WIGNER'S 'UNREASONABLE EFFECTIVENESS OF MATHEMATICS' AS A PHILOSOPHICAL PROBLEM

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"Mathematics is a part of physics. Physics is an experimental science, a part of natural science. Mathematics is the part of physics where experiments are cheap." – V.I. Arnold.

1. Wigner's 'unreasonable effectiveness of mathematics'

In a famous essay[1] the physicist Eugene Wigner contemplates what he calls the 'unreasonable effectiveness of mathematics'. Mathematics is effective in the way it enables us do very detailed calculations leading to predictions that are consistent with experiment to incredible precision. However, this effectiveness is unreasonable: the mathematics needed for these calculation does not bear any evident relationship to our every-day experience, nor was it designed for the purpose of these applications. In Wigner's own words:

> "The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve."

Wigner's essay, which is based on a lecture for a general audience, illustrates the quote with some examples which can be understood without too much expert knowledge. To give some further depth to it we will begin by giving three examples of the way mathematics seems to "impose itself" upon physics.[2]

Example 1. The first example is connected with a famous theorem of differential geometry, due to Emmy Noether, stating that to every symmetry there corresponds a conserved quantity.[3] The proof of this theorem actually constructs the preserved quantity from the symmetry. The best-known examples in classical mechanics are the following:

[1] E. Wigner, The unreasonable effectiveness of mathematics in the natural sciences, Comm. Pure Appl. Math. 13 (1) 1960: 1–14.

[2] A disclaimer seems appropriate: the author is neither a physicist nor a philosopher, nor can he claim any expert knowledge in the foundations of mathematics. This article is just a collection of very personal reflections based on a rather modest understanding of some of these issues.

[3] For a systematic discussion of the role of symmetry in modern theoretical physics, a good starting point is David Gross's paper: "The role of symmetry in fundamental physics", Proc. Natl. Acad. Sci. USA 93 (1996), 14256–14259.

symmetry	conserved quantity
translation in space	momentum
rotation in space	angular momentum
translation in time	energy

Here the 'symmetries' are the Galilean symmetries of the space-time $\mathbb{R}^3 \times \mathbb{R}$ of classical physics preserving length of line segments in space and time intervals in time. Thus the physical conserved quantities correspond in a one-to-one fashion to the mathematical symmetries of $\mathbb{R}^3 \times \mathbb{R}$. This by itself is remarkable, for 'symmetry' is a mathematical notion, while the conserved quantities momentum and energy are definitely physical ones.

Even more interesting is that the discovery of the theory of special relativity can be understood in terms of symmetry groups. After Maxwell had formulated his famous equations unifying electricity and magnetism in 1861, it was quickly realised that these equations are *not* invariant under the Galilean symmetry group of $\mathbb{R}^3 \times \mathbb{R}$. Maxwell's equations feature a certain absolute constant c, interpreted as the speed of light, but invariance under the Galilean symmetries precludes the existence of such a constant: it forces the observed speed of light to depend on the relative speed of the observer and the light source. If a light source on a riding train would emit light in the direction of travel, an observer on the ground would measure the speed of light $c_{\rm ground} = c_{\rm train} + v$, where $c_{\rm train}$ is the speed of light relative to the train and v is the speed of the train. This was put to test in the famous Michelson-Morley experiment in 1887. To the surprise of many, however, the outcome was unequivocal: in agreement with Maxwell's equations, the speed of light appeared to be an absolute constant, independent of the relative speeds of observer and source, and in subsequent experiments this has been confirmed with the precision $\Delta c/c \le 10^{-17}$.[4] About the same time it was noted by Poincaré and Lorentz that, curiously, Maxwell's equations are invariant under a different group, nowadays called the Lorentz group. Attempts to explain this on physical grounds failed for being artificial. It required the genius of Einstein to simply *postulate* that the Lorentz group is the correct symmetry group of space-time and to work out the mathematical consequences of this assumption – the theory of special relativity.

