ON CLOSABILITY OF DIRECTIONAL GRADIENTS

B. GOLDYS, F. GOZZI, AND J.M.A.M. VAN NEERVEN

ABSTRACT. Let μ be a centred Gaussian measure on a separable real Banach space E, and let H be a Hilbert subspace of E. We provide necessary and sufficient conditions for closability in $L^p(E,\mu)$ of the gradient D_H in the direction of H. These conditions are further elaborated in case when the gradient D_H corresponds to a bilinear form associated with a certain nonsymmetric Ornstein-Uhlenbeck operator. Some natural examples of closability and nonclosability are presented.

1. INTRODUCTION

Let E be a separable real Banach space with norm $\|\cdot\|_E$ and let (i,H) be a Hilbert subspace of E, that is, H is a separable real Hilbert space and $i: H \hookrightarrow E$ is a continuous inclusion. Let μ be a Gaussian measure on E with the covariance operator C. In this paper we study the closability in $L^p(E,\mu)$ of the gradient operator D_H corresponding to differentiation in the direction of H.

If D_H is closable in $L^p(E,\mu)$, we can define the Sobolev space $W_H^{1,p}(E,\mu)$ as the domain of the closure $\overline{D_H}$ endowed with the graph norm

$$\|\phi\|_{1,p} = \left(\|\phi\|_p^p + \left\|\overline{D_H}\phi\right\|_p^p\right)^{1/p},$$

and the space $W_H^{1,p}(E,\mu)$ may be identified with a subspace of $L^p(E,\mu)$.

The question whether D_H is closable in $L^p(E,\mu)$ is of some importance in the theory of diffusion processes and associated second order parabolic PDE's in finite and infinite dimensions. In particular, if D_H is closable in $L^2(E,\mu)$ then, under some additional assumptions, a symmetric Dirichlet form can be associated to it and the corresponding symmetric diffusion process can be constructed, see [17] for a thorough exposition of this theory. By perturbations, this question is also important for the study of nonsymmetric diffusions, see for example [21] and [11]. Another application arises in the theory of optimal control of stochastic partial differential equations and it was in fact the main motivation of this paper, see [15] for details. Consider a controlled stochastic differential equation

$$\begin{cases} dX^u(t,x) = \left(AX^u(t,x) + a\left(X^u(t,x)\right) - u(t)\right)dt + dW_H(t), \\ X^u(0,x) = x \in E, \end{cases}$$

in a separable Hilbert space E, where u is a bounded control taking values in H and W is a standard cylindrical Wiener process with H being its reproducing kernel. A control u

---

2000 *Mathematics Subject Classification.* Primary: 35R15, 60H15; Secondary: 46G05, 58D25, 47D03.

Key words and phrases. Gaussian measure, gradient operator, closability, Ornstein-Uhlenbeck process, invariant measure, C_0-semigroup.

This work was partially supported by the Small ARC Grant Scheme.

should be chosen so as to minimize the cost functional

$$J(T,x,u)=\mathbb{E}\left(\int_0^T\left(f\left(t,X^u(t,x)\right)+h(u(t))\,dt+\phi\left(X^u(T,x)\right)\right)\right).$$

A well known approach to this problem is to show the existence and uniqueness of solutions to the Hamilton-Jacobi equation

$$\left\{\begin{array}{l}\frac{\partial u}{\partial t}(t,x)=Lu(t,x)-F\left(D_Hu(t,x)\right)+f(t,x),\\ u(0,x)=\phi(x),\end{array}\right. \tag{1.1}$$

where L is a generator of the linearized process Z obtained for $a=u=0$ and F is the Hamiltonian of the system and then to use the Dynamic Programming Principle to identify the optimal control and the optimal cost. If we assume that there exists a nondegenerate invariant measure μ for Z then equation (1.1) can be studied in the space $L^2\left(E,\mu\right)$ and if D_H is closable then the existence and uniqueness of solutions to (1.1) follows from the Fixed Point Theorem, even for irregular data f and ϕ. This argument can be extended to other, non Gaussian cases, and again the closability of D_H is an important ingredient.

Despite the importance of the closability question it seems that there are not too many definitive results in the infinite-dimensional case. A very general necessary and sufficient condition may be found in [1], see also the discussion of this problem in Section II.3 of [17]. However, this condition is rather difficult to check in particular cases.

After some preliminaries in Section 2 we provide in Section 3 necessary and sufficient conditions for the closability of D_H in $L^p(E,\mu)$ for $p\in[1,\infty)$ and a formula for the divergence operator D_H^*. Let us note that the sufficient condition for closability and the formula for D_H^* is known to specialists, but not easily available in the literature in the formulation given here. The necessity of the condition seems to be new.

In Section 4 this condition is used to prove closability of D_H in case we have $H_C\subseteq H$ with dense inclusion, where H_C is a certain Hilbert space canonically associated with C; see Section 2. In this situation we also obtain necessary and sufficient conditions for compactness of the embedding $W_H^{1,2}(E,\mu)\hookrightarrow L^2(E,\mu)$.

In Section 5 we will be concerned with the case when the measure μ arises as an invariant measure of the Ornstein-Uhlenbeck process $\{Z(t)\}_{t\geqslant 0}$ solving a linear equation

$$\left\{\begin{array}{l}dZ(t)=AZ(t)dt+dW_H(t),\\ Z(0)=x\in E,\end{array}\right. \tag{1.2}$$

on E; here A is assumed to be the generator of a C_0-semigroup $\{S(t)\}_{t\geqslant 0}$ on E and $\{W_H(t)\}_{t\geqslant 0}$ is a standard cylindrical Wiener process with Cameron-Martin space H.

Under appropriate assumptions, formulated in Section 5, equation (1.2) has a unique solution given by

$$Z(t,x)=S(t)x+\int_0^t S(t-s)\,dW_H(s). \tag{1.3}$$

The process $\{Z(t,x)\}_{t\geqslant 0}$ is Markovian, and if it has an invariant measure μ, then the transition semigroup

$$R(t)f(x)=\mathbb{E}\,f(Z(t,x))$$

defines a C_0-semigroup of contractions on $L^p(E,\mu)$ for all $p\in[1,\infty)$. The symmetric bilinear form associated with the generator L of the semigroup $\{R(t)\}_{t\geqslant 0}$ in $L^2(E,\mu)$ is given by the formula

$$\mathscr{E}(f,g) = \frac{1}{2}\int_E f(x)Lg(x) + g(x)Lf(x)\,d\mu(x) = -\frac{1}{2}\int_E [D_H f(x), D_H g(x)]_H\,d\mu(x).$$

Therefore the question whether the form $\mathscr{E}$ is closable in $L^2(E,\mu)$, (see pp. 28-31 of [17] for details) is equivalent to the problem of closability of D_H in $L^2(E,\mu)$, hence in $L^p(E,\mu)$ by the result in Section 3.

In Section 6 we reformulate the general results in the important case when E is a Hilbert space.

Finally, in Section 7 we apply the general theory developed in previous sections to present some natural examples of closability and nonclosability of Ornstein-Uhlenbeck operators. In particular, we show that D_H is not closable if H is one-dimensional and A generates the semigroup of left shifts in $L^2(0,\infty)$, or if (1.2) is the abstract formulation of a finite dimensional stochastic equation with delays.

2. PRELIMINARIES

Let E be a separable real Banach space and let $C \in \mathscr{L}(E^*,E)$ be positive and symmetric, that is $\langle Cx^*, x^*\rangle \geqslant 0$ and $\langle Cx^*, y^*\rangle = \langle Cy^*, x^*\rangle$ for all $x^*, y^* \in E^*$. On the range of C, the formula

$$[Cx^*, Cy^*] := \langle Cx^*, y^*\rangle, \qquad x^*, y^* \in E^*,$$

defines an inner product. The completion of range C with respect to this inner product is denoted by H_C. This is a separable real Hilbert space, the *reproducing kernel Hilbert space* (RKHS) associated with C. The inclusion mapping from range C into E extends to a continuous inclusion, denoted by i_C, of H_C into E. Thus, (i_C, H_C) is a Hilbert subspace of E, and we have the operator identity

$$C = i_C \circ i_C^*.$$

The inner product of H_C will be denoted by $[\cdot,\cdot]_{H_C}$.

