COMPACTNESS IN VECTOR-VALUED BANACH FUNCTION SPACES

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Abstract. We give a new proof of a recent characterization by Diaz and Mayoral of compactness in the Lebesgue-Bochner spaces L^p_X, where X is a Banach space and $1 \leqslant p < \infty$, and extend the result to vector-valued Banach function spaces E_X, where E is a Banach function space with order continuous norm.

Let X be a Banach space. The problem of describing the compact sets in the Lebesgue-Bochner spaces L^p_X, $1 \leqslant p < \infty$, goes back to the work of Riesz, Fréchet, Vitali in the scalar-valued case, cf. [7], and has been considered by many authors, cf. [2, 4, 5, 11, 12]. In a recent paper, Diaz and Mayoral [5] proved that if the underlying measure space is finite, then a subset K of L^p_X is relatively compact if and only if K is uniformly p-integrable, scalarly relatively compact, and either uniformly tight or flatly concentrated. Their proof relies on the Diestel-Ruess-Schachermayer characterization [6] of weak compactness in L^1_X and the notion of Bocce oscillation, which was studied recently by Girardi [8] and Balder-Girardi-Jalby [3] in the context of compactness in L^1_X. The purpose of this note is to present an extension of the Diaz-Mayoral result to vector-valued Banach function spaces E_X, with a proof based on Prohorov's tightness theorem.

We begin with some preliminaries on Banach lattices and Banach function spaces. Our terminology is standard and follows [9].

A Banach lattice E is said to have *order continuous norm* if every net in E which decreases to 0 converges to 0. Every separable Banach function space E has this property. Indeed, because such spaces are Dedekind complete [9, Lemma 2.6.1] and cannot contain an isomorphic copy of l^∞, this follows from [9, Corollary 2.4.3].

A subset F of a Banach lattice E is called *almost order bounded* if for every $\varepsilon > 0$ there exists an element $x_\varepsilon \in E_+$ such that $F \subseteq [-x_\varepsilon, x_\varepsilon] + B(\varepsilon)$, where $[-x_\varepsilon, x_\varepsilon] := \{y \in E : -x_\varepsilon \leqslant y \leqslant x_\varepsilon\}$ and $B(\varepsilon) := \{x \in X : \|x\| < \varepsilon\}$. It follows from [9, Theorem 2.4.2] that every almost order bounded set in a Banach lattice with order continuous norm is relatively weakly compact.

Lemma 1. *Let E be a Banach lattice and let I be a dense ideal in E. If the set $A \subseteq E^+$ is almost order bounded, then for every $\varepsilon > 0$ there exists an element $x_\varepsilon \in I^+$ such that $A \subseteq [0, x_\varepsilon] + B(\varepsilon)$.*

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Proof. Fix $\varepsilon > 0$ and choose $y_\varepsilon \in E^+$ such that $A \subseteq [-y_\varepsilon, y_\varepsilon] + B(\frac12\varepsilon)$. Choose $x_\varepsilon \in I$ such that $0 \leqslant x_\varepsilon \leqslant y_\varepsilon$ and $\|y_\varepsilon - x_\varepsilon\| < \frac12\varepsilon$.

Fix $a \in A$, say $a = y + b$ with $y \in [-y_\varepsilon, y_\varepsilon]$ and $\|b\| < \frac12\varepsilon$. With $z_\varepsilon := y_\varepsilon + |b|$ we have $\|z_\varepsilon - x_\varepsilon\| \leqslant \|y_\varepsilon - x_\varepsilon\| + \|b\| < \varepsilon$. From $a \leqslant z_\varepsilon$ we infer $(a - x_\varepsilon)^+ \leqslant (z_\varepsilon - x_\varepsilon)^+ = z_\varepsilon - x_\varepsilon$ and hence $\|(a - x_\varepsilon)^+\| \leqslant \|z_\varepsilon - x_\varepsilon\| < \varepsilon$. It follows that $a = a \wedge x_\varepsilon + (a - x_\varepsilon)^+ \in [0, x_\varepsilon] + B(\varepsilon)$. □

If E is a Banach function space with order continuous norm, then for all $f \in E$ we have $\lim_{r\to\infty} \|\mathbf{1}_{\{|\phi|>r\}}\phi\|_E = 0$. Motivated by this we shall call a subset F of E *uniformly E–integrable* if

$$\lim_{r\to\infty} \sup_{\phi\in F} \big\|\mathbf{1}_{\{|\phi|>r\}}\phi\big\|_E = 0.$$

For $E = L^p$ with $1 \leqslant p < \infty$, this definition reduces to the classical definition of uniform p-integrability.