Example 2. In classical mechanics, the equations of motion for objects satisfying suitable constraints are modelled on a smooth manifold M which serves as the configuration space. A *Lagrangian* is a smooth real-valued function L on $TM \times \mathbb{R}$, where TM is the tangent bundle of M (the points of which are (q, v) with $q \in M$ and $v \in T_q(M)$, the tangent space at the point M). Given a Lagrangian L, the *action* along a smooth path $\gamma : [t_0, t_1] \to M$ is defined as

$$S := \int_{t_0}^{t_1} L(\gamma(t), \gamma'(t), t)\,{\rm d}t.$$

The *principle of least action* states that the motion of the system is given by the paths γ that are the critical points of the action functional S. Somewhat informally,

[4] S. Herrmann, A. Senger, L. Möhle, W. Nagel, E.V. Kovalchuk, A. Peters, A., "Rotating optical cavity experiment testing Lorentz invariance at the 10^{-17} level", Physical Review D. 80 (2009), no. 100, 105011.

by the latter we mean that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} S(\gamma_\varepsilon) = 0$$

for any "parametrised perturbation" γ_ε of γ with a small parameter ε. This procedure can easily be made rigorous using coordinate charts.[5] A simple derivation shows that the equation of motion of L is the partial differential equation

$$\frac{\partial L}{\partial q}(q(t), v(t), t) = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial L}{\partial v}(q(t), v(t), t)\Big) = 0,$$

the so-called *Euler-Lagrange equation*.

An example may illustrate this. Consider a free point particle with mass m moving in $M = \mathbb{R}^3$ through a potential V. Taking

$$L := \frac{1}{2}mv^2 - V(q),$$

a simple computation shows that the Euler-Lagrange equations for L reduce to

$$ma = -\frac{\partial V}{\partial x},$$

where $a = \mathrm{d}v/\mathrm{d}t$ denotes acceleration. This is Newton's equation of motion for a point mass m subject to the *force* $-\partial V/\partial x$.

Perhaps surprisingly, the Maxwell equations can also be cast in the form of a principle of least action. This is not the place to develop this in detail, so we only summarise the main steps.[6] The first is to interpret, in the language of differential geometry, the electric and magnetic fields E and B as a 2-form and a 1-form, respectively, and to define the *electromagnetic field*

$$F := B + E \wedge \mathrm{d}t.$$

Now there is a duality mapping, the so-called *Hodge star operator*, which provides a canonical way of associating, in the context of an d-dimensional Riemannian manifold, a $(d-k)$-form $\star\omega$ to any k-form ω. In our case the dimension equals $d = 3$ and the Hodge star operator associates a 1-form $\star F$ to the 2-form F. Their wedge product $F \wedge \star F$ is a 3-form which can be integrated over M and the Maxwell equations are recovered as the Euler-Lagrange equations for the "action"

$$S := -\frac{1}{2}\int_M F \wedge \star F.$$

Referring once more to the language of differential geometry, the electromagnetic field F can be interpreted as the *curvature* of a suitable *connection* associated with F. The marvellous thing, discovered by Hilbert, is that Einstein's equations of general relativity have the same form: they can be written as the Euler-Lagrange equations corresponding to the action functional given by the scalar curvature of space-time!

[5] See Chapter 9 of M. Spivak, "A Comprehensive Introduction to Differential Geometry", Vol. I, Publish or Perish, 1999.

[6] A lucid and fully self-contained treatment of what follows is given in J. Baez and J.P. Muniain, "Gauge Fields, Knots and Gravity", World Scientific, 1994.

Example 3. This example will be even more condensed. The passage from classical mechanics to quantum mechanics essentially consists of replacing measured quantities, such as position, momentum, energy, etc., by the operation of measuring them. At the risk of oversimplifying things, instead of asking (classically) whether a particle finds itself in a region R of its configuration space M (which can be answered by 'no' or 'yes') we may consider the multiplication operator

$$\pi_R : f \mapsto \mathbf{1}_R f$$

on the Hilbert space $L^2(M)$, where $\mathbf{1}_R$ is the indicator function of R. This is a self-adjoint projection on $L^2(M)$ whose spectrum consists of two eigenvalues, 0 and 1, with eigenfunctions $f_0 = \mathbf{1}_{\complement R}$ and $f_1 = \mathbf{1}_R$ corresponding to 'no' and 'yes', respectively. More generally, in the mathematical formulation of quantum mechanics, observables are self-adjoint operators and their spectral values the potential outcomes of measurement. The spectral theorem for self-adjoint operators, which states in a precise way how self-adjoint operators can be assembled from self-adjoint projections, can then be interpreted as saying that every question that we can pose to Nature can be constructed from yes-no questions.