If μ is a centred Gaussian measure on E, then its covariance operator $C \in \mathscr{L}(E^*,E)$ is positive and symmetric, as may be seen from the identity

$$\int_E \langle x, x^*\rangle\,\langle x, y^*\rangle\,d\mu(x) = \langle Cx^*, y^*\rangle\,, \quad x^*, y^* \in E^*.$$

The mapping

$$\varphi : i_C^* x^* \mapsto \langle\,\cdot\,, x^*\rangle \in L^2(E,\mu)$$

is well defined and extends to an isometric isomorphism from H_C onto a closed linear subspace $\mathscr{H}_C$ of $L^2(E,\mu)$. Instead of $\varphi(h)$ we shall write φ_h.

For $p \in [1,\infty)$ we denote by $L^p(E,\mu)$ the Banach space of p-integrable functions $f : E \to \mathbb{R}$ endowed with the norm

$$\|f\|_p = \left(\int_E \|f(x)\|^p\,d\mu(x)\right)^{1/p}.$$

For $f, g \in L^2(E,\mu)$ we write

$$[f,g] = \int_E f(x)g(x)\,d\mu(x).$$

If H is a real Hilbert space, then in a similar way we define the space $L^p(E,\mu;H)$ of H-valued p-integrable functions on E. We use the notation

$$\langle f,g\rangle_H = \int_E [f(x),g(x)]_H \, d\mu(x)$$

for the duality between $L^p(E,\mu;H)$ and $L^q(E,\mu;H)$, $\frac{1}{p}+\frac{1}{q}=1$. For $f\in L^p(E,\mu;H)$ and $h\in H$ we write $[f,h]_H$ to denote the function whose value at the point $x\in E$ is $[f(x),h]_H$. The inner product of $L^2(E,\mu;H)$ is also denoted by $[\cdot,\cdot]_H$.

3. THE ABSTRACT RESULTS

As before we let E be a separable real Banach space and μ a centred Gaussian measure on E with covariance operator C. Let (i,H) be a Hilbert subspace of E.

Lemma 3.1. *For each $x^*\in E^*$, the function $\langle\,\cdot\,,x^*\rangle$ is Fréchet differentiable in the direction of H. Its derivative $D_H\langle\,\cdot\,,x^*\rangle : E\to H$ is the constant function*

$$D_H\langle\,\cdot\,,x^*\rangle = \mathbf{1}\otimes i^*x^*.$$

Proof. For all $x\in E$ and $h\in H$ we have, upon identifying h and ih,

$$\langle x+h,x^*\rangle - \langle x,x^*\rangle = \langle h,x^*\rangle = [h,i^*x^*]_H,$$

which shows that $\big(D_H\langle\,\cdot\,,x^*\rangle\big)(x) = i^*x^*$. ∎

Let $p\in[1,\infty)$ be fixed. By $\mathscr{F}C_b^1(E)$ we denote the linear subspace of $L^p(E,\mu)$ consisting of all functions $\Phi: E\to\mathbb{R}$ of the form

$$\Phi(x) = \phi(\langle x,x_1^*\rangle,\,\dots,\,\langle x,x_k^*\rangle) \tag{3.1}$$

for certain $k\geqslant 1$, $x_1^*,\dots x_k^*\in E^*$ and $\phi\in C_b^1(\mathbb{R}^k)$, the space of continuously differentiable bounded functions on $\mathbb{R}^k$ with bounded derivative. The space $\mathscr{F}C_b^1(E)$ is a dense subspace of $L^p(E,\mu)$; as a subspace of $L^\infty(E,\mu)$ it is weak*-dense.

If $\Phi\in\mathscr{F}C_b^1(E)$ is given by (3.1), then Φ is Fréchet differentiable in the direction of H with derivative

$$\big(D_H\Phi\big)(x) = \sum_{j=1}^k \frac{\partial\phi}{\partial x_j}\big(\langle x,x_1^*\rangle,\,\dots,\,\langle x,x_k^*\rangle\big)\otimes i^*x_j^*. \tag{3.2}$$

Thus we can define a densely defined linear operator $(D_H,\mathscr{D}(D_H))$ from $L^p(E,\mu)$ into $L^p(E,\mu;H)$ with domain

$$\mathscr{D}(D_H) := \mathscr{F}C_b^1(E)$$

by putting

$$\big(D_H(\Phi)\big)(x) := (D_H\Phi)(x), \qquad x\in E, \quad \Phi\in\mathscr{D}(D_H).$$

Let $Q\in\mathscr{L}(E^*,E)$ be defined by $Q := i\circ i^*$. Throughout this section we make the following

Assumption 3.2. $\ker C\subseteq\ker Q$.

This assumption is for instance satisfied if μ is nondegenerate, in which case we have $\ker C=\{0\}$.

We define a densely defined operator $(V,\mathscr{D}(V)) : H_C\to H$ by

$$\mathscr{D}(V) := \{i_C^*x^* :\ x^*\in E^*\},$$
$$V(i_C^*x^*) := i^*x^*, \qquad i_C^*\,x^*\in\mathscr{D}(V).$$

Thanks to Assumption 3.2 and the identities $i_C \circ i_C^* = C$ and $i \circ i^* = Q$, the operator V is well defined.

Lemma 3.3. *For all $\Phi \in \mathscr{D}(D_H)$ and $h \in \mathscr{D}(V^*)$ we have*

$$\int_E [D_H\Phi, h]_H \, d\mu = \int_E \Phi \, \varphi_{V^*h} \, d\mu.$$

Proof. Let $x^* \in E^*$ be given. Upon identifying V^*h and $i_C(V^*h)$ we have

$$\begin{aligned}\partial_{V^*h}\langle \cdot\,, x^*\rangle(x) &= \lim_{t\downarrow 0}\langle x + tV^*h, x^*\rangle - \langle x, x^*\rangle \\ &= \langle V^*h, x^*\rangle = [V^*h, i_C^*x^*]_{H_C} = [i^*x^*, h]_H.\end{aligned}$$

Let Φ be given by (3.1). Then,

$$\begin{aligned}\partial_{V^*h}\Phi(x) &= \sum_{j=1}^k \frac{\partial\phi}{\partial x_j}\big(\langle x, x_1^*\rangle, \, \dots, \, \langle x, x_k^*\rangle\big)\partial_{V^*h}\langle \cdot\,, x_j^*\rangle(x) \\ &= \sum_{j=1}^k \frac{\partial\phi}{\partial x_j}\big(\langle x, x_1^*\rangle, \, \dots, \, \langle x, x_k^*\rangle\big)[i^*x_j^*, h]_H = [D_H\Phi(x), h]_H.\end{aligned}$$

Hence by [2, Theorem 5.1.8],

$$\int_E [D_H\Phi, h]_H \, d\mu = \int_E \partial_{V^*h}\Phi \, d\mu = \int_E \Phi \, \varphi_{V^*h} \, d\mu. \tag{3.3}$$

∎

We will be interested in conditions under which $(D_H, \mathscr{D}(D_H))$ is closable as operator from $L^p(E,\mu)$ into $L^p(E,\mu;H)$. The following general criterion for closability can be found in many textbooks.

Proposition 3.4. *Let X and Y be Banach spaces. A linear operator $(S, \mathscr{D}(S))$ from X into Y is closable if there exists a weak*-densely defined linear operator $(T, \mathscr{D}(T))$ from Y^* into X^* which is adjoint to S in the sense that*

$$\langle Sx, y^*\rangle = \langle x, Ty^*\rangle, \qquad x \in \mathscr{D}(S), \; y^* \in \mathscr{D}(T).$$

Conversely, if $(S, \mathscr{D}(S))$ is closable and densely defined, then its adjoint $(S^*, \mathscr{D}(S^*))$ is weak*-densely defined.

After these preparations we can prove the main result of this section.

