Remark on a theorem of Riddle, Saab and Uhl

J.M.A.M. van Neerven[1]

CWI
P.O. Box 4079,
1009 AB Amsterdam,
The Netherlands

We give a short proof of the Riddle-Saab-Uhl theorem on Pettis integrability in dual Banach spaces, avoiding the use of Banach space techniques and extending it to certain locally convex spaces.

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0. Introduction

In this short note we propose to give a simplified proof of the Riddle-Saab-Uhl theorem [RSU] on Pettis integrability (Corollary 0.3 below). The original proof relies on subtle Banach space theory in combination with measure theoretic results concerning pointwise limits of certain classes of functions. In our approach we only use the latter, thereby avoiding all Banach space techniques. This allows extension of the RSU-theorem to separable metrizable locally convex spaces.

Let X be a (real) locally convex topological vector space (lcs). As usual, we let X^{**} denote the dual of $(X^*, \beta(X^*, X))$, where $\beta(X^*, X)$ denotes the strong topology of X^*. Let $j : X \to X^{**}$ denote the natural map. Our main result is:

Theorem 0.1. *Let F be a countable bounded subset of an lcs X, let x^{**} be a weak*-cluster point of jF and let Q be compact Hausdorff. If $\phi : Q \to X^*$ is $\beta(X^*, X)$-bounded, universally weak*-integrable and universally scalarly measurable with respect to every weak*-cluster point of jF, then*

$$\langle x^{**}, weak^* \int_Q \phi(\sigma)\ d\mu(\sigma)\rangle = \int_Q \langle x^{**}, \phi(\sigma)\rangle\ d\mu(\sigma).$$

Let us say that X has *Property C* if for all $x^{**} \in X^{**}$ there is a *Countable* bounded set $F \subset X$ such that x^{**} is a weak*-cluster point of jF. Every metrisable separable lcs has property. We do not know whether every separable lcs has property C; the problem is that an arbitrary closed subspace of a separable lcs need not be separable [LS].

[1] Present address: California Institute of Technology, Alfred P. Sloan Laboratory of Mathematics, Pasadena, CA 91125, U.S.A.

Corollary 0.2. *Let X be a lcs with property C and let Q be compact Hausdorff. If $\phi : Q \to X^*$ is universally scalarly measurable and the convex hull of its range is relatively weak*-compact, then ϕ is universally Pettis integrable.*

Proof: By [S, IV.5.1] ϕ is $\beta(X^*, X)$-bounded and by [Si] ϕ is weak*-integrable. The rest follows from Theorem 0.1. ////

Corollary 0.3 [Riddle-Saab-Uhl]. *Let X be a separable Banach space and let Q be compact Hausdorff. If $\phi : Q \to X^*$ is bounded and universally scalarly measurable, then ϕ is universally Pettis integrable.*

1. The proof

Let Q be compact Hausdorff. If $x^{**} \in X^{**}$, then we say that a function $\phi : Q \to X^*$ is *universally scalarly measurable with respect to* x^{**} if $\langle x^{**}, \phi(\cdot)\rangle$ is μ-measurable for every Radon measure μ on Q. We say that ϕ is *Pettis integrable* with respect to μ if ϕ is Pettis integrable as a function $(Q, \mu) \to (X^*, \beta(X^*, X))$. By τ_p we denote the topology of pointwise convergence.

Lemma 1.1. *Let $\phi : Q \to X^*$ be $\beta(X^*, X)$-bounded and weak*-integrable with respect to a Radon measure μ and let $x^{**} \in X^{**}$. If for every $\varepsilon > 0$ there is a measurable Q_ε such that $\mu(Q \backslash Q_\varepsilon) < \varepsilon$ and*

$$\langle x^{**}, weak^* \int_{Q_\varepsilon} \phi(\sigma)\ d\mu(\sigma)\rangle = \int_{Q_\varepsilon} \langle x^{**}, \phi(\sigma)\rangle\ d\mu(\sigma),$$

then

$$\langle x^{**}, weak^* \int_{Q} \phi(\sigma)\ d\mu(\sigma)\rangle = \int_{Q} \langle x^{**}, \phi(\sigma)\rangle\ d\mu(\sigma).$$

Proof: If not, then

$$\Big|\langle x^{**}, weak^* \int_{Q} \phi(\sigma)\ d\mu(\sigma)\rangle - \int_{Q} \langle x^{**} \phi(\sigma)\rangle\ d\mu(\sigma)\Big| = \delta > 0.$$

Choose a bounded set $F \subset X$ such that x^{**} is a weak*-cluster point of jF (this is always possible: for instance, take the $\langle E, E^*\rangle$-polar of the $\langle E^*, E^{**}\rangle$ polar of $\{x^{**}\}$) and let $K := \sup_{q \in Q, x \in F} |\langle \phi(q), x\rangle|$. This supremum exists since the range of ϕ is $\beta(X^*, X)$-bounded. For $\varepsilon = \delta/3K$ choose Q_ε as indicated. Then

$$\begin{aligned}
\delta &= |\langle x^{**}, weak^* \int_{Q} \phi(\sigma)\ d\mu(\sigma)\rangle - \int_{Q} \langle x^{**}, \phi(\sigma)\rangle\ d\mu(\sigma)| \\
&= |\langle x^{**}, weak^* \int_{Q \backslash Q_\varepsilon} \phi(\sigma)\ d\mu(\sigma)\rangle - \int_{Q \backslash Q_\varepsilon} \langle x^{**}, \phi(\sigma)\rangle\ d\mu(\sigma)| \\
&\leqslant K\varepsilon + K\varepsilon = \frac{2\delta}{3}.
\end{aligned}$$