In the first example we have seen the importance of symmetries. One may now ask whether the symmetries of M can be similarly implemented on a Hilbert space. This is indeed possible and the way to do it is to associate with every symmetry σ of M a unitary operator U_σ acting on a Hilbert space H in such a way that $\sigma \mapsto U_\sigma$ is a homomorphism of groups (note that both the symmetries of M and the unitary operators on H form a group). Such a mapping is called a *unitary representation* of the symmetry group of M. By taking direct sums one can add unitary representations. A unitary representation is said to *irreducible* if it cannot be decomposed as a sum of smaller unitary representations.

Eugene Wigner[7] proposed that an *elementary particle* may now be defined mathematically as a unitary representation of the inhomogeneous Lorentz group, and went on to show that they are classified by two numbers: a continuous parameter $m \geq 0$ ('mass') and a half-integer parameter s ('spin'). This indeed corresponds to the observed elementary particles of Nature, albeit that quantities such as 'charge' are not captured in this framework. They, however, appear by the same mechanism once the external symmetry group of space-time is augmented by a group of internal symmetries. Now something truly spectacular happens: if one chooses $U(1) \times SU(2) \times SU(3)$ for the internal symmetries, the irreducible unitary representations precisely correspond the known particles of particle physics![8] Roughly speaking, the groups $U(1)$, $SU(2)$, $SU(3)$ correspond to the symmetries of electromagnetism, the weak force and the strong force, respectively. Nobody knows why Nature choose these particular groups; this is an empirical fact unveiled by particle accelerators at CERN, Fermilab and other facilities.

2. … AS A PHILOSOPHICAL PROBLEM[9]

[7]E. Wigner, "On unitary representations of the inhomogeneous Lorentz group", Annals of Math. 40 (1939), no. 1, 149–204.

[8]A detailed historical account of the discoveries leading up to this realisation is presented in D. Griffiths's book "Introduction to Elementary Particles", 2nd revised edition, Wiley, 2008.

[9]The title of this paragraph, and indeed of the article, is inspired by Mark Steiner's book "The Applicability of Mathematics as a Philosophical Problem", Harvard University Press, 1998.

These are but three examples of what appears to be a profound unity of mathematics and physics. That this unity should exist is deeply mysterious, if only because physics is the science of exploring the laws governing the external world whereas mathematics explores a mathematical universe created by our collective minds. If one considers mathematics as the art of deducing statements from a predefined collection of axioms by making use of predetermined deduction rules, one must admit that the choice of those axioms and deduction rules is in a sense arbitrary. Other choices are possible and lead to different mathematics. The French mathematician Jean Dieudonné has compared the formalist view of mathematics with a game of chess: it, too, has an alphabet (the pieces), axioms (an initial configuration) and rules of inference (the chess rules).[10]

Since their formulation in the early 20th century the axioms of set theory (the so-called ZF axioms, named after their inventors Zermelo and Fraenkel) have been widely accepted as the 'standard' axiomatisation of mathematics. For most working mathematicians, however, these axioms appear technical if not bizarre, and they seem to have little or no connection with everyday intuitions. Apart from set theorists, only few mathematicians seem to actually *know* the ZF axioms! How, then, can it be explained that the edifice we call 'mathematics' which is built upon these axioms is so useful in describing the external world around us? Why would elementary particles care about the axioms of set theory? A game of chess cannot teach us anything about the external world, so why would the axioms of set theory?