Theorem 3.5. *Let $1 \leqslant p < \infty$. The following assertions are equivalent:*

1. *The operator $(D_H, \mathscr{D}(D_H))$ is closable from $L^p(E,\mu)$ into $L^p(E,\mu;H)$;*
2. *The operator $(V, \mathscr{D}(V))$ is closable from H_C into H.*

If these conditions hold, then $\mathscr{D}(D_H) \otimes \mathscr{D}(V^*) \subseteq \mathscr{D}(D_H^*)$ and

$$D_H^*(\Psi \otimes h) = \Psi \, \varphi_{V^*h} - [D_H\Psi, h]_H, \qquad \Psi \in \mathscr{D}(D_H), \; h \in D(V^*).$$

Proof. 1⇒2: Suppose $h_n \to 0$ in H_C, with $h_n \in \mathscr{D}(V)$ for all n, and $Vh_n \to g$ in H. We have to show that $g = 0$.

Write $h_n = i_C^*x_n^*$ with $x_n^* \in E^*$. Then

$$i^*x_n^* = Vh_n \to g \;\text{ in }\; H$$

and

$$\langle \cdot\,, x_n^*\rangle = \varphi_{i_C^*x_n^*} = \varphi_{h_n} \to 0 \;\text{ in }\; L^2(E,\mu).$$

After passing to a subsequence if necessary, we may assume that $\lim_{n\to\infty}\langle x, x_n^*\rangle = 0$ for μ-almost all $x \in E$. Let $\phi \in C_b^1(\mathbb{R})$ satisfy $\phi(0) = 0$ and $\phi'(0) = 1$. Then for the function $\Phi_n := \phi(\langle\,\cdot\,, x_n^*\rangle) \in \mathscr{D}(D_H)$ we have

$$\lim_{n\to\infty}\Phi_n(x) = \lim_{n\to\infty}\phi(\langle x, x_n^*\rangle) = \phi(0) = 0 \quad \text{for } \mu\text{-almost all } x \in E,$$

and therefore

$$\lim_{n\to\infty}\Phi_n = 0 \ \text{ in } \ L^p(E,\mu)$$

by dominated convergence. On the other hand,

$$D_H\Phi_n = \phi'(\langle\,\cdot\,, x_n^*\rangle) \otimes i^*x_n^*.$$

Hence,

$$\lim_{n\to\infty}\|D_H\Phi_n - \mathbf{1}\otimes g\|_{L^p(E,\mu;H)}^p = \lim_{n\to\infty}\int_E \big\|\phi'(\langle x, x_n^*\rangle)\otimes i^*x_n^* - g\big\|_H^p\,d\mu(x) = 0$$

by dominated convergence, where we used that $\lim_{n\to\infty}\phi'(\langle\,\cdot\,, x_n^*\rangle) = \phi'(0) = 1$ μ-a.e. and $\lim_{n\to\infty} i^*x_n^* = g$. But D_H being closable, this forces that $g = 0$.

2⇒1: From (3.3) we deduce that for all $h \in \mathscr{D}(V^*)$ and $\Phi, \Psi \in \mathscr{D}(D_H)$,

$$\begin{aligned}\langle\Phi, \Psi\,\varphi_{V^*h}\rangle &= \int_E \Phi\Psi\,\varphi_{V^*h}\,d\mu\\ &= \int_E [D_H(\Phi\Psi), h]_H\,d\mu\\ &= \int_E [(D_H\Phi)\Psi, h]_H\,d\mu + \int_E [\Phi D_H(\Psi), h]_H\,d\mu\\ &= [D_H\Phi, \Psi\otimes h]_H + \langle\Phi, [D_H\Psi, h]_H\rangle.\end{aligned} \tag{3.4}$$

Hence,

$$\langle D_H\Phi, \Psi\otimes h\rangle = \langle\Phi, \Psi\,\varphi_{V^*h}\rangle - \langle\Phi, [D_H\Psi, h]_H\rangle = \langle\Phi, T(\Psi\otimes h)\rangle,$$

where

$$T(\Psi\otimes h) := \Psi\varphi_{V^*h} - [D_H\Psi, h]_H, \qquad \Psi \in \mathscr{D}(D_H),\ h \in D(V^*).$$

It follows that the operator $(T, \mathscr{D}(T))$, with domain $\mathscr{D}(T) = \mathscr{D}(D_H)\otimes\mathscr{D}(V^*)$, from $L^q(E,\mu;H)$ into $L^q(E,\mu)$ ($\frac{1}{p}+\frac{1}{q} = 1$) is adjoint to $(D_H, \mathscr{D}(D_H))$.

Since V is closable, $\mathscr{D}(V^*)$ is densely defined. Therefore the domain $\mathscr{D}(T)$ is weak*-dense in $L^q(E,\mu;H)$. It follows from Proposition 3.4 that $(D_H, \mathscr{D}(D_H))$ is closable. ∎

Example 3.6. It is well known that the directional derivative D_{H_C}, the so-called *Malliavin derivative* associated with μ, is closable. This follows immediately from the theorem by taking $H = H_C$, in which case we have $V = I$.

Example 3.7. Let $b \in E$, $\|b\| = 1$, be given and let H be the one-dimensional subspace of E spanned by b. Denoting the inclusion mapping of H into E by i, we have $i\circ i^* = b\otimes b$. Assuming that $\ker C \subseteq \ker Q$, we claim that D_H *is closable if and only if* $b \in H_C$ (in order to simplify notation we identify both H and H_C with linear subspaces of E). This will be used in Section 7.

To see this, first note that every densely defined linear operator on H is everywhere defined. Therefore by Proposition 3.4, D_H is closable if and only there exists a linear operator $V^* : H \to H_C$ with

$$[Vh, b]_H = [h, V^*b]_{H_C}, \qquad h \in H_C.$$

If such V^* exists, then $V \circ i_C^* = i^*$ implies $(i_C \circ V^*)b = ib = b$. Hence $b = V^*b \in H_C$.

Conversely, if $b \in H_C$, then $V^*b := b$ defines an operator from H into H_C which is adjoint to V.

We close this section by commenting on the rôle of $\mathscr{F}C_b^1(E)$ as the initial domain on which D_H is defined. Clearly, if D_H is closable on the domain $\mathscr{F}C_b^1(E)$, then D_H is also closable on any smaller domain, e.g. on $\mathscr{F}C_c^\infty(E)$. Moreover, by a modification of the arguments above, D_H is also closable on the domain $\mathscr{F}P(E)$ consisting of all cylindrical polynomials.

4. COMPACTNESS OF THE EMBEDDING $W_H^{1,2}(E,\mu) \hookrightarrow L^2(E,\mu)$

Throughout this section *we assume that Assumption 3.2 holds*. We recall that this is for instance the case if μ is nondegenerate. We also fix $p \in [1,\infty)$.

In this section we will analyze the case where we have an inclusion $H_C \subseteq H$. In our first result, which is rather general, we identify H with a linear subspace of E.

Theorem 4.1. *Suppose there exists a linear subspace Y of E^* such that $C(Y)$ is dense in H. Then the operator $(D_H, \mathscr{D}(D_H))$ is closable from $L^p(E,\mu)$ into $L^p(E,\mu;H)$.*

Proof. To check that $(V, \mathscr{D}(V))$ is closable we show that its adjoint is densely defined.

Let j denote the linear operator from Y into H given by $jx^* := Cx^*$ for $x^* \in Y$. Note that for all $x^* \in Y$ we have $ijx^* = Cx^*$. Hence for all $x^* \in Y$ and $y^* \in E^*$,

$$[jx^*, V(i_C^* y^*)]_H = [jx^*, i^*y^*]_H = \langle ijx^*, y^*\rangle = \langle Cx^*, y^*\rangle = [i_C^* x^*, i_C^* y^*]_{H_C}.$$

This shows that $j(Y) \subseteq \mathscr{D}(V^*)$ and

$$V^*(jx^*) = i_C^* x^*, \qquad x^* \in Y.$$

By assumption, $j(Y)$ is dense in H, so V^* is densely defined. ∎

It is well known (cf. [7, Appendix B]) that $H_C \subseteq H$ (as subsets of E) if and only if there exist a constant $K \geqslant 0$ such that

$$\langle Cx^*, x^*\rangle \leqslant K\langle Qx^*, x^*\rangle, \qquad x^* \in E^*,$$

or equivalently,

$$\|i_C^* x^*\|_{H_C}^2 \leqslant K\|i^*x^*\|_H^2, \qquad x^* \in E^*.$$

In this situation the map $V^{-1} : i^*x^* \mapsto i_C^* x^*$ is well defined and extends to a bounded linear operator, also denoted by V^{-1}, from H into H_C.

