedding (5.6) is actually onto (pass to the limit $T \rightarrow \infty$ in the corresponding result for finite time intervals in [78, Theorem 3.5]) and we obtain an isomorphism of Banach spaces

$$I : L^p_{\mathcal{F}^W}(\Omega; \gamma(L^2(\mathbb{R}_+; H), X)) \simeq L^p(\Omega, \mathcal{F}_\infty^W; X).$$

This result contains a martingale representation theorem: every \mathcal{F}_∞^W -measurable random variable in $L^p(\Omega; X)$ is the stochastic integral of a suitable element of $L^p_{\mathcal{F}^W}(\Omega; \gamma(L^2(\mathbb{R}_+; H), X))$.

We continue with a description of $L^p_{\mathcal{F}}(\Omega; \gamma(L^2(\mathbb{R}_+; H), X))$ (the definition of which extends to $p \in [0, \infty)$ in the obvious way). A proof can be found in [78, Proposition 2.10].

Proposition 5.6. *Let $p \in [0, \infty)$. For an element $\phi \in L^p(\Omega; \gamma(L^2(\mathbb{R}_+; H), X))$ the following assertions are equivalent:*

- (1) $\phi \in L^p_{\mathcal{F}}(\Omega; \gamma(L^2(\mathbb{R}_+; H), X))$;
- (2) *the random variable $\langle \phi(\mathbf{1}_{[0,t]}f), x^* \rangle \in L^p(\Omega)$ is \mathcal{F}_t -measurable for all $t \in \mathbb{R}_+$, $f \in L^2(\mathbb{R}_+; H)$, and $x^* \in X^*$.*

In particular if $\phi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X)$ is H -strongly measurable and adapted, in the sense that for all $h \in H$ the X -valued process $\phi h : \mathbb{R}_+ \times \Omega \rightarrow X$ is strongly measurable and adapted, then $\phi \in L^p(\Omega; \gamma(L^2(\mathbb{R}_+; H), X))$ implies $\phi \in L^p_{\mathcal{F}}(\Omega; \gamma(L^2(\mathbb{R}_+; H), X))$. Indeed, in that case, for all $h \in H$ and $x^* \in X^*$ the process $[h, \phi^*x^*]$ is measurable and adapted. Since $\phi : \Omega \rightarrow \gamma(L^2(\mathbb{R}_+; H), X)$ is strongly measurable and elements in $\gamma(L^2(\mathbb{R}_+; H))$ are separably supported (see [76, Section 3]), we may assume that H is separable, and then the Pettis measurability theorem implies that the H -valued process ϕ^*x^* is strongly measurable and adapted. Passing to a progressively measurable version of ϕ^*x^* (see [87] for a short existence proof), we see that $\langle \phi(\mathbf{1}_{[0,t]}f), x^* \rangle$ is equal almost surely to a strongly \mathcal{F}_t -measurable random variable.

In the special case $X = L^q(\mu)$ with $1 < q < \infty$, combination of (5.5) with (3.2) gives the following two-sided inequality for a measurable and

adapted processes $\phi : \mathbb{R}_+ \times \Omega \rightarrow L^q(\mu; H)$: if $\phi \in L^p(\Omega; L^q(\mu; L^2(\mathbb{R}_+; H)))$ for some $0 < p < \infty$, then

$$\mathbb{E} \sup_{t \geq 0} \left\| \int_0^{\cdot} \phi dW \right\|_{L^q(\mu)}^p \underset{p,q}{\approx} \mathbb{E} \|\phi\|_{L^q(\mu; L^2(\mathbb{R}_+; H))}^p.$$

The next step in the construction of the UMD stochastic integral consists in a localisation argument. The process

$$\zeta := \int_0^{\cdot} \phi dW$$

is a continuous martingale, and by standard stopping time techniques (see [94, Lemma 4.6]) one proves the following inequalities, valid for all $\delta > 0$ and $\varepsilon > 0$:

$$\begin{aligned} \mathbb{P}(\|\zeta\|_{C_b(\mathbb{R}_+; X)} > \varepsilon) \\ \leq \varepsilon^{-p} C_{p,X}^p \mathbb{E}(\delta^p \wedge \|\phi\|_{\gamma^p(L^2(\mathbb{R}_+; H), X)}^p) + \mathbb{P}(\|\phi\|_{\gamma^p(L^2(\mathbb{R}_+; H), X)} \geq \delta), \end{aligned}$$

where $C_{p,X} = \frac{p}{p-1} \beta_{p,X}$, and

$$\begin{aligned} \mathbb{P}(\|\phi\|_{\gamma^p(L^2(\mathbb{R}_+; H), X)} > \varepsilon) \\ \leq \varepsilon^{-p} \beta_{p,X}^p \mathbb{E}(\delta^p \wedge \|\zeta\|_{C_b(\mathbb{R}_+; X)}^p) + \mathbb{P}(\|\zeta\|_{C_b(\mathbb{R}_+; X)} \geq \delta). \end{aligned}$$

A direct consequence is that the stochastic integral $I : \phi \mapsto \int_0^{\cdot} \phi dW$ uniquely extends to a continuous linear embedding

$$I : L_{\mathcal{F}}^0(\Omega; \gamma(L^2(\mathbb{R}_+; H), X)) \rightarrow L^0(\Omega; C_b(\mathbb{R}_+; X)).$$

For the details we refer to [78]. We call $I\phi$ the stochastic integral of ϕ with respect to W and write

$$\int_0^t \phi dW = I\phi(t), \quad t \geq 0, \quad \phi \in L_{\mathcal{F}}^0(\Omega; \gamma(L^2(\mathbb{R}_+; H), X)).$$

Remark 5.7. Fix $p \in (1, \infty)$ and $0 < q < p$. Taking $\varepsilon = \delta$ in the above estimates and integrating with respect to $d\varepsilon^q$ one obtains that (see [94, Proposition 4.7] for a similar argument)

$$\begin{aligned} \left(\frac{p-q}{\beta_{p,X} p}\right)^{1/q} \|\phi\|_{L^q(\Omega; \gamma^p(L^2(\mathbb{R}_+; H), X))} \\ \leq \left(\mathbb{E} \sup_{t \geq 0} \left\| \int_0^t \phi dW \right\|^q\right)^{1/q} \leq \left(\frac{p C_{p,X}}{p-q}\right)^{1/q} \|\phi\|_{L^q(\Omega; \gamma^p(L^2(\mathbb{R}_+; H), X))}. \end{aligned}$$

Up to this point we have set up the abstract stochastic integral by a density argument, starting from adapted elementary processes. The next result, taken from [88, Theorem 4.1], gives a criterion which enables one to decide whether a given operator-valued stochastic process belongs to the closure of the adapted elementary processes. Earlier versions of this result, as well as related characterisations, can be found in [77, 78].

Theorem 5.8. *Let X be a UMD Banach space. Let $\phi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X)$ be an H -strongly measurable adapted process such that $\phi^* x^* \in L^0(\Omega; L^2(\mathbb{R}_+; H))$ for all $x^* \in X^*$. Let $\zeta : \mathbb{R}_+ \times \Omega \rightarrow X$ be a process whose paths are almost surely bounded. If for all $x^* \in X^*$ almost surely, one has*

$$\int_0^t \phi^* x^* dW = \langle \zeta_t, x^* \rangle, \quad t \in \mathbb{R}_+,$$

then ϕ represents an element in $L^0_{\mathcal{F}}(\Omega; \gamma(L^2(\mathbb{R}_+; H), X))$, and almost surely one has

$$\int_0^t \phi dW = \zeta_t, \quad t \in \mathbb{R}_+.$$

Moreover, ζ is a local martingale with continuous paths almost surely.

This theorem is contrasted by the following example [88, Theorem 2.1].

Example. If X is an infinite-dimensional Hilbert space, then there exists a strongly measurable adapted process $\phi : (0, 1) \times \Omega \rightarrow X$ with the following properties:

- (i) for all $x \in X$, the real-valued process $[\phi, x]$ belongs to $L^0(\Omega; L^2(0, 1))$ and we have

$$\int_0^1 [\phi, x] dW = 0, \quad \text{almost surely;}$$

- (ii) $\|\phi\|_{L^2(0,1;X)} = \infty$ almost surely.

In particular, ϕ does not define an element of $L^0(\Omega; L^2(0, 1; X))$.

Concerning the necessity of the UMD condition we have the following result due to Garling [37]. Suppose that for a Banach space X and an exponent $p \in (1, \infty)$ the estimates of Theorem 5.5 hold for all adapted elementary processes $\phi : \mathbb{R}_+ \times \Omega \rightarrow X$ (we take $H = \mathbb{R}$):

$$\frac{1}{c_p} \|\phi\|_{L^p(\Omega; \gamma^p(L^2(\mathbb{R}_+), X))} \leq \left\| \int_0^\infty \phi dB \right\|_{L^p(\Omega; X)} \leq C_p \|\phi\|_{L^p(\Omega; \gamma^p(L^2(\mathbb{R}_+), X))}. \quad (5.7)$$

Then X is a UMD space, with constant $\beta_{p,X} \leq c_p C_p$. This result shows that the scope of Theorem 5.5 is naturally restricted to the class of UMD spaces.

