A SEMIGROUP APPROACH TO STOCHASTIC DELAY EQUATIONS IN SPACES OF CONTINUOUS FUNCTIONS

JAN VAN NEERVEN AND MARKUS RIEDLE

Dedicated to Rainer Nagel on the occasion of his 65th birthday

Abstract. We present a semigroup approach to stochastic delay equations of the form

$$\begin{aligned} dX(t) &= \Big(\int_{-h}^{0} X(t+s)\,d\mu(s)\Big)\,dt + dW(t) \quad \text{for } t \ge 0,\\ X(t) &= f(t) \quad \text{for } t \in [-h,0], \end{aligned}$$

in the space of continuous functions $C[-h,0]$. We represent the solution as a $C[-h,0]$-valued process arising from a stochastic weak*-integral in the bidual $C[-h,0]^{**}$ and show how this process can be interpreted as a mild solution of an associated stochastic abstract Cauchy problem. We obtain a necessary and sufficient condition guaranteeing the existence of an invariant measure.

1. Introduction

In this paper we study the stochastic linear delay differential equation

$$\begin{aligned} dX(t) &= \Big(\int_{-h}^{0} X(t+s)\,d\mu(s)\Big)\,dt + dW(t) \quad \text{for } t \ge 0,\\ X(t) &= f(t) \quad \text{for } t \in [-h,0], \end{aligned} \tag{1.1}$$

where μ is a finite signed Borel measure on $[-h,0]$ and $W = (W(t))_{t\ge 0}$ is a standard Brownian motion. Following a well known approach in the theory of deterministic delay equations, we lift the equation to an abstract stochastic Cauchy problem in the space of continuous functions $C[-h,0]$ of the form

$$\begin{aligned} dU(t) &= AU(t)\,dt + B\,dW(t) \qquad \text{for } t \ge 0,\\ U(0) &= f. \end{aligned} \tag{1.2}$$

Here A is the generator of the C_0-semigroup $(T(t))_{t\ge 0}$ on $C[-h,0]$ canonically associated with the deterministic part of (1.1). In contrast to the deterministic situation, B is an element of the bidual space $C[-h,0]^{**}$ which is defined by

$$\langle \mu, B\rangle = \mu(\{0\}), \quad \mu \in M[-h,0] = C[-h,0]^*.$$

Although the problem (1.2) is formulated in the bidual space $C[-h,0]^{**}$, it turns out that its unique mild solution U takes its values in $C[-h,0]$ almost surely. A solution to

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the problem (1.1) is then obtained by putting

$$X(t,f) := (U(t,f))(0), \quad t \ge 0.$$

The semigroup approach to deterministic delay equations in $C[-h,0]$ used here is presented in detail in the monographs of Diekmann–van Gils–Verduyn Lunel–Walther [6] and Engel–Nagel [7], where further references to the literature can be found. With the role of $C[-h,0]$ replaced by $L^2(-h,0)$, stochastic delay equations were studied by Chojnowska-Michalik [4] and, more recently, in the monograph by Da Prato–Zabczyk [5]. The reason for taking $L^2(-h,0)$ comes from the fact that the Itô stochastic calculus extends readily to Hilbert spaces. Recently, a theory of stochastic integration in Banach spaces has been developed and applied to abstract stochastic Cauchy problems in [3, 11]. This theory cannot be applied to study the problem (1.1) in $C[-h,0]$, however, since the functional B belongs to $C[-h,0]^{**}$ rather than $C[-h,0]$. To overcome this problem, in Section 2 we formulate a simple extension of the theory of [3, 11] to locally convex spaces and apply it to dual Banach spaces in their weak*-topology. In the case of a bidual Banach space E^{**}, we are particularly interested in conditions ensuring that the weak*-stochastic integral takes its values in E almost surely (Theorem 2.3). In Section 3 we verify these conditions for the problem (1.2) and prove the existence of a mild solution with values in $C[-h,0]$ (Theorem 3.3). Furthermore, necessary and sufficient conditions for the existence of an invariant measure are obtained in terms of the resolvent function associated with the deterministic problem (Theorem 3.4).

2. Stochastic integration in locally convex spaces

2.1. **Gaussian Radon measures