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ON INDIVIDUAL STABILITY OF C_0−SEMIGROUPS

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ABSTRACT. Let $\{T(t)\}_{t\geqslant 0}$ be a C_0−semigroup with generator A on a Banach space X. Let $x_0 \in X$ be a fixed element. We prove the following individual stability results.

(i) Suppose X is an ordered Banach space with weakly normal closed cone C and assume there exists $t_0 \geqslant 0$ such that $T(t)x_0 \in C$ for all $t \geqslant t_0$. If the local resolvent $\lambda \mapsto (\lambda - A)^{-1}x_0$ admits a bounded analytic extension to the right half-plane $\{\mathrm{Re}\,\lambda > 0\}$, then for all $\mu \in \varrho(A)$ and $x^* \in X^*$ we have
$$\lim_{t\to\infty} \langle T(t)(\mu - A)^{-1}x_0, x^*\rangle = 0.$$

(ii) Suppose E is a rearrangement invariant Banach function space over $[0,\infty)$ with order continuous norm. If $x_0^* \in X^*$ is an element such that $t \mapsto \langle T(t)x_0, x_0^*\rangle$ defines an element of E, then for all $\mu \in \varrho(A)$ and $\beta \geqslant 1$ we have
$$\lim_{t\to\infty} \langle T(t)(\mu - A)^{-\beta}x_0, x_0^*\rangle = 0.$$

For an $n \times n$ matrix A, the linear differential equation

$$\begin{aligned} u'(t) &= Au(t) \qquad (t \geqslant 0) \\ u(0) &= u_0 \end{aligned} \tag{1}$$

is solved by $u(t) = T(t)u_0$, where $T(t) := e^{tA}$. The classical Lyapunov theorem asserts that $\{T(t)\}_{t\geqslant 0}$ decays at exponential rate to 0 if and only if all eigenvalues of A are located in the open left halfplane. The importance of this result resides in the fact that it enables one to derive *a priori* information about the asymptotic behaviour of the solutions of (1) from the spectral properties of A.

If A is the infinitesimal generator of a C_0−semigroup $\{T(t)\}_{t\geqslant 0}$ of bounded linear operators on an infinite-dimensional Hilbert space, it may happen that the semigroup is unstable although

$$s(A) := \sup\{\mathrm{Re}\,\lambda : \ \lambda \in \sigma(A)\} < 0.$$

The first example of such a semigroup was given by Zabczyk [16]. As it turns out, in contrast to the finite-dimensional case we have to take into account the growth of the resolvent operators $R(\lambda, A) = (\lambda - A)^{-1}$. Indeed, if $\mathbb{C}_+ := \{\lambda \in \mathbb{C} : \ \mathrm{Re}\,\lambda > 0\}$ belongs to the resolvent set $\varrho(A)$ and

$$\sup_{\lambda\in\mathbb{C}_+} \|R(\lambda, A)\| < \infty, \tag{2}$$

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then there exist constants $\omega > 0$ and $M \geqslant 1$ such that

$$\|T(t)\| \leqslant Me^{-\omega t}, \qquad \forall t \geqslant 0.$$

This was proved by Gearhart [5] for contraction semigroups, whereas the general case is due to Herbst [6] and Prüss [11].

For C_0-semigroups on an infinite-dimensional Banach space, the Gearhart-Herbst-Prüss theorem fails. The following counterexample, due to Arendt [1], is particularly concise. Let $1 \leqslant p < q < \infty$ and consider the C_0-semigroup $\{T(t)\}_{t\geqslant 0}$ on $L^p(1,\infty) \cap L^q(1,\infty)$ defined by

$$(T(t)f)(s) = e^{\frac{t}{q}} f(se^t), \qquad t \geqslant 0, \quad s \in (1,\infty).$$

Then (2) holds, but $\|T(t)\| \geqslant 1$ for all $t \geqslant 0$.

