NULL CONTROLLABILITY AND THE
ALGEBRAIC RICCATI EQUATION IN BANACH SPACES[*]

J. M. A. M. VAN NEERVEN[†]

Abstract. By a recent result of Priola and Zabczyk, a null controllable linear system

$$y'(t) = Ay(t) + Bu(t)$$

in a Hilbert space E is null controllable with vanishing energy if and only if it is null controllable and the only positive self-adjoint solution of the associated algebraic Riccati equation

$$XA + A^*X - XBB^*X = 0$$

is the trivial solution $X = 0$. In this paper we extend this result to Banach spaces with an elementary proof which uses only reproducing kernel Hilbert space techniques. We also show that null controllability with vanishing energy implies null controllability.

Key words. null controllability with vanishing energy, algebraic Riccati equation, reproducing kernel Hilbert space

AMS subject classifications. Primary, 93B05; Secondary, 47D06, 93B03, 93C25

DOI. 10.1137/S0363012903437058

Let A be the generator of a C_0-semigroup on a real Banach space E and let B be a bounded linear operator from a real Hilbert space H into E. The pair (A, B) is said to be *null controllable with vanishing energy* if for all $x \in E$ and all $\varepsilon > 0$ there exists a time $t > 0$ and a function $u \in L^2(0, t; H)$ satisfying $\|u\|_{L^2(0,t;H)} < \varepsilon$ such that the mild solution $y^{u,x}$ of the linear control problem

$$(0.1) \qquad \begin{aligned} y'(s) &= Ay(s) + Bu(s) \qquad (s \in [0, t]), \\ y(0) &= x \end{aligned}$$

satisfies $y^{u,x}(t) = 0$. The pair (A, B) is said to be *null controllable in finite time* if there exists a fixed time $t_0 > 0$ such that for all $x \in E$ there exists a function $u \in L^2(0, t_0; H)$ such that the mild solution of the problem (0.1) satisfies $y^{u,x}(t_0) = 0$.

For Hilbert spaces E, Priola and Zabczyk recently proved that a pair (A, B), which is null controllable in finite time, is null controllable with vanishing energy if and only if the only positive self-adjoint solution to the algebraic Riccati equation

$$XA + A^*X - XBB^*X = 0$$

is the trivial solution $X = 0$ [10]. One of the main ingredients of the proof is the fact that a certain differential Riccati equation is solved in terms of a minimal energy functional. In this paper we extend the Priola–Zabczyk result to Banach spaces with a different proof which is based on reproducing kernel Hilbert space techniques, and we

show that null controllability with vanishing energy in fact implies null controllability in finite time. Our approach relies upon the identification of the space H_t of points that are reachable in time t as the reproducing kernel Hilbert space associated with the operator $Q_t \in \mathcal{L}(E^*, E)$ defined by

$$Q_t x^* := \int_0^t S(s) BB^* S^*(s) x^* \, ds \qquad (x^* \in E^*).$$

The square norm $\|h\|_{H_t}^2$ can be interpreted as the minimal energy needed to reach the state $h \in H_t$ in time t starting from the origin. The basic problem is then to understand how this minimal energy varies with h and t. Our main result in this direction is Theorem 2.5, which describes the instantaneous rate of change of the minimal energy along curves in H_t as time progresses. It is used to obtain an explicit positive symmetric solution $X(t)$ for a differential Riccati equation. As in [10], the weak operator limit $X = \lim_{t \to \infty} X(t)$ then turns out to be the maximal positive symmetric solution of the algebraic Riccati equation, and null controllability with vanishing energy is equivalent to the condition that $X = 0$.

For more information about null controllability and Riccati equations as well as applications to various control systems we refer to [1, 2, 3, 4, 7, 8, 12, 13].

1. Reachable states and reproducing kernels. The mild solution of the problem (0.1) will be denoted by $y^{u,x}$. Thus,

$$y^{u,x}(s) := S(s)x + \int_0^s S(s-r) Bu(r) \, dr \qquad (s \in [0, t]).$$

An element $h \in E$ is *reachable in time t* if there exists a control $u \in L^2(0, t; H)$ such that $y^{u,0}(t) = h$. The collection H_t of all elements that are reachable in time t is a linear subspace of E which is a Hilbert space with norm

$$\|h\|_{H_t}^2 := \inf\{\|u\|_{L^2(0,t;H)}^2 : u \in L^2(0, t; H), \ y^{u,0}(t) = h\}.$$

Thus, $\|h\|_{H_t}^2$ is the minimal energy needed to steer the system from 0 to h in time t. Notice that H_t equals the range of the operator $L_t \in \mathcal{L}(L^2(0, t; H), E)$ defined by

$$L_t f := \int_0^t S(t-s) Bf(s) \, ds.$$

It is easy to check that $L_t^* x^* = B^* S^*(t - \cdot) x^*$ for all $x^* \in E^*$. Consequently, $L_t \circ L_t^* = Q_t$, where $Q_t \in \mathcal{L}(E^*, E)$ is defined by

$$(1.1) \qquad Q_t x^* := \int_0^t S(s) BB^* S^*(s) x^* \, ds.$$

It follows from this that H_t can be identified with the reproducing kernel Hilbert space of Q_t. Denoting the inclusion operator $H_t \hookrightarrow E$ by i_t, we have the operator identity

$$(1.2) \qquad i_t \circ i_t^* = Q_t.$$

Moreover, by general results on reproducing kernel Hilbert spaces, the range of i_t^* is dense in H_t.

We insert a simple result on controls with minimal energy. It will not be needed in what follows and is included for reasons of completeness only. We write Λ_t for the L_t when we regard it as an operator from $L^2(0, t; H)$ onto H_t.

PROPOSITION 1.1 (control with minimal energy). *For all $h \in H_t$ we have* $\Lambda_t \Lambda_t^* h = h$ *and* $\|\Lambda_t^* h\|_{L^2(0,t;H)}^2 = \|h\|_{H_t}^2$.

Upon identifying $h \in H_t$ with $i_t h \in E$, we have $L_t(\Lambda_t^* h) = h$. Thus, the lemma states that the control $\Lambda_t^* h$ steers 0 to h in time t with minimal energy.

Proof. For all $x^* \in E^*$ we have $\Lambda_t^* i_t^* x^* = L_t^* x^* = B^* S^*(t - \cdot) x^*$. Hence,

$$i_t \Lambda_t \Lambda_t^* i_t^* x^* = L_t \Lambda_t^* i_t^* x^* = \int_0^t S(t - s) B B^* S^*(t - s) x^* \, ds = Q_t x^* = i_t i_t^* x^*.$$

Since i_t is injective and the range of i_t^* is dense in H_t, this implies that $\Lambda_t \Lambda_t^* h = h$ for all $h \in H_t$. This proves the first assertion. The second follows from

$$\|\Lambda_t^* i_t^* x^*\|_{L^2(0,t;H)}^2 = \|L_t^* x^*\|_{L^2(0,t;H)}^2 = \langle L_t L_t^* x^*, x^* \rangle = \langle Q_t x^*, x^* \rangle = \|i_t^* x^*\|_{H_t}^2$$

and another density argument.   □

It will be helpful to recall some elementary facts about the spaces H_t; for the proofs we refer to [9, 13]. The inequality $\langle Q_t x^*, x^* \rangle \leqslant \langle Q_{t+s} x^*, x^* \rangle$, valid for all $x^* \in E^*$, $t > 0$ and $s \geqslant 0$, implies that $H_t \subseteq H_{t+s}$ (as subsets of E) with a contractive inclusion mapping

$$i_{t,t+s} : H_t \hookrightarrow H_{t+s}, \quad i_{t,t+s} h = h \quad (h \in H_t).$$

Moreover, $S(s)$ restricts to a contraction from H_t into H_{t+s}. We will denote this restriction by $S_{t,t+s}(s)$. Thus,

$$S_{t,t+s}(s) : H_t \to H_{t+s}, \quad S_{t,t+s}(s) h = S(s) h \quad (h \in H_t).$$

2. Null controllability. The pair (A, B) is said to be *null controllable in finite time* if there exists a time $t_0 > 0$ such that for any $x \in E$ there exists a control $u \in L^2(0, t_0; H)$ such that $y^{u,x}(t_0) = 0$. If we want to stress the role of t_0, we say that (A, B) is *null controllable in time t_0*.

