Continuity and representation of Gaussian Mehler semigroups

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Abstract - We present sufficient conditions on a Gaussian Mehler semigroup on a reflexive Banach space E to be induced by a single positive symmetric operator $Q \in \mathcal{L}(E^*, E)$, and give a counterexample which shows that this representation theorem is false in every non-reflexive Banach space with a Schauder basis. We also show that the transition semigroup of a Gaussian Mehler semigroup on a separable Banach space E acts in a pointwise continuous way, uniformly on compact subsets of E, in the space $BUC(E)$ of bounded uniformly continuous real-valued funtions on E. The transition semigroup is shown to be strongly continuous on $BUC(E)$ if and only if $S(t) = I$ for all $t \geqslant 0$.

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0. Introduction

Let E be a real Banach space and let $\mu = \{\mu_t\}_{t \geqslant 0}$ be a one-parameter family of probability measures defined on the $\sigma-$algebra Σ generated by the *cylindrical* subsets of E, i.e. sets V of the form

$$V = \{x \in E : \ (\langle x, x_1^* \rangle, \ \ldots \ \langle x, x_n^* \rangle) \in B\},$$

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where $x_1^*, \ldots x_n^* \in E^*$ and $B \subset \mathbb{R}^n$ is a Borel set. Let $\mathbf{S} = \{S(t)\}_{t\geqslant 0}$ be a strongly continuous semigroup of linear operators (briefly, a C_0-semigroup) on E. The pair $(\mathbf{S}, \mu)$ is called a *Mehler semigroup* on E if on Σ we have

$$\mu_{t+s} = (S(s)\mu_t) * \mu_s, \qquad t, s \geqslant 0, \tag{0.1}$$

where $S(s)\mu_t$ denotes the image measure of μ_t under $S(s)$. Mehler semigroups were introduced by Bogachev, Röckner, and Schmuland [BRS] in an axiomatic approach to transition semigroups of non-symmetric Ornstein-Uhlenbeck processes.

If E is separable it is easy to see [BRS, Proposition 2.2] that $(\mathbf{S}, \mu)$ is a Mehler semigroup if and only if the so-called Mehler formula

$$P(t)f(x) := \int_E f(S(t)x - y)\, d\mu_t(y), \qquad t \geqslant 0,\, x \in E, \tag{0.2}$$

defines a semigroup $\{P(t)\}_{t\geqslant 0}$ of bounded linear operators on $B_b(E)$, the space of bounded real-valued Borel functions on E. In the application to stochastic abstract Cauchy problems, this semigroup can be interpreted as the transition semigroup of the Ornstein-Uhlenbeck process solving the Cauchy problem. We refer to [BRS] for more details.

A pair $(\mathbf{S}, \mu)$, where $\mathbf{S}$ is a C_0-semigroup on E and μ is a one-parameter family of *cylindrical* probability measures on the field $\mathcal{F}$ generated by the cylindrical sets in E, will be called a *cylindrical Mehler semigroup* on E if (0.1) holds on $\mathcal{F}$. If for each $t \geqslant 0$ there exists a bounded linear operator $Q_t \in \mathcal{L}(E^*, E)$, the *covariance* of μ_t, such that the Fourier transform of μ_t is given by

$$\hat{\mu_t}(x^*) = \exp\left(-\frac{1}{2}\langle Q_t x^*, x^*\rangle\right), \qquad x^* \in E^*, \tag{0.3}$$

then $(\mathbf{S}, \mu)$ will be called a *Gaussian* cylindrical Mehler semigroup.

For more information concerning C_0-semigroups and cylindrical probability measures, the reader is referred to [Pa] and [VTC], where also unexplained terminology may be found. A compresensive treatment of stochastic abstract Cauchy problems in infinite-dimensional spaces is presented in [DZ].

There is a canonical way of constructing Gaussian cylindrical Mehler semigroups as follows. An operator $Q \in \mathcal{L}(E^*, E)$ is called *positive* if $\langle Qx^*, x^*\rangle \geqslant 0$ for all $x^* \in E^*$, and *symmetric* if $\langle Qx^*, y^*\rangle = \langle Qy^*, x^*\rangle$ for all $x^* \in E^*$ and $y^* \in E^*$. One can show [Ne2] that $s \mapsto S(s)QS^*(s)x^*$ is a strongly measurable function on $[0, \infty)$, and therefore for $t \geqslant 0$ we may define positive symmetric operators $Q_t \in \mathcal{L}(E^*, E)$ by