This time the problem can be attributed to the failure of the Plancherel theorem for Banach space-valued functions. Using the notion of Fourier type, Wrobel and Weis [14] recently obtained a generalization of the Gearhart-Herbst-Prüss theorem for Banach spaces with Fourier type $p \in [1,2]$. As a corollary to the Wrobel-Weis theorem we have the following result, valid for arbitrary C_0-semigroups on a Banach space: if (2) holds, then there exist constants $\omega > 0$ and $M \geqslant 1$ such that for all $x \in \mathcal{D}(A)$ we have

$$\|T(t)x\| \leqslant Me^{-\omega t}\|x\|_{\mathcal{D}(A)}, \qquad \forall t \geqslant 0. \tag{3}$$

Here $\|x\|_{D(A)} = \|x\| + \|Ax\|$ denotes the graph norm. For an overview of these results we refer to the monograph [10].

In [9], the following generalization of (3) to individual orbits was obtained. If $x_0 \in X$ is an element whose local resolvent $\lambda \mapsto R(\lambda, A)x_0$ admits a bounded analytic extension to $\mathbb{C}_+$, then for all $\mu \in \varrho(A)$ there exists a constant $M \geqslant 0$ such that

$$\|T(t)R(\mu, A)x_0\| \leqslant M(1+t), \qquad \forall t \geqslant 0. \tag{4}$$

By a simple resolvent expansion argument, if (2) holds and (4) holds for all $x_0 \in X$, then (3) holds for all $x \in \mathcal{D}(A)$. A generalization of (4) to the context of Laplace transforms of arbitrary exponentially bounded vector-valued functions was obtained subsequently by Batty and Blake [3].

The estimate in (4) raises the natural question whether linear growth is optimal. This has been the object of some investigations, the main results of which we summarize next.

In [7] it is shown that if X has Fourier type $p \in (1,2]$, then bounded analytic extendability of the local resolvent implies

$$\lim_{t\to\infty} \|T(t)R(\mu, A)x_0\| = 0.$$

In the same paper an example is given which shows that this result is false for arbitrary Banach spaces. In this example, however, we do have

$$\lim_{t\to\infty} \langle T(t)R(\mu, A)x_0, x^*\rangle = 0, \qquad \forall x^* \in X^*,$$

thereby leaving open the possibility that bounded analytic extendability of the local resolvent always implies boundedness of $T(\cdot)R(\mu, A)x_0$ or even its weak convergence to 0.

In [4], it is shown that if $\mathbf{T}$ is a C_0-semigroup on an ordered normal Banach space X with cone C, and $x_0 \in C$ is such that $T(t)x_0 \in C$ for all $t \geqslant 0$ and the local resolvent $\lambda \mapsto R(\lambda, A)x_0$ has a bounded analytic extension to $\mathbb{C}_+$, then for all $\mu \in \varrho(A)$ there exists a constant $M \geqslant 1$ such that

$$\|T(t)R(\mu, A)x_0\| \leqslant M(1 + \ln t), \qquad \forall t \geqslant 0. \tag{5}$$

We shall now give a surprisingly simple proof that in the situation of (5) we actually have weak convergence to 0:

$$\lim_{t\to\infty} \langle T(t)R(\mu, A)x_0, x^*\rangle = 0, \qquad \forall x^* \in X^*.$$

In particular, by the uniform boundedness theorem this implies that

$$\sup_{t\geqslant 0} \|T(t)R(\mu, A)x_0\| < \infty.$$

Recall that the *abscissa of improper convergence* of the Laplace transform of a function $f \in L^1_{loc}[0, \infty)$ is defined as the infimum $\omega(f)$ of all $\omega \in \mathbb{R}$ such that the integral

$$\mathcal{L}f(\lambda) := \int_0^\infty e^{-\lambda t} f(t)\, dt$$

converges as an improper integral for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$. The Laplace transform $\mathcal{L}f$ is analytic in the open half-plane $\{\operatorname{Re}\lambda > \omega(f)\}$. The Pringsheim-Landau theorem [13, Theorem II.5b], [10, Theorem 1.3.4], asserts that for *positive* f, the function $\mathcal{L}f$ cannot be extended analytically to a neighbourhood of the point $\omega(f)$.