From the trivial identity $y^{u,x}(t_0) = S(t_0) x + y^{u,0}(t_0)$ we see that (A, B) is null controllable in time t_0 if and only if

$$S(t_0) x \in H_{t_0} \quad \text{for all } x \in E.$$

As an operator from E into H_{t_0}, we shall denote $S(t_0)$ by $\Sigma(t_0)$. Thus,

$$(2.1) \qquad\qquad\qquad S(t_0) = i_{t_0} \circ \Sigma(t_0).$$

If (A, B) is null controllable in time t_0, then (A, B) is null controllable in time t for all $t \geqslant t_0$. Indeed, from $S(t_0) x \in H_{t_0}$ and the fact that $S(t - t_0)$ maps H_{t_0} into H_t we see that $S(t) x \in H_t$ for all $x \in E$. As subsets of E, the spaces of reachable points agree:

$$H_t = H_{t_0} \quad \text{with equivalent norms.}$$

The inclusion $H_{t_0} \hookrightarrow H_t$ always holds. To prove the converse inclusion $H_t \hookrightarrow H_{t_0}$, we first note that (1.1) implies the operator identity

$$(2.2) \qquad\qquad Q_t = Q_{t_0} + S(t_0) Q_{t-t_0} S^*(t_0) \qquad (t \geqslant t_0).$$

Using this identity, for all $t \geqslant t_0$ and $x^* \in E^*$ we have

$$
\begin{aligned}
\langle Q_t x^*, x^* \rangle &= \langle Q_{t_0} x^*, x^* \rangle + \langle Q_{t-t_0} S^*(t_0) x^*, S^*(t_0) x^* \rangle \\
&= \langle Q_{t_0} x^*, x^* \rangle + \langle Q_{t-t_0} \Sigma^*(t_0) i_{t_0}^* x^*, \Sigma^*(t_0) i_{t_0}^* x^* \rangle \\
&\leqslant \langle Q_{t_0} x^*, x^* \rangle + \|Q_{t-t_0}\| \cdot \|\Sigma(t_0)\|^2 \cdot \|i_{t_0}^* x^*\|_{H_{t_0}}^2 \\
&= \left(1 + \|Q_{t-t_0}\| \cdot \|\Sigma(t_0)\|^2\right) \cdot \langle Q_{t_0} x^*, x^* \rangle.
\end{aligned}
$$

The inclusion $H_t \hookrightarrow H_{t_0}$ now follows from [9, Proposition 1.1]. In general, H_{t_0} and H_t will be different as Hilbert spaces, and for this reason we will distinguish between these spaces carefully.

For the rest of this section we fix $t_0 > 0$ and assume that the pair (A, B) is null controllable in time t_0.

Since (A, B) is null controllable in any time $t \geqslant t_0$, for $t \geqslant t_0$ we define $\Sigma(t)$ as $S(t)$, regarded as an operator from E into H_t. Notice that $\|\Sigma(t)x\|_{H_t}^2$ is the minimal energy to steer from x to 0 in time t. The function $t \mapsto \|\Sigma(t)x\|_{H_t}^2$ is nonincreasing on $[t_0, \infty)$: this follows from

$$
(2.3) \qquad \|\Sigma(t+s)x\|_{H_{t+s}}^2 = \|S_{t,t+s}(s)\Sigma(t)x\|_{H_{t+s}}^2 \leqslant \|\Sigma(t)x\|_{H_t}^2.
$$

By a similar argument, for each $t \geqslant t_0$ the function $s \mapsto \|i_{t,t+s}\Sigma(t)x\|_{H_{t+s}}^2$ is nonincreasing on $[0, \infty)$. The main result of this section, Theorem 2.5, will show that this function is in fact differentiable at $s = 0$, and its derivative will be computed explicitly.

To prepare for the proof we need a series of lemmas. The first uses the identity

$$
(2.4) \qquad i_{t+s}^* = i_{t,t+s} i_t^* + i_{t,t+s} \Sigma(t) Q_s \Sigma^*(t) i_t^*,
$$

which follows from (2.2) by using (1.2), (2.1), the trivial identity $i_t = i_{t+s} \circ i_{t,t+s}$, and the injectivity of i_{t+s}.

LEMMA 2.1. *For all $h \in H_{t_0}$ the function $t \mapsto i_{t_0,t}^* i_{t_0,t} h$ is continuous on the interval $[t_0, \infty)$.*

Proof. Fix $t' \geqslant t \geqslant t_0$ arbitrary. Since $\|i_{t_0,t}\| \leqslant 1$, for all $h \in H_{t_0}$ we have

$$
\|i_{t_0,t'}^* i_{t_0,t'} h - i_{t_0,t}^* i_{t_0,t} h\|_{H_{t_0}} = \|i_{t_0,t}^* (i_{t,t'}^* i_{t,t'} - I) i_{t_0,t} h\|_{H_{t_0}} \leqslant \|(i_{t,t'}^* i_{t,t'} - I) i_{t_0,t} h\|_{H_t}.
$$

Hence it suffices to prove that $\lim_{t'-t \downarrow 0} \|i_{t,t'}^* i_{t,t'} g - g\|_{H_t} = 0$ for all $g \in H_t$. We first take $g = i_t^* x^*$ with $x^* \in E^*$. Then by (2.4),

$$
i_{t,t'}^* i_{t,t'} g = i_{t,t'}^* \left(i_{t'}^* x^* - i_{t,t'} \Sigma(t) Q_{t'-t} \Sigma^*(t) i_t^* x^* \right) = g - i_{t,t'}^* i_{t,t'} \Sigma(t) Q_{t'-t} \Sigma^*(t) g.
$$

Since the range of i_t^* is dense in H_t, a limiting argument shows that this identity holds for all $g \in H_t$. Using (2.3), for all $g \in H_t$ we have

$$
\begin{aligned}
\|i_{t,t'}^* i_{t,t'} g - g\|_{H_t} &= \|i_{t,t'}^* i_{t,t'} \Sigma(t) Q_{t'-t} \Sigma^*(t) g\|_{H_t} \\
&\leqslant \|\Sigma(t)\|^2 \|Q_{t'-t}\| \|g\|_{H_t} \leqslant \|\Sigma(t_0)\|^2 \|Q_{t'-t}\| \|g\|_{H_t}.
\end{aligned}
$$

Since $\lim_{t'-t \downarrow 0} \|Q_{t'-t}\| = 0$, this proves that $\lim_{t' \downarrow t} \|i_{t,t'}^* i_{t,t'} g - g\|_{H_t} = 0$. $\square$

The adjoint T^* of a C_0-semigroup T on a Banach space X may fail to be strongly continuous on X^*. To overcome this problem, one defines

$$
X^\odot := \left\{ x^* \in X^* : \lim_{t \downarrow 0} \|T^*(t) x^* - x^*\| = 0 \right\}.
$$

This is a norm closed, weak*-dense, S^*-invariant subspace of X^*, and the restricted semigroup $T^\odot = T^*\big|_{X^\odot}$ is strongly continuous on $X^\odot$. If X is reflexive, then $X^\odot$ is norm closed and weakly dense in X^*, and therefore we have $X^\odot = X^*$.