To address this problem we will have to consider more closely the nature of mathematics. This has two aspects: What is the structure of mathematics and what are its object of investigation? The first question leads straight into the foundations of mathematics and constitutes the subject matter of mathematical logic, which treats mathematics and its various sub-disciplines as formal languages. Present-day mathematics is based on the ZF axioms of set theory, but alternative foundations can be given.[11] Interestingly, all these systems seem to reproduce at least those parts of mathematics that are relevant for physics (we don't try to define what this means!). We will focus on the second question, which is a question about the metaphysical status of mathematical concepts. For example, number theory investigates numbers, but what are "numbers" really? Are they merely patterns of neural activity in our brains, or do they have some real "existence" independently

[10] In his article "Modern axiomatic method and the foundations of mathematics" (in: *Great Currents of Mathematical Thought*, 1971) he writes:

> "Mathematics becomes a *game*, whose *pieces* are graphic symbols distinguished from each other by their forms; with these symbols we make groupings which will be called *relationships* or *terms* according to their forms. By virtue of certain rules, certain relationships are described as *true*; other rules permit the construction of true relationships either from any relationships whatsoever or from other true relationships. The essential point is that these rules are of such a nature that in order to verify that they are being observed, it is sufficient to examine the *form* of these groupings which come into play."

[11] Not only do different axiomatisations of set theory exist. e.g. by von Neumann-Bernays-Gödel (see K. Kunen, "Set Theory", College Publications, 2nd edition, 2011), there are also very different proposals based on topos theory (see F.W. Lawvere and R. Rosebrugh, "Sets for Mathematicians", Cambridge University Press, 2003; an accessible introduction is the paper by T. Leinster, "Rethinking Set Theory", Amer. Math. Monthly 121 (2014), no. 5, 403–415) and Voevodski's univalence programme (see "Homotopy Type Theory: Univalent Foundations of Mathematics", The Univalent Foundations Program, Institute for Advanced Study, 2013)

of us? Both positions (and various intermediate variants) have passionate adherents and declared opponents.

The delightful book "Conversations on Mind, Matter, and Mathematics" records a discussion between the neurologist Jean-Pierre Changeux and Field medallist Alain Connes defending, respectively, the former and the latter position.[12] Changeux's reductionist point of view seems difficult to reconcile with our every-day experience that we (mathematicians) are able to meaningfully communicate with each other about mathematical objects, given that each one of us has a different brain with different neuronal connections. All mathematicians will agree that mathematics is really "about something" which does not depend on the specific shape and wiring of one's brain. To dismiss mathematical objects as patterns of neuronal activity seems to deny the very essence of mathematics. What is more, in this view mathematics could not have existed before man arrived on the planet, and will cease to exist when (and if) mankind would ever be wiped out. The situation bears some resemblance with the ontological status of, say, Beethoven's 9th symphony. Nobody would sensibly claim it "existed" before Beethoven wrote it, and it is imaginable that at some point in the future the collective memory of this great work could be wept out completely – in which case one could reasonably say it then no longer "exists". There is one marked difference, however, in that mathematics, or at least a good part of it, is likely to be discovered (perhaps phrased in a different formal language) by any intelligent beings studying the laws of physics – something which can be hardly said of Beethoven's 9th symphony.

Let us, then, consider the platonist view that mathematical objects "exist" independently of us. In which "universe" do they exist? Obviously, they do not exist as material objects in our physical universe. Nobody has ever encountered a 'two' on a walk in the park; at best one sees two trees or two birds. But if mathematical objects "exist" in an "immaterial universe" "beyond the space and time",[13] how can they causally interact with physical objects within space and time, such as

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[12]Jean-Pierre Changeux and Alain Connes, "Conversations on Mind, Matter, and Mathematics", Princeton University Press, 1995.

[13]This view is held by several prominent mathematicians. Kurt Gödel, in his article "What is Cantor's continuum problem" (in: "Philosophy of Mathematics, Selected Readings", Prentice-Hall, 1964) writes:

> "(...) the objects of transfinite set theory (...) clearly do not belong to the physical world and even their indirect connection with physical experience is very loose (...) But, despite their remoteness from sense experience, we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves on us as being true. I don't see any reason why we should have less confidence in this kind of perception, i.e., in mathematical intuition, than in sense perception, which induces us to build up physical theories and to expect that future sense perceptions will agree with them and, moreover, to believe that a question not decidable now has meaning and may be decided in the future. The set-theoretical paradoxes are hardly any more troublesome for mathematics than deceptions of the senses are for physics."