Lemma 1. *Let $0 \leqslant f \in L^1_{loc}[0, \infty)$ be a function whose Laplace transform admits an analytic extension F to $\mathbb{C}_+$ satisfying $\sup_{\lambda\in\mathbb{C}_+} |F(\lambda)| \leqslant M$ for some constant $M \geqslant 0$. Then $f \in L^1[0, \infty)$ and $\|f\|_{L^1[0,\infty)} \leqslant M$.*

Proof. By the Pringsheim-Landau theorem, $\omega(f) \leqslant 0$. Hence by analytic continuation, for all $\operatorname{Re}\lambda > 0$ we have

$$\int_0^\infty e^{-\lambda t} f(t)\, dt \;=\; F(\lambda).$$

Noting that $|F(\lambda)| \leqslant M$ for all $\operatorname{Re}\lambda > 0$, upon letting $\lambda \downarrow 0$ from the Monotone Convergence theorem we obtain

$$\int_0^\infty f(t)\, dt \;=\; \lim_{\lambda\downarrow 0} \int_0^\infty e^{-\lambda t} f(t)\, dt \;\leqslant\; M.$$

∎

We will say that a function $g \in L^1_{loc}[0, \infty)$ is *asymptotically positive* if there exists a function $0 \leqslant h \in L^1[0, \infty)$ such that $g + h \geqslant 0$ a.e.

Theorem 2. *Let $\mathbf{T}$ be a C_0−semigroup on a Banach space X. Let $x_0 \in X$ and $x_0^* \in X^*$ be such that the following two conditions are satisfied:*

(i) *$t \mapsto \langle T(t)x_0, x_0^*\rangle$ is asymptotically positive;*
(ii) *$\lambda \mapsto \langle R(\lambda, A)x_0, x_0^*\rangle$ admits a bounded analytic extension to $\mathbb{C}_+$.*

Then for all $\mu \in \mathbb{C}$ with $\operatorname{Re}\mu > \max\{\omega_0(\mathbf{T}), 0\}$ we have

$$\lim_{t\to\infty} \langle T(t)R(\mu, A)x_0, x_0^*\rangle \;=\; 0.$$

Proof. Choose $0 \leqslant h \in L^1[0,\infty)$ such that $f(t) := \langle T(t)x_0, x_0^*\rangle + h(t) \geqslant 0$ for almost all $t \geqslant 0$. Then $\mathcal{L}f(\lambda) = \langle R(\lambda, A)x_0, x_0^*\rangle + \mathcal{L}h(\lambda)$ for $\operatorname{Re}\lambda$ large enough, and $\mathcal{L}f$ admits a bounded analytic extension to $\mathbb{C}_+$. Hence by Lemma 1, $f \in L^1[0,\infty)$ and therefore also $\langle T(\cdot)x_0, x_0^*\rangle \in L^1[0,\infty)$. Now let $\operatorname{Re}\mu > \max\{\omega_0(\mathbf{T}), 0\}$. Then,

$$|\langle T(t)R(\mu, A)x_0, x_0^*\rangle| \leqslant \int_0^\infty e^{-\operatorname{Re}\mu s}|\langle T(t+s)x_0, x_0^*\rangle|\, ds \leqslant \int_t^\infty |\langle T(s)x_0, x_0^*\rangle|\, ds.$$

Since the right hand side tends to 0 as $t \to \infty$, this proves the theorem. ∎

Recall that if X is an ordered Banach space with weakly normal closed cone C, cf. [12, V.3], then every real $x^* \in X^*$ admits a decomposition $x^* = x_+^* - x_-^*$ with $\langle x, x_+^*\rangle \geqslant 0$ and $\langle x, x_-^*\rangle \geqslant 0$ for all $x \in C$. Examples of such spaces are:

- Banach lattices with the cone of positive elements;
- C^*-algebras with the cone positive selfadjoint elements.