LEMMA 2.2. *For all $t \geqslant t_0$ the space H_t is S-invariant and the restricted semigroup $S_t := S|_{H_t}$ is strongly continuous on H_t.*

Proof. Invariance follows from the fact that $S(s)$ maps H_t into H_{t+s} and the fact that both H_t and H_{t+s} equal H_{t_0} as subsets of E.

Let $\delta > 0$ be arbitrary and fixed. For all $x^* \in E^*$ and $s \in [0, \delta]$ we have

$$
\begin{aligned}
\|S_t^*(s)i_t^*x^*\|_{H_t}^2 &= \|i_t^*S^*(s)x^*\|_{H_t}^2 \\
&= \langle Q_{t+s}x^*, x^*\rangle - \langle Q_s x^*, x^*\rangle \\
&\leqslant \langle Q_{t+s}x^*, x^*\rangle \\
&= \langle Q_t x^*, x^*\rangle + \int_0^s \langle BB^*S^*(t+r)x^*, S^*(t+r)x^*\rangle \, dr \\
&= \|i_t^*x^*\|_{H_t}^2 + \int_0^s \langle BB^*S^*(r)\Sigma^*(t)i_t^*x^*, S^*(r)\Sigma^*(t)i_t^*x^*\rangle \, dr \\
&\leqslant \left(1 + \delta \cdot \|BB^*\| \cdot \|\Sigma(t)\|^2 \cdot \sup_{r \in [0,\delta]} \|S(r)\|^2\right) \cdot \|i_t^*x^*\|_{H_t}^2.
\end{aligned}
$$

Hence,

$$
\limsup_{s\downarrow 0} \|S_t(s)\| \leqslant \left(1 + \delta \cdot \|BB^*\| \cdot \|\Sigma(t)\|^2 \cdot \sup_{r \in [0,\delta]} \|S(r)\|^2\right).
$$

On the other hand, for all $h \in H_t$ and $x^* \in E^*$ we have

$$
\lim_{s\downarrow 0}[S_t(s)h - h, i_t^*x^*]_{H_t} = \lim_{s\downarrow 0}\langle S(t)i_t h - i_t h, x^*\rangle = 0.
$$

It follows that S_t is weakly continuous. By a general result from semigroup theory, this implies that S_t is strongly continuous. □

We note two immediate consequences of this lemma.

LEMMA 2.3. *For all $x \in E$ the function $t \mapsto \Sigma^*(t)\Sigma(t)x$ is continuous on the interval $[t_0, \infty)$.*

Proof. By the observations preceding Lemma 2.2, the adjoint semigroup S_t^* is strongly continuous. The lemma now follows from the identity

$$
\Sigma^*(t)\Sigma(t)x = \Sigma^*(t_0)S_{t_0}^*(t - t_0)i_{t_0,t}^* i_{t_0,t}S_{t_0}(t - t_0)\Sigma(t_0)x
$$

and Lemmas 2.1 and 2.2. □

LEMMA 2.4. *For all $h \in H_t$ we have $\Sigma^*(t)h \in E^\odot$.*

Proof. This follows from

$$
\lim_{s\downarrow 0}\|S^*(s)\Sigma^*(t)h - \Sigma^*(t)h\| = \lim_{s\downarrow 0}\big\|\Sigma^*(t)\big(S_t^*(s)h - h\big)\big\| = 0,
$$

where we used again the strong continuity of S_t^*. □

We are now ready for the main result of this section, which describes the instantaneous rate of the change of the minimal energy along curves in the space of reachable states as time progresses.

THEOREM 2.5 (rate of change of minimal energy). *Let the pair (A, B) be null controllable in time t_0. Fix $t \geqslant t_0$ and let $f : [0, \infty) \to H_t$ be differentiable at 0. The function $\phi : [0, \infty) \to [0, \infty)$ defined by*

$$\phi(s) := \|i_{t,t+s} f(s)\|^2_{H_{t+s}}$$

is differentiable at 0, with derivative

$$\phi'(0) = 2[f'(0), f(0)]_{H_t} - \|B^* \Sigma^*(t) f(0)\|^2_H.$$

Notice that the first term on the right-hand side accounts for the speed and direction of leaving $f(0)$, while the second term describes the energy savings resulting from the extra time available.

Proof. Upon writing $f(s) = f(0) + s f'(0) + g(s)$ with $\lim_{s \downarrow 0} g(s)/s = 0$ we have

$$\lim_{s \downarrow 0} \frac{1}{s} \Big[\|f(s)\|^2_{H_t} - \|f(0)\|^2_{H_t} \Big]$$

$$= \lim_{s \downarrow 0} \frac{1}{s} \Big[2[s f'(0) + g(s), f(0)]_{H_t} + \|s f'(0) + g(s)\|^2_{H_t} \Big] = 2[f'(0), f(0)]_{H_t}.$$

Consequently, it remains to prove that

$$\lim_{s \downarrow 0} \frac{1}{s} \Big[\|i_{t,t+s} f(s)\|^2_{H_{t+s}} - \|f(s)\|^2_{H_t} \Big] = -\|B^* \Sigma^*(t) f(0)\|^2_H.$$

Let $x^* \in E^*$ be fixed. Noting that

$$\|i^*_{t+s} x^*\|^2_{H_{t+s}} - \|i^*_t x^*\|^2_{H_t} = \langle Q_{t+s} x^*, x^* \rangle - \langle Q_t x^*, x^* \rangle = \langle Q_s \Sigma^*(t) i^*_t x^*, \Sigma^*(t) i^*_t x^* \rangle,$$

from identity (2.4) we have

$$\|i_{t,t+s} i^*_t x^*\|^2_{H_{t+s}} - \|i^*_t x^*\|^2_{H_t}$$
$$= \|i^*_{t+s} x^*\|^2_{H_{t+s}} - \|i^*_t x^*\|^2_{H_t}$$
$$\quad - 2[i^*_{t+s} x^*, i_{t,t+s} \Sigma(t) Q_s \Sigma^*(t) i^*_t x^*]_{H_{t+s}} + \|i_{t,t+s} \Sigma(t) Q_s \Sigma^*(t) i^*_t x^*\|^2_{H_{t+s}}$$
$$= \langle Q_s \Sigma^*(t) i^*_t x^*, \Sigma^*(t) i^*_t x^* \rangle$$
$$\quad - 2[i^*_t x^*, \Sigma(t) Q_s \Sigma^*(t) i^*_t x^*]_{H_t} + \|i_{t,t+s} \Sigma(t) Q_s \Sigma^*(t) i^*_t x^*\|^2_{H_{t+s}}.$$