Remark 5.9. In [20, 23, 38] the class of Banach spaces in which the right-hand side inequality of (5.7) holds for all adapted elementary processes ϕ is investigated. This class includes all UMD spaces, but also some non-UMD spaces such as the spaces $L^1(\mu)$. By extrapolation techniques from [39], (see [23, Remark 3.2]) this implies that for all $1 \leq p \leq q < \infty$,

$$\left\| \int_0^\cdot \phi dW \right\|_{L^q(\Omega; C_b(\mathbb{R}_+; X))} \leq C_{X,p,q} \|\phi\|_{L^q(\Omega; \gamma^p(L^2(\mathbb{R}_+; H), X))}.$$

This shows that an estimate with linear dependence in q holds.

Remark 5.10. In the case when X is a Hilbert space or $X = L^p(\mu)$ with $p \geq 1$, it is known that

$$\left\| \int_0^\cdot \phi dW \right\|_{L^p(\Omega; C_b(\mathbb{R}_+; X))} \leq C_{p,X} \|\phi\|_{L^p(\Omega; \gamma^p(L^2(\mathbb{R}_+; H), X))}$$

holds with a constant $C_{p,X} \leq C_X$. In particular,

$$\left\| \int_0^\cdot \phi dW \right\|_{L^p(\Omega; C_b(\mathbb{R}_+; X))} \leq C_{p,X} \|\phi\|_{L^p(\Omega; \gamma(L^2(\mathbb{R}_+; H), X))}$$

holds with a constant $C_{p,X} \leq C'_X \sqrt{p}$ (for Hilbert spaces this also follows from Seidler's result quoted earlier, and for L^p from Fubini's theorem). It would be interesting to know whether this remains true for arbitrary (UMD) Banach spaces X . This problem is open even in the case $X = L^q$ with $q \in (1, \infty) \setminus \{2, p\}$.

We continue with a version of Itô's lemma taken from [12]. Let X, Y, Z be Banach spaces and let $(h_n)_{n \geq 1}$ be an orthonormal basis of H . Let $R \in \gamma(H, X)$, $S \in \gamma(H, Y)$ and $T \in \mathcal{L}(X, \mathcal{L}(Y, Z))$ be given. It is not hard to show that the sum

$$\mathrm{tr}_{R,S} T := \sum_{n \geq 1} (TRh_n)(Sh_n)$$

converges in Z and does not depend on the choice of the orthonormal basis. Moreover,

$$\|\mathrm{tr}_{R,S} T\| \leq \|T\| \|R\|_{\gamma(H,X)} \|S\|_{\gamma(H,Y)}.$$

If $X = Y$ we shall write $\mathrm{tr}_R := \mathrm{tr}_{R,R}$.

Proposition 5.11 (Itô lemma). *Let X and Y be UMD spaces. Assume that $f : \mathbb{R}_+ \times X \rightarrow Y$ is of class $C^{1,2}$ on every bounded interval. Let $\phi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X)$ be H -strongly measurable and adapted and assume that ϕ locally defines an element of $L^0(\Omega; \gamma(L^2(\mathbb{R}_+; H), X)) \cap L^0(\Omega; L^2(\mathbb{R}_+; \gamma(H, X)))$. Let $\psi : \mathbb{R}_+ \times \Omega \rightarrow X$ be strongly measurable and adapted with locally integrable paths almost surely. Let $\xi : \Omega \rightarrow X$ be strongly \mathcal{F}_0 -measurable. Define $\zeta : \mathbb{R}_+ \times \Omega \rightarrow X$ by*

$$\zeta = \xi + \int_0^\cdot \psi_s ds + \int_0^\cdot \phi_s dW_s.$$

Then $s \mapsto D_2 f(s, \zeta_s) \phi_s$ is stochastically integrable and almost surely we have, for all $t \geq 0$,

$$\begin{aligned} f(t, \zeta_t) - f(0, \zeta_0) &= \int_0^t D_1 f(s, \zeta_s) ds + \int_0^t D_2 f(s, \zeta_s) \psi_s ds \\ &\quad + \int_0^t D_2 f(s, \zeta_s) \phi_s dW_s + \frac{1}{2} \int_0^t \mathrm{tr}_{\phi_s} (D_2^2 f(s, \zeta_s)) ds. \end{aligned}$$

The first two integrals and the last integral are almost surely defined as a Bochner integral.

As a special case, let X be a UMD space, let $X_1 = X$, $X_2 = X^*$, and set

$$(\zeta_i)_t = \xi_i + \int_0^t (\psi_i)_s ds + \int_0^t (\phi_i)_s dW_s, \quad i = 1, 2,$$

where $\phi_i : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X_i)$, $\psi_i : \mathbb{R}_+ \times \Omega \rightarrow X_i$ and $\xi_i : \Omega \rightarrow X_i$ satisfy the assumptions of Itô's lemma. Then, almost surely, for all $t \geq 0$ we have

$$\begin{aligned} \langle (\zeta_1)_t, (\zeta_2)_t \rangle - \langle (\zeta_1)_0, (\zeta_2)_0 \rangle &= \int_0^t \langle (\zeta_1)_s, (\psi_2)_s \rangle + \langle (\psi_1)_s, (\zeta_2)_s \rangle ds \\ &\quad + \int_0^t \langle (\zeta_1)_s, (\phi_2)_s \rangle + \langle (\phi_1)_s, (\zeta_2)_s \rangle dW_s \\ &\quad + \int_0^t \sum_{n \geq 1} \langle (\phi_1)_s h_n, (\phi_2)_s h_n \rangle ds. \end{aligned}$$

6. Malliavin calculus

The techniques of the previous section lend themselves very naturally to set up a Malliavin calculus in UMD Banach spaces.

Let \mathcal{H} be a Hilbert space and let $W : \mathcal{H} \rightarrow L^2(\Omega)$ be an isonormal Gaussian process (cf. Definition 2.1). The *Malliavin derivative* of an X -valued smooth random variable of the form

$$F = f(Wh_1, \dots, Wh_n) \otimes x$$

with $f \in C_b^\infty(\mathbb{R}^n)$, $h_1, \dots, h_n \in \mathcal{H}$ and $x \in X$, is the random variable $DF : \Omega \rightarrow \gamma(\mathcal{H}, X)$ defined by

$$DF = \sum_{j=1}^n \partial_j f(Wh_1, \dots, Wh_n) \otimes (h_j \otimes x).$$

Here, ∂_j denotes the j -th partial derivative. The definition extends by linearity. Thanks to the integration by parts formula

$$\mathbb{E}\langle DF(h), G \rangle = \mathbb{E}\langle Wh \langle F, G \rangle \rangle - \mathbb{E}\langle F, DG(h) \rangle,$$

valid for smooth random variables F and G with values in X and X^* , respectively, the operator D is closable as a densely defined linear operator from $L^p(\Omega; X)$ into $L^p(\Omega; \gamma(\mathcal{H}, X))$, $1 \leq p < \infty$ (see [64, Proposition 3.3]). The domain of its closure in $L^p(\Omega; X)$ is denoted by $\mathbb{D}^{1,p}(\Omega; X)$. This is a Banach space endowed with the norm

$$\|F\|_{\mathbb{D}^{1,p}(\Omega; X)} := (\|F\|_{L^p(\Omega; X)}^p + \|DF\|_{L^p(\Omega; \gamma(\mathcal{H}, X))}^p)^{1/p}.$$

Let $(H_m)_{m \geq 0}$ denote the Hermite polynomials, given by $H_0(x) = 1$, $H_1(x) = x$, and the recurrence relation $(m+1)H_{m+1}(x) = xH_m(x) - H_{m-1}(x)$. Let

$$\mathcal{H}_m = \overline{\text{lin}}\{H_m(Wh) : \|h\| = 1\}, \quad m \geq 0.$$

The Wiener-Itô decomposition theorem asserts that

$$L^2(\Omega, \mathcal{G}) = \bigoplus_{m \geq 0} \mathcal{H}_m,$$

where \mathcal{G} is the σ -algebra generated by W . Let P be the Ornstein-Uhlenbeck semigroup on $L^2(\Omega, \mathcal{G})$,

$$P(t) := \sum_{m \geq 0} e^{-mt} J_m,$$

where J_m is the orthogonal projection onto \mathcal{H}_m . The semigroup $P \otimes I_X$ extends to a strongly continuous semigroup of contractions on $L^2(\Omega, \mathcal{G}; X)$. Its generator will be denoted by L_X .

The following result is due Pisier [91].