$$Q_t x^* := \int_0^t S(s)QS^*(s)x^*\, ds, \qquad x^* \in E^*. \tag{0.4}$$

By [VTC, Section IV.3.1], for each $t \geqslant 0$ there exists a unique Gaussian cylindrical probability measure μ_t on $\mathcal{F}$ whose Fourier transform is given by (0.3). It is easy to check that $(\mathbf{S}, \mu)$, where $\mu = \{\mu_t\}_{t\geqslant 0}$, is a cylindrical Mehler semigroup on E. Gaussian cylindrical Mehler semigroups arising in this way will be called *canonical*. Note that if $(\mathbf{S}, \mu)$ is canonical, then

$$\limsup_{t\downarrow 0} \frac{1}{t}\|Q_t\| < \infty. \tag{0.5}$$

It turns out that if E is reflexive, (0.5) is also sufficient for a Gaussian cylindrical Mehler semigroup to be canonical; this is the main result of Section 1. It improves [BRS, Proposition 4.3], where it was proved that $(\mathbf{S}, \mu)$ is canonical if E is a separable Hilbert space and the functions $t \mapsto \langle Q_t x^*, x^*\rangle$ are differentiable at $t = 0$ for all $x^* \in E^*$.

We will further show how to construct non-canonical Gaussian cylindrical Mehler semigroups verifying (0.5) in any non-reflexive Banach space with a Schauder basis. Thus, the reflexivity condition in our representation theorem is close to being necessary.

Let us now suppose that E is separable and that $(\mathbf{S}, \mu)$ is a Gaussian Mehler semigroup on E. Let the semigroup $\mathbf{P} = \{P(t)\}_{t\geqslant 0}$ on $B_b(E)$ be defined by (0.2). It is easy to see that each operator $P(t)$ maps $BUC(E)$, the space of bounded uniformly continuous real-valued functions on E, into itself. The restriction of $\mathbf{P}$ to $BUC(E)$ will be called the *transition semigroup* of $(\mathbf{S}, \mu)$ and, by slight abuse of notation, will also be denoted by $\mathbf{P}$.

Under the additional assumptions that E is a Hilbert space and $(\mathbf{S}, \mu)$ is canonial, the following results are well-known:

(i) The transition semigroup $\mathbf{P}$ is pointwise continuous, uniformly on compact subsets of E [Ce];
(ii) The transition semigroup $\mathbf{P}$ is strongly continuous if and only if for all $t \geqslant 0$ we have $S(t) = I$, the identity operator on E [NZ].

For arbitrary Gaussian Mehler semigroups on a separable real Hilbert space E it is known that $\mathbf{P}$ is pointwise continuous on $BUC(E)$ (in fact, even on $C_b(E)$, the space of bounded continuous real-valued functions on E) [BRS, Lemma 2.1 and Proposition 4.1].

In Section 2 we will extend these results by showing that the transition semigroup of a Gaussian Mehler semigroup on a separable real Banach space E is always pointwise continuous, uniformly on compact subsets of E. The main step in the proof is to show that the measure-valued function $t \mapsto \mu_t$ is weakly continuous on $[0, \infty)$. This was also the main step in [BRS, Lemma 2.1], but our proof is simpler in that it avoids considerations involving the Sazonov topology. We will also show that strong continuity of the transition semigroup is equivalent to $\mathbf{S}$ being the identity semigroup.

1. Canonical Gaussian cylindrical Mehler semigroups

Let $(\mathbf{S},\mu)$ be a Gaussian cylindrical Mehler semigroup on a real Banach space E. In this section we do not assume that E is separable.

By taking Fourier transforms in (0.1), for the covariance operators of the cylindrical measures μ_t we obtain the identity (cf. [BRS, Proposition 4.1])

$$Q_{t+s} = Q_s + S(s)Q_tS^*(s), \qquad t,s \geqslant 0. \tag{1.1}$$

Conversely, if $\{Q_t\}_{t\geqslant 0}$ is a family of positive symmetric operators in $\mathcal{L}(E^*,E)$ satisfying (1.1), and if μ_t are the Gaussian cylindrical probability measures on $\mathcal{F}$ with covariances Q_t, then $(\mathbf{S},\mu)$ is a Gaussian cylindrical Mehler semigroup.

In Theorem 1.2 below, we present sufficient conditions on a Gaussian cylindrical Mehler semigroup $(\mathbf{S},\mu)$ to be canonical, i.e. to be induced by a single positive symmetric operator $Q \in \mathcal{L}(E^*,E)$ through the formula (0.4). This result improves [BRS, Proposition 4.3], where it was proved that $(\mathbf{S},\mu)$ is canonical if E is separable Hilbert and the functions $t \mapsto \langle Q_tx^*,x^*\rangle$ are differentiable at $t=0$ for all $x^* \in E^*$. In fact, under these assumptions the operator Q in (0.4) is uniquely determined by the relation

$$\langle Qx^*,x^*\rangle = \frac{d}{dt}\langle Q_tx^*,x^*\rangle\big|_{t=0}.$$

Notice that by polarization, differentiability at $t=0$ of $\langle Q_tx^*,x^*\rangle$ for all $x^* \in E^*$ implies differentiability at $t=0$ of $\langle Q_tx^*,y^*\rangle$ for all $x^* \in E^*$ and $y^* \in E^*$. Then by applying the uniform boundedness theorem twice one sees that (0.5) holds.