By approximation, for all $s \geqslant 0$ we obtain

$$\|i_{t,t+s} f(s)\|^2_{H_{t+s}} - \|f(s)\|^2_{H_t}$$
$$= \langle Q_s \Sigma^*(t) f(s), \Sigma^*(t) f(s) \rangle$$
$$\quad - 2[f(s), \Sigma(t) Q_s \Sigma^*(t) f(s)]_{H_t} + \|i_{t,t+s} \Sigma(t) Q_s \Sigma^*(t) f(s)\|^2_{H_{t+s}}.$$

Next, for any $y^\odot \in E^\odot$ we have, by strong continuity,

$$\lim_{s \downarrow 0} \frac{1}{s} Q_s y^\odot = \lim_{s \downarrow 0} \frac{1}{s} \int_0^s S(r) B B^* S^*(r) y^\odot \, dr = B B^* y^\odot.$$

Hence, using the continuity of f at 0, the fact that $\limsup_{s\downarrow 0}\frac{1}{s}\|Q_s\| < \infty$, and the fact that $\Sigma^*(t)f(0) \in E^\odot$ by Lemma 2.4, we obtain

$$\limsup_{s\downarrow 0}\left\|\frac{1}{s}Q_s\Sigma^*(t)f(s) - BB^*\Sigma^*(t)f(0)\right\|$$

$$\leqslant \limsup_{s\downarrow 0}\left\|\frac{1}{s}Q_s\Sigma^*(t)f(s) - \frac{1}{s}Q_s\Sigma^*(t)f(0)\right\|$$

$$+ \limsup_{s\downarrow 0}\left\|\frac{1}{s}Q_s\Sigma^*(t)f(0) - BB^*\Sigma^*(t)f(0)\right\|$$

$$\leqslant \|\Sigma^*(t)\| \cdot \limsup_{s\downarrow 0}\left(\frac{1}{s}\|Q_s\|\right) \cdot \limsup_{s\downarrow 0}\|f(s) - f(0)\|$$

$$+ \limsup_{s\downarrow 0}\left\|\frac{1}{s}Q_s\Sigma^*(t)f(0) - BB^*\Sigma^*(t)f(0)\right\| = 0.$$

It follows that

$$\lim_{s\downarrow 0}\frac{1}{s}Q_s\Sigma^*(t)f(s) = BB^*\Sigma^*(t)f(0).$$

As a consequence,

$$\lim_{s\downarrow 0}\frac{1}{s}\Big[\langle Q_s\Sigma^*(t)f(s), \Sigma^*(t)f(s)\rangle - 2[f(s), \Sigma(t)Q_s\Sigma^*(t)f(s)]_{H_t}$$

$$+ \|i_{t,t+s}\Sigma(t)Q_s\Sigma^*(t)f(s)\|^2_{H_{t+s}}\Big]$$

$$= \lim_{s\downarrow 0}\left\langle\frac{1}{s}Q_s\Sigma^*(t)f(s), \Sigma^*(t)f(s)\right\rangle - 2\lim_{s\downarrow 0}\left[f(s), \Sigma(t)\left(\frac{1}{s}Q_s\Sigma^*(t)f(s)\right)\right]_{H_t}$$

$$+ \lim_{s\downarrow 0} s\left\|i_{t,t+s}\Sigma(t)\left(\frac{1}{s}Q_s\Sigma^*(t)f(s)\right)\right\|^2_{H_{t+s}}$$

$$= \langle BB^*\Sigma^*(t)f(0), \Sigma^*(t)f(0)\rangle - 2[f(0), \Sigma(t)BB^*\Sigma^*(t)f(0)]_{H_t} + 0$$

$$= -\|B^*\Sigma^*(t)f(0)\|^2_H;$$

in the next to last step we used that $\|i_{t,t+s}\| \leqslant 1$.   □

For the convenience of those readers familiar with the Hilbert space formalism as used, e.g., in [10], we add a reformulation of Theorem 2.5 for Hilbert spaces E. In this setting we identify E and its dual in the usual way and identify Q_t with a positive self-adjoint operator on E. As is well known, the reproducing kernel Hilbert space of Q_t is then given by

$$(2.5) \qquad\qquad i_t(H_t) = \operatorname{Im} Q_t^{1/2}.$$

In what follows we identify $i_t(H_t)$ and H_t and abuse notation by regarding both $Q_t^{1/2}$ and Q_t as operators from E to H_t whenever this is convenient. Denoting the closure of H_t in E by E_t, it follows from (2.5) and a standard argument that $Q_t^{1/2}$ is unitary as an operator from E_t to H_t.

By (2.5), the pair (A, B) is null controllable in time t_0 if and only if $\operatorname{Im} S(t_0) \subseteq \operatorname{Im} Q_{t_0}^{1/2}$. Since the restriction of $Q_{t_0}^{1/2}$ to E_{t_0} is injective, the inverse $Q_{t_0}^{-1/2}$ is well-defined on the linear subspace H_{t_0} of E. Then by null controllability, the operator

$\Gamma(t_0) := Q_{t_0}^{-1/2} S(t_0)$ is well-defined as a bounded operator from E to E_{t_0}. For all $h = Q_{t_0}^{1/2} y \in H_{t_0}$ we have

$$[\Gamma(t_0)x, h]_E = [S(t_0)x, y]_E = [x, S^*(t_0)y]_E = [x, S^*(t_0)Q_{t_0}^{-1/2}h]_E.$$

Since H_{t_0} is dense in E_{t_0} we see that $\Gamma^*(t_0) := (\Gamma(t_0))^*$ is the unique extension of $S^*(t_0)Q_{t_0}^{-1/2}$ to a bounded operator from E_{t_0} to E.

COROLLARY 2.6. *Let the pair (A, B) be null controllable in time t_0. Fix $t \geqslant t_0$ and let $g : [0, \infty) \to E_t$ be differentiable at 0. The function $\phi : [0, \infty) \to [0, \infty)$ defined by*

$$(2.6) \qquad\qquad \phi(s) := \|Q_t^{1/2} g(s)\|_{H_{t+s}}^2$$

is differentiable at 0, with derivative

$$\phi'(0) = 2[g'(0), g(0)]_E - [Q\Gamma^*(t)g(0), \Gamma^*(t)g(0)]_E.$$

Note some further abuse of notation in (2.6), where $Q_t^{1/2} g(s)$ is regarded as an element of H_{t+s}.

Proof. Let $f : [0, \infty) \to H_t$ be defined by $f(s) = Q_t^{1/2} g(s)$. Since $Q_t^{1/2}$ is unitary as an operator from E_t to H_t, f is differentiable at 0 with derivative $f'(0) = Q_t^{1/2} g'(0)$. Let $Q := BB^*$. By Theorem 2.5,

$$\phi(s) := \|i_{t,t+s}f(s)\|_{H_{t+s}}^2 = \|Q_t^{1/2} g(s)\|_{H_{t+s}}^2$$

is differentiable at 0 with derivative

$$
\begin{aligned}
(2.7) \qquad \phi'(0) &= 2[f'(0), f(0)]_{H_t} - \|B^*\Sigma^*(t)f(0)\|_H^2 \\
&= 2[Q_t^{1/2} g'(0), Q_t^{1/2} g(0)]_{H_t} - [Q\Gamma^*(t)g(0), \Gamma^*(t)g(0)]_E \\
&= 2[g'(0), g(0)]_E - [Q\Gamma^*(t)g(0), \Gamma^*(t)g(0)]_E.
\end{aligned}
$$

In the second identity of (2.7) we used that $\Gamma^*(t)$ extends $S^*(t)Q_t^{-1/2}$ on E_t and that for all $h = Q_t y \in H_t$ we have

$$[B^*\Sigma^*(t)h, B^*\Sigma^*(t)h]_H = [Q\Sigma^*(t)i_t^* y, \Sigma^*(t)i_t^* y]_E = [Q^*S(t)y, S^*(t)y]_E,$$

recalling that we identify $Q_t y = i_t i_t^* y$ and $i_t^* y$. In the third identity of (2.7) we used that $Q_t^{1/2}$ is unitary from E_t to H_t. □

3. Null controllability with vanishing energy. Following Priola and Zabczyk [10] we call the pair (A, B) *null controllable with vanishing energy* if for all $\varepsilon > 0$ and $x \in E$ there exists a time $t > 0$ and a control $u \in L^2(0, t; H)$ with $y^{u,x}(t) = 0$ and $\|u\|_{L^2(0,t;H)} < \varepsilon$. Clearly, null controllability with vanishing energy implies null controllability with bounded energy.