Theorem 6.1 (Meyer inequalities). *Let X be a UMD space and let $1 < p < \infty$. Then*

$$D_p((-L_X)^{1/2}) = \mathbb{D}^{1,p}(\Omega; X)$$

and for all $F \in \mathbb{D}^{1,p}(\Omega; X)$ we have an equivalence of the homogeneous norms

$$\|DF\|_{L^p(\Omega; \gamma(\mathcal{H}, X))} \approx_{p,X} \|(L \otimes I_X)^{1/2} F\|_{L^p(\Omega; X)}.$$

An extension to higher order derivatives was obtained by Maas [63]. We refer the reader to this paper for more on history of vector-valued Malliavin calculus.

From now on we assume that X is a UMD space. Since UMD spaces are K -convex, trace duality establishes a canonical isomorphism

$$\gamma(\mathcal{H}, X^*) \approx (\gamma(\mathcal{H}, X))^*.$$

See [48, 92] for a proof. We apply this with X replaced by X^* and note that X , being a UMD space, is K -convex. Starting from the Malliavin derivative D on $L^{p'}(\Omega; X^*)$ with $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, we define the Skorohod integral δ as the adjoint of D ; thus, δ is a densely defined closed linear operator from $L^p(\Omega; \gamma(\mathcal{H}, X))$ into $L^p(\Omega; X)$, $1 < p < \infty$. The domain of its closure will be denoted by $D_p(\delta)$.

So far, \mathcal{H} has been an arbitrary Hilbert space. We now specialise to $\mathcal{H} = L^2(\mathbb{R}_+; H)$ and let $(\mathcal{F}_t)_{t \geq 0}$ be the filtration induced by W (see Section 4.2). The following result has been proved in [64]:

Theorem 6.2. *Let X be a UMD space and let $1 < p < \infty$ be given. The space $L^p_{\mathcal{F}}(\Omega; \gamma(L^2(\mathbb{R}_+; H), X))$ is contained in $D_p(\delta)$ and*

$$\delta(\phi) = \int_0^\infty \phi dW, \quad \phi \in L^p_{\mathcal{F}}(\Omega; \gamma(L^2(\mathbb{R}_+; H), X)).$$

Let \mathcal{F}^W denote the filtration generated by W and define step functions $f : \mathbb{R}_+ \rightarrow \gamma(H, L^p(\Omega; X))$ with bounded support,

$$(P_{\mathcal{F}^W} f)(t) := \mathbb{E}(f(t) | \mathcal{F}_t^W),$$

where $\mathbb{E}(\cdot | \mathcal{F}_t^W)$ is considered as a bounded operator on $\gamma(H, L^p(\Omega; X))$. It is shown in [64] that if $1 < p, q < \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, then the mapping $P_{\mathcal{F}^W}$

extends to a bounded operator on $\gamma(L^2(\mathbb{R}_+; H), L^p(\Omega; X))$. Moreover, as a bounded operator on $L^p(\Omega; \gamma(L^2(\mathbb{R}_+; H), X))$, $P_{\mathcal{F}^W}$ is a projection onto the closed subspace $L^p_{\mathcal{F}^W}(\Omega; \gamma(L^2(\mathbb{R}_+; H), X))$.

Theorem 6.3 (Clark-Ocone representation, [64]). *Let X be a UMD space. The operator $P_{\mathcal{F}^W} \circ D$ has a unique extension to a continuous operator from $L^1(\Omega, \mathcal{F}^W_\infty; X)$ to $L^0_{\mathcal{F}^W}(\Omega; \gamma(L^2(\mathbb{R}_+; H), X))$, and for all $F \in L^1(\Omega, \mathcal{F}^W_\infty; X)$ we have the representation*

$$F = \mathbb{E}(F) + I((P_{\mathcal{F}^W} \circ D)F),$$

where I is the stochastic integral with respect to W . Moreover, $(P_{\mathcal{F}^W} \circ D)F$ is the unique element $\phi \in L^0_{\mathcal{F}^W}(\Omega; \gamma(L^2(\mathbb{R}_+; H), X))$ satisfying $F = \mathbb{E}(F) + I(\phi)$.

The UMD Malliavin calculus has been pushed further in the recent paper [93], where in particular the authors obtained an Itô formula for the Skorohod integral.

7. Stochastic maximal L^p -regularity

Applications of the theory of stochastic integration in UMD spaces have been worked out in a number of papers; see [12, 13, 18, 21, 22, 24, 30, 57, 56, 58, 79, 80, 81, 93, 96] and the references therein. Here we will limit ourselves to the maximal regularity theorem for stochastic convolutions from [81] which is obtained by combining Theorem 7.1 and 7.3 below, and which crucially depends on the sharp two-sided inequality of Theorem 5.5.

As before we let (Ω, \mathbb{P}) be a probability space, let W be an H -cylindrical Brownian motion defined on it, and let the filtration \mathcal{F} be as before. For an operator A admitting a bounded H^∞ -calculus, we denote by $(S(t))_{t \geq 0}$ the bounded analytic semigroup generated by $-A$. For detailed treatments of the H^∞ -calculus we refer to [29, 41, 55].

The main result of [81] is formulated for L^q -spaces with $q \in [2, \infty)$, but inspection of the proof shows that can be restated for UMD spaces satisfying a certain hypothesis which will be explained in detail below.

Theorem 7.1. *Let $p \in [2, \infty)$ and let X be a UMD Banach space with type 2 which satisfies Hypothesis (H_p) . Suppose the operator A admits a bounded H^∞ -calculus of angle less than $\pi/2$ on X and let $(S(t))_{t \geq 0}$ denote the bounded analytic semigroup on X generated by $-A$. For all $G \in L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H, X))$ the stochastic convolution process*

$$U(t) = \int_0^t S(t-s)G_s dW_s, \quad t \geq 0, \quad (7.1)$$

is well defined in X , takes values in the fractional domain $D(A^{1/2})$ almost surely, and we have the stochastic maximal L^p -regularity estimate

$$\mathbb{E}\|A^{1/2}U\|_{L^p(\mathbb{R}_+; X)}^p \leq C^p \mathbb{E}\|G\|_{L^p(\mathbb{R}_+; \gamma(H, X))}^p \quad (7.2)$$

with a constant C independent of G . If, in addition to the above assumptions, we have $0 \in \rho(A)$, then

$$\mathbb{E}\|U\|_{BUC(\mathbb{R}_+; (L^q(\mathcal{O}), D(A))_{\frac{1}{2} - \frac{1}{p}, p})}^p \leq C^p \mathbb{E}\|G\|_{L^p(\mathbb{R}_+; L^q(\mathcal{O}; H))}^p. \quad (7.3)$$

In the special case of $X = L^q(\mathcal{O})$, where (\mathcal{O}, μ) is a σ -finite measure space and $q \in [2, \infty)$, Hypothesis (H_p) is fulfilled for all $p \in (2, \infty)$; the value $p = 2$ is allowed if $q = 2$ (see Theorem 7.3 below). In this special case, (7.2) is equivalent to the estimate

$$\mathbb{E}\|A^{1/2}U\|_{L^p(\mathbb{R}_+; L^q(\mathcal{O}))}^p \leq C^p \mathbb{E}\|G\|_{L^p(\mathbb{R}_+; L^q(\mathcal{O}; H))}^p.$$

The convolution process U defined by (7.1) is the mild solution of the abstract SPDE

$$dU(t) + AU(t) dt = G_t dW_t, \quad t \geq 0,$$

and therefore Theorem 7.1 can be interpreted as a maximal L^p -regularity result for such equations. As is well-known [8, 26, 51], stochastic maximal regularity estimates can be combined with fixed point arguments to obtain existence, uniqueness and regularity results for solutions to more general classes of nonlinear stochastic PDEs. For the setting considered here this has been worked out in detail in [80], where an application is included for Navier-Stokes equation with multiplicative gradient-type noise.

Theorem 7.1 generalises previous results due to Krylov [50, 51, 52, 53] who proved the estimate for second-order uniformly elliptic operators on $X = L^q(\mathbb{R}^d)$ with $2 \leq q \leq p$, where $D = \mathbb{R}^d$ or D is a smooth enough bounded domain in \mathbb{R}^d . Using PDE arguments, Krylov was able to prove his result for operators with coefficients which may be both time-dependent and random in an adapted and measurable way. These results were extended to half-spaces and bounded domains by Kim [49].

The proof of Theorem 7.1 for $X = L^q(\mathcal{O})$ in [81] consists of three main steps:

- (i) The H^∞ -calculus of A is used to obtain a reduction to an estimate for stochastic convolutions of scalar-valued kernels;
- (ii) This estimate is then proved using Hypothesis (H_p) .
- (iii) Hypothesis (H_p) is verified for $X = L^q(\mathcal{O})$.

In this section we shall present a proof of Theorem 7.1 which replaces (i) and (ii) by a simpler H^∞ -functional calculus argument.