If E is a Banach function space containing the constant function $\mathbf{1}$, then every uniformly E–integrable subset of E is almost order bounded. From Lemma 1 we deduce the following converse:

Lemma 2. *Let E be a Banach function space with order continuous norm over a σ-finite measure space (S, ν). If $F \subseteq E^+$ is almost order bounded, then F is uniformly E-integrable.*

Proof. Let $\varepsilon > 0$ be fixed. By Lemma 1, applied to $I := E \cap L^\infty(S, \nu)$, we may choose $x_\varepsilon \in E^+$ and real numbers $R_\varepsilon \geqslant 0$ such that $0 \leqslant x_\varepsilon \leqslant R_\varepsilon$ ν-almost everywhere and $F \subseteq [0, x_\varepsilon] + B(\varepsilon)$. Keeping $\phi \in F$ fixed for the moment, we can write $\phi = x + b$ with $x \in [0, x_\varepsilon]$ and $\|b\|_E < \varepsilon$. Then, for all $r > 0$,

$$\begin{aligned}
\big\|\mathbf{1}_{\{\phi>r\}}\phi\big\|_E &\leqslant \big\|\mathbf{1}_{\{\phi>r\}}x\big\|_E + \big\|\mathbf{1}_{\{\phi>r\}}b\big\|_E \\
&\leqslant \big\|\mathbf{1}_{\{x>\frac12 r\}}x\big\|_E + \big\|\mathbf{1}_{\{|b|>\frac12 r\}}x\big\|_E + \|b\|_E \\
&\leqslant \big\|\mathbf{1}_{\{x_\varepsilon>\frac12 r\}}x_\varepsilon\big\|_E + \frac{2R_\varepsilon}{r}\big\|b\big\|_E + \varepsilon,
\end{aligned}$$

where in the last step we used that ν-almost everywhere we have

$$0 \leqslant \tfrac12 r\mathbf{1}_{\{|b|>\frac12 r\}}x \leqslant |b|x \leqslant |b|x_\varepsilon \leqslant R_\varepsilon|b|.$$

The lemma immediately follows from this. □

The next lemma gives a sufficient condition for norm convergence in almost order bounded sets. Recall that an element $x^* \in E^*$ in the dual of a Banach lattice E is called *stricly positive* if $\langle |x|, x^*\rangle = 0$ implies $x = 0$.

Lemma 3. *Let E be a Banach lattice with order continuous norm and let F be an almost order bounded subset of E. If $(x_j)_{j\geqslant1}$ is a sequence in F such that $\lim_{j\to\infty}\langle |x_j|, x^*\rangle = 0$ for some strictly positive element $x^* \in E^*$, then $\lim_{j\to\infty} x_j = 0$ in E.*

Proof. Assume the contrary and choose sequences $j_n \to \infty$ and a number $\delta > 0$ such that $\|x_{j_n}\|_E \geqslant \delta$ for all n. We have

$$\lim_{m,n\to\infty}\langle |x_{j_m} - x_{j_n}|, x^*\rangle \leqslant \lim_{m\to\infty}\langle |x_{j_m}|, x^*\rangle + \lim_{n\to\infty}\langle |x_{j_n}|, x^*\rangle = 0$$

and therefore, by [10, Lemma 3.8], $\lim_{n\to\infty} x_{j_n} = x$ for some $x \in E$. Then $\|x\| \geqslant \delta$ and $0 = \lim_{n\to\infty}\langle |x_{j_n}|, x^*\rangle = \langle |x|, x^*\rangle$. This contradicts the fact that x^* is strictly positive. □

Let X be a Banach space. A set M of Radon probability measures on X is called *uniformly tight* if for every $\varepsilon > 0$ there exists a compact set K in X such that

$$\mu(K) \geqslant 1 - \varepsilon \quad \forall \mu \in M.$$

By Prohorov's theorem for Radon measures [13, Theorem I.3.6], M is uniformly tight if and only if M relatively weakly compact, i.e., every sequence $(\mu_n)_{n\geqslant 1}$ has a subsequence $(\mu_{n_k})_{k\geqslant 1}$ such that for some Radon probability measure μ we have

$$\lim_{k\to\infty} \int_X f\, d\mu_{n_k} = \int_X f\, d\mu \ \text{ for all } f \in C_b(X),$$

where $C_b(X)$ is the space of all scalar-valued bounded continuous functions on X.