Theorem 3. *Let $\mathbf{T}$ be a C_0-semigroup on an ordered Banach space X with weakly normal closed cone C. Let $x_0 \in X$ be such that:*

(i) *There exists $t_0 \geqslant 0$ such that $T(t)x_0 \in C$ for $t \geqslant t_0$;*
(ii) *$\lambda \mapsto R(\lambda, A)x_0$ admits a bounded analytic extension to $\mathbb{C}_+$.*

Then for all $\mu \in \varrho(A)$ and $x^* \in X^*$ we have

$$\lim_{t\to\infty} \langle T(t)R(\mu, A)x_0, x^*\rangle = 0.$$

Proof. Let $x^* \in X^*$ be fixed. By decomposing x^* if necessary into real and imaginary parts and representing each of these as the difference of two elements that are positive on C, we may assume that $\langle x, x^*\rangle \geqslant 0$ for all $x \in C$. In particular, $\langle T(t)x_0, x^*\rangle \geqslant 0$ for all $t \geqslant t_0$. It follows that the function $\langle T(\cdot)x_0, x^*\rangle$ is asymptotically positive, and therefore by Theorem 2, for all $\mu \in \mathbb{C}$ with $\operatorname{Re}\mu > \max\{\omega_0(\mathbf{T}), 0\}$ we have

$$\lim_{t\to\infty} \langle T(t)R(\mu, A)x_0, x^*\rangle = 0.$$

For general $\mu \in \varrho(A)$, choose an arbitrary $\mu_0 \in \mathbb{C}$ with $\mu_0 > \max\{\omega_0(\mathbf{T}), 0\}$ and apply the resolvent identity to conclude that

$$\begin{aligned}\lim_{t\to\infty} &\langle T(t)R(\mu, A)x_0, x^*\rangle \\ &= \lim_{t\to\infty} \langle T(t)R(\mu_0, A)x_0, x^*\rangle + (\mu_0 - \mu)\lim_{t\to\infty} \langle T(t)R(\mu_0, A)x_0, R(\mu, A^*)x^*\rangle = 0.\end{aligned}$$

∎

In the course of the proof of Theorem 2 it was shown that from

$$\int_0^\infty |\langle T(t)x_0, x^*\rangle|\, dt < \infty \tag{6}$$

it follows that

$$\lim_{t\to\infty} \langle T(t)R(\mu_0, A)x_0, x^*\rangle = 0.$$

We will give several extensions of this simple observation, where the rôle of $L^1[0,\infty)$ is replaced by certain Banach function spaces over $[0,\infty)$.

Let us first observe, however, that from (6) it does not necessarily follow that $\lim_{t\to\infty} \langle T(t)x_0, x^*\rangle = 0$. In fact, there exist unbounded C_0-semigroups for which

$$\int_0^\infty |\langle T(t)x, x^*\rangle|\, dt < \infty, \qquad \forall x \in X,\ x^* \in X^*;$$

see e.g., [10]. Then by the uniform boundedness theorem, there exist $x_0 \in X$ and $x_0^* \in X^*$ such that $t \mapsto \langle T(t)x_0, x_0^*\rangle$ is unbounded.

In what follows we need some terminology concerning Banach function spaces. We refer to [8] and [15] for an explanation of the terminology we are using.

A Banach function space E has *order continuous norm* if every net in E that decreases to 0 is convergent to 0. Every separable Banach function space has order continuous norm. If E is a rearrangement invariant Banach function space over $[0,\infty)$, then E has order continuous norm if and only if the simple functions are dense and

$$\lim_{t\downarrow 0} \phi_E(t) = 0,$$

where the *fundamental function* of E is defined as

$$\phi_E(t) := \|\chi_{H_t}\|_E, \qquad t \geqslant 0;$$

here H_t is any measurable subset of $[0,\infty)$ of measure t and χ_{H_t} is its indicator function.

Lemma 4. *If E is a rearrangement invariant Banach function space over $[0,\infty)$ with order continuous norm, then the semigroup $\mathbf{S}$ of left shifts on E,*

$$S(t)f(s) = f(s+t), \qquad f \in E, \quad s,t \in [0,\infty),$$

is a C_0-contraction semigroup which is strongly stable, i.e.

$$\lim_{t\to\infty} \|S(t)f\|_E = 0, \qquad \forall f \in E.$$

Proof. It is a trivial consequence of the rearrangement invariance that each operator $S(t)$ is a contraction.