Let us first turn to the precise formulation of Hypothesis (H_p) . Let \mathcal{K} be the set of all absolutely continuous functions $k : \mathbb{R}_+ \rightarrow \mathbb{R}$ such that $\lim_{t \rightarrow \infty} k(t) = 0$ and

$$\int_0^\infty t^{1/2} |k'(t)| dt \leq 1.$$

Fix $p \in [2, \infty)$ and let X be an arbitrary Banach space. For $k \in \mathcal{K}$ and adapted elementary processes $G : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X)$ we define the process

$I(k)G : \mathbb{R}_+ \times \Omega \rightarrow X$ by

$$(I(k)G)_t := \int_0^t k(t-s)G_s dW_s, \quad t \geq 0.$$

Since G is an adapted elementary process, the Itô isometry for scalar-valued processes shows that these stochastic integrals are well-defined for all $t \geq 0$; no condition on X is needed for this. If X has martingale type 2 (in particular, when X is UMD with type 2), then by Theorem 4.6 and Young's inequality it is easy to see that $I(k)$ extends to a bounded operator from $L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H, X))$ into $L^p(\mathbb{R}_+ \times \Omega; X)$ and that the family

$$\mathcal{I} := \{I(k) : k \in \mathcal{K}\}$$

is uniformly bounded. We will need that this family has the stronger property of being R -bounded.

A family \mathcal{S} of bounded linear operators from a Banach space X_1 into another Banach space X_2 is called R -bounded if there exists a constant $C \geq 0$ such that for all finite sequences $(x_n)_{n=1}^N$ in X_1 and $(T_n)_{n=1}^N$ in \mathcal{S} we have

$$\mathbb{E} \left\| \sum_{n=1}^N r_n T_n x_n \right\|^2 \leq C^2 \mathbb{E} \left\| \sum_{n=1}^N r_n x_n \right\|^2.$$

Every R -bounded family is uniformly bounded; the converse holds if (and only if, see [1]) X has cotype 2 and Y has type 2. In particular, the converse holds if X_1 and X_2 are Hilbert spaces. The notion of R -boundedness has been first studied systematically in [19]; for further results and historical remarks see [29, 55].

Now we are ready to formulate Hypothesis (H_p) :

(H_p) Each of the operators $I(k)$, $k \in \mathcal{K}$, extends to a bounded operator from $L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H, X))$ into $L^p(\mathbb{R}_+ \times \Omega; X)$, and the family

$$\mathcal{I} = \{I(k) : k \in \mathcal{K}\}$$

is R -bounded from $L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H, X))$ into $L^p(\mathbb{R}_+ \times \Omega; X)$.

One can show that if the operators $I(k)$ extend to a uniformly bounded family of bounded operators from $L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H, X))$ into $L^p(\mathbb{R}_+ \times \Omega; X)$, then $p \geq 2$ and X has type 2 (see [82]). If X satisfies (H_p) and Y is isomorphic to a closed subspace of X , then Y satisfies (H_p) as well.

Hypothesis (H_p) admits various equivalent formulations. We present one of them, implicit in [81]; for a systematic study we refer the reader to [82]. Let B be a real-valued Brownian motion.

Proposition 7.2. *Hypothesis (H_p) holds if and only if the family $\{I_t : t > 0\}$ of stochastic convolution operators defined by*

$$I_t g(s) := \int_0^t \frac{1}{\sqrt{t}} \mathbf{1}_{(0,t)}(s-r) g(r) dB_r, \quad s \geq 0,$$

is well defined and R -bounded from $L^p(\mathbb{R}_+; X)$ into $L^p(\mathbb{R}_+ \times \Omega; X)$.

Stated differently, in order to verify (H_p) it suffices to take $H = \mathbb{R}$ and to consider the kernels $\frac{1}{\sqrt{t}}\mathbf{1}_{(0,t)}$, $t > 0$.

The following theorem gives sufficient conditions for (H_p) in case $X = L^q(\mathcal{O})$.

Theorem 7.3. *Let X be isomorphic to a closed subspace of a space $L^q(\mathcal{O})$ with $q \in [2, \infty)$. Then (H_p) holds for all $p \in (2, \infty)$. The same result holds when $p = q = 2$.*

This is a non-trivial result which has been proved in [81] using the Fefferman–Stein maximal theorem; it is here that the full force of Theorem 5.5 is needed. By the above remarks, (H_p) also holds for Sobolev spaces $W^{\alpha,p}(\mathcal{O})$ as long as $p \in [2, \infty)$. It is an open problem to describe the class of Banach spaces X to which the result of Theorem 7.3 can be extended. A sufficient condition for Hypothesis (H_p) for any $p > 2$ is that X be a UMD Banach function space for which the norm can be written as $\|x\|_X = \| |x|^2 \|_F$, and F is another UMD Banach function space [82].

In order to set the stage for the proof of Theorem 7.1 we need to introduce some terminology. Let X be a Banach space and let $\Sigma_\sigma = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \sigma\}$ denote the open sector of angle σ about the positive real axis in the complex plane. Let A be a sectorial operator on X with a bounded $H^\infty(\Sigma_\sigma)$ -calculus. Following [47] and [55, Chapter 12], we denote by \mathcal{A} the sub-algebra of $\mathcal{L}(X)$ of all operators commuting with the resolvent $R(\lambda, A) = (\lambda - A)^{-1}$. For $\nu > \sigma$, the space of all bounded analytic functions $f : \Sigma_\nu \rightarrow \mathcal{A}$ with R -bounded range is denoted by $RH^\infty(\Sigma_\nu, \mathcal{A})$. By $RH_0^\infty(\Sigma_\nu, \mathcal{A})$ we denote the functions in $RH^\infty(\Sigma_\nu, \mathcal{A})$ whose operator norm is dominated by $|\lambda|^\varepsilon / (1 + |\lambda|)^{2\varepsilon}$ for some $\varepsilon > 0$. For such f we may define

$$f(A) = \frac{1}{2\pi i} \int_{\partial\Sigma_{\sigma'}} f(\lambda)R(\lambda, A) d\lambda$$

as an absolutely convergent Bochner integral in $\mathcal{L}(X)$ for $\sigma < \sigma' < \nu$. By [47, Theorem 4.4] (see also [55, Theorem 12.7]), the mapping $f \mapsto f(A)$ extends to a bounded algebra homomorphism from $RH^\infty(\Sigma_\nu, \mathcal{A})$ to $\mathcal{L}(X)$ which is unique in the sense that it has the following convergence property: if (f_n) is a bounded sequence in $RH^\infty(\Sigma_\nu, \mathcal{A})$ (in the sense that the corresponding R -bounds are uniformly bounded) and $f_n(\lambda)x \rightarrow f(\lambda)x$ for some $f \in RH^\infty(\Sigma_\nu, \mathcal{A})$ and all $\lambda \in \Sigma_\nu$ and $x \in X$, then $f_n(A)x \rightarrow f(A)x$ for all $x \in X$.

Proof of Theorem 7.1. Let X be a UMD Banach space with type 2 satisfying Hypothesis (H_p) for some fixed $p \in [2, \infty)$. For adapted elementary processes $G : \mathbb{R}_+ \times \Omega \rightarrow \gamma(H, X)$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > 0$ define

$$(L_\lambda G)_t := \int_0^t \lambda^{1/2} e^{-\lambda(t-s)} G_s dW_s, \quad t > 0.$$

The functions $k_\lambda(t) := \lambda^{1/2} e^{-\lambda t}$ are uniformly bounded in the norm of $L^2(\mathbb{R}_+)$ and therefore Young’s inequality and Theorem 4.7 show that the

operators L_λ are bounded from $L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H, X))$ to $L^p(\mathbb{R}_+ \times \Omega; X)$. Moreover, the substitution $t\operatorname{Re} \lambda = s$ gives, for $\lambda \in \Sigma_\nu$,

$$\int_0^\infty t^{1/2} |k'_\lambda(t)| dt \leq \frac{1}{\sqrt{\cos \nu}} \int_0^\infty s^{1/2} e^{-s} ds = \frac{1}{2} \sqrt{\frac{\pi}{\cos \nu}}.$$

This shows that the functions k_λ , $\lambda \in \Sigma_\nu$, belong to \mathcal{K} after scaling by a constant depending only on ν . Hence, by Hypothesis (H_p) , for any $0 \leq \nu < \frac{1}{2}\pi$ the family $\{L_\lambda : \lambda \in \Sigma_\nu\}$ is R -bounded from $L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H, X))$ to $L^p_{\mathcal{F}}(\mathbb{R} \times \Omega; X)$.

In order to view the operators L_λ as bounded operators on $\tilde{X} := L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H, X))$ we think of X as being embedded isometrically as a closed subspace of $\gamma(H, X)$ by identifying each $x \in X$ with the rank one operator $h_0 \otimes x$, where $h_0 \in H$ is an arbitrary but fixed unit vector. Using this identification, $L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; X)$ is isometric to a closed subspace of \tilde{X} and we may identify L_λ with a bounded operator \tilde{L}_λ on \tilde{X} ; the resulting family $\{\tilde{L}_\lambda : \lambda \in \Sigma_\nu\}$ is R -bounded on \tilde{X} .