We shall formulate the main result of this paper for Banach function spaces E over a probability space $(\Omega, \mathbb{P})$. This is done merely for convenience; the result extends to arbitrary finite measure spaces by a trivial normalization argument.

The space E_X of all strongly $\mathbb{P}$-measurable functions $\phi : \Omega \to X$ such that $\omega \mapsto \|\phi(\omega)\|$ belongs to E is a Banach space with respect to the norm

$$\|\phi\|_{E_X} := \big\| \|\phi\| \big\|_E.$$

Here, as usual, we identify functions that are equal $\mathbb{P}$-almost everywhere. It follows from [9, Proposition 2.6.3] that $\lim_{n\to\infty}\phi_n = \phi$ in E_X implies that for some subsequence we have $\lim_{k\to\infty}\phi_{n_k}(\omega) = \phi(\omega)$ in X for $\mathbb{P}$-almost all $\omega \in \Omega$.

The *distribution* of a function $\phi \in E_X$ is the Radon probability measure μ_ϕ on X defined by

$$\mu_\phi(B) = \mathbb{P}\{\phi \in B\} \ \text{ for } B \subseteq X \text{ Borel}.$$

This definition is independent of the representative of ϕ used to define μ_ϕ.

We call a subset F of E_X:

- *almost order bounded*, if $\{\|\phi\| : \ \phi \in F\}$ is almost order bounded in E;
- *scalarly relatively compact*, if $\{\langle \phi, x^*\rangle : \ \phi \in F\}$ is relatively norm compact in E for all $x^* \in E^*$;
- *uniformly tight*, if $\{\mu_\phi : \ \phi \in F\}$ is uniformly tight.

Lemma 4. *Let F be a subset of E_X. If F is almost order bounded, then also $F - F$ is almost order bounded.*

Proof. Fix $\varepsilon > 0$. Using Lemma 1 we choose $x_\varepsilon \in E^+$ such that $\|\phi\| \in [0, x_\varepsilon] + B(\frac{1}{2}\varepsilon)$ for all $\phi \in F$.

Step 1 – We claim that each $\phi \in F$ can be written as $\phi = f + g$ with $\|f\| \in [0, x_\varepsilon]$ and $g \in B(\frac{1}{2}\varepsilon)$. Indeed, we have

$$\phi = \Big(\mathbf{1}_{\{\|\phi\|\leqslant x_\varepsilon\}}\phi + \mathbf{1}_{\{\|\phi\|> x_\varepsilon\}}\frac{x_\varepsilon}{\|\phi\|}\phi\Big) + \mathbf{1}_{\{\|\phi\|> x_\varepsilon\}}\frac{(\|\phi\| - x_\varepsilon)}{\|\phi\|}\phi.$$

For the first term on the right hand side we have

$$\Big\|\mathbf{1}_{\{\|\phi\|\leqslant x_\varepsilon\}}\phi + \mathbf{1}_{\{\|\phi\|> x_\varepsilon\}}\frac{x_\varepsilon}{\|\phi\|}\phi\Big\| \in [0, x_\varepsilon].$$

Writing $\|\phi\| = a + b$ with $a \in [0, x_\varepsilon]$ and $\|b\|_E < \frac{1}{2}\varepsilon$, for the second term we have

$$\Big\| \mathbf{1}_{\{\|\phi\|>x_\varepsilon\}} \frac{(\|\phi\| - x_\varepsilon)}{\|\phi\|} \phi \Big\| = \mathbf{1}_{\{\|\phi\|>x_\varepsilon\}} (a + b - x_\varepsilon) \leqslant \mathbf{1}_{\{\|\phi\|>x_\varepsilon\}} b,$$

which shows that

$$\Big\| \mathbf{1}_{\{\|\phi\|>x_\varepsilon\}} \frac{(\|\phi\| - x_\varepsilon)}{\|\phi\|} \phi \Big\|_{E_X} \leqslant \|b\|_E < \tfrac{1}{2}\varepsilon.$$

This proves the claim.