In order to prove strong continuity of $\mathbf{S}$ we note that from $\lim_{t\downarrow 0} \phi_E(t) = 0$ it follows that $\lim_{t\downarrow 0} \|S(t)f - f\|_E = 0$ for simple functions f. These functions being dense in E, the strong continuity of $\mathbf{S}$ follows by a density argument.

It remains to prove strong stability. Let $f \in E$ be fixed. By considering positive and negative parts separately, we may assume that $f \geqslant 0$. By rearrangement invariance, $\|S(t)f\|_E = \|f \cdot \chi_{[t,\infty)}\|_E$. Since $f \cdot \chi_{[t,\infty)} \downarrow 0$ as $t \to \infty$, the order continuity of the norm implies

$$\lim_{t\to\infty} \|S(t)f\|_E = \lim_{t\to\infty} \|f \cdot \chi_{[t,\infty)}\|_E = 0.$$

∎

In what follows, $\mathbf{T}$ is a C_0-semigroup with generator A on a Banach space X, Y is another Banach space and $P \in \mathcal{L}(X,Y)$ is a bounded linear operator. For instance, Y could be the scalar field and $P = x_0^*$ an element of X^*.

Lemma 5. *Let E be a rearrangement invariant Banach function space over $[0,\infty)$ with order continuous norm. If $x_0 \in X$ is an element such that*

$$t \mapsto \|PT(t)x_0\|$$

belongs to E, then for all $\beta > 0$ and $\mu \in \mathbb{C}$ with $\operatorname{Re}\mu > \max\{\omega_0(\mathbf{T}), 0\}$ the function

$$t \mapsto \|PT(\mu - A)^{-\beta}x_0\|$$

belongs to E.

Proof. *Case 1.* First we prove the lemma for $\beta = 1$.

Define $f(t) := \|PT(t)R(\mu, A)x_0\|$. For almost all $t \in [0,\infty)$ we have

$$\begin{aligned} \|PT(t)R(\mu, A)x_0\| &\leqslant \int_0^\infty e^{-\mathrm{Re}\,\mu s}\|PT(t+s)x_0\|\,ds \\ &= \int_0^\infty e^{-\mathrm{Re}\,\mu s} S(s)f(t)\,ds = R(\mathrm{Re}\,\mu, B)f(t), \end{aligned} \tag{7}$$

where B denotes the generator of the left shift semigroup $\mathbf{S}$ in E. It follows that $\|PT(\cdot)R(\mu, A)x_0\| \in E$.

Case 2. Next let $\beta \in (0,1)$. With the above notation,

$$\int_0^\infty s^{-\beta}\|R(\mathrm{Re}\,\mu + s, B)f\|_E\,ds \leqslant \int_0^\infty s^{-\beta}(\mathrm{Re}\,\mu + s)^{-1}\|f\|_E\,ds < \infty.$$

This estimate shows that the integral $\int_0^\infty s^{-\beta}R(\mathrm{Re}\,\mu + s, B)f\,ds$ exists as a Bochner integral in E. Using the identity

$$(\mu - A)^{-\beta}x_0 = \frac{\sin\pi\beta}{\pi}\int_0^\infty s^{-\beta}R(\mu + s, A)x_0\,ds$$

and (7), for almost all t we have

$$\begin{aligned} \|PT(t)(\mu - A)^{-\beta}x_0\| &\leqslant \frac{\sin\pi\beta}{\pi}\int_0^\infty s^{-\beta}\|PT(t)R(\mu + s, A)x_0\|\,ds \\ &\leqslant \frac{\sin\pi\beta}{\pi}\int_0^\infty s^{-\beta}R(\mathrm{Re}\,\mu + s, B)f(t)\,ds. \end{aligned}$$

Hence,

$$\|PT(\cdot)(\mu - A)^{-\beta}x_0\| \leqslant \frac{\sin\pi\beta}{\pi}\int_0^\infty s^{-\beta}R(\mathrm{Re}\,\mu + s, B)f\,ds$$

almost everywhere, and therefore $\|PT(\cdot)(\mu - A)^{-\beta}x_0\| \in E$.