Suppose A has a bounded $H^\infty(\Sigma_\sigma)$ -calculus on X for some $\sigma \in [0, \frac{1}{2}\pi)$. Let \tilde{A} denote the induced operator on $\tilde{X} = L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H, X))$, given by $(\tilde{A}G)_t := A(G_t)$ for $G \in L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H, \mathcal{D}(A)))$. It is routine to check that \tilde{A} has a bounded $H^\infty(\Sigma_\sigma)$ -calculus on \tilde{X} and

$$(\varphi(\tilde{A})G)_t = \varphi(A)(G_t).$$

Noting that the operators \tilde{L}_λ and $R(\lambda, \tilde{A})$ commute, the above-mentioned result from [47], applied to the function

$$f(\lambda) = \tilde{L}_\lambda, \quad \lambda \in \Sigma_\nu,$$

shows that the operator

$$G \mapsto f(\tilde{A})G = \int_{\partial\Sigma_{\sigma'}} R(\lambda, \tilde{A})\tilde{L}_\lambda G d\lambda$$

with $\sigma < \sigma' < \nu$, is well defined and bounded on \tilde{X} . It follows that the operator

$$G \mapsto f(\tilde{A})G = \int_{\partial\Sigma_{\sigma'}} R(\lambda, \tilde{A})L_\lambda G d\lambda$$

with $\sigma < \sigma' < \nu$, is well defined and bounded from \tilde{X} to $L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; X)$ (cf. [47, Theorem 4.5]). By the stochastic Fubini theorem, for adapted elementary processes $G : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, \mathcal{D}(A))$ we have, for all $t > 0$,

$$\begin{aligned} (f(\tilde{A})G)_t &= \int_{\partial\Sigma_{\sigma'}} R(\lambda, \tilde{A})L_\lambda G_t d\lambda \\ &= \int_{\partial\Sigma_{\sigma'}} \int_0^t \lambda^{1/2} e^{-\lambda(t-s)} R(\lambda, A)G_s dW_s d\lambda \\ &= \int_0^t \int_{\partial\Sigma_{\sigma'}} \lambda^{1/2} e^{-\lambda(t-s)} R(\lambda, A)G_s d\lambda dW_s \end{aligned}$$

$$= \int_0^t A^{1/2} e^{-(t-s)A} G_s dW_s.$$

Putting the results together we obtain

$$\begin{aligned} \left\| t \mapsto \int_0^t A^{1/2} S(t-s) G_s dW_s \right\|_{L^p(\mathbb{R}_+ \times \Omega; X)} &= \|f(\tilde{A})G\|_{L^p(\mathbb{R}_+ \times \Omega; X)} \\ &\leq C^p \|G\|_{L^p(\mathbb{R}_+ \times \Omega; \gamma(H, X))} \end{aligned}$$

This proves Theorem 7.1. \square

Next we deduce a variant of Theorem 7.1 for processes with mixed integrability assumptions. Its proof is a straightforward application of the two-sided estimates for stochastic integrals in UMD spaces.

Corollary 7.4. *Let the assumptions of Theorem 7.1 be satisfied, and let $G \in L^r_{\mathcal{F}}(\Omega; L^p(\mathbb{R}_+; \gamma(H, X)))$ with $r \in (0, \infty)$ be given. If U is defined as in (7.1), then*

$$\mathbb{E} \|A^{1/2} U\|_{L^p(\mathbb{R}_+; X)}^r \leq C^r \mathbb{E} \|G\|_{L^p(\mathbb{R}_+; \gamma(H, X))}^r \quad (7.4)$$

with a constant C independent of G .

Proof. By Proposition 5.1 and Theorem 7.1, applied to deterministic functions $G \in L^p(\mathbb{R}_+; \gamma(H, D(A)))$, we have

$$\|s \mapsto A^{1/2} S(t-s) \mathbf{1}_{[0,t](s)} G_s\|_{\gamma(L^2(\mathbb{R}_+; H), L^p(\mathbb{R}_+; X))} \leq C \|G\|_{L^p(\mathbb{R}_+; \gamma(H, D(A)))}. \quad (7.5)$$

Next let $G \in L^r_{\mathcal{F}}(\Omega; L^p(\mathbb{R}_+; \gamma(H, D(A))))$. By Theorem 5.5 (or rather, by its extension to the closure of the elementary adapted processes, cf. (5.6)) applied to the UMD space $L^p(\mathbb{R}_+; X)$ we obtain

$$\begin{aligned} &\|A^{1/2} U\|_{L^r(\Omega; L^p(\mathbb{R}_+; X))} \\ &\quad \approx \|s \mapsto A^{1/2} S(t-s) \mathbf{1}_{[0,t](s)} G_s\|_{L^r(\Omega; \gamma(L^2(\mathbb{R}_+; H), L^p(\mathbb{R}_+; X)))}. \end{aligned}$$

Now (7.4) follows by applying the estimate (7.5) pointwise in Ω . \square

Remark 7.5. A variation of the notion of stochastic maximal L^p -regularity, in which the $L^p(\mathbb{R}_+; X)$ -norm over the time variable is replaced by the $\gamma(L^2(\mathbb{R}_+), X)$ -norm, has been studied in [75]. With this change, a stochastic maximal L^p -regularity result holds for arbitrary UMD Banach spaces with Pisier's property (α) and all exponents $0 < p < \infty$. In this situation the trace inequality (7.3) holds with $(X, D(A))_{\frac{1}{2} - \frac{1}{p}, p}$ replaced by X .

8. Poisson stochastic integration

Up to this point we have been exclusively concerned with the Gaussian case. Here we shall briefly address the problem of extending Theorem 5.5 to more general classes of integrators. More specifically, with an eye towards the Lévy case, a natural question is whether similar two-sided estimates as in Theorem 5.5 can be given in the Poissonian case. This question has been addressed

recently by Dirksen [35], who was able to work out the correct norms in the special case $X = L^q(\mathcal{O})$.

We begin by recalling some standard definitions. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let (E, \mathcal{E}) be a measurable space. We write $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$.

Definition 8.1. A *random measure* is a mapping $N : \Omega \times \mathcal{E} \rightarrow \overline{\mathbb{N}}$ with the following properties:

- (i) For all $B \in \mathcal{E}$ the mapping $N(B) : \omega \mapsto N(\omega, B)$ is measurable;
- (ii) For all $\omega \in \Omega$, the mapping $B \mapsto N(\omega, B)$ is a measure.

The measure $\mu(B) := \mathbb{E}N(B)$ is called the *intensity measure* of N .

Definition 8.2. A random measure $N : \Omega \times \mathcal{E} \rightarrow \overline{\mathbb{N}}$ with intensity μ is called a *Poisson random measure* if the following conditions are satisfied:

- (iii) For all pairwise disjoint sets B_1, \dots, B_n in \mathcal{E} the random variables $N(B_1), \dots, N(B_n)$ are independent;
- (iv) For all $B \in \mathcal{E}$ with $\mu(B) < \infty$ the random variable $N(B)$ is Poisson distributed with parameter $\mu(B)$.

Recall that a random variable $f : \Omega \rightarrow \mathbb{N}$ is *Poisson distributed* with parameter $\lambda > 0$ if

$$\mathbb{P}(f = n) = \frac{\lambda^n}{n!} e^{-\lambda}, \quad n \in \mathbb{N}.$$

For $B \in \mathcal{E}$ with $\mu(B) < \infty$ we write

$$\tilde{N}(B) := N(B) - \mu(B).$$

It is customary to call \tilde{N} the *compensated Poisson random measure* associated with N (even it is not a random measure in the sense of Definition 8.1, as it is defined on the sets of finite μ -measure only).

Let (J, \mathcal{J}, ν) be a σ -finite measure space and let N be a Poisson random measure on $(\mathbb{R}_+ \times J, \mathcal{B}(\mathbb{R}_+) \times \mathcal{J}, dt \times \nu)$. Throughout this section we let \mathcal{F} be the filtration generated by the random variables $\{\tilde{N}((s, u] \times A) : 0 \leq s < u \leq t, A \in \mathcal{J}\}$.

An *adapted elementary process* $\phi : \Omega \times \mathbb{R}_+ \times J \rightarrow X$ is a linear combination of processes of the form $\phi = \mathbf{1}_F \mathbf{1}_{(s, t] \times A} \otimes x$, with $0 \leq s < t < \infty$, $A \in \mathcal{J}$ satisfying $\nu(A_j) < \infty$, $F \in \mathcal{F}_s$, and $x \in X$. For an adapted elementary process ϕ and a set $B \in \mathcal{J}$ we define the (*compensated*) *Poisson stochastic integral* by

$$\int_{\mathbb{R}_+ \times B} \mathbf{1}_F \mathbf{1}_{(s, t] \times A} \otimes x d\tilde{N} := \mathbf{1}_F \tilde{N}((s, t] \times (A \cap B)) \otimes x$$

and extend this definition by linearity.