In our first theorem we call an operator $Q \in \mathcal{L}(E,E^*)$ *positive*, resp. *symmetric*, if for all $x \in E$ and $y \in E$ we have $\langle Qx,y\rangle = \langle Qy,x\rangle$, resp. $\langle Qx,x\rangle \geqslant 0$.

Theorem 1.1. *Let $\{S(t)\}_{t\geqslant 0}$ be a C_0-semigroup on a real Banach space E, and let $\{Q_t\}_{t\geqslant 0}$ be a family of positive symmetric operators in $\mathcal{L}(E,E^*)$ such that $Q_{t+s} = Q_s + S^*(s)Q_tS(s)$ for all $t \geqslant 0$ and $s \geqslant 0$. Then the following assertions are equivalent:*

(i) $\limsup_{t\downarrow 0} \frac{1}{t}\|Q_t\| < \infty$;

(ii) There exists a positive symmetric operator $Q \in \mathcal{L}(E,E^*)$ such that

$$Q_tx = \int_0^t S^*(s)QS(s)x\,ds, \qquad \forall\, t \geqslant 0, \quad x \in E.$$

In this situation, Q is the unique operator with these properties.

Remark - The integral is to be understood is the weak*–sense.

Proof: We only need to prove the implication (i)$\Rightarrow$(ii), the converse implication being trivial. For $x \in E$ and $y \in E$, we define $f_{(x,y)} : [0,\infty) \to \mathbb{R}$ by

$$f_{(x,y)}(t) := \langle Q_tx,y\rangle, \qquad t \geqslant 0.$$

For all $t \geqslant 0$, $x \in E$, and $y \in E$ we have

$$\frac{1}{h}(f_{(x,y)}(t+h) - f_{(x,y)}(t)) = \frac{1}{h}\langle Q_hS(t)x,S(t)y\rangle \qquad h > 0,$$

which shows that $f_{(x,y)}$ is locally Lipschitz on $[0,\infty)$.

Fix $x \in E$ and $y \in E$, let $(q_n)_{n\in\mathbb{N}}$ be an enumeration of $\mathbb{Q}_{\geqslant 0}$, and let $\mathcal{N}(x,y)$ be the complement in $[0,\infty)$ of the set of all $t \geqslant 0$ for which the right derivative

$$D^+ f_{(S(q_n)x,y)}(t) := \lim_{h\downarrow 0} \frac{1}{h}(\langle Q_{t+h}S(q_n)x, y\rangle - \langle Q_t S(q_n)x, y\rangle)$$

exists for all $n \in \mathbb{N}$. Since locally Lipschitz functions are differentiable almost everywhere, $\mathcal{N}(x,y)$ is a null set. A $3\varepsilon-$argument shows that $[0,\infty)\backslash\mathcal{N}(x,y)$ is precisely the set of all $t \geqslant 0$ for which the right derivative $D^+ f_{(S(r)x,y)}(t)$ exists for all $r \geqslant 0$. For $t \notin \mathcal{N}(x,y)$ and $s \geqslant 0$ we have, for all $r \geqslant 0$,

$$\begin{aligned}
\lim_{h\downarrow 0} \frac{1}{h}(\langle Q_{t+s+h}S(r)x,y\rangle &- \langle Q_{t+s}S(r)x, y\rangle) \\
&= \lim_{h\downarrow 0} \frac{1}{h}\langle S^*(t+s)Q_h S(t+s+r)x, y\rangle \\
&= \lim_{h\downarrow 0} \frac{1}{h}\langle S^*(s)(Q_{t+h} - Q_t)S(s+r)x, y\rangle \\
&= D^+ f_{(S(s+r)x, S(s)y)}(t).
\end{aligned}$$

This implies $t + s \notin \mathcal{N}(x,y)$ and

$$D^+ f_{(S(r)x,y)}(t+s) = D^+ f_{(S(s+r)x,S(s)y)}(t), \qquad r \geqslant 0.$$

Thus, $t \notin \mathcal{N}(x,y)$ implies $\mathcal{N}(x,y) \cap [t,\infty) = \varnothing$. Since $\mathcal{N}(x,y)$ is null it follows that $\mathcal{N}(x,y) \subset \{0\}$. This being true for all $x \in E$ and $y \in E$ we conclude that $t \mapsto \langle Q_t x, y\rangle$ is differentiable from the right on $(0,\infty)$ for all $x \in E$ and $y \in E$.