Corollary 4.2. *Assume that $H_C \subseteq H$. The following assertions are equivalent:*

(1) *The operator $(D_H, \mathscr{D}(D_H))$ is closable from $L^p(E,\mu)$ into $L^p(E,\mu;H)$;*
(2) *H_C is dense in H.*

Proof. Noting that the set $\{i_C^* x^* : x^* \in E^*\}$ is dense in H_C, the implication $2 \Rightarrow 1$ follows immediately from Theorem 4.1. It remains to prove that 1 implies 2. Let us denote the inclusion mapping $H_C \hookrightarrow H$ by j. For all $x^*, y^* \in E^*$ we have

$$\begin{aligned}
[j(i_C^* x^*), i^*y^*]_H &= \langle (i \circ j)(i_C^* x^*), y^*\rangle \\
&= \langle i_C(i_C^* x^*), y^*\rangle \\
&= [i_C^* x^*, i_C^* y^*]_{H_C} \\
&= [i_C^* x^*, V^{-1}(i^*y^*)]_{H_C}.
\end{aligned}$$

It follows that $j = (V^{-1})^*$. Let us suppose now that H_C is not dense in H. Then there exists $0 \neq g \in H$ such that $[jh, g]_H = 0$ for all $h \in H_C$. Choose a sequence (x_n^*) in E^* such that $\lim_{n\to\infty} i^* x_n^* = g$ in H. For all $h \in H_C$ we have

$$\lim_{n\to\infty}[h, i_C^* x_n^*]_{H_C} = \lim_{n\to\infty}[h, V^{-1}(i^* x_n^*)]_{H_C} = \lim_{n\to\infty}[jh, i^* x_n^*]_H = [jh, g]_H = 0.$$

This shows that $\lim_{n\to\infty} i_C^* x_n^* = 0$ weakly in H_C. By the Hahn-Banach theorem we may choose convex combinations y_n^* of the elements from the sequence (x_n^*) such that

$$\lim_{n\to\infty} i_C^* y_n^* = 0$$

strongly in H_C; by choosing y_n^* in the convex hull of $\{x_k^* : \ k \geqslant n\}$ we further arrange that $\lim_{n\to\infty} i^* y_n^* = g$. But then

$$\lim_{n\to\infty} V(i_C^* y_n^*) = \lim_{n\to\infty} i^* y_n^* = g \neq 0,$$

and it follows that V is not closable. ∎

Under the assumption that $H_C \subseteq H$ densely, the following result gives necessary and sufficient conditions for the embedding $W_H^{1,2}(E,\mu) \hookrightarrow L^2(E,\mu)$ to be compact:

Theorem 4.3. *Let $H_C \subseteq H$ with dense inclusion. Then the following assertions are equivalent:*

(1) *The inclusion $W_H^{1,2}(E,\mu) \hookrightarrow L^2(E,\mu)$ is compact;*
(2) *The inclusion $H_C \hookrightarrow H$ is compact;*
(3) *The operator $V^{-1} : H \to H_C$ is compact.*

Proof. 1⇒3: If V^{-1} is not compact we can find a bounded sequence (h_n) in H such that $(V^{-1}h_n)$ fails to be totally bounded in H_C. Define $f_n = \varphi_{V^{-1}h_n}$. Then $f_k \in \mathscr{D}(\overline{D_H})$, $\overline{D_H} f_n = 1 \otimes h_n$, and and

$$\|f_n\|^2_{W_H^{1,2}(E,\mu)} = \|\varphi_{V^{-1}h_n}\|^2_{L^2(E,\mu)} + \|1 \otimes h_n\|^2_{L^2(E,\mu;H)} = \|V^{-1}h_n\|^2_{H_C} + \|h_n\|^2_H.$$

This shows that (f_n) is bounded in $W_H^{1,2}(E,\mu)$. On the other hand, from

$$\|f_n - f_m\|_{L^2(E,\mu)} = \|V^{-1}h_n - V^{-1}h_m\|_{H_C}$$

it follows that (f_n) is not totally bounded in $L^2(E,\mu)$. We conclude that the inclusion $W_H^{1,2}(E,\mu) \subseteq L^2(E,\mu)$ fails to be compact.

3⇒2: Let us denote the inclusion mapping $H_C \hookrightarrow H$ by j. As we have seen in the proof of Corollary 4.2 we have $j^* = V^{-1}$. The compactness of V^{-1} therefore implies that j^*, and hence j, is compact.

2⇒1: This follows from [19, Theorem 3.1]. ∎

For Hilbert spaces E and $p \in (1,\infty)$ it is proved in [5] that compactness of the embedding $W_H^{1,p}(E,\mu) \hookrightarrow L^p(E,\mu)$ is equivalent to the above conditions 2 and 3. If E is a Hilbert space and each eigenvector of C is also an eigenvector for Q, our characterization reduces to the one obtained in [9]. In both papers the result is formulated in terms of the operator $D_{Q^{\frac12}}$ introduced in Section 6. Further embedding results in the Hilbert space case may be found in [12, 13].

We conclude this section with a result that in some sense parallels Theorem 4.1.

Theorem 4.4. *Suppose there exists a linear subspace Y of E^* such that $Q(Y)$ is dense in H_C. Then the operator $(D_H, \mathscr{D}(D_H))$ is closable from $L^p(E,\mu)$ into $L^p(E,\mu;H)$.*

Proof. To check that $(V, \mathscr{D}(V))$ is closable we show that its adjoint is densely defined.

Let j denote the linear operator from Y into H_C given by $jx^* := Qx^*$ for $x^* \in Y$. Note that for all $x^* \in Y$ we have $i_C jx^* = Qx^*$. Hence all $x^* \in Y$ and $y^* \in E^*$,

$$[i^*x^*, V(i_C^* y^*)]_H = [i^*x^*, i^*y^*]_H = \langle Qx^*, y^*\rangle = [jx^*, i_C^* y^*]_{H_C}.$$

This shows that $i^*(Y) \subseteq \mathscr{D}(V^*)$ and

$$V^*(i^*x^*) = jx^*, \qquad x^* \in Y.$$

By the assumption on Y, this shows that V^* is densely defined. ∎

For Hilbert spaces E, this result was obtained by Fuhrman [14]. Theorem 4.1 may be more useful, however, as the practical applicability of Theorem 4.4 seems to be restricted mainly to the case where E is Hilbertian and $Q = I$ (in which case Theorem 2.6 applies as well).

5. GRADIENTS ASSOCIATED TO ORNSTEIN-UHLENBECK OPERATORS

As before, E is a separable real Banach space and (i, H) is a Hilbert subspace of E. In this section we apply our abstract results to the situation where μ is an invariant measure of the stochastic Cauchy problem

$$\begin{cases} dZ(t) = AZ(t)\,dt + dW_H(t), \\ Z(0) = x. \end{cases} \tag{5.1}$$

Here $\{W_H(t)\}_{t\geqslant 0}$ is a standard cylindrical Wiener process with Cameron-Martin space H and A is the generator of a C_0-semigroup $\mathbf{S} = \{S(t)\}_{t\geqslant 0}$ on E. A *weak solution* of equation (5.1) is a predictable $E-$valued stochastic process $\{Z(t)\}_{t\geqslant 0}$ such that for all $x^* \in D(A^*)$ the function $s \mapsto \langle Z(s), A^*x^*\rangle$ is almost surely integrable on $[0, T]$ and

$$\langle Z(t), x^*\rangle = \langle S(t)x, x^*\rangle + \int_0^t \langle Z(s), A^*x^*\rangle\,ds + [W_H(t), i^*x^*]_H, \qquad t \geqslant 0.$$

For more details we refer to [3], where it is shown that equation (5.1) has a unique weak solution (for some, and hence for all, $x \in E$) if and only if for all $t \geqslant 0$ the operator $Q_t \in \mathscr{L}(E^*, E)$ defined by

$$Q_t\,x^* = \int_0^t S(s)QS^*(s)x^*\,ds, \qquad x^* \in E^*,$$

where $Q = i \circ i^*$, is the covariance operator of a centred Gaussian measure μ_t on E.