The next two theorems, taken from [36], give an upper and lower bound for the Poisson stochastic integral of an elementary adapted process in the presence of non-trivial martingale type and finite martingale cotype, respectively. Theorem 8.3 may be regarded as a Poisson analogue of Theorem 4.6.

We write

$$D_{s,X}^p := L^p(\Omega; L^s(\mathbb{R}_+ \times J; X)).$$

Theorem 8.3. *Let ϕ be an elementary adapted process with values in a Banach space X with martingale type $s \in (1, 2]$.*

(1) *If $1 < s \leq p < \infty$ we have, for all $B \in \mathcal{J}$,*

$$\left(\mathbb{E} \sup_{t \geq 0} \left\| \int_{[0,t] \times B} \phi \, d\tilde{N} \right\|^p \right)^{1/p} \lesssim_{p,s,X} \|\mathbf{1}_B \phi\|_{D_{s,X}^p \cap D_{p,X}^p}.$$

(2) *If $1 \leq p < s$ we have, for all $B \in \mathcal{J}$,*

$$\left(\mathbb{E} \sup_{t \geq 0} \left\| \int_{[0,t] \times B} \phi \, d\tilde{N} \right\|^p \right)^{1/p} \lesssim_{p,s,X} \|\mathbf{1}_B \phi\|_{D_{s,X}^p + D_{p,X}^p}.$$

Theorem 8.3 extends several known vector-valued inequalities in the literature. In the special case where $X = \mathbb{R}^n$ and $2 \leq p < \infty$, the estimate (1) was obtained in [54, p. 335, Corollary 2.12] by a completely different argument based on Itô's formula. An estimate for Hilbert spaces X and $2 \leq p < \infty$ was obtained in [66, Lemma 3.1]. The estimate (1) is slightly stronger in this case. In [67, Lemma 4], a slightly weaker inequality than (1) was obtained in the special case $X = L^s(\mu)$ and $p = s \geq 2$. This result was deduced from the corresponding scalar-valued inequality via Fubini's theorem. Finally, in [43], the inequality (1) was obtained in the special case when $p = s^n$ for some integer $n \geq 1$. Using a different approach, Theorem 8.3 has been obtained independently by Zhu [100].

The following 'dual' version of Theorem 8.3 holds for Banach spaces with martingale cotype.

Theorem 8.4. *Let ϕ be an elementary adapted process with values in a Banach space X with martingale cotype $s \in [2, \infty)$.*

(1) *If $s \leq p < \infty$ we have, for all $B \in \mathcal{J}$ and $t \geq 0$,*

$$\|\mathbf{1}_{[0,t] \times B} \phi\|_{D_{s,X}^p \cap D_{p,X}^p} \lesssim_{p,s,X} \left(\mathbb{E} \left\| \int_{[0,t] \times B} \phi \, d\tilde{N} \right\|^p \right)^{1/p}.$$

(2) *If $1 < p < s$ we have, for all $B \in \mathcal{J}$ and $t \geq 0$,*

$$\|\mathbf{1}_{[0,t] \times B} \phi\|_{D_{s,X}^p + D_{p,X}^p} \lesssim_{p,s,X} \left(\mathbb{E} \left\| \int_{[0,t] \times B} \phi \, d\tilde{N} \right\|^p \right)^{1/p}.$$

For Hilbert spaces X , Theorems 8.3 and 8.4 combine to yield two-sided estimates for the L^p -norm of the stochastic integral with respect to a compensated Poisson random measure.

Corollary 8.5. *Let H be a Hilbert space and let ϕ be an elementary adapted H -valued process.*

(1) *If $2 \leq p < \infty$, then for all $B \in \mathcal{J}$ we have*

$$\left(\mathbb{E} \sup_{t \geq 0} \left\| \int_{[0,t] \times B} \phi \, d\tilde{N} \right\|^p \right)^{1/p} \simeq_p \|\mathbf{1}_B \phi\|_{D_{s,H}^p \cap D_{p,H}^p}.$$

(2) If $1 < p < 2$, then for all $B \in \mathcal{J}$ we have

$$\left(\mathbb{E} \sup_{t \geq 0} \left\| \int_{[0,t] \times B} \phi \, d\tilde{N} \right\|^p \right)^{1/p} \simeq_p \| \mathbf{1}_B \phi \|_{D_{s,H}^p + D_{p,H}^p}.$$

For the spaces $X = L^q(\mathcal{O})$, where $(\mathcal{O}, \Sigma, \mu)$ is an arbitrary measure space, sharp two-sided bounds for the Poisson stochastic integral can be proved. This result, due Dirksen [35], may be regarded as the Poisson analogue of Theorem 5.5 for $X = L^q(\mathcal{O})$. An alternative proof has been obtained subsequently by Marinelli [65]. We write

$$\begin{aligned} S_q^p &:= L^p(\Omega; L^q(\mathcal{O}; L^2(\mathbb{R}_+ \times J))), \\ D_{s,q}^p &:= L^p(\Omega; L^s(\mathbb{R}_+ \times J; L^q(\mathcal{O}))). \end{aligned}$$

Theorem 8.6. *Let $1 < p, q < \infty$. For any $B \in \mathcal{J}$ and for any elementary adapted $L^q(\mathcal{O})$ -valued process ϕ ,*

$$\left(\mathbb{E} \sup_{t \geq 0} \left\| \int_{[0,t] \times B} \phi \, d\tilde{N} \right\|_{L^q(\mathcal{O})}^p \right)^{1/p} \simeq_{p,q} \| \mathbf{1}_B \phi \|_{I_{p,q}}, \quad (8.1)$$

where $I_{p,q}$ is given by

$$\begin{aligned} S_q^p \cap D_{q,q}^p \cap D_{p,q}^p &\text{ if } 2 \leq q \leq p; \\ S_q^p \cap (D_{q,q}^p + D_{p,q}^p) &\text{ if } 2 \leq p \leq q; \\ (S_q^p \cap D_{q,q}^p) + D_{p,q}^p &\text{ if } p \leq 2 \leq q; \\ (S_q^p + D_{q,q}^p) \cap D_{p,q}^p &\text{ if } q \leq 2 \leq p; \\ S_q^p + (D_{q,q}^p \cap D_{p,q}^p) &\text{ if } q \leq p \leq 2; \\ S_q^p + D_{q,q}^p + D_{p,q}^p &\text{ if } p \leq q \leq 2. \end{aligned}$$

It is also shown that the estimate $\lesssim_{p,q}$ in (8.1) remains valid if $q = 1$. A non-commutative version of Theorem 8.6 in a more general abstract setting can be found in [35, Section 7].

In contrast to the Gaussian case, where one expression for the norm suffices for all $1 < p, q < \infty$, in the Poisson case 6 different expressions are obtained depending on the mutual positions of the numbers p , q , and 2. This also suggests that the problem of determining sharp two-sided bounds for elementary adapted processes with values in a general UMD space X seems to be a very challenging one.

Noting that $X = L^q(\mathcal{O})$ has martingale type $q \wedge 2$ and martingale cotype $q \vee 2$, Theorems 8.3 and 8.4 are applicable as well; for $q \neq 2$ the bound obtained from these theorems are weaker than the ones obtained from Theorem 8.6.

Acknowledgment

We thank Markus Antoni for carefully reading an earlier draft of this paper.