Let F denote the linear span in E of the set $\{S(t)x : \ t > 0, \ x \in E\}$. Note that F is dense in E. For $x \in F$ and $y \in F$, say $x := \sum_{j=1}^N a_j S(t_j)x_j$ and $y := \sum_{j=1}^{N'} b_j S(\tau_j)y_j$, we have

$$\begin{aligned}
\lim_{h\downarrow 0} \frac{1}{h}\langle Q_h x, y\rangle &= \lim_{h\downarrow 0} \frac{1}{h} \sum_{j=1}^{N}\sum_{k=1}^{N'} a_j b_k \langle S^*(\delta)Q_h S(\delta)S(t_j - \delta)x_j, S(\tau_k - \delta)y_k\rangle \\
&= \sum_{j=1}^{N}\sum_{k=1}^{N'} a_j b_k D^+ f_{(S(t_j-\delta)x_j, S(\tau_k-\delta)y_k)}(\delta),
\end{aligned}$$

where $\delta := \min\{t_1, ..., t_N, \tau_1, ..., \tau_{N'}\} > 0$. Moreover,

$$\lim_{h\downarrow 0} \frac{1}{h}|\langle Q_h x, y\rangle| \leqslant M\|x\|\,\|y\|,$$

where $M := \limsup_{t\downarrow 0} \frac{1}{t}\|Q_t\|$. By a $3\varepsilon-$argument, this estimate implies that the limit $\lim_{h\downarrow 0} \frac{1}{h}\langle Q_h x, y\rangle$ exists for all $x \in E$ and $y \in E$. Given a fixed $x \in E$ we now define $Qx \in E^*$ by

$$\langle Qx, y\rangle := \lim_{h\downarrow 0} \frac{1}{h}\langle Q_h x, y\rangle, \qquad y \in E.$$

It is clear that $Q \in \mathcal{L}(E, E^*)$, and Q is positive and symmetric. By dominated convergence and the continuity of $s \mapsto \langle Q_s x, y\rangle$ we finally have, for all $t \geqslant 0$, $x \in E$, and $y \in E$,

$$
\begin{aligned}
\int_0^t \langle S^*(s)QS(s)x, y\rangle \, ds &= \lim_{h\downarrow 0} \frac{1}{h} \int_0^t \langle Q_h S(s)x, S(s)y\rangle \, ds \\
&= \lim_{h\downarrow 0} \frac{1}{h} \int_0^t \langle Q_{s+h}x - Q_s x, y\rangle \, ds \\
&= \lim_{h\downarrow 0} \frac{1}{h} \left(\int_t^{t+h} \langle Q_s x, y\rangle \, ds - \int_0^h \langle Q_s x, y\rangle \, ds \right) \\
&= \langle Q_t x, y\rangle.
\end{aligned}
$$

Uniqueness of Q follows from the identity

$$
\langle Qx, y\rangle = \lim_{s\downarrow 0} \left(\frac{d}{dt}\Big|_{t=s} \langle Q_t x, y\rangle \right).
$$

∎

By a standard result from semigroup theory [Pa, Corollary 1.10.6], the adjoint semigroup $\mathbf{S}^*$ on E^* of a C_0−semigroup $\mathbf{S}$ on a reflexive Banach space E is strongly continuous, and we may apply Theorem 1.1 to $\mathbf{S}^*$. This leads to the following result, which, when rephrased in terms of Gaussian cylindrical Mehler semigroups, shows that in reflexive spaces (0.5) is a sufficient condition for canonicity.

Theorem 1.2. *Let $\{S(t)\}_{t\geqslant 0}$ be a C_0−semigroup on a reflexive real Banach space E, and let $\{Q_t\}_{t\geqslant 0}$ be a family of positive symmetric operators in $\mathcal{L}(E^*, E)$ such that $Q_{t+s} = Q_s + S(s)Q_t S^*(s)$ for all $t \geqslant 0$ and $s \geqslant 0$. Then the following assertions are equivalent:*

(i) $\limsup_{t\downarrow 0} \frac{1}{t}\|Q_t\| < \infty$;

(ii) There exists a positive symmetric operator $Q \in \mathcal{L}(E^*, E)$ such that

$$
Q_t x = \int_0^t S(s)QS^*(s)x^* \, ds, \qquad \forall\, t \geqslant 0, \quad x^* \in E^*.
$$

In this situation, Q is the unique operator with these properties.

This following example shows that the reflexivity assumption in Theorem 1.2 is close to being necessary, in the sense that the theorem fails in every non-reflexive Banach space E with a Schauder basis.