Throughout this section we make the following

Assumption 5.1. *Equation* (5.1) *has an invariant measure* μ_∞*, whose covariance operator* Q_∞ *is given by the improper integral*

$$\langle Q_\infty x^*, y^*\rangle = \int_0^\infty \langle S(s)QS^*(s)x^*, y^*\rangle\,ds, \qquad x^*, y^* \in E^*.$$

This assumption is satisfied whenever the family of measures $\{\mu_t\}_{t\geqslant 0}$ is tight; in this case we have $\mu_t \to \mu_\infty$ weakly.

The following lemma shows that Assumption 5.1 implies Assumption 3.2 for $C = Q_\infty$.

Lemma 5.2. $\ker Q_\infty \subseteq \ker Q$.

Proof. Let (i_∞, H_∞) denote the RKHS associated with the operator Q_∞; cf. Section 2. Suppose $Q_\infty x^* = 0$ for some $x^* \in E^*$. Then $i_\infty^* x^* = 0$ and from

$$\|i_\infty^* x^*\|_{H_\infty}^2 = \langle Q_\infty x^*, x^*\rangle = \int_0^\infty \langle S(s)QS^*(s)x^*, x^*\rangle\, ds = \int_0^\infty \|i^*S^*(s)x^*\|_H^2\, ds$$

it follows that $i^*S^*(s)x^* = 0$ for almost all $s \geqslant 0$. But $\ker i^*$ is weak*-closed, and therefore the weak*-continuity of $\mathbf{S}^*$ shows that $i^*x^* = 0$. Hence $Qx^* = 0$. ∎

In the results that follow we will derive various sufficient conditions for closability of the gradient D_H in $L^p(E, \mu_\infty)$. We will always assume $p \in [1, \infty)$ to be fixed.

The next result is concerned with the case when $L = -\frac{1}{2}D_H^* D_H$ in $L^2(E, \mu_\infty)$ which is studied in [6] and [16]. Hence L is symmetric in $L^2(E, \mu_\infty)$ and the closability of D_H follows from the general theory as presented for example in [17]. Here we provide a short and independent argument.

Theorem 5.3. *If $S(t)Q = QS^*(t)$ for all $t \geqslant 0$, then $(D_H, \mathscr{D}(D_H))$ is closable as an operator from $L^p(E, \mu_\infty)$ into $L^p(E, \mu_\infty; H)$.*

Proof. We need to check that the operator $(V, \mathscr{D}(V))$ is closable from H_∞ into H. Suppose that (x_n^*) is a sequence in E^* such that $i_\infty^* x_n^* \to 0$ in H and $i^*x_n^* = V(i_\infty^* x_n^*) \to g$ in H. Then $Qx_n^* = i(i^*x_n^*) \to ig$ in E and therefore

$$\begin{aligned}
\int_0^1 \|S(s)ig\|^2\, ds &= \lim_{n\to\infty} \int_0^1 \|S(s)Qx_n^*\|^2\, ds \\
&= \lim_{n\to\infty} \int_0^1 \|QS^*(s)x_n^*\|^2\, ds \\
&\leqslant \|i\|^2 \limsup_{n\to\infty} \int_0^1 \|i^*S^*(s)x_n^*\|_H^2\, ds \\
&\leqslant \|i\|^2 \limsup_{n\to\infty} \int_0^\infty \|i^*S^*(s)x_n^*\|_H^2\, ds \\
&= \|i\|^2 \limsup_{n\to\infty} \|i_\infty^* x_n^*\|_{H_\infty}^2 \\
&= 0.
\end{aligned}$$

Hence $\|S(s)ig\| = 0$ for all $s \in [0, 1]$, and hence $g = 0$ since $\mathbf{S}$ is strongly continuous and i is injective. ∎

Next we will analyze what happens if H is $\mathbf{S}-$invariant.

Lemma 5.4. *Suppose $S(t)H \subseteq H$ for all $t \geqslant 0$ and define the operators $S_H(t) : H \to H$ by restriction. Define*

$$\begin{aligned}
\mathscr{D}(T) &:= \{i^*x^* : \ x^* \in E^*\}, \\
T(i^*x^*)(t) &:= S_H^*(t)i^*x^*, \qquad t > 0,\ i^*x^* \in \mathscr{D}(T).
\end{aligned}$$

For each $i^*x^* \in \mathscr{D}(T)$, the function $T(i^*x^*)$ belongs to $L^2((0, \infty); H)$, and the operator $(T, \mathscr{D}(T))$ is closable from H into $L^2((0, \infty); H)$.

Proof. The closed graph theorem shows that $S_H(t)$ is bounded for each $t \geqslant 0$. By dualizing, from $i \circ S_H(t) = S(t) \circ i$ it follows that

$$S_H^*(t)i^*x^* = i^*S^*(t)x^*, \qquad t \geqslant 0,\ x^* \in E^*.$$

Hence,

$$\int_0^\infty \|T(i^*x^*)(t)\|_H^2\,dt = \int_0^\infty \|i^*S^*(t)x^*\|_H^2\,dt = \|i_\infty^*x^*\|_{H_\infty}^2. \tag{5.2}$$

Hence $T(i^*x^*) \in L^2((0,\infty);H)$ and $\|T(ix^*)\|_{L^2((0,\infty);H)} = \|i_\infty^*x^*\|_{H_\infty}$ for all $i^*x^* \in \mathscr{D}(T)$.

Next suppose $i^*x_n^* \to 0$ in H and $T(i^*x_n^*) \to f$ in $L^2((0,\infty);H)$. Passing if necessary to a pointwise a.e. convergent subsequence, we have for almost all $t > 0$:

$$f(t) = \lim_{n\to\infty} T(i^*x_n^*)(t) = \lim_{n\to\infty} S_H^*(t)i^*x_n^* = 0.$$

∎

With the notation introduced in this lemma we have the following result:

Theorem 5.5. *Suppose $S(t)H \subseteq H$ for all $t \geqslant 0$. The following assertions are equivalent:*

(1) *The operator $(D_H, \mathscr{D}(D_H))$ is closable from $L^p(E,\mu_\infty)$ into $L^p(E,\mu_\infty;H)$;*
(2) $\ker \overline{T} = \{0\}$.

Proof. Suppose $x^* \in \ker Q$. Then $x^* \in \ker i^*$, and in view of $i^*S^*(t)x^* = S_H^*(t)i^*x^* = 0$ it follows from the second identity in (5.2) that $x^* \in \ker Q_\infty$. In combination with Lemma 5.2 we conclude that $\ker Q_\infty = \ker Q$. It follows that V is a bijection from $\mathscr{D}(V)$ onto $\mathscr{D}(T)$, and for its inverse $V^{-1} : \mathscr{D}(T) \to \mathscr{D}(V)$ we find from (5.2)

$$\|T(i^*x^*)\|_{L^2((0,\infty);H)} = \|V^{-1}i^*x^*\|_{H_\infty}, \qquad i^*x^* \in \mathscr{D}(T).$$

Since $(T, \mathscr{D}(T))$ is closable, so is $(V^{-1}, \mathscr{D}(T))$; moreover we see that $\ker \overline{T} = \ker \overline{V^{-1}}$.

1⇒2: Since $(D_H, \mathscr{D}(D_H))$ is closable by assumption, $(V, \mathscr{D}(V))$ is closable. Suppose $h \in \ker \overline{V^{-1}}$. Then there is a sequence $i^*x_n^* \to h$ in H with $V^{-1}i^*x_n^* \to 0$. From $V(V^{-1}i^*x_n^*) = i^*x_n^* \to h$ and the closability of V it follows that $0 = \overline{V}0 = h$. Hence, $\ker \overline{T} = \ker \overline{V^{-1}} = \{0\}$.