Step 2 – Let $\phi_1, \phi_2 \in F$ be given, and write $\phi_k = f_k + g_k$, where $\|f_k\| \in [0, x_\varepsilon]$ and $g_k \in B(\frac{1}{2}\varepsilon)$ for $k = 1, 2$. Then

$$\|\phi_1 - \phi_2\| = \|f_1 - f_2\| + \big(\|\phi_1 - \phi_2\| - \|f_1 - f_2\| \big),$$

with $\|f_1 - f_2\| \in [0, 2x_\varepsilon]$ and

$$\big| \|\phi_1 - \phi_2\| - \|f_1 - f_2\| \big| \leqslant \|g_1 - g_2\|,$$

which shows that $\big\| \|\phi_1 - \phi_2\| - \|f_1 - f_2\| \big\|_E < \varepsilon$. □

Theorem 5. *Let E be a Banach function space with order continuous norm over a probability space $(\Omega, \mathbb{P})$. Let X a Banach space. For a subset F of E_X the following assertions are equivalent:*

1. *The set F is relatively compact;*
2. *The set F is uniformly tight, almost order bounded, and scalarly relatively compact.*

As has been mentioned above, every separable Banach function space has order continuous norm.

Proof. Without loss of generality we may assume that E is *saturated*, i.e., that $f \equiv 0$ on A for all $f \in E$ implies $\mathbb{P}(A) = 0$ [14, Section 67].

(1)⇒(2): It is clear that the relative compactness of F implies its almost order boundedness and scalar relative compactness.

To prove the uniform tightness of F, by Prohorov's theorem it suffices to show that every sequence $(\phi_n)_{n\geqslant 1}$ in F has a subsequence $(\phi_{n_j})_{j\geqslant 1}$ whose distributions converge weakly.

Let us write $\mu_n := \mu_{\phi_n}$ for simplicity. Since F is compact we may assume, by passing to a subsequence, that $(\phi_n)_{n\geqslant 1}$ converges in E_X to an element $\phi \in E_X$. By passing to a further subsequence we may also assume that the convergence takes place almost surely. Let $\mu := \mu_\phi$ be the distribution of ϕ. Then for all $f \in C_b(X)$ we have, by dominated convergence,

$$\lim_{n\to\infty} \int_X f \, d\mu_n = \lim_{n\to\infty} \int_\Omega f \circ \phi_n \, d\mathbb{P} = \int_\Omega f \circ \phi \, d\mathbb{P} = \int_X f \, d\mu.$$

(2)⇒(1): Let $(\phi_n)_{n\geqslant 1}$ be a sequence in F. We shall prove that some subsequence $(\phi_{n_j})_{j\geqslant 1}$ converges in E_X.

Step 1 – Let $\nu_{n,m}$ denote distribution of the random variable $\phi_n - \phi_m$. We claim that the family $(\nu_{n,m})_{n,m\geqslant 1}$ is uniformly tight. The proof is standard and runs as follows. Fix some $\varepsilon > 0$. Since $(\mu_n)_{n\geqslant 1}$ is uniformly tight we may choose a compact set $K \subseteq X$ such that $\mu_n(K) \geqslant 1 - \varepsilon$ for all $n \geqslant 1$. The set $L = \{x - y : x, y \in K\}$ is compact as well, being the image of the compact set $K \times K$ under the continuous map $(x, y) \mapsto x - y$. Noting that $\phi_n(\omega) \in K$ and $\phi_m(\omega) \in K$ implies $\phi_n(\omega) - \phi_m(\omega) \in L$, the claim now follows from

$$\begin{aligned}\nu_{n,m}(L) &\geqslant \mathbb{P}\{\phi_n \in K,\ \phi_m \in K\} \\ &\geqslant 1 - \big(\mathbb{P}\{\phi_n \in \complement K\} + \mathbb{P}\{\phi_m \in \complement K\}\big) = 1 - \big(\mu_n(K) + \mu_m(K)\big) \geqslant 1 - 2\varepsilon.\end{aligned}$$

Step 2 – Since F is uniformly tight, we may assume X to be separable. Let $(x_m^*)_{m\geqslant 1}$ be a sequence in X^* whose intersection with every ball is weak*-dense. As before we let μ_n denote the distribution of ϕ_n. Prohorov's theorem implies the existence of a weakly convergent subsequence $(\mu_{n_j})_{j\geqslant 1}$. By passing to a subsequence we may assume that the limit $\psi_m := \lim_{j\to\infty}\langle \phi_{n_j}, x_m^*\rangle$ exists in E for all m and that the convergence happens almost surely.