THEOREM 3.1. *If the pair (A, B) is null controllable with vanishing energy, then it is null controllable in finite time.*

Proof. For $n = 1, 2, \ldots$, let E_n denote the set of all $x \in E$ for which there exists a control $u \in L^2(0, n; H)$ with $y^{u,x}(n) = 0$ and $\|u\|_{L^2(0,n;H)} \leqslant 1$. Notice that $\bigcup_{n \geqslant 1} E_n = E$.

We claim that each E_n is closed. To see this, fix $n \geqslant 1$ and let $\lim_{k \to \infty} x_k = x$ in E with all $x_k \in E_n$. We must check that $x \in E_n$. For each k we choose a

control $u_k \in L^2(0, n; H)$ with $y^{u_k, x_k}(n) = 0$ and $\|u_k\|_{L^2(0,n;H)} \leqslant 1$. After passing to a subsequence, we may assume that there exists a control $u \in L^2(0, n; H)$ with $\|u\|_{L^2(0,n;H)} \leqslant 1$ such that $\lim_{k\to\infty} u_k = u$ weakly in $L^2(0, n; H)$. Then for all $x^* \in E^*$ we have

$$\langle y^{u,x}(n), x^* \rangle = \langle S(n)x, x^* \rangle + \int_0^n [u(s), B^* S^*(n-s)x^*]_H \, ds$$

$$= \lim_{k\to\infty} \left(\langle S(n)x_k, x^* \rangle + \int_0^n [u_k(s), B^* S^*(n-s)x^*]_H \, ds \right)$$

$$= \lim_{k\to\infty} \langle y^{u_k, x_k}(n), x^* \rangle = 0.$$

Hence $y^{u,x}(n) = 0$ and $x \in E_n$.

By the Baire category theorem, at least one E_{n_0} has a nonempty interior. Fix an arbitrary x_0 in the interior of E_{n_0} and consider the set $E_{n_0} - x_0$. This is a neighborhood of 0 consisting of elements that can be steered to 0 in time n_0. By linearity it follows that every $x \in E$ can be steered to 0 in time n_0. This means that the pair (A, B) is null controllable in time n_0.   □

Recall that if (A, B) is null controllable in time t_0, then for all $t \geqslant t_0$ the square norm $\|\Sigma(t)x\|_{H_t}^2$ is the minimal energy to steer from x to 0 in time t. Hence the following observation is a straightforward consequence of (2.3) and the above theorem.

COROLLARY 3.2. *The following assertions are equivalent:*
1. *The pair (A, B) is null controllable with vanishing energy.*
2. *The pair (A, B) is null controllable in finite time and $\lim_{t\to\infty} \|\Sigma(t)x\|_{H_t} = 0$ for all $x \in E$.*

We proceed with two simple examples of systems that are null controllable with vanishing energy.

Example 3.3. If (A, B) is null controllable in finite time and the semigroup S generated by A is *strongly stable*, i.e., if $\lim_{t\to\infty} S(t)x = 0$ for all $x \in E$, then (A, B) is null controllable with vanishing energy. Indeed, if (A, B) is null controllable in time t_0, then for all $t \geqslant t_0$ we have

$$\|\Sigma(t)x\|_{H_t} = \|i_{t_0,t}\Sigma(t_0)S(t - t_0)x\|_{H_t} \leqslant \|\Sigma(t_0)\| \, \|S(t - t_0)x\|.$$

Example 3.4. The range of B is a Hilbert space with norm

$$\|Bh\|_{\text{range } B} = \inf\{\|h'\|_H : \ Bh' = Bh\}.$$

With this norm, the range of B equals the reproducing kernel Hilbert space of the operator BB^*. Accordingly we shall denote the range of B by H_{BB^*}. If S restricts to a C_0-semigroup S_B on H_{BB^*}, then it follows from [6, Theorem 3.5] that the reachable spaces H_t for the pair (A, B) coincide with the reproducing kernel space of the operators $R_t \in \mathcal{L}(H_{BB^*})$ defined by

$$R_t h := \int_0^t S_B(s) S_B^*(s) h \, ds \qquad (h \in H_{BB^*}),$$

and the pair (S_B, I_B) is null controllable for all times $t > 0$. Here I_B denotes the identity operator on H_{BB^*}. It follows from the same reference that for all $h \in \text{range } B$ and $t > 0$ we have an estimate

$$\|\Sigma_B(t)h\|_{H_t}^2 \leqslant \frac{1}{t^2} \int_0^t \|S_B(s)h\|_{H_{BB^*}}^2 \, ds \qquad (h \in H_{BB^*}).$$

Here, $\Sigma_B(t)$ denotes $S_B(t)$, regarded as an operator from H_{BB^*} into H_t. In particular the pair (S_B, I_B) is null controllable with vanishing energy if the semigroup S_B is uniformly bounded on H_{BB^*}.

In [10], under the assumption that E is a Hilbert space it was shown by control theoretic methods that a pair (A, B) which is null controllable in finite time is null controllable with vanishing energy if and only if the algebraic Riccati equation

$$(3.1) \qquad\qquad XA + A^*X - XBB^*X = 0$$

admits $X = 0$ as its only positive self-adjoint solution. A *solution* of (3.1) is a bounded operator $X \in \mathcal{L}(E)$ such that

$$(3.2) \qquad \langle XAx, y \rangle + \langle Xx, Ay \rangle - \langle XBB^*Xx, y \rangle = 0 \quad \text{for all } x, y \in D(A).$$

In this identity the brackets denote the scalar product of E.

In this section we shall prove an extension of this result to Banach spaces E. It shares with [10] the strategy of first solving a differential Riccati equation and obtaining the final characterization from a maximality argument, but both steps are accomplished in a completely different way. In the Banach space setting, a *solution* of (3.1) is a bounded operator $X \in \mathcal{L}(E, E^*)$ such that (3.2) holds for all $x, y \in D(A)$; this time the brackets denote the duality pairing between E^* and E. The notions of positivity and self-adjointness extend as follows: we call $X \in \mathcal{L}(E, E^*)$ *positive* if $\langle Xx, x \rangle \geqslant 0$ for all $x \in E$ and *symmetric* if $\langle Xx, y \rangle = \langle Xy, x \rangle$ for all $x, y \in E$.

We begin with a result which states that the operator function $t \mapsto \Sigma^*(t)\Sigma(t)$ solves, in some appropriate sense, the differential Riccati equation

$$\frac{d}{dt} X(t) = X(t)A + A^*X(t) - X(t)BB^*X(t)$$

on the interval $[t_0, \infty)$.