2⇒1: If $\ker \overline{T} = \{0\}$, then $\ker \overline{V^{-1}} = \{0\}$. Now suppose $i_\infty^*x_n^* \to 0$ in H_∞ and $Vi_\infty^*x_n^* = i^*x_n^* \to y$ in H. From $V^{-1}i^*x_n^* = i_\infty^*x_n^* \to 0$ it follows that $y \in \mathscr{D}(\overline{V}^{-1})$ and $\overline{V^{-1}}y = 0$. Hence $y = 0$. This shows that V, and therefore D_H, is closable. ∎

Corollary 5.6. *Suppose $S(t)H \subseteq H$ for all $t \geqslant 0$ and assume that the semigroup $\mathbf{S}_H$ is strongly continuous on H. Then the operator $(D_H, \mathscr{D}(D_H))$ is closable from $L^p(E,\mu_\infty)$ into $L^p(E,\mu_\infty;H)$.*

Proof. Suppose $h \in \mathscr{D}(\overline{T})$ and let $i^*x_n^* \to h$ in H be such that $T(i^*x_n^*) \to \overline{T}h$ in $L^2((0,\infty);H)$. Passing if necessary to a pointwise a.e. convergent subsequence, we see that for almost all $t > 0$,

$$\overline{T}h(t) = \lim_{n\to\infty} T(i^*x_n^*)(t) = \lim_{n\to\infty} S_H^*(t)i^*x_n^* = S_H^*(t)h.$$

Hence $\overline{T}h(t) = S_H^*(t)h$ for almost all $t > 0$. If $\overline{T}h = 0$, then the strong continuity of $\mathbf{S}_H^*$ implies $h = 0$, and therefore $\ker \overline{T} = \{0\}$. ∎

Remark 5.7. Mutatis mutandis, the results of this section can be extended to the problem

$$\begin{cases} dZ(t) = AZ(t)dt + BdW_H(t), \\ Z(0) = x, \end{cases}$$

where now H is an arbitrary separable real Hilbert space and $B : H \to E$ is bounded and linear. The rôle of H as being the RKHS associated with $Q = i \circ i^*$ is then taken over by the RKHS H_R of the operator $R = B \circ B^*$. As a Hilbert subspace of E, H_R equals the range of B, with norm given by the formula

$$\|Bh\|_{H_R} = \|Ph\|_H, \qquad h \in H,$$

where P is the orthogonal projection in H onto $(\ker B)^\perp$.

6. THE CASE WHEN E IS A HILBERT SPACE

In this section we specify the previous results in the important case when E is a Hilbert space. We will use the framework introduced in books [7, 10].

Let E be a Hilbert space. One may consider the operator $\big(D_{Q^{\frac12}}, \mathscr{D}(D_{Q^{\frac12}})\big)$ acting from $L^p(E,\mu)$ into $L^p(E,\mu;E)$ defined by $\mathscr{D}(D_{Q^{\frac12}}) = \mathscr{F}C_b^1(E)$ and

$$D_{Q^{\frac12}}\Phi = Q^{\frac12} \circ D\Phi, \qquad \Phi \in \mathscr{D}(D_{Q^{\frac12}}),$$

where D is the Fréchet derivative (into the direction of E). If

$$\Phi(x) = \phi(\langle x, x_1^*\rangle, \dots, \langle x, x_k^*\rangle)$$

for certain $x_1^*, \dots, x_k^* \in E$ and $\phi \in C_b^1(\mathbb{R}^k)$, then

$$D_{Q^{\frac12}}\Phi(x) = \sum_{j=1}^k \frac{\partial\phi}{\partial x_j}\big(\langle x, x_1^*\rangle, \dots, \langle x, x_k^*\rangle\big) \otimes Q^{\frac12}x_j^*,$$

whereas

$$D_H\Phi(x) = \sum_{j=1}^k \frac{\partial\phi}{\partial x_j}\big(\langle x, x_1^*\rangle, \dots, \langle x, x_k^*\rangle\big) \otimes i^*x_j^*.$$

It follows that $D_H = R \circ D_{Q^{\frac12}}$, where $R := Q^{\frac12}$ as an operator from E onto H. Since for all $x^* \in E$ we have

$$\|Q^{\frac12}x^*\|_E^2 = \langle Qx^*, x^*\rangle = \|i^*x^*\|_H^2$$

we see that $D_{Q^{\frac12}}$ *is closable as an operator from* $L^p(E,\mu)$ *into* $L^p(E,\mu;E)$ *if and only if* D_H *is closable from* $L^p(E,\mu)$ *into* $L^p(E,\mu;H)$.

We can also reformulate the closability of V in terms of the operators $C^{\frac12}$ and $Q^{\frac12}$. To this end we define an operator $(W, \mathscr{D}(W)) : E \to E$ by

$$\mathscr{D}(W) := \{C^{\frac12}x : \ x \in E\} = H_C,$$
$$W(C^{\frac12}x) := Q^{\frac12}x \qquad (x \in E).$$

Identifying E and its dual, we have

$$Q^{\frac12}W(C^{\frac12}x) = Qx = i(i^*)x = iV(i_C^*x), \qquad x \in E.$$

If we think of $C^{\frac12}$ and $Q^{\frac12}$ as bounded operators from E onto $H_C = \mathrm{Im}\,(C^{\frac12})$ and $H = \mathrm{Im}\,(Q^{\frac12})$ respectively, we can rewrite this as

$$Q^{\frac12}Wh = VC^{\frac12}h, \qquad h \in \mathrm{Im}\,(C^{\frac12}).$$

We claim: *if* $\ker Q = \{0\}$*, then* W *is closable in* E *if and only if* V *is closable from* H_C *into* H.

Assume first that W is closable, and suppose that $h_n \to 0$ in H_C and $Vh_n \to y$ in H. Since $\ker C \subset \ker Q = \{0\}$ by Assumption 3.2, the operator $C^{\frac{1}{2}} : E \to H_C$ is an isomorphism. Denoting $x_n := C^{-\frac{1}{2}} h_n$ we have $x_n \to 0$ in E and $Q^{\frac{1}{2}} W x_n = V C^{\frac{1}{2}} x_n = V h_n \to y$ in H. The operator $Q^{\frac{1}{2}} : E \to H$ is an isomorphism as well, and therefore $W x_n \to Q^{-\frac{1}{2}} y$ in E. The closability of W gives $Q^{-\frac{1}{2}} y = 0$, so $y = 0$.

Assume now that V is closable, and suppose that $x_n \to 0$ and $W x_n \to y$ in E. Then $C^{\frac{1}{2}} x_n \to 0$ in H_C and $V C^{\frac{1}{2}} x_n = Q^{\frac{1}{2}} W x_n \to Q^{\frac{1}{2}} y$ in H. The closability of V then gives $Q^{\frac{1}{2}} y = 0$. It follows that $y \in \ker Q = \{0\}$, so $y = 0$. This proves the claim.

Combining what we just proved with Theorem 3.5, we obtain:

Theorem 6.1. *Assume that* $\ker Q = \{0\}$*. Then the following assertions are equivalent:*

(1) *The operator* D_H *is closable from* $L^p(E, \mu)$ *into* $L^p(E, \mu; H)$*;*
(2) *The operator* $D_{Q^{\frac{1}{2}}}$ *is closable from* $L^p(E, \mu)$ *into* $L^p(E, \mu; E)$*;*
(3) *The operator* W *is closable in* E*.*

In a similar way we can reformulate the conditions for the compact imbedding of $W_H^{1,2}(E, \mu)$ into $L^2(E, \mu)$.

Let us finally consider the equation

$$\begin{cases} dZ(t) = AZ(t)dt + dW_H(t), \\ Z(0) = x, \end{cases}$$

on E. If E is a Hilbert space, a weak solution is given by the variation of constants formula

$$Z(t) = S(t) + \int_0^t S(t-s)dW_H(s), \quad t \geqslant 0,$$

provided the operators Q_t are of trace class on E and the existence of the invariant measure μ_∞ amounts to the assumption that Q_∞ is of trace class as well. In this case we may take $C = Q_\infty$ and all the results of Section 5 hold as well.

7. EXAMPLES

In this section we use the previously developed theory to study the closability and nonclosability of some Ornstein-Uhlenbeck operators. In particular we provide two examples of nonclosability (a first order equation and a delay equation) which show that such a 'bad' behaviour may appear quite naturally. We also present an example where closability follows from the results of Section 3.