For Alain Connes (in: Alain Connes, André Lichnerowicz, Marcel Paul Schützenberger: *Triangle of Thoughts*, American Mathematical Society, 2000), arithmetical truth describe a "primordial reality" (*"réalité mathématique archaïque"*):

> "By this intentionally imprecise term, I lump together the vast continent of arithmetical truths."

Axiomatic theories of mathematics are nothing but a tool to explore this reality:

our brains? For the very moment we practice mathematics, this is exactly what happens and the interaction leads to tangible consequences in our physical world, such as articles in journals and lectures at universities.

This objection to Platonism appears to have been brought up first by the philosopher Paul Benacerraf.[14] He has a point if one accepts the sharp distinction between the external physical world, of which our brain is a part, and the mental world of our consciousness, of which mathematics is assumed to be a part. But exactly this could be questioned on the basis that it presupposes the 'scientific realist' position – this is the position that an external world exists independently of our observation and that its workings can be unravelled through scientific investigation. Reconciling this position with 'mathematical realism' seems indeed problematic.

Can we be sure of the objective existence of an external world? This problem, known as the *Ding an sich* problem of the German philosopher Immanuel Kant, is not so easily dismissed. After all, we perceive the sense-data presented in our consciousness, but not the "objects themselves". Even the realist will admit that when we "see a table", we do not actually see "the table itself". At best we see the photons that are reflected from the table. And in fact we do not even see those: the image we perceive is formed in our brain only after retinal nerve pulses have

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> "We cannot avoid discussing in greater detail a distinction that is quite simple in the science of matter, but which turns out to be more subtle in the case of mathematics: it is the distinction between tools that are invented and the objects that they uncover. For example, the structure of DNA was discovered thanks to the electron microscope. The electron microscope is clearly a tool. Nobody would question that this tool, unlike DNA, was created by man. (...) As long as a tool has not proved itself by lifting in a significant way the veil that conceals primordial mathematical reality, it can rightly be considered sterile and nonexistent. (...) If we look at the sequence of prime numbers, for example, it appears at first glance to be as bizarre and disorderly as external reality. But it happens that, by developing an instrument of observation, by inventing appropriate concepts, we gradually succeed in guessing some of the regularities that lie within this seemingly disorganized reality. By trying to understand the geometric structure of the "arithmetic site", that is, the set of prime numbers, we manage little by little to perceive the extraordinary fundamental organization of this reality. (...) For me, the properties that are true characterize the object in its primordial reality, whereas those that are provable are the ones that our brain perceives through its instruments of observation."

[14] P. Benacerraf, "Mathematical Proof" (in: "Philosophy of Mathematics, Selected Readings", Prentice-Hall, 1964). After elaborating his view that

> "I favor a causal account of knowledge on which for X to know that S is true requires some causal relation to obtain between X and the referents of the names, predicates, and quantifiers of S."

he writes:

> "It will come as no surprise that this has been a preamble to pointing out that combining this [the causal – JvN] view of knowledge with the "standard" view of mathematical truth makes it difficult to see how mathematical knowledge is possible. If, for example, numbers are the kinds of entities they are normally taken to be, then the connection between the truth conditions for the statements of number theory and any relevant events connected with the people who are supposed to have mathematical knowledge cannot be made out. It will be impossible to account for how anyone knows any properly number-theoretical propositions."

made their way from the eye to the visual cortex. In the end, what we really see is the electrical activity of our visual cortex. The same reasoning applies to all other senses. How then can we be sure that there really "is" a table, even when nobody is watching? The same question can be asked about all objects around us, including our own brain! One could consistently argue that the "external world" is just a working hypothesis of our consciousness that serves to explain the patterns of our sense-data.