In the Hilbert space literature, existence of a solution is usually derived from a fixed point argument. Here, we obtain it as a direct consequence of Theorem 2.5.

PROPOSITION 3.5. *Let the pair (A, B) be null controllable in time t_0. For all $x, y \in D(A)$ the function $t \mapsto \langle \Sigma^*(t)\Sigma(t)x, y \rangle$ is differentiable on the interval $[t_0, \infty)$, with derivative*

$$\frac{d}{dt} \langle \Sigma^*(t)\Sigma(t)x, y \rangle$$
$$= \langle \Sigma^*(t)\Sigma(t)Ax, y \rangle + \langle \Sigma^*(t)\Sigma(t)x, Ay \rangle - \langle BB^*\Sigma^*(t)\Sigma(t)x, \Sigma^*(t)\Sigma(t)y \rangle.$$

Proof. Since both BB^* and $\Sigma^*(t)\Sigma(t)$ are symmetric operators, by polarization it suffices to prove that for all $x \in D(A)$ and $t \geqslant t_0$ we have

$$\frac{d}{dt} \langle \Sigma^*(t)\Sigma(t)x, x \rangle = 2\langle \Sigma^*(t)\Sigma(t)Ax, x \rangle - \langle BB^*\Sigma^*(t)\Sigma(t)x, \Sigma^*(t)\Sigma(t)x \rangle.$$

For this, in turn, it suffices to prove right differentiability. Indeed, by Lemma 2.3 the functions $\langle \Sigma^*(t)\Sigma(t)x, x \rangle$ and $2\langle \Sigma^*(t)\Sigma(t)Ax, x \rangle - \langle BB^*\Sigma^*(t)\Sigma(t)x, \Sigma^*(t)\Sigma(t)x \rangle$ are continuous functions of $t \in [t_0, \infty)$, and by elementary calculus a continuous function that is right differentiable with continuous right derivative is differentiable; cf. [13].

Fix $x \in D(A)$ and $t \geqslant t_0$. By Theorem 2.5 applied to $f(s) = \Sigma(t)S(s)x$ we have

$$\lim_{s \downarrow 0} \frac{1}{s}\Big(\langle \Sigma^*(t+s)\Sigma(t+s)x, x\rangle - \langle \Sigma^*(t)\Sigma(t)x, x\rangle\Big)$$

$$= \lim_{s \downarrow 0} \frac{1}{s}\Big(\|i_{t,t+s}\Sigma(t)S(s)x\|^2_{H_{t+s}} - \|\Sigma(t)x\|^2_{H_t}\Big)$$

$$= 2[\Sigma(t)Ax, \Sigma(t)x]_{H_t} - \|B^*\Sigma^*(t)\Sigma(t)x\|^2_H$$

$$= 2\langle \Sigma^*(t)\Sigma(t)Ax, x\rangle - \langle BB^*\Sigma^*(t)\Sigma(t)x, \Sigma^*(t)\Sigma(t)x\rangle. \qquad \square$$

Remark 3.6. In the special case where E is a Hilbert space, instead of using Theorem 2.5 we could apply Corollary 2.6 to the E_t-valued function $g(s) := \Gamma(t)S(s)$; note that $Q_t^{1/2}g(s) = \Sigma(t)S(s)x = f(s)$.

From Proposition 3.5 we obtain the following.

PROPOSITION 3.7. *Let the pair* (A, B) *be null controllable in time* t_0. *For all* $x, y \in E$ *the limit* $\lim_{t \to \infty} \langle \Sigma^*(t)\Sigma(t)x, y\rangle$ *exists, and the operator* $X \in \mathcal{L}(E, E^*)$ *defined by*

$$(3.3) \qquad\qquad \langle Xx, y\rangle := \lim_{t \to \infty} \langle \Sigma^*(t)\Sigma(t)x, y\rangle$$

defines a positive symmetric solution of the algebraic Riccati equation

$$XA + A^*X - XBB^*X = 0.$$

Proof. For all $x \in E$ we have $\langle \Sigma^*(t)\Sigma(t)x, x\rangle = \|\Sigma(t)x\|^2_{H_t}$, which is a nonincreasing function of $t \geqslant t_0$. In particular, for all $x \in E$ the limit $\lim_{t \to \infty} \langle \Sigma^*(t)\Sigma(t)x, x\rangle$ exists. Since each $\Sigma^*(t)\Sigma(t)$ is positive and symmetric, by polarization it follows that for all $x, y \in E$ the limit $\lim_{t \to \infty} \langle \Sigma^*(t)\Sigma(t)x, y\rangle$ exists, and then (3.3) defines a positive and symmetric operator X.

Since $t \mapsto \Sigma^*(t)\Sigma(t)$ solves the differential Riccati equation, a standard argument implies that X solves the algebraic Riccati equation. $\qquad \square$

Our next aim is to show that the weak operator limit $X = \lim_{t \to \infty} \Sigma^*(t)\Sigma(t)$ is in fact the *maximal* symmetric solution of the algebraic Riccati equation. More precisely we have the following.

THEOREM 3.8. *Let the pair* (A, B) *be null controllable at time* $t_0 > 0$. *If* Y *is a symmetric solution of the algebraic Riccati equation, then for all* $x \in E$ *we have* $\langle Yx, x\rangle \leqslant \langle Xx, x\rangle$.

Proof. Fix $t \geqslant t_0$ and $x \in E$, and let $u \in L^2(0, t; H)$ be any control steering x to 0 in time t:

$$y^{u,x}(t) = S(t)x + \int_0^t S(t-s)Bu(s)\, ds = 0.$$

We will show that the function $f_u : [0, t] \to \mathbb{R}$ defined by

$$f_u(s) := \int_0^s \|u(r)\|^2_H\, dr + \langle Yy^{u,x}(s), y^{u,x}(s)\rangle$$

is nondecreasing. To prove this we shall show that f_u is almost everywhere differentiable with nonnegative derivative.

Let us first consider the function $g_u(s) := \langle Yy^{u,x}(s), y^{u,x}(s)\rangle$. In order to show that g_u is differentiable we introduce a regularization operator as follows. For $\lambda > 0$ large enough, put $E_\lambda := \lambda(\lambda - A)^{-1}$ and define

$$g_{u,\lambda}(s) := \langle YE_\lambda y^{u,x}(s), E_\lambda y^{u,x}(s)\rangle.$$

Then, by the symmetry of Y and the fact that this operator solves the algebraic Riccati equation,

$$
\begin{aligned}
g'_{u,\lambda}(s) &= 2\Big\langle Y E_\lambda y^{u,x}(s), \frac{d}{ds} E_\lambda y^{u,x}(s) \Big\rangle \\
&= 2\Big\langle Y E_\lambda y^{u,x}(s), \frac{d}{ds}\Big(S(s) E_\lambda x + \int_0^s S(s-r) E_\lambda B u(r)\, dr \Big) \Big\rangle \\
&= 2\Big\langle Y E_\lambda y^{u,x}(s), A\Big(S(s) E_\lambda x + \int_0^s S(s-r) E_\lambda B u(r)\, dr \Big) + E_\lambda B u(s) \Big\rangle \\
&= \langle Y B B^* Y E_\lambda y^{u,x}(s), E_\lambda y^{u,x}(s) \rangle + 2\langle Y E_\lambda y^{u,x}(s), E_\lambda B u(s) \rangle \\
&=: G_{u,\lambda}(s).
\end{aligned}
$$