7.1. **Stochastic PDE of first order.**

Let us consider the stochastic partial differential equation

$$\begin{cases} dy(t) = \frac{\partial y}{\partial x}\, dt + b\, dB(t), \\ y(0) = y_0, \end{cases} \tag{7.1}$$

where $\{B(t)\}_{t\geqslant 0}$ is a standard Brownian motion and $b, y_0 \in E = L^2(0, \infty)$. Without loss of generality we may assume that $\|b\| = 1$. This process is closely related to the Gaussian Musiela model of interest rates, see [18] for more details.

In order to apply our results of the previous section we let (i, H) denote the one-dimensional Hilbert subspace of E spanned by the function b. Then $b\,B(t)$ may be identified in the natural way with a standard Wiener process $W_H(t)$ whose Cameron-Martin space is H. Let

$$S(t)x(\zeta) = x(t+\zeta)$$

denote the semigroup of left shifts on E. Then the E-valued process

$$Z(t) = S(t)y_0 + \int_0^t S(t-s)b\,dB(s) = S(t)y_0 + \int_0^t \big(S(t-s)\circ i\big)\,dW_H(s)$$

is the solution to (7.1). Let us make the following assumptions:

$$\int_0^\infty \|S(t)b\|^2 dt < \infty, \tag{7.2}$$

$$\overline{\operatorname{lin}\{S(t)b : t \geqslant 0\}} = E. \tag{7.3}$$

It follows from the results in [7, chapter 11], that (7.2) and (7.3) are necessary and sufficient for the process $\{Z(t)\}_{t\geqslant 0}$ to have a unique nondegenerate invariant measure μ_∞, whose covariance operator Q_∞ is given by

$$Q_\infty f = \int_0^\infty S(t)(b\otimes b)S^*(t)f\,dt.$$

We will show that D_H is not closable from $L^p(E,\mu_\infty)$ into $L^p(E,\mu;H)$ for any $p \in [1,\infty)$.

Assume, for a contradiction, that D_H is closable for some p. Then by Example 3.7, $b \in H_\infty$. By the result in [7, Appendix B] this implies the existence of a function $u \in L^2(0,\infty)$ such that

$$\int_0^\infty S(s)b\,u(s)\,ds = b.$$

Then for almost all $\zeta > 0$ we have

$$b(\zeta) = \int_0^\infty b(\zeta+s)u(s)\,ds \tag{7.4}$$

and hence by the Cauchy-Schwarz inequality,

$$|b(\zeta)| \leqslant \left(\int_\zeta^\infty b^2(s)\,ds\right)^{\frac12} \|u\|_2\,. \tag{7.5}$$

It follows that $b \in L^\infty(0,\infty)$. By a similar estimate, for almost all $0 < \zeta_1 < \zeta_2$ we have

$$\begin{aligned}|b(\zeta_2) - b(\zeta_1)| &\leqslant \int_0^\infty |b(\zeta_2+s) - b(\zeta_1+s)|\,|u(s)|\,ds\\ &\leqslant \left(\int_0^\infty |b(\zeta_2-\zeta_1+v) - b(v)|^2\,dv\right)^{\frac12}\|u\|_2\,,\end{aligned}$$

which shows that b has a continuous representative. Finally, by (7.5),

$$\lim_{\zeta\to\infty} |b(\zeta)| = 0.$$

Therefore, (7.4) can be rewritten in the form

$$\langle S(\zeta)b, \delta_0 - u\rangle = 0, \quad t \geqslant 0.$$

Since $\operatorname{lin}\{S(t)b : t\geqslant 0\}$ is dense in $L^2(0,\infty)$ the functional $\delta_0 - u$ extends to the zero functional on $L^2(0,\infty)$, a contradiction.

Remark 7.1. More generally the above argument implies that each $x \in H_\infty$ can be redefined on a set of measure zero as to become a continuous function on $[0, \infty)$ vanishing at infinity.

7.2. **Delay Equations.** Let us consider the following stochastic differential equation with a delay $r > 0$:

$$\begin{cases} dx(t) = (a_0 x(t) + a_1 x(t-r))\, dt + dB(t), \\ x(0) = x_0,\ x(\theta) = x_1(\theta),\ \theta \in [-r, 0). \end{cases} \tag{7.6}$$

This equation may be rewritten as a stochastic evolution equation in the space $E = \mathbb{R} \times L^2(-r, 0)$. To this end note first that for $h = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \in E$ the equation

$$\begin{cases} \dot{y}(t) = a_0 y(t) + a_1 y(t-r), \\ y(0) = x_0,\ y(\theta) = x_1(\theta), \quad \theta \in [-r, 0), \end{cases} \tag{7.7}$$

can be rewritten as

$$\begin{cases} \dot{Y}(t) = AY(t), \\ Y(0) = h, \end{cases} \tag{7.8}$$

where the operator A on E is defined as

$$\mathscr{D}(A) = \left\{ \begin{pmatrix} f(0) \\ f \end{pmatrix} : \quad f \in W^{1,2}(-r, 0) \right\}$$

$$Af = \begin{pmatrix} a_0 f(0) + a_1 f(-r) \\ df/d\theta \end{pmatrix}, \qquad f \in \mathscr{D}(A).$$

It is well known that the operator A generates a strongly continuous semigroup $\{S(t)\}_{t \geqslant 0}$ in E and that $Y(t) := S(t)h$ is a mild solution of (7.8). Concerning the stochastic equation (7.6) it may be shown (see [4] and [10, Chapter 10]) that it has a unique solution x and the process

$$Z(t) = \begin{pmatrix} x(t) \\ x(t + \cdot) \end{pmatrix} \in E, \quad t \geqslant 0,$$

is the unique mild solution of the stochastic linear evolution equation

$$\begin{cases} dZ(t) = AZ(t)dt + b\, dB(t), \\ Z(0) = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \in E, \end{cases} \tag{7.9}$$

where

$$b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{7.10}$$

We assume that

$$a_0 < 1, \quad a_0 < -a_1 < \sqrt{\gamma^2 + a_0^2}.$$

where $\gamma \in (0, \pi)$ and $\gamma \coth \gamma = a_0$. Under this condition (see e.g. [10, Chapter 10]) equation (7.9) has a unique invariant measure μ_∞ with nondegenerate covariance operator Q_∞. Moreover, the semigroup $\{S(t)\}_{t \geqslant 0}$ is uniformly exponentially stable, implying that the solution $y(\cdot)$ of (7.7) belongs to $L^2(0, \infty)$.

We will show that the operator D_H is not closable on $L^2(E,\mu_\infty)$. By the result in Example 3.7, D_H is closable if and only if $b\in H_\infty$, the RKHS associated with Q_∞. By the result in [7, Appendix B] $b\in H_\infty$ if and only if

$$\int_0^\infty S(s)b\,u(s)ds=b \tag{7.11}$$

for a certain $u\in L^2(0,\infty)$. Consider equation (7.7) with $x_0=1$ and $x_1=0$. It is easy to check that the solution is continuous after time $t=0$ (see also [4]) and

$$S(t)b=\begin{pmatrix} y(t)\\ y(t+\cdot)\end{pmatrix}. \tag{7.12}$$

Assume now that there exists $u\in L^2(0,\infty)$ such that (7.11) holds. Then by (7.12) and (7.10),

$$\int_0^\infty y(s)u(s)ds=1,$$

and

$$\int_0^\infty y(s+\theta)u(s)ds=0,\quad \text{for a.a. } \theta\in[-r,0).$$

Finally, taking into account the strong continuity of the C_0-semigroup of translations in $L^2(0,\infty)$ and the fact that $y\in L^2(0,\infty)$ we see that this is impossible.

Remark 7.2. For simplicity of presentation, we considered a one dimensional case of stochastic delay equations. In fact the same result holds in the following more general situation. Take a linear d-dimensional stochastic delay equation of the following type

$$\begin{cases} dx(t)=\left[a_0x(t)+\sum_{i=1}^N a_ix(t+\theta_i)\right]dt+b\,dB(t),\\ x(0)=x_0,\ x(\theta)=x_1(\theta),\quad \theta\in[-r,0),\end{cases} \tag{7.13}$$

with a finite number of delays

$$-r=\theta_1<\theta_2<\dots<\theta_N<0.$$

We assume that $a_0,a_1,\dots,a_N$ are suitable $d\times d$ matrices, $\{B(t)\}_{t\geqslant 0}$ is an m-dimensional Brownian motion and b is a $d\times m$ matrix. If there exists an invariant measure μ_∞ for the above equation (7.13) (see [10, Chapter 10] for conditions that guarantees this property), then D_H is again not closable by an argument similar to that in the above proof.