Example 1.3. Let E be a non-reflexive real Banach space with a Schauder basis. By a theorem of Zippin [Zi], E has a (possibly different) Schauder basis $(\xi_n)_{n\geqslant 1}$ of norm one vectors which is not boundedly complete. This means that there exists a bounded scalar sequence $(\alpha_n)_{n\geqslant 1}$ such that the sequence $(x_n)_{n\geqslant 1}$ defined by

$$x_n := \sum_{j=1}^{n} \alpha_j \xi_j$$

is bounded in E but fails to converge in E. Let $x_0^{**} \in E^{**}$ be a weak*–limit point of $(x_n)_{n\geqslant 1}$, regarded as a sequence in E^{**}. Note that $x_0^{**} \notin E$. To see this, assume for a moment that $x_0^{**} \in E$. Since $(\xi_n)_{n\geqslant 1}$ is a Schauder basis in E we have an expansion $x_0^{**} = \sum_{j=1}^{\infty} \beta_j \xi_j$. Denoting by $(\xi_n^*)_{n\geqslant 1}$ the sequence of coordinate functionals in E^*, for all $j \geqslant 1$ we have $\langle x_0^{**}, \xi_j^* \rangle = \beta_j$. On the other hand, by definition of x_0^{**}, for each $j \geqslant 1$ we can find a subsequence $(x_{n_k})_{k\geqslant 1}$ such that

$$\langle x_0^{**}, \xi_j^* \rangle = \lim_{k\to\infty} \langle x_{n_k}, \xi_j^* \rangle = \alpha_j.$$

It follows that $\alpha_j = \beta_j$ for all $j \geqslant 1$, and therefore

$$x_0^{**} = \sum_{j=1}^{\infty} \beta_j \xi_j = \sum_{j=1}^{\infty} \alpha_j \xi_j = \lim_{n\to\infty} x_n,$$

a contradiction.

Define $Q \in \mathcal{L}(E^*, E^{**})$ by

$$Qx^* := \langle x^*, x_0^{**} \rangle x_0^{**}.$$

Note that $\langle Qx^*, x^* \rangle \geqslant 0$ and $\langle Qx^*, y^* \rangle = \langle Qy^*, x^* \rangle$ for all $x^* \in E^*$ and $y^* \in E^*$. By [Ne1, Theorem 1.5.2],

$$S(t)\xi_n := e^{-n^3 t} \xi_n, \qquad t \geqslant 0, \quad n \geqslant 1,$$

defines a C_0-semigroup on E.

For $x^* \in E^*$ and $t \geqslant 0$ we now define

$$Q_t x^* := \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{\alpha_k \alpha_n \langle \xi_n, x^* \rangle}{k^3 + n^3} \left(1 - e^{-(k^3+n^3)t}\right) \xi_k.$$

This double sum is absolutely convergent in E. In this way we obtain a bounded linear operator $Q_t \in \mathcal{L}(E^*, E)$ and by dominated convergence it is immediate that Q_t is sequentially continuous from the weak*-topology to the weak topology. Denoting as before the $n-$th coordinate functional in E^* by ξ_n^*, for finite sums $x^* = \sum_{n=1}^{N} b_n \xi_n^*$ and $y^* = \sum_{n=1}^{N} c_n \xi_n^*$ we have the representation

$$\langle Q_t x^*, y^* \rangle = \sum_{n=1}^{N} \sum_{m=1}^{N} b_n c_m \int_0^t \langle QS^*(s)\xi_n^*, S^*(s)\xi_m^* \rangle \, ds = \int_0^t \langle QS^*(s)x^*, S^*(s)y^* \rangle \, ds. \tag{1.2}$$

Hence the properties of Q imply that for such x^* and y^* we have:

(i) $\langle Q_t x^*, x^*\rangle \geqslant 0$,
(ii) $\langle Q_t x^*, y^*\rangle = \langle Q_t y^*, x^*\rangle$,
(iii) $Q_{t+s}x^* = Q_s x^* + S(s)Q_t S^*(s)x^*$ for all $t, s \geqslant 0$,
(iv) For all $\delta \geqslant 0$ and $t \in [0,\delta]$ we have $|\langle Q_t x^*, y^*\rangle| \leqslant tM_\delta^2 \|Q\|\,\|x^*\|\,\|y^*\|$, where $M_\delta = \sup_{t\in[0,\delta]} \|S(t)\|$.

By weak*–to–weak sequential continuity it follows from (i) and (ii) that Q is positive and symmetric and from (iii) that (1.1) holds. Thus if μ_t is the Gaussian cylindrical measure on E with covariance operator Q_t, it follows that the pair $(\mathbf{S}, \mu)$ is a Gaussian cylindrical Mehler semigroup on E, where $\mu = \{\mu_t\}_{t\geqslant 0}$. Denoting by π_n the projection in E onto the first n coordinates, (iv) implies that

$$\limsup_{t\downarrow 0} \frac{1}{t}\|Q_t\| \leqslant M^2 N^2 \|Q\|,$$

where $M = \limsup_{\delta\downarrow 0} M_\delta$ and $N = \limsup_{n\to\infty} \|\pi_n\|$. This shows that (0.5) holds.