From $\lim_{\lambda\to\infty} E_\lambda = I$ strongly we have $\lim_{\lambda\to\infty} g_{u,\lambda} = g_u$ and

$$
\lim_{\lambda\to\infty} G_{u,\lambda} = \langle Y B B^* Y y^{u,x}(s), y^{u,x}(s) \rangle + 2\langle Y y^{u,x}(s), B u(s) \rangle
$$

uniformly on $[0, t]$ (notice that $y^{u,x}$ is continuous on $[0, t]$). The closedness of the first derivative now implies that g_u is differentiable, with derivative

$$
g'_u(s) = \langle Y B B^* Y y^{u,x}(s), y^{u,x}(s) \rangle + 2\langle Y y^{u,x}(s), B u(s) \rangle.
$$

It follows that f_u is almost everywhere differentiable, with derivative

$$
\begin{aligned}
f'_u(s) &= \|u(s)\|_H^2 + \langle Y B B^* Y y^{u,x}(s), y^{u,x}(s) \rangle + 2\langle Y y^{u,x}(s), B u(s) \rangle \\
&= \|u(s)\|_H^2 + \|B^* Y y^{u,x}(s)\|_H^2 + 2[B^* Y y^{u,x}(s), u(s)]_H \\
&\geqslant \|u(s)\|_H^2 + \|B^* Y y^{u,x}(s)\|_H^2 - 2\|B^* Y y^{u,x}(s)\|_H \|u(s)\|_H \\
&= \big(\|u(s)\|_H - \|B^* Y y^{u,x}(s)\|_H \big)^2,
\end{aligned}
$$

which is nonnegative.

By what has been shown so far, we have

$$
\begin{aligned}
\|u\|_{L^2(0,t;H)}^2 &= \int_0^t \|u(r)\|_H^2\, dr = \int_0^t \|u(r)\|_H^2\, dr + \langle Y y^{u,x}(t), y^{u,x}(t) \rangle \\
&= f_u(t) \geqslant f_u(0) = \langle Y y^{u,x}(0), y^{u,x}(0) \rangle = \langle Y x, x \rangle.
\end{aligned}
$$

Taking the infimum over all admissible controls we obtain

$$
\|\Sigma(t)x\|_{H_t}^2 \geqslant \langle Y x, x \rangle.
$$

Finally, letting $t \to \infty$, this gives

$$
\langle X x, x \rangle = \lim_{t\to\infty} \|\Sigma(t)x\|_{H_t}^2 \geqslant \langle Y x, x \rangle. \qquad \square
$$

The preceding two results may now be combined to prove the following characterization of null controllability with vanishing energy, which extends the corresponding Hilbert space result of [10] to Banach spaces.

THEOREM 3.9. *The following assertions are equivalent:*
1. *The pair (A, B) is null controllable with vanishing energy.*
2. *The pair (A, B) is null controllable in finite time and the only positive symmetric solution of the algebraic Riccati equation $XA + A^*X - XBB^*X = 0$ is the trivial solution $X = 0$.*

Proof. We will use Corollary 3.2.

(1)$\Rightarrow$(2): Let Y be any positive symmetric solution of the algebraic Riccati equation. Then for all $x \in E$ we have

$$0 \leqslant \langle Yx, x \rangle \leqslant \langle Xx, x \rangle = \lim_{t \to \infty} \|\Sigma(t)x\|_{H_t}^2 = 0,$$

which implies that $Y = 0$.

(2)$\Rightarrow$(1): Since $X = \lim_{t \to \infty} \Sigma^*(t)\Sigma(t)$ is a positive symmetric solution of the algebraic Riccati equation, it follows that $\lim_{t \to \infty} \|\Sigma(t)x\|_{H_t}^2 = \langle Xx, x \rangle = 0$ for all $x \in E$.   □

Under additional spectral assumptions (which are satisfied, e.g., if S is eventually compact), it is shown in [10] that the pair (A, B) is null controllable with vanishing energy if and only if $\sup\{\mathrm{Re}\,\lambda : \lambda \in \sigma(A)\} \leqslant 0$. This result is applied in [11], where it is used to obtain necessary and sufficient conditions for the validity of Liouville's theorem for the Ornstein–Uhlenbeck operator associated with the pair (A, B).

As an application of Theorem 3.9 we give a sufficient condition for null controllability with vanishing energy in the symmetric case.

THEOREM 3.10. *Let the pair (A, B) be null controllable at time $t_0 > 0$. Assume furthermore that*
  - *(nondegeneracy) B has dense range,*
  - *(BB*-symmetry) $S(t)BB^* = BB^*S^*(t)$ for all $t \geqslant 0$.*
If the limit $Q_\infty := \lim_{t \to \infty} Q_t$ exists in the weak operator topology, then (A, B) is null controllable with vanishing energy.

Without any nondegeneracy condition on B, the assumptions of the theorem imply that S restricts to a strongly stable C_0-semigroup of contractions S_B on the range of B [6, Theorem 4.5]. By Examples 3.3 and 3.4, the pair (S_B, I_B) is null controllable with vanishing energy.

Proof. We shall use the fact that $Q_\infty := \lim_{t \to \infty} Q_t$ exists in the weak operator topology if and only if there exists a positive symmetric solution in $\mathcal{L}(E^*, E)$ of the Lyapunov equation

$$AY + YA^* + BB^* = 0$$

and that in this case Q_∞ is the minimal positive symmetric solution of this equation [6, Theorem 4.4]. In this context, a bounded operator $Y \in \mathcal{L}(E^*, E)$ is called *positive* if $\langle Yx, x \rangle \geqslant 0$ for all $x \in E$ and *symmetric* if $\langle Yx, y \rangle = \langle Yy, x \rangle$ for all $x, y \in E$.

Assume now that $X \in \mathcal{L}(E, E^*)$ is a positive symmetric solution of the algebraic Riccati equation. We have to show that $X = 0$.

Since B is assumed to have dense range, it is an easy consequence of the Hahn–Banach theorem that BB^* is injective and has dense range as well. From this it follows that Q_∞ is injective and has dense range [5, Lemma 5.2].

By the same argument as in the proof of [6, Theorem 4.5], the assumption $S(t)BB^* = BB^*S(t)$ implies that the semigroup S_t on H_t is self-adjoint for all $t \geqslant t_0$. Moreover, for all $x \in D(A)$ we have $\Sigma(t)x \in D(A_t)$ and $A_t\Sigma(t)x = \Sigma(t)Ax$. Similarly, for all $h \in D(A_t^*)$ we have $\Sigma^*(t)h \in D(A^*)$ and $A^*\Sigma^*(t)h = \Sigma^*(t)A_t^*h$. Using these facts, for all $x, y \in D(A)$ we obtain

$$\begin{aligned}
\langle Xx, Ay \rangle &= \lim_{t \to \infty} \langle \Sigma^*(t)\Sigma(t)x, Ay \rangle = \lim_{t \to \infty} \langle \Sigma^*(t)A_t^*\Sigma(t)x, y \rangle \\
&= \lim_{t \to \infty} \langle \Sigma^*(t)A_t\Sigma(t)x, y \rangle = \lim_{t \to \infty} \langle \Sigma^*(t)\Sigma(t)Ax, y \rangle = \langle XAx, y \rangle.
\end{aligned}$$