Case 3. If $\beta > 1$ we write $\beta = m + \gamma$ with $m \in \mathbb{N}$ and $\gamma \in [0,1)$ and apply the Cases 1 and 2. ∎

Let E be a Banach function space over $[0,\infty)$. The *associate space* of E, notation E', consists of all measurable functions f on $[0,\infty)$ such that

$$\|f\|_{E'} := \sup\left\{\int_0^\infty |f(t)g(t)|\,dt : \ g \in E, \ \|g\|_E \leqslant 1\right\}$$

is finite.

We say that E is *monotone complete* if for all nonnegative measurable functions f and all sequences $0 \leqslant f_1 \leqslant f_2 \leqslant \ldots \uparrow f$ in E with $\sup_n \|f_n\|_E < \infty$ it follows that $f \in E$. It is well known [15, Theorem 15.71.3] that E is monotone complete if and only if $E = E''$ with equivalent norms. Lemma 5 remains true if the order continuity of the norm is replaced by monotone completeness. Instead of a domination argument we now integrate against functions from the associate space E'; estimates similar to the ones we did above then show that $\|PT(\cdot)(\mu - A)^{-\beta}x_0\|$ defines an element of E'', and hence of E. We will not need this result and leave the details to the reader.

Theorem 6. *Let E be a rearrangement invariant Banach function space over $[0,\infty)$ with order continuous norm. If $x_0 \in X$ is a fixed element for which the* *function* $t \mapsto \|PT(t)x_0\|$ *belongs to* E*, then for all* $\beta \geqslant 1$ *and* $\mu \in \mathbb{C}$ *with* $\operatorname{Re}\mu > \max\{\omega_0(\mathbf{T}), 0\}$ *we have*

$$\lim_{t\to\infty} \|PT(t)(\mu - A)^{-\beta}x_0\| = 0.$$

Proof. Fix $\mu \in \mathbb{C}$ with $\operatorname{Re}\mu > \max\{\omega_0(\mathbf{T}), 0\}$. In view of Lemma 5 it is enough to prove the theorem for $\beta = 1$.

It is an easy consequence of the rearrangement invariance of E' that the function $s \mapsto e^{-\operatorname{Re}\mu s}$ defines an element of E'; cf. [10, Lemma 4.6.1]. Therefore,

$$\begin{aligned}\|PT(t)R(\mu, A)x_0\| &\leqslant \int_0^\infty e^{-\operatorname{Re}\mu s}\|PT(t+s)x_0\|\,ds \\ &\leqslant \left\|e^{-\operatorname{Re}\mu(\cdot)}\right\|_{E'} \big\| \, \|PT(t+\cdot)x_0\| \, \big\|_E.\end{aligned}$$

The right hand side tends to 0 as $t \to \infty$ by Lemma 4. ∎

If E is rearrangement invariant, then E' is rearrangement invariant as well, and we have

$$\phi_E(t)\phi_{E'}(t) = t, \qquad t \geqslant 0. \tag{8}$$

We shall use this identity to give an improvement of Theorem 6 under a growth assumption on $\phi_E(t)$.

Theorem 7. *Let* E *be a rearrangement invariant Banach function space over* $[0,\infty)$ *with order continuous norm, and assume that there exists an* $\alpha \in [0, 1)$ *such that*

$$\liminf_{t\downarrow 0} \frac{\phi_E(t)}{t^\alpha} > 0.$$

If $x_0 \in X$ *is a fixed element for which the function* $t \mapsto \|PT(t)x_0\|$ *belongs to* E*, then for all* $\beta > \alpha$ *and all* $\mu \in \mathbb{C}$ *with* $\operatorname{Re}\mu > \max\{\omega_0(\mathbf{T}), 0\}$ *we have*

$$\lim_{t\to\infty} \|PT(t)(\mu - A)^{-\beta}x_0\| = 0.$$

Proof. For $\beta \geqslant 1$ this has already been proved, so we shall assume that $\alpha < \beta < 1$.