References

- [1] W. Arendt and S. Bu. The operator-valued Marcinkiewicz multiplier theorem and maximal regularity. *Math. Z.*, 240(2):311–343, 2002.
- [2] Ya.I. Belopol'skaya and Yu.L. Daletskiĭ. Itô equations and differential geometry. *Uspekhi Mat. Nauk*, 37(3(225)):95–142, 224, 1982.
- [3] Ya.I. Belopol'skaya and Yu.L. Daletskiĭ. *Stochastic equations and differential geometry*, volume 30 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1990.
- [4] V.I. Bogachev. *Gaussian measures*, volume 62 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1998.
- [5] J. Bourgain. Some remarks on Banach spaces in which martingale difference sequences are unconditional. *Ark. Mat.*, 21(2):163–168, 1983.
- [6] J. Bourgain. Vector-valued singular integrals and the H^1 -BMO duality. In *Probability theory and harmonic analysis (Cleveland, Ohio, 1983)*, volume 98 of *Monogr. Textbooks Pure Appl. Math.*, pages 1–19. Dekker, New York, 1986.
- [7] J.K. Brooks and N. Dinculeanu. *Stochastic integration in Banach spaces*, volume 81 of *Advances in Mathematics*. Academic Press, 1990.
- [8] Z. Brzeźniak. Stochastic partial differential equations in M-type 2 Banach spaces. *Potential Anal.*, 4(1):1–45, 1995.
- [9] Z. Brzeźniak. On stochastic convolution in Banach spaces and applications. *Stochastics Stochastics Rep.*, 61(3-4):245–295, 1997.
- [10] Z. Brzeźniak. Some remarks on Itô and Stratonovich integration in 2-smooth Banach spaces. In *Probabilistic methods in fluids*, pages 48–69. World Sci. Publishing, River Edge, NJ, 2003.
- [11] Z. Brzeźniak and A. Millet. On the stochastic Strichartz estimates and the stochastic nonlinear Schrödinger equation on a compact Riemannian manifold. *Potential Analysis*, 2013. Online First.
- [12] Z. Brzeźniak, J.M.A.M. van Neerven, M.C. Veraar, and L.W. Weis. Itô's formula in UMD Banach spaces and regularity of solutions of the Zakai equation. *J. Differential Equations*, 245(1):30–58, 2008.
- [13] Z. Brzeźniak and M.C. Veraar. Is the stochastic parabolicity condition dependent on p and q ? *Electron. J. Probab.*, 17(56):1–24, 2012.
- [14] D.L. Burkholder. A geometrical characterization of Banach spaces in which martingale difference sequences are unconditional. *Ann. Probab.*, 9(6):997–1011, 1981.
- [15] D.L. Burkholder. Martingales and Fourier analysis in Banach spaces. In *Probability and analysis (Varenna, 1985)*, volume 1206 of *Lecture Notes in Math.*, pages 61–108. Springer, Berlin, 1986.
- [16] D.L. Burkholder. Martingales and singular integrals in Banach spaces. In *Handbook of the geometry of Banach spaces, Vol. I*, pages 233–269. North-Holland, Amsterdam, 2001.
- [17] A. Chojnowska-Michalik. Stochastic differential equations in Hilbert spaces. In *Probability theory (Papers, VIIth Semester, Stefan Banach Internat. Math. Center, Warsaw, 1976)*, volume 5 of *Banach Center Publ.*, pages 53–74. PWN, Warsaw, 1979.

- [18] P.A. Cioica, K.-H. Kim, K. Lee, and F. Lindner. On the $l^q(l^p)$ -regularity and Besov smoothness of stochastic parabolic equations on bounded Lipschitz domains. *Electron. J. Probab.*, 18(82):1–41, 2013.
- [19] Ph. Clément, B. de Pagter, F.A. Sukochev, and H. Witvliet. Schauder decompositions and multiplier theorems. *Studia Math.*, 138(2):135–163, 2000.
- [20] S.G. Cox and S. Geiss. On relations between decoupling inequalities in Banach spaces. in preparation.
- [21] S.G. Cox and M. Górajski. Vector-valued stochastic delay equations – a semi-group approach. *Semigroup Forum*, 82(3):389–411, 2011.
- [22] S.G. Cox and J.M.A.M. van Neerven. Pathwise Hölder convergence of the implicit Euler scheme for semi-linear SPDEs with multiplicative noise. *Numer. Math.*, 125(2):259–345, 2013.
- [23] S.G. Cox and M.C. Veraar. Vector-valued decoupling and the Burkholder-Davis-Gundy inequality. *Illinois J. Math.*, 55(1):343–375, 2011.
- [24] P. Crewe. Infinitely delayed stochastic evolution equations in UMD Banach spaces. *J. Math. Anal. Appl.*, 415(1):325–345, 2014.
- [25] R.F. Curtain. *Stochastic differential equations in a Hilbert space*. PhD thesis, 1969. Thesis (Ph.D.)—Brown University.
- [26] G. Da Prato and J. Zabczyk. *Stochastic equations in infinite dimensions*, volume 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1992.
- [27] G. Da Prato and J. Zabczyk. *Ergodicity for Infinite-Dimensional Systems*, volume 229 of *LMS Lecture Note Series*. London Mathematical Society, 1996.
- [28] Yu.L. Daletskiĭ. Stochastic differential geometry. *Akad. Nauk Ukrain. SSR Inst. Mat. Preprint*, (46):48, 1982.
- [29] R. Denk, M. Hieber, and J. Prüss. R -boundedness, Fourier multipliers and problems of elliptic and parabolic type. *Mem. Amer. Math. Soc.*, 166(788), 2003.
- [30] G. Desch and S.-O. Londen. Maximal regularity for stochastic integral equations. *Journal of Applied Analysis*, 19(1):125–140, 2013.
- [31] E. Dettweiler. Stochastic integration relative to Brownian motion on a general Banach space. *Doga Mat*, 15(2):58–97, 1991.
- [32] C. Di Girolami, G. Fabbri, and F. Russo. The covariation for Banach space valued processes and applications. *Metrica (Online First)*. arXiv:1301.5715.
- [33] C. Di Girolami and F. Russo. Generalized covariation for Banach valued processes and Itô formula. *Osaka J. Math.* To appear, HAL-INRIA Preprint, 2010.
- [34] J. Diestel, H. Jarchow, and A. Tonge. *Absolutely summing operators*, volume 43 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995.
- [35] S. Dirksen. Itô isomorphisms for L^p -valued Poisson stochastic integrals. *Ann. Probab.* To appear, arXiv:1208.3885.
- [36] S. Dirksen, J. Maas, and J.M.A.M. van Neerven. Poisson stochastic integration in Banach spaces. *Electron. J. Probab.*, 18(100):1–28, 2013.

- [37] D.J.H. Garling. Brownian motion and UMD-spaces. In *Probability and Banach spaces (Zaragoza, 1985)*, volume 1221 of *Lecture Notes in Math.*, pages 36–49. Springer, Berlin, 1986.
- [38] D.J.H. Garling. Random martingale transform inequalities. In *Probability in Banach spaces 6 (Sandbjerg, 1986)*, volume 20 of *Progr. Probab.*, pages 101–119. Birkhäuser Boston, Boston, MA, 1990.
- [39] S. Geiss. BMO_ψ -spaces and applications to extrapolation theory. *Studia Math.*, 122(3):235–274, 1997.
- [40] M. Girardi and L. Weis. Operator-valued Fourier multiplier theorems on $L_p(X)$ and geometry of Banach spaces. *J. Funct. Anal.*, 204(2):320–354, 2003.
- [41] M.H.A. Haase. *The functional calculus for sectorial operators*, volume 169 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 2006.
- [42] R. Haller, H. Heck, and A. Noll. Mikhlin’s theorem for operator-valued Fourier multipliers in n variables. *Math. Nachr.*, 244:110–130, 2002.
- [43] E. Hausenblas. Maximal inequalities of the Itô integral with respect to Poisson random measures or Lévy processes on Banach spaces. *Potential Anal.*, 35(3):223–251, 2011.
- [44] J. Hoffmann-Jørgensen and G. Pisier. The law of large numbers and the central limit theorem in Banach spaces. *Ann. Probability*, 4(4):587–599, 1976.
- [45] T.P. Hytönen. An operator-valued Tb theorem. *J. Funct. Anal.*, 234(2):420–463, 2006.
- [46] N. Kalton, J.M.A.M. van Neerven, M.C. Veraar, and L.W. Weis. Embedding vector-valued Besov spaces into spaces of γ -radonifying operators. *Math. Nachr.*, 281(2):238–252, 2008.
- [47] N.J. Kalton and L.W. Weis. The H^∞ -calculus and sums of closed operators. *Math. Ann.*, 321(2):319–345, 2001.
- [48] N.J. Kalton and L.W. Weis. The H^∞ -calculus and square function estimates. Preprint, 2004.
- [49] K.-H. Kim. Sobolev space theory of SPDEs with continuous or measurable leading coefficients. *Stochastic Process. Appl.*, 119(1):16–44, 2009.
- [50] N.V. Krylov. A generalization of the Littlewood-Paley inequality and some other results related to stochastic partial differential equations. *Ulam Quart.*, 2(4):16, 1994.
- [51] N.V. Krylov. An analytic approach to SPDEs. In *Stochastic partial differential equations: six perspectives*, volume 64 of *Math. Surveys Monogr.*, pages 185–242. Amer. Math. Soc., Providence, RI, 1999.
- [52] N.V. Krylov. SPDEs in $L_q((0, \tau], L_p)$ spaces. *Electron. J. Probab.*, 5(13):1–29, 2000.
- [53] N.V. Krylov. On the foundation of the L_p -theory of stochastic partial differential equations. In *Stochastic partial differential equations and applications—VII*, volume 245 of *Lect. Notes Pure Appl. Math.*, pages 179–191. Chapman & Hall/CRC, Boca Raton, FL, 2006.
- [54] H. Kunita. Stochastic integrals based on martingales taking values in Hilbert space. *Nagoya Math. J.*, 38:41–52, 1970.
- [55] P.C. Kunstmann and L.W. Weis. Maximal L_p -regularity for parabolic equations, Fourier multiplier theorems and H^∞ -functional calculus. In *Functional*