Now suppose the family $\{Q_t\}_{t\geqslant 0}$ is generated, in the sense of (0.4), by a positive symmetric operator $\tilde{Q} \in \mathcal{L}(E^*, E)$. Regarding $\tilde{Q}$ as an $E^{**}-$valued operator, (1.2) and the uniqueness part of Theorem 1.1 imply that $Q = \tilde{Q}$. It follows that Q is $E-$valued, a contradiction. Therefore $(\mathbf{S}, \mu)$ is not canonical. ■

The following simple example shows that the limes superior conditions in Theorems 1.1 and 1.2 are not already implied by the other assumptions, even in the case where E is a separable Hilbert space.

Example 1.4. For each $n \geqslant 1$ let $\mathbf{S}_n$ denote the nilpotent shift semigroup on $H_n := L^2(0, \frac{1}{n})$, i.e.

$$S_n(t)f(s) := \begin{cases} f(s-t) & \text{if } s \geqslant t, \\ 0, & \text{otherwise,} \end{cases} \qquad t \geqslant 0, \quad s \in (0, \tfrac{1}{n}).$$

For each $n \geqslant 1$ and $t \geqslant 0$ let $Q_t^{(n)} \in \mathcal{L}(H_n)$ be the positive symmetric operator defined by

$$Q_t^{(n)} h := \sqrt{n} \int_0^t S_n(s)S_n^*(s)h\,ds, \qquad h \in H_n.$$

Since $S_n(t) = 0$ for $t \geqslant \frac{1}{n}$ we have $\|Q_t^{(n)}\| \leqslant 1/\sqrt{n}$ for all $t \geqslant 0$. Moreover, by considering the indicator function of small enough subintervals of $(0, \frac{1}{n})$ it is easy to see that $\|Q_t^{(n)}\| = t\sqrt{n}$ for $t \in (0, \frac{1}{n})$. By the first estimate, on the direct sum $H := \oplus_{n\geqslant 1} H_n$ the operator $Q_t := \oplus_{n\geqslant 1} Q_t^{(n)}$ is a well-defined contraction. Clearly, $Q_{t+s}^{(n)} = Q_s^{(n)} + S_n(s)Q_t^{(n)}S_n^*(s)$ for all $n \geqslant 1$, $s \geqslant 0$, and $t \geqslant 0$. Therefore on H we have $Q_{t+s} = Q_s + S(s)Q_t S^*(s)$ for all $s \geqslant 0$ and $t \geqslant 0$, where $S(t) := \oplus_{n\geqslant 1} S_n(t)$. On the other hand, if $t \in [\frac{1}{N}, \frac{1}{N+1})$ then

$$\frac{1}{t}\|Q_t\| \geqslant \frac{1}{t}\|Q_t^{(N+1)}\| = \sqrt{N+1} \geqslant \sqrt{1 + \frac{1}{t}},$$

which shows that $\limsup_{t\downarrow 0} \frac{1}{t}\|Q_t\| = \infty$. ■

2. Continuity of the transition semigroup

Let E be a separable real Banach space and $(\mathbf{S},\mu)$ a Gaussian Mehler semigroup on E. Let $\mathbf{P}=\{P(t)\}_{t\geqslant 0}$ denote its transition semigroup acting on $BUC(E)$. In this section we will show that $\mathbf{P}$ is pointwise continuous, uniformly on compact subsets of E. Our approach is based upon Anderson's inequality, which we recall first.

Proposition 2.1 [An]. *Let X and Y be two $\mathbb{R}^n-$valued centered Gaussian random variables and assume that for all $x\in\mathbb{R}^n$ we have*

$$\mathbb{E}\,(\langle Y,x\rangle^2)\leqslant\mathbb{E}\,(\langle X,x\rangle^2).$$

Then, for any convex set $C\subset\mathbb{R}^n$,

$$\mathbb{P}\{Y\notin C\}\leqslant 2\,\mathbb{P}\{X\notin C\}.$$

This result has the following simple consequence (cf. [LT, pp. 73-74]): *if E is a separable real Banach space and if X_n $(n\geqslant 1)$ and X are $E-$valued centered Gaussian random variables such that for all $x^*\in E^*$ and $n\geqslant 1$ we have*

$$\mathbb{E}\,(\langle X_n,x^*\rangle^2)\leqslant\mathbb{E}\,(\langle X,x^*\rangle^2),$$

then the sequence $(X_n)_{n\geqslant 1}$ is tight.