For $t \geqslant 0$ we have

$$\begin{aligned}PT(t)(\mu - A)^{-\beta}x_0 &= PT(t)\left(\frac{\sin\pi\beta}{\pi}\int_0^\infty s^{-\beta}R(\mu+s, A)x_0\,ds\right) \\ &= \frac{\sin\pi\beta}{\pi}\int_0^\infty s^{-\beta}\int_0^\infty e^{-(\mu+s)r}PT(t+r)x_0\,dr\,ds.\end{aligned}$$

By [10, Lemma 4.6.1],

$$\left\|\int_0^\infty e^{-(\mu+s)r}PT(t+r)x_0\,dr\right\| \leqslant c\,\phi_{E'}\left((\operatorname{Re}\mu + s)^{-1}\right) \big\| \, \|PT(t+\cdot)x_0\| \, \big\|_E,$$

where $c := (1 - e^{-1})^{-1}$. Using (8) and the assumption on α, for s large enough we have

$$\phi_{E'}\left((\operatorname{Re}\mu + s)^{-1}\right) = \frac{1}{(\operatorname{Re}\mu + s)\,\phi_E\left((\operatorname{Re}\mu+s)^{-1}\right)} \leqslant K(\operatorname{Re}\mu + s)^{\alpha-1}$$

for some finite constant $K \geqslant 0$. In view of $0 \leqslant \alpha < \beta < 1$, this implies

$$C_\mu := \int_0^\infty s^{-\beta}(\operatorname{Re}\mu + s)^{\alpha-1}\,ds < \infty.$$

Combining this estimate with the above one, this yields

$$\begin{aligned}
&\|PT(t)(\mu-A)^{-\beta}x_0\| \\
&\qquad\leqslant \frac{c\sin\pi\beta}{\pi}\int_0^\infty s^{-\beta}\phi_{E'}\left((\operatorname{Re}\mu+s)^{-1}\right)\,ds\cdot\Big\|\,\|PT(t+\cdot)x_0\|\,\Big\|_E \\
&\qquad\leqslant \frac{c\,C_\mu K\sin\pi\beta}{\pi}\Big\|\,\|PT(t+\cdot)x_0\|\,\Big\|_E,
\end{aligned}$$

By Lemma 4, the right hand side tends to 0 as $t\to\infty$. ∎

For all $p\in[1,\infty)$, the space $E=L^p[0,\infty)$ is a rearrangement Banach function space with order continuous norm. Moreover, $\phi_E(t)=t^{\frac{1}{p}}$ for all $t\geqslant 0$. From Theorem 7 we obtain:

Corollary 8. *Let $P\in\mathcal{L}(X,Y)$ be a bounded linear operator. If $x_0\in E$ is a fixed element such that*

$$\int_0^\infty \|PT(t)x_0\|^p\,dt<\infty$$

for some $p\in(1,\infty)$, then for all $\beta>\frac{1}{p}$ and all $\mu\in\mathbb{C}$ with $\operatorname{Re}\mu>\max\{\omega_0(\mathbf{T}),0\}$ we have

$$\lim_{t\to\infty}\|PT(t)(\mu-A)^{-\beta}x_0\|=0.$$

For $p\in(1,2]$, a global version of this result was obtained in [7] as a consequence of the Hausdorff-Young inequality for the Fourier transform and an approximation argument.

If E is a rearrangement invariant Banach function space over $[0,\infty)$ with the property that $\lim_{t\to\infty}\phi_E(t)=\infty$, and if $\langle T(\cdot)x,x^*\rangle\in E$ for all $x\in X$ and $x^*\in X^*$, then the resolvent $R(\lambda,A)$ admits a bounded analytic extension to $\mathbb{C}_+$ [10, Theorem 4.6.2]; this provides a link between Theorems 6 and 7 on the one hand and Theorem 2 on the other. The proof of this result, however, is global in nature, and in contrast to the more direct method adopted here, it does not lead to individual stability results.

Added in proof: After this paper had been submitted for publication, Charles Batty found an example which shows that linear growth in (4) is optimal [2]. In this example, $\{T(t)\}_{t\geqslant 0}$ is a positive C_0−semigroup on a Banach lattice X. It follows that the eventual positivity condition in Theorem 3 cannot be omitted.

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