- analytic methods for evolution equations*, volume 1855 of *Lecture Notes in Math.*, pages 65–311. Springer, Berlin, 2004.
- [56] M.C. Kunze. Stochastic reaction-diffusion systems with Hölder continuous multiplicative noise. *arXiv:1209.4821*, 2012.
- [57] M.C. Kunze. On a class of martingale problems on Banach spaces. *Electron. J. Probab.*, 18(104):1–30, 2013.
- [58] M.C. Kunze and J.M.A.M. van Neerven. Continuous dependence on the coefficients and global existence for stochastic reaction diffusion equations. *J. Differential Equations*, 253(3):1036–1068, 2012.
- [59] S. Kwapien and W.A. Woyczyński. Tangent sequences of random variables: basic inequalities and their applications. In *Almost everywhere convergence (Columbus, OH, 1988)*, pages 237–265. Academic Press, Boston, MA, 1989.
- [60] R. Latała and K. Oleszkiewicz. Gaussian measures of dilatations of convex symmetric sets. *Ann. Probab.*, 27(4):1922–1938, 1999.
- [61] M. Ledoux and M. Talagrand. *Probability in Banach spaces: isoperimetry and processes*, volume 23 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, Berlin, 1991.
- [62] E. Lenglar. Relation de domination entre deux processus. *Ann. Inst. H. Poincaré Sect. B (N.S.)*, 13(2):171–179, 1977.
- [63] J. Maas. Malliavin calculus and decoupling inequalities in Banach spaces. *J. Math. Anal. Appl.*, 363(2):383–398, 2010.
- [64] J. Maas and J.M.A.M. van Neerven. A Clark-Ocone formula in UMD Banach spaces. *Electron. Commun. Probab.*, 13:151–164, 2008.
- [65] C. Marinelli. On maximal inequalities for purely discontinuous L_q -valued martingales. *arXiv:1311.7120*, 2013.
- [66] C. Marinelli, C. Prévôt, and M. Röckner. Regular dependence on initial data for stochastic evolution equations with multiplicative Poisson noise. *Journal of Functional Analysis*, 258(2):616–649, 2010.
- [67] C. Marinelli and M. Röckner. Well-posedness and asymptotic behavior for stochastic reaction-diffusion equations with multiplicative Poisson noise. *Electron. J. Probab.*, 15:1528–1555, 2010.
- [68] B. Maurey. Système de Haar. In *Séminaire Maurey-Schwartz 1974–1975: Espaces L^p , applications radonifiantes et géométrie des espaces de Banach, Exp. Nos. I et II*, page 26 pp. Centre Math., École Polytech., Paris, 1975.
- [69] T.R. McConnell. On Fourier multiplier transformations of Banach-valued functions. *Trans. Amer. Math. Soc.*, 285(2):739–757, 1984.
- [70] T.R. McConnell. Decoupling and stochastic integration in UMD Banach spaces. *Probab. Math. Statist.*, 10(2):283–295, 1989.
- [71] T.R. McConnell. Decoupling and stochastic integration in UMD Banach spaces. *Probab. Math. Statist.*, 10(2):283–295, 1989.
- [72] M. Métivier. *Reelle und vektorwertige Quasimartingale und die Theorie der stochastischen Integration*. Lecture Notes in Mathematics, Vol. 607. Springer-Verlag, Berlin, 1977.
- [73] M. Métivier. *Stochastic partial differential equations in infinite-dimensional spaces*. Scuola Normale Superiore di Pisa, Quaderni. Scuola Normale Superiore, Pisa, 1988. With a preface by G. Da Prato.

- [74] M. Métivier and J. Pellaumail. *Stochastic integration*. Academic Press, New York, 1980. Probability and Mathematical Statistics.
- [75] Jan van Neerven, Mark Veraar, and Lutz Weis. Maximal γ -regularity. *arXiv preprint arXiv:1209.3782*, 2012.
- [76] J.M.A.M. van Neerven. γ -Radonifying operators—a survey. In *Spectral Theory and Harmonic Analysis (Canberra, 2009)*, volume 44 of *Proc. Centre Math. Anal. Austral. Nat. Univ.*, pages 1–62. Austral. Nat. Univ., Canberra, 2010.
- [77] J.M.A.M. van Neerven, M.C. Veraar, and L.W. Weis. Conditions for stochastic integrability in UMD Banach spaces. In *Banach spaces and their applications in analysis (in honor of Nigel Kalton's 60th birthday)*, pages 127–146. De Gruyter Proceedings in Mathematics, De Gruyter, 2007.
- [78] J.M.A.M. van Neerven, M.C. Veraar, and L.W. Weis. Stochastic integration in UMD Banach spaces. *Ann. Probab.*, 35(4):1438–1478, 2007.
- [79] J.M.A.M. van Neerven, M.C. Veraar, and L.W. Weis. Stochastic evolution equations in UMD Banach spaces. *J. Funct. Anal.*, 255(4):940–993, 2008.
- [80] J.M.A.M. van Neerven, M.C. Veraar, and L.W. Weis. Maximal L^p -regularity for stochastic evolution equations. *SIAM J. Math. Anal.*, 44(3):1372–1414, 2012.
- [81] J.M.A.M. van Neerven, M.C. Veraar, and L.W. Weis. Stochastic maximal L^p -regularity. *Ann. Probab.*, 40(2):788–812, 2012.
- [82] J.M.A.M. van Neerven, M.C. Veraar, and L.W. Weis. On the R -boundedness of stochastic convolution operators. To appear in *Positivity*, arXiv:1404.3353, 2014.
- [83] J.M.A.M. van Neerven and L.W. Weis. Stochastic integration of functions with values in a Banach space. *Studia Math.*, 166(2):131–170, 2005.
- [84] J.M.A.M. van Neerven and L.W. Weis. Weak limits and integrals of Gaussian covariances in Banach spaces. *Probab. Math. Statist.*, 25(1, Acta Univ. Wratislav. No. 2784):55–74, 2005.
- [85] D. Nualart. *Malliavin calculus and its applications*, volume 110 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2009.
- [86] M. Ondreját. Uniqueness for stochastic evolution equations in Banach spaces. *Dissertationes Math. (Rozprawy Mat.)*, 426:63, 2004.
- [87] M. Ondreját and J. Seidler. On existence of progressively measurable modifications. *Electron. Commun. Probab.*, 18:1–6, 2013.
- [88] M. Ondreját and M.C. Veraar. Weak characterizations of stochastic integrability and Dudley's theorem in infinite dimensions. *J. Theoretical Probab.*, 2013. Online First.
- [89] I. Pinelis. Optimum bounds for the distributions of martingales in Banach spaces. *The Annals of Probability*, 22(4):1679–1706, 1994. Correction: *ibid.* 27:2119, 1999.
- [90] G. Pisier. Martingales with values in uniformly convex spaces. *Israel J. Math.*, 20(3-4):326–350, 1975.
- [91] G. Pisier. Riesz transforms: a simpler analytic proof of P.A. Meyer's inequality. *Séminaire de probabilités XXII*, pages 485–501, 1988.

- [92] G. Pisier. *The volume of convex bodies and Banach space geometry*, volume 94 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1989.
- [93] M. Pronk and M.C Veraar. Tools for Malliavin calculus in UMD Banach spaces. *Potential Analysis*, 40:307–344, 2014.
- [94] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1991.
- [95] J. Rosiński and Z. Suchanecki. On the space of vector-valued functions integrable with respect to the white noise. *Colloq. Math.*, 43(1):183–201 (1981), 1980.
- [96] R. Schnaubelt and M.C. Veraar. Structurally damped plate and wave equations with random point force in arbitrary space dimensions. *Differential Integral Equations*, 23(9-10):957–988, 2010.
- [97] J. Seidler. Exponential estimates for stochastic convolutions in 2-smooth Banach spaces. *Electron. J. Probab.*, 15:no. 50, 1556–1573, 2010.
- [98] Ž. Štrkalj and L. W. Weis. On operator-valued Fourier multiplier theorems. *Trans. Amer. Math. Soc.*, 359(8):3529–3547 (electronic), 2007.
- [99] M. Yor. Sur les intégrales stochastiques à valeurs dans un espace de Banach. *Ann. Inst. H. Poincaré Sect. B (N.S.)*, 10:31–36, 1974.
- [100] J. Zhu. Maximal inequalities for stochastic convolutions driven by Lévy processes in Banach spaces. Work in progress.
- [101] F. Zimmermann. On vector-valued Fourier multiplier theorems. *Studia Math.*, 93(3):201–222, 1989.

Jan van Neerven
Delft Institute of Applied Mathematics
Delft University of Technology
P.O. Box 5031
2600 GA Delft
The Netherlands
e-mail: J.M.A.M.vanNeerven@tudelft.nl

Mark Veraar
Delft Institute of Applied Mathematics
Delft University of Technology
P.O. Box 5031
2600 GA Delft
The Netherlands
e-mail: M.C.Veraar@tudelft.nl

Lutz Weis
Karlsruhe Institute of Technology
Department of Mathematics
76128 Karlsruhe
Germany
e-mail: Lutz.Weis@kit.edu