The philosophical debate about the nature of existence of things has recently moved from the philosophy departments to the physics laboratory,[15] often with surprising conclusions. Thus, relativity theory teaches us that distances and time intervals are not absolute but differ from observer to observer, depending on his/her frame of reference. In the same vein, quantum mechanics tells us that observables do not have a definite objective value before being measured and that different observers will give different (but consistent) accounts of reality. In the words of the physicist John Wheeler[16]

> "It has no sense to speak of what [the particle] was doing except as it is observed or calculable from what is observed. More generally we would seem forced to say that no phenomenon is a phenomenon until – by observation, or some proper combination of theory and observation – it is an observed phenomenon. The universe does not 'exist, out there' independent of all acts of observation. Instead, it is in some sense a strange participatory universe."

A subsequent series of wonderful experiments performed by Alain Aspect, Anton Zeilinger and others has confirmed this view and showed that our naive ideas about the reality of things are wrong.[17] Let us mention just two of them: the experimental violation of Bell's inequality is commonly interpreted as forcing us to either give up realism or locality (the principle that there is no causation-at-a-distance), and the experimental realisation of Wheeler's delayed choice experiment forces us to choose between realism and backward causation. In reaction to such experiments, new interpretations of quantum mechanics have emerged which to some degree do away with notion of objective reality in favour of an operationalist view that 'performing measurements' is 'posing questions and getting answers', such as Griffith's consistent histories interpretation, Zeilinger's information-theoretical interpretation, and Rovelli's relational interpretation.[18] In some sense, all this is wonderfully in line with the positivist's tenet that it is meaningless to speak of things that in principle cannot be investigated empirically – 'states of particles before they are measured' are precisely that.

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[15]The well-known physicist Abner Shimony has reportedly called quantum mechanics 'experimental metaphysics'.

[16]In: "Mathematical Foundations of Quantum Theory", Elsevier, 1978.

[17]A detailed description of the most important ones has been given in the book "The Quantum Divide: Why Schrödinger's Cat is Either Dead or Alive" by C. Gerry and K. Bruno (Oxford University Press, 2013).

[18]R.B. Griffiths, "Consistent Quantum Theory", Cambridge University Press, 2002; A. Zeilinger, A foundational principle for quantum mechanics, Foundations of Physics 29 (1999), no. 4, pp 631–643; C. Rovelli, Relational quantum mechanics, Int. J. Theor. Phys. 35 (1996), no. 8, 1637–1678; M. Smerlak and C. Rovelli, "Relational EPR", Found. Phys. 37, (2007), no. 3, 427–445.

3. Structuralism versus relationalism

We have argued that the notions of 'mathematical reality' and 'physical reality', if taken too naive or literal, are both problematic, especially when treated in connection with the problem of applicability of mathematics in physics. In both cases the problem is about ontology: in mathematics it is unclear "where" and "how" mathematical objects could exist and interact with the physical world, and physics has revealed that it is problematic to assign objective existence to space, time, and states.

A modern view in mathematics is that one can dispense with ontology altogether and view mathematics as describing structures rather than objects. This view, known as *structuralism* and popularised through the works of Bourbaki, maintains that mathematical objects are exhaustively described by the relations between them and do not have any "intrinsic" properties whatsoever. This is exemplified by the pervasive use of implicit definitions in present-day mathematics, which define structures rather than objects. For example, the definition of a 'group' doesn't present us with actual groups; rather, it lists what it takes to be a group. 'Group theory' doesn't require the existence of actual groups: it studies what can be said *if* one is presented with a group.