7.3. **Ornstein-Uhlenbeck Process in Chaotic Environment.** In this subsection we consider the so-called Ornstein-Uhlenbeck process in a random environment, see [10]. We shall use the framework considered in [20]. Let $\vartheta\in C^\infty\left(\mathbb{R}^d\right)$ be an even and strictly positive function such that $\vartheta(\zeta)=e^{-|\zeta|}$ for $|\zeta|\geqslant 1$. For $\rho\in\mathbb{R}$ we put $\vartheta_\rho(\zeta)=\vartheta^\rho(\zeta)$. We will denote by L^2_ρ the weighted L^2-space endowed with the norm

$$|x|_\rho=\left(\int_{\mathbb{R}^d}|x(\zeta)|^2\vartheta_\rho(\zeta)\,d\zeta\right)^{1/2}.$$

For $\rho=0$ we write L^2 instead of L^2_ρ. If $\rho>0$, then $L^2\subseteq L^2_\rho$ with continuous inclusion. Let

$$A_0x(\zeta)=\sum_{|\alpha|\leqslant 2m}a_\alpha(\zeta)D^\alpha x(\zeta),$$

where $m<2d$ and the functions $a_\alpha\in C^1\left(\mathbb{R}^d\right)$ are bounded. We assume that A_0 is uniformly elliptic in L^2. Then by the result in [20] A_0 has a unique extension to a generator

A of an analytic C_0-semigroup $\{S(t)\}_{t\geqslant 0}$ in L^2_ρ for every $\rho > 0$. In the space L^2_ρ we will consider a linear equation

$$\begin{cases} dZ(t) = AZ(t)dt + dW(t), \\ Z(0) = x \in L^2_\rho, \end{cases} \tag{7.14}$$

where $\{W(t)\}_{t\geqslant 0}$ is a standard cylindrical Wiener process on L^2. By the results in [20] the variation of constants formula (1.3) is meaningful and defines a solution to (7.14) in L^2_ρ.

Assume now that there exists an invariant measure μ_∞ for (7.14) in L^2_ρ. It is well known that then μ_∞ is nondegenerate. Noting that L^2 is invariant under $\{S(t)\}_{t\geqslant 0}$ and that $\{S(t)\}_{t\geqslant 0}$ restricts to a C_0-semigroup on L^2, it follows from Corollary 5.6 that the gradient operator D_{L^2} is closable in $L^p(L^2_\rho, \mu_\infty)$ for all $p \in [1,\infty)$ and $\rho > 0$.

Remark 7.3. Let $a = (a_{ij})$. If for a certain $\omega > 0$,

$$\vartheta_{\rho/2} A \vartheta_{-\rho/2} - \operatorname{div}\left(\vartheta_{\rho/2} a \nabla \vartheta_{\rho/2}\right) \leqslant -\omega$$

in sense of distributions, then the semigroup $\{S(t)\}_{t\geqslant 0}$ is uniformly exponentially stable in L^2_ρ and therefore there exists a unique invariant measure for $\{Z(t)\}_{t\geqslant 0}$. In particular, this is true if $d = 1$, $A = \Delta - m$ with $m > \frac{\rho^2}{2}$, see p. 192 of [10].

REFERENCES

[1] Albeverio S. and Röckner M.: Classical Dirichlet forms on topological vector spaces - closability and a Cameron-Martin formula, *J. Funct. Anal.* **88** (1990), 395-435

[2] Bogachev V.I.: GAUSSIAN MEASURES, Mathematical Surveys and Monographs, Vol. 62, AMS, 1998

[3] Brzeźniak Z. and van Neerven J.M.A.M.: Stochastic convolution in separable Banach spaces and the stochastic linear Cauchy problem, *Studia Math.* **143** (2000), 43-74

[4] Chojnowska-Michalik A.: Representation theorem for general stochastic delay equations, *Bull. Acad. Pol. Sci.* Ser. Sci. Math. **26** (1978), 634-641

[5] Chojnowska-Michalik A. and Goldys B.: Nonsymmetric Ornstein-Uhlenbeck generators, in: Ph. Clément, F. den Hollander, J. van Neerven and B. de Pagter (Eds.): INFINITE DIMENSIONAL STOCHASTIC ANALYSIS, Royal Netherlands Academy of Arts and Sciences, Amsterdam, 2000, pp. 99-116

[6] Chojnowska-Michalik and Goldys B.: Symmetric Ornstein-Uhlenbeck generators: characterizations and identification of domains, in preparation

[7] Da Prato G. and Zabczyk J.: STOCHASTIC EQUATIONS IN INFINITE DIMENSIONS, Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge (UK) 1992

[8] Da Prato G. and Zabczyk J.: Evolution equations with white noise boundary conditions, *Stochastics Stochastics Rep.* **42** (1993), 167-182

[9] Da Prato G. and Zabczyk J.: Regular densities of invariant measures in Hilbert spaces, *J. Funct. Anal.* **130** (1995), 427-449

[10] Da Prato G. and Zabczyk J.: ERGODICITY FOR INFINITE DIMENSIONAL SYSTEMS, Cambridge University Press, Cambridge (UK) 1996

[11] Eberle A.: UNIQUENESS AND NON-UNIQUENESS OF SEMIGROUPS GENERATED BY SINGULAR DIFFUSION OPERATORS, Lecture Notes in Mathematics, Vol. 1718, Springer-Verlag, Berlin, 1999

[12] Frolov N.N.: Embedding theorems for functions of a countable number of variables and their applications to the Dirichlet problem, *Soviet Math. Dokl.* **13** (1972), 346-349

[13] Frolov N.N.: Embedding theorems for spaces of functions of a countable number of variables and their applications to the Dirichlet problem, *Siberian Math. J.* **22** (1981), 638-652

[14] Fuhrman M.: Analyticity of transition semigroups and closability of bilinear forms in Hilbert spaces, *Studia Math.* **115** (1995), 53-71

[15] Goldys B. and Gozzi F.: Second order Hamilton-Jacobi equations in Hilbert spaces and stochastic control: L^2-approach, in preparation

[16] Goldys B. and van Neerven J. M. A. M.: Symmetric and nonsymmetric Ornstein-Uhlenbeck semigroups in Banach spaces and their reproducing kernel Hilbert spaces, in preparation.

[17] Ma Z. M. and Röckner M., INTRODUCTION TO THE THEORY OF (NONSYMMETRIC) DIRICHLET FORMS, Springer-Verlag, 1992
[18] Musiela M.: Stochastic PDEs and term structure models, Journées Internationales de Finance, IGR-AFFI, La Baule 1993
[19] Peszat S.: Sobolev spaces of functions on an infinite-dimensional domain. In: Stochastic processes and related topics (Siegmundsberg, 1994), 103-116, Stochastics Monogr. 10, Gordon and Breach, Yverdon, 1996
[20] Peszat S. and Zabczyk J.: Stochastic evolution equations with a spatially homogeneous Wiener process, *Stochastic Process. Appl.* **72** (1997), 187-204
[21] Stannat W.: (Nonsymmetric) Dirichlet operators on L^1: existence, uniqueness and associated Markov processes, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **28** (1999), 99–140.

SCHOOL OF MATHEMATICS, THE UNIVERSITY OF NEW SOUTH WALES, SYDNEY 2052, AUSTRALIA
E-mail address: B.Goldys@unsw.edu.au

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI PISA, VIA F. BUONARROTI N.2, 56127 PISA, ITALY
E-mail address: gozzi@dm.unipi.it

DEPARTMENT OF APPLIED MATHEMATICAL ANALYSIS, TECHNICAL UNIVERSITY OF DELFT, P.O. BOX 5031, 2600 GA DELFT, THE NETHERLANDS
E-mail address: J.vanNeerven@its.tudelft.nl