It follows that $Xx \in D(A^*)$ and $A^*Xx = XAx$. Thus, $A^*X = XA$. Similarly one proves that $AQ_\infty = Q_\infty A^*$. As X and Q_∞ are symmetric and solve the algebraic Riccati equation and the Lyapunov equation, respectively, for all $x^*, y^* \in D(A^*)$ we obtain

$$
\begin{aligned}
0 &= \langle A^*XQ_\infty x^*, Q_\infty y^* \rangle + \langle XAQ_\infty x^*, Q_\infty y^* \rangle - \langle XBB^*XQ_\infty x^*, Q_\infty y^* \rangle \\
&= \langle XQ_\infty A^* x^*, Q_\infty y^* \rangle + \langle XAQ_\infty y^*, Q_\infty x^* \rangle - \langle XBB^*XQ_\infty x^*, Q_\infty y^* \rangle \\
&= -\langle XBB^* x^*, Q_\infty y^* \rangle - \langle XBB^*XQ_\infty x^*, Q_\infty y^* \rangle \\
&= -\langle Q_\infty XBB^* x^*, y^* \rangle - \langle Q_\infty XBB^*XQ_\infty x^*, y^* \rangle.
\end{aligned}
$$

Thus,

(3.4) $$\langle Q_\infty XBB^*(I + XQ_\infty)x^*, y^* \rangle = 0$$

for all $x^*, y^* \in D(A^*)$. Since $D(A^*)$ is weak*-dense, it follows that

(3.5) $$Q_\infty XBB^*(I + XQ_\infty)x^* = 0$$

for all $x^* \in D(A^*)$. Furthermore, by the symmetry of Q_∞, X, and BB^*, from (3.4) we obtain

$$\langle (I + Q_\infty X)BB^*XQ_\infty y^*, x^* \rangle = 0$$

for all $x^*, y^* \in D(A^*)$. Since $D(A^*)$ is weak*-dense, it follows that

(3.6) $$(I + Q_\infty X)BB^*XQ_\infty y^* = 0$$

for all $y^* \in D(A^*)$. Taking $y^* = x^*$ and subtracting (3.5) and (3.6), we find

$$BB^*XQ_\infty x^* = Q_\infty XBB^* x^*$$

for all $x^* \in D(A^*)$. Hence, by (3.6),

$$(I + Q_\infty X)Q_\infty XBB^* x^* = 0$$

for all $x^* \in D(A^*)$. Since $D(A^*)$ is weak*-dense and BB^* is weak*-to-weakly continuous and has weakly dense range, this implies that

$$(I + Q_\infty X)Q_\infty X = 0$$

or, equivalently, $P(I - P) = 0$, where $P := -Q_\infty X$. Thus, P is a projection in E.

For any $x \in \ker P$ we have $Q_\infty Xx = 0$ and therefore $Xx = 0$ by the injectivity of Q_∞.

For any $x \in \ker(I - P)$ we have $-Q_\infty Xx = x$ and therefore

$$0 \leqslant \langle Xx, x \rangle = -\langle Xx, Q_\infty Xx \rangle = -\langle Q_\infty Xx, Xx \rangle \leqslant 0$$

by the positivity of Q_∞. It follows that $\langle Q_\infty Xx, Xx \rangle = \|i_\infty^* Xx\|_{H_\infty}^2 = 0$, where $i_\infty : H_\infty \hookrightarrow E$ denotes the reproducing kernel Hilbert space of Q_∞. Since $Q_\infty = i_\infty \circ i_\infty^*$ is injective, i_∞^* is injective, and we conclude that $Xx = 0$.

Combining the facts just proved, we obtain that $Xx = 0$ for all $x \in E$, i.e., $X = 0$. □

It is worthwhile to point out that Theorem 3.10 is not covered by Example 3.3, since the existence of Q_∞ does not imply strong stability of the semigroup S.

Example 3.11. Let $E = \mathbb{R}^2$ and $S(t) = \left(\begin{smallmatrix} e^{-t} & 0 \\ 0 & 1 \end{smallmatrix} \right)$. The semigroup S is not strongly stable. Taking $H = \mathbb{R}$ and $Bh = (h, 0)$, the limit $Q_\infty = \lim_{t\to\infty} Q_t$ exists: we have

$$\lim_{t\to\infty} Q_t = \lim_{t\to\infty} \int_0^t \left(\begin{array}{cc} e^{-2s} & 0 \\ 0 & 0 \end{array} \right) ds = \left(\begin{array}{cc} \frac{1}{2} & 0 \\ 0 & 0 \end{array} \right).$$

Let us finally observe that in Theorem 3.10 the condition on existence of Q_∞ is not a necessary one (take $E = H = \mathbb{R}$, $B = I$, and $S(t) = I$), nor can it be dropped (take $E = H = \mathbb{R}$, $B = I$, and $S(t) = e^t I$).

Acknowledgment. The author thanks Professors Jerzy Zabczyk and Ben Goldys for stimulating discussions and kind hospitality.

REFERENCES

[1] V. Barbu and G. Da Prato, *Hamilton-Jacobi Equations in Hilbert Spaces*, Research Notes in Math., Pitman, Boston, MA, 1986.

[2] A. Bensoussan, G. Da Prato, M. C. Delfour, and S. K. Mitter, *Representation and Control of Infinite-Dimensional Systems, Vol.* 1, Systems Control Found. Appl., Birkhäuser Boston, Boston, MA, 1992.

[3] A. Bensoussan, G. Da Prato, M. C. Delfour, and S. K. Mitter, *Representation and Control of Infinite-Dimensional Systems, Vol.* 2, Systems Control Found. Appl., Birkhäuser Boston, Boston, MA, 1993.

[4] R. Curtain and H. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory*, Texts Appl. Math. 21, Springer-Verlag, New York, 1995.

[5] B. Goldys, F. Gozzi, and J. M. A. M. van Neerven, *On closability of directional gradients*, Potential Anal., 18 (2003), pp. 289–310.

[6] B. Goldys and J. M. A. M. van Neerven, *Transition semigroups of Banach space valued Ornstein-Uhlenbeck processes*, Acta Appl. Math., 76 (2003), pp. 283–330.

[7] I. Lasiecka and R. Triggiani, *Control Theory for Partial Differential Equations: Continuous and Approximation Theories.* I, Encyclopedia Math. Appl. 74, Cambridge University Press, Cambridge, UK, 2000.

[8] I. Lasiecka and R. Triggiani, *Control Theory for Partial Differential Equations: Continuous and Approximation Theories.* II, Encyclopedia Math. Appl. 75, Cambridge University Press, Cambridge, UK, 2000.

[9] J. M. A. M. van Neerven, *Nonsymmetric Ornstein-Uhlenbeck semigroups in Banach spaces*, J. Funct. Anal., 155 (1998), pp. 495–535.

[10] E. Priola and J. Zabczyk, *Null controllability with vanishing energy*, SIAM J. Control Optim., 42 (2003), pp. 1013–1032.

[11] E. Priola and J. Zabczyk, *Liouville Theorems in Finite and Infinite Dimensions*, Preprint 9/2003, Scuola Normale Superiore, Pisa, 2003.

[12] M. Sîrbu, *A Riccati equation approach to the null controllability of linear systems*, Commun. Appl. Anal., 6 (2002), pp. 163–177.

[13] J. Zabczyk, *Mathematical Control Theory: An Introduction*, Birkhäuser Boston, Boston, MA, 1992.