The estimate is obtained by noting that its holds when x^{**} is replaced by an arbitrary $x \in F$ and using that x^{**} is a weak*-cluster point of jF. Thus we have a contradiction. ////

Proof of Theorem 0.1: Let $F = (x_n)$ and define the functions $f_n, f : Q \to \mathbb{R}$ by

$$f_n(s) = \langle \phi(s), x_n \rangle, \qquad f(s) := \langle x^{**}, \phi(s) \rangle.$$

Let $\varepsilon > 0$ be arbitrary and fix a Radon measure μ on Q. By Lusin's theorem and the regularity of μ there is a compact subset $Q_\varepsilon \subset Q$ such that $\mu(Q \backslash Q_\varepsilon) < \varepsilon$ and such that $\langle \phi(\cdot), x_n \rangle$ is continuous on Q_ε for every n.

Since x^{**} is a weak*-cluster point of (x_n), there is a subsequence (x_{n_m}), depending on ε, such that

$$\langle x^{**}, weak^* \int_{Q_\varepsilon} \phi(\sigma)\ d\mu(\sigma) \rangle = \lim_{m \to \infty} \langle weak^* \int_{Q_\varepsilon} \phi(\sigma)\ d\mu(\sigma), x_{n_m} \rangle.$$

Let $Z := \{f_{n_m} : m \in \mathbb{N}\}$ and note that these functions are uniformly bounded. Let $K \subset Q_\varepsilon$ be compact. We claim that every sequence in $Z|_K$ has a subsequence converging pointwise $\mu|_K$-a.e. Suppose the contrary. Noting that $(K, \mu|_K)$ is a perfect measure space, by Fremlin's subsequence theorem [F] there is a sequence in $Z|_K$, say $(f_{n_{m_i}}|_K)$, all of whose τ_p-cluster points in $\mathbb{R}^K$ are non-$\mu|_K$-measurable. Let $(x_{n_{m_i}})$ be the corresponding subsequence of (x_{n_m}) and let x_0^{**} be a weak*-cluster point of $j(x_{n_{m_i}})$ in X^{**}. Such a cluster point exists since jB is relatively weak*-compact for every bounded $B \subset X$. Clearly x_0^{**} is also a weak*-cluster point of (x_n). Moreover, the function $g(s) := \langle x_0^{**}, \phi(s) \rangle$ is a τ_p-cluster point of $(f_{n_{m_i}})$, hence non-$\mu|_K$-measurable as a function on K. This is a contradiction to the scalar μ-measurability of ϕ. This proves the claim.

In particular it follows that if μ is Radon on Q and $K \subset Q_\varepsilon$ is compact, then every sequence in $Z|_K$ has a $\mu|_K$-measurable τ_p-cluster point. By Theorem 2.7 (vii)$\Rightarrow$(ii) of [BFT] and taking $K = Q_\varepsilon$ it follows that every sequence in $Z|_{Q_\varepsilon}$ has a pointwise convergent subsequence. By Theorem 3.11 of [vD], $Z|_{Q_\varepsilon}$ is t-stable and then Proposition 3.10 of [vD] implies that on Q_ε the function f defined above is the pointwise limit of a subsequence $(f_{n_{m_k}})$ of (f_{n_m}). Let $(x_{n_{m_k}})$ be the corresponding subsequence of (x_{n_m}). Combining everything we see from the dominated convergence theorem that

$$\langle x^{**}, weak^* \int_{Q_\varepsilon} \phi(\sigma)\ d\mu(\sigma) \rangle = \lim_{k \to \infty} \int_{Q_\varepsilon} \langle \phi(\sigma), x_{n_{m_k}} \rangle\ d\mu(\sigma) = \int_{Q_\varepsilon} \langle x^{**}, \phi(\sigma) \rangle\ d\mu(\sigma),$$

and we can apply the lemma.

Let X be an arbitrary Banach space. If x^{**} is the weak*-*limit* of some sequence (x_n) in X, then it follows immediately from the dominated convergence theorem that x^{**} commutes with every weak*-integral. The following corollary shows that it is enough that x^{**} be a weak*-*cluster* point of (x_n), provided the integrated function has good measurability properties.

Corollary 1.2. *Let X be a Banach space, $\phi : Q \to X^*$ be bounded and universally scalarly measurable. Suppose x^{**} is a weak*- cluster point of a bounded sequence in X. Then*

$$\langle x^{**}, weak^* \int_Q \phi(\sigma)\ d\mu(\sigma) \rangle = \int_Q \langle x^{**}, \phi(\sigma) \rangle\ d\mu(\sigma).$$

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