The following lemma shows that $\mathbf{P}$ is pointwise continuous on the subspace of bounded continuous cylindrical functions on E. Its proof is a simple application of Lévy's continuity theorem and is presented, for canonical Gaussian Mehler semigroups, in [Ne2]; the argument carries over *verbatim* to the present situation.

Lemma 2.2. *Let $\tilde f:\mathbb{R}^n\to\mathbb{R}$ be a bounded continuous function, let $x_1^*,...,x_n^*\in E^*$, and define $f:E\to\mathbb{R}$ by*

$$f(x):=\tilde f(\langle x,x_1^*\rangle,...,\langle x,x_n^*\rangle),\qquad x\in E.$$

Then for all $x\in E$ we have $\lim_{t\downarrow 0}P(t)f(x)-f(x)=0$.

After these preparations we can prove:

Proposition 2.3. $\lim_{t\downarrow 0}\mu_t=\delta_0$ *(the Dirac measure at* 0*) weakly.*

Proof: Let $t_n\downarrow 0$; we claim that the sequence (μ_{t_n}) is tight. To see this, note that for all $x^*\in E^*$ and $n\geqslant 1$ we have, using (1.1) (cf. [Ne2, Section 7]),

$$\mathbb{E}_{\mu_{t_n}}(\langle\,\cdot\,,x^*\rangle^2)=\langle Q_{t_n}x^*,x^*\rangle\leqslant\langle Q_{t_1}x^*,x^*\rangle=\mathbb{E}_{\mu_{t_1}}(\langle\,\cdot\,,x^*\rangle^2).$$

Therefore tightness follows from Anderson's inequality.

Extract a subsequence $t_{n_k}\downarrow 0$ such that $(\mu_{t_{n_k}})$ is weakly convergent. If f is a bounded continuous cylindrical function on E, then by Lemma 2.2 applied to the function $g(y):=f(-y)$,

$$\lim_{k\to\infty}\int_E f(y)\,d\mu_{t_{n_k}}(y)=\lim_{k\to\infty}P(t_{n_k})g(0)=g(0)=f(0)=\delta_0(f).$$

Since the bounded continuous cylindrical functions separate the points of the set $M_1(E)$ of Borel probability measures on E, it follows that $\lim_{t\downarrow 0}\mu_{t_{n_k}}=\delta_0$ weakly.

We have shown that every sequence $t_n\downarrow 0$ contains a subsequence $t_{n_k}\downarrow 0$ such that $\lim_{t\downarrow 0}\mu_{t_{n_k}}=\delta_0$ weakly. This proves the proposition. ∎

Remark - The same argument can be used to show that the function $t \mapsto \mu_t$ is continuous on $[0, \infty)$ with respect to the weak topology.

We are now prepared to state and prove the main result of this section:

Theorem 2.4. *Let $f \in BUC(E)$ be fixed. Then for all compact sets $K \subset E$ we have*

$$\lim_{t \downarrow 0} \left(\sup_{x \in K} |P(t)f(x) - f(x)| \right) = 0.$$

Proof: First we show that for every $x \in E$,

$$\lim_{t \downarrow 0} |P(t)f(x) - f(x)| = 0. \tag{2.1}$$

Fix $\varepsilon > 0$ and $x \in E$ arbitrary. Using the uniform continuity of f and the strong continuity of $\mathbf{S}$, we choose $t_0 > 0$ small enough such that for all $y \in E$ we have

$$\sup_{t \in [0, t_0]} |f(S(t)x - y) - f(x - y)| \leqslant \varepsilon.$$

Define the function $f_x \in BUC(E)$ by $f_x(y) := f(x - y)$. The weak convergence $\mu_t \to \delta_0$ then implies

$$\begin{aligned}
\limsup_{t \downarrow 0} |P(t)f(x) - f(x)| &= \limsup_{t \downarrow 0} \left| \int_E f(S(t)x - y)\, d\mu_t(y) - f(x) \right| \\
&\leqslant \varepsilon + \limsup_{t \downarrow 0} \left| \int_E f(x - y)\, d\mu_t(y) - f(x) \right| \\
&= \varepsilon + \limsup_{t \downarrow 0} |\langle f_x, \mu_t - \delta_0 \rangle| = \varepsilon.
\end{aligned}$$