Likewise, a modern view in physics is to view it as the study of 'generally covariant' quantities, i.e., those quantities that can be defined in a coordinate-free way and agreed upon by different observers. An 'observer' is to be understood in a general sense and includes inanimate measuring devices. For example, in special relativity the length of a time interval is not a generally covariant quantity: two observers in different inertial reference frames will measure time intervals differently. The generally covariant quantities of special relativity are precisely those that are invariant under the Lorentz group, such as the speed of light, the four-dimensional Minkowski distance between space-time points (but not space and time distances separately), energy-momentum (but not energy and momentum separately), the electromagnetic field (but not the electric and magnetic fields separately), etc. In general relativity, generally covariant properties have to be invariant under arbitrary space-time diffeomorphisms, and the famous 'hole argument' by Einstein demonstrates that because of this it becomes entirely meaningless to talk about 'points in space-time' altogether. Only invariant descriptions of *relations* between space-time points have physical meaning. In Einstein's own words:

> All our space-time verifications invariably amount to a determination of space-time coincidences. If, for example, events consisted merely in the motion of material points, then ultimately nothing would be observable but the meeting of two or more of these points.[19]

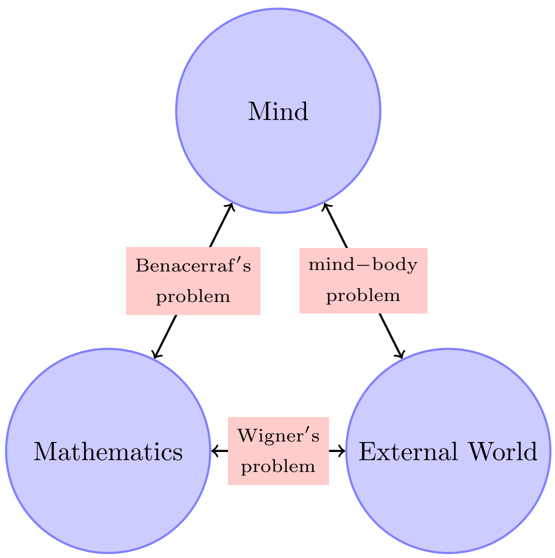
Arguments such as these have convinced many physicists that the laws of physics should be formulated in a 'background free' manner. Here, under a 'background' one understands the "empty arena" in which the events unfold. Classical mechanics and special relativity do have such backgrounds, namely Galilean space-time $\mathbb{R}^3 \times \mathbb{R}$

[19] A. Einstein, "The Foundation of the General Theory of Relativity" 1916, p. 117. An especially lucid discussion of the hole argument is given in C. Rovelli, "Quantum Gravity", Cambridge University Press, 2004. Here one also finds a precise analysis of the physical meaning of statements such as "the event A happened at time B" as the coincidence of A with the event that the pointer of a clock points at B.

and Minkowski space-time $\mathbb{R}^4$ respectively, but general relativity is a background free theory. This point of view is sometimes called *relationalism* and can be viewed as the modern version of Leibniz's relationist view of Galilean space and time.[20]

4. Conclusion

Roger Penrose, in his book "The Road to Reality", organises the three basic problems of reality in his mind/mathematics/external world triangle:



the philosophical problem of existence of mathematical objects (we take the liberty of calling this *Benacerraf's problem*) is about the relation [mathematics vs. mind], the mind-body problem is about the relation [mind vs. external world], and Wigner's problem concerns the relation [external world vs. mathematics]. Given the striking resemblance between mathematical structuralism and physical relationalism, which both reduce the ontological content of their respective domains to the bare minimum, one may go a step further by *defining* the physicist's objective reality as the mathematical consistency of the individually observed realities presented to the minds of different observers. Once we have accepted that mathematics and physics can be stripped from their ontological burdens, this seems a reasonable proposal which may represent a step towards resolving all three problems in Penrose's triangle, in that it reconciles mathematics and physics as two intrinsically related aspects of our description of the reality presented to our minds. To paraphrase Arnold's motto: Mathematics is physics and physics is mathematics.

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[20] In his debate with Newton, who held the view that space and time are absolute, Leibniz already put forward the principle of *identity of indiscernibles*, which states that two things are equal if and only if they have the same properties. If absolute space and time existed (this was the view held by Newton), a different universe would result if all of its contents were translated or rotated by the same amount - but Galilean invariance means that no experiment could ever distinguish between these two universes. The same argument applies to translations in time. Therefore, Leibniz argues, absolute space and time are to be rejected. For an excellent account of the Leibniz-Newton controversy we recommend Sklar, "Space, Time, and Space-time", University of California Press, 1974.