We claim that $\nu_{n_j,n_k} \to \delta_0$ weakly as $j, k \to \infty$, where δ_0 denotes the Dirac measure concentrated at 0. Let $j_l \to \infty$ and $k_l \to \infty$. By Step 1 we may pass to a subsequence of the indices l and assume that $\nu_{n_{j_l},n_{k_l}} \to \nu$ for some Radon probability measure ν on X. By taking Fourier transforms, from the almost sure convergence $\lim_{l\to\infty}\langle \phi_{n_{j_l}}, x_m^*\rangle = \lim_{l\to\infty}\langle \phi_{n_{k_l}}, x_m^*\rangle = \psi_m$ we see that for all m,

$$\widehat{\nu}(x_m^*) = \lim_{l\to\infty} \widehat{\nu_{n_{j_l},n_{k_l}}}(x_m^*) = \lim_{l\to\infty}\int_\Omega \exp(-i\langle \phi_{n_{j_l}} - \phi_{n_{k_l}}, x_m^*\rangle)\,d\mathbb{P} = 1 = \widehat{\delta_0}(x_m^*)$$

by dominated convergence. Noting that the weak*-topology of every ball in X^* is metrizable, combined with the fact that the Fourier transforms of Radon probability measures are weak*-sequentially continuous, it follows that $\widehat{\nu} = \widehat{\delta_0}$. Therefore $\nu = \delta_0$ by the uniqueness of the Fourier transform. Since the sequences j_l and k_l were arbitrary, this proves the claim.

Step 3 – It remains to show that the sequence $(\phi_{n_j})_{j\geqslant 1}$ is Cauchy in E_X.

For $j, k \geqslant 1$ define the functions $g_{jk} \in E$ by

$$g_{jk} := \|\phi_{n_j} - \phi_{n_k}\|.$$

For $n \geqslant 1$ choose $r_n \geqslant 0$ so large that

$$\|\mathbf{1}_{\{g_{jk}>r_n\}} g_{jk}\|_E < \tfrac{1}{n} \quad \text{for all } j, k \geqslant 1.$$

This is possible since $F - F$ is almost order bounded by Lemma 4. By Lemma 2, $\|F - F\|$ is uniformly E-integrable.

Let $f \in C_b(\mathbb{R})$ be arbitrary. By Step 2 and Prohorov's theorem,

$$\lim_{j,k\to\infty}\int_\Omega f \circ g_{jk}\, d\mathbb{P} = f(0).$$

Keeping $n \geqslant 1$ fixed for the moment and taking $f(t) = |t| \wedge r_n$, it follows that there exists an index $N_n \geqslant 1$ such that

$$\int_\Omega g_{jk} \wedge r_n \, d\mathbb{P} < \tfrac{1}{n} \quad \text{for all } j, k \geqslant N_n.$$

Let $0 \leqslant \psi_0 \leqslant \mathbf{1}$ be a $\mathbb{P}$-almost everywhere strictly positive function belonging to the associate space E', which is defined as the space of all ν-measurable functions ψ on S such that

$$\|\psi\|_{E'} := \sup_{\|\phi\|_E \leqslant 1}\int_\Omega |\phi\psi|\, d\mathbb{P} < \infty.$$

Such a function exists since E is assumed to be saturated. Note that ψ_0 is strictly positive as element of E^*. For $j, k \geqslant N_n$,

$$0 \leqslant \langle g_{jk}, \psi_0\rangle \leqslant \langle g_{jk} \wedge r_n, \psi_0\rangle + \langle \mathbf{1}_{\{g_{jk}>r_n\}} g_{jk}, \psi_0\rangle < \tfrac{1}{n}(1 + \|\psi_0\|_{E'}).$$

It follows that $\lim_{j,k\to\infty}\langle g_{jk},\psi_0\rangle = 0$. Now Lemma 3 shows that $\lim_{j,k\to\infty} g_{jk} = 0$ in E. $\square$

As in [5], the uniform tightness assumption in assertion (2) may be replaced by flat concentration. This follows from Prohorov's theorem in combination with the well known result of de Acosta [1], see also [13, Theorem I.3.7], that a family M of Radon probability measures on X is uniformly tight if and only if M is flatly concentrated and for all $x^* \in E^*$ the set of image measures $\langle M, x^*\rangle = \{\langle \mu, x^*\rangle : x^* \in E^*\}$ is uniformly tight.

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