This proves (2.1). Next we note that the family $\{P(t)f : t \in [0, 1]\}$ is uniformly equicontinuous on E. Indeed, using the uniform continuity of f we may choose, for any fixed $\varepsilon > 0$, a $\delta_0 > 0$ such that $|f(u_1) - f(u_2)| \leqslant \varepsilon$ whenever $u_1, u_2 \in E$ satisfy $\|u_1 - u_2\| < \delta_0$. Then for all $t \in [0, 1]$, all $y \in E$, and all $v_1, v_2 \in E$ with $\|v_1 - v_2\| < \delta := \delta_0 / \sup_{t \in [0,1]} \|S(t)\|$ we have $\|(S(t)v_1 - y) - (S(t)v_2 - y)\| < \delta_0$, and hence

$$|P(t)f(v_1) - P(t)f(v_2)| \leqslant \int_E |f(S(t)v_1 - y) - f(S(t)v_2 - y)|\, d\mu_t(y) \leqslant \varepsilon.$$

This gives the equicontinuity. By the Arzela-Ascoli theorem, we conclude that the set $\{(P(t)f)|_K : t \in [0, 1]\}$ is relatively compact in the space $C(K)$. Hence every sequence $t_n \downarrow 0$ contains a subsequence $t_{n_k} \downarrow 0$ such that the sequence $(P(t_{n_k})f)|_K$ converges, uniformly on K, to some limit $g \in C(K)$. The proof is complete once we know that $g = f|_K$. But this follows from (2.1): for all $x \in K$ we have $g(x) = \lim_{k \to \infty} P(t_{n_k})f(x) = f(x)$. ∎

Semigroups on $BUC(E)$ which are pointwise continuous, uniformly on compact subsets of E, have been studied from an abstract point of view in [Ce] and [CG].

Under a compactness assumption we have the following stronger continuity result:

Corollary 2.5. *Let $f \in BUC(E)$ and $t_0 > 0$ be fixed. If $S(t_0)$ is a compact operator, then for all bounded sets $B \subset E$ we have*

$$\lim_{h\downarrow 0}\left(\sup_{x\in B}|P(t_0+h)f(x)-P(t_0)f(x)|\right)=0.$$

Proof: Given $\varepsilon > 0$ and a bounded set $B \subset E$, let $K_0 := \overline{S(t_0)B}$ and let $K_1 \subset E$ be a compact set such that $\mu_{t_0}(K_1) > 1-\varepsilon$. Writing $g_h := P(h)f - f$, we have $\lim_{h\downarrow 0} g_h = 0$ uniformly on the compact set $\{y_0 - y_1 : y_0 \in K_0, y_1 \in K_1\}$ by Theorem 2.4, and hence

$$\begin{aligned}
\lim_{h\downarrow 0}&\left(\sup_{x\in B}|P(t_0+h)f(x)-P(t_0)f(x)|\right)\\
&= \lim_{h\downarrow 0}\left(\sup_{x\in B}\left|\int_E g_h(S(t_0)x-y)\,d\mu_{t_0}(y)\right|\right)\\
&\leqslant 2\varepsilon\,\|f\| + \lim_{h\downarrow 0}\left(\sup_{x\in B}\int_{K_1} g_h(S(t_0)x-y)\,d\mu_{t_0}(y)\right)\\
&= 2\varepsilon\,\|f\|.
\end{aligned}$$

∎

For canonical Gaussian Mehler semigroups on a Hilbert space E, the subspace of all $f \in BUC(E)$ on which $\mathbf{P}$ acts in a strongly continuous way was investigated in [DL]. The following corollary, which is a straightforward application of Theorem 2.4, extends the criterion obtained there to gauusian Mehler semigroups in a Banach space setting.

Corollary 2.6. *For a function $f \in BUC(E)$ the following assertions are equivalent:*

(i) $\lim_{t\downarrow 0}\left(\sup_{x\in E}|P(t)f(x)-f(x)|\right)=0;$

(ii) $\lim_{t\downarrow 0}\left(\sup_{x\in E}|f(S(t)x)-f(x)|\right)=0.$

This characterization of strong continuity was used in [NZ] to show that strong continuity of $\mathbf{P}$ on all of $BUC(E)$, with E a separable real Hilbert space and $(\mathbf{S},\mu)$ is canonical, implies that $S(t) = I$ for all $t \geqslant 0$. This implies that $Q_t = tQ$ for some positive symmetric $Q \in \mathcal{L}(E)$ which is necessarily the covariance of a centered Gaussian Borel measure μ on E. In fact we have $\mu(B) = \mu_t(\sqrt{t}B)$ for all Borel sets $B \subset E$ and all $t \geqslant 0$. Thus we see that $\mathbf{P}$ is the Wiener semigroup corresponding to μ.

By virtue of Corollary 2.6, the proof in [NZ] extends to the more general setting considered here and we obtain:

Corollary 2.7. *If $\mathbf{P}$ is strongly continuous on $BUC(E)$, then $S(t) = I$ for all $t \geqslant 0$, and there is a centered Gaussian Borel measure μ on E such that $\mu(B) = \mu_t(\sqrt{t}B)$ for all Borel sets $B \subset E$ and all $t \geqslant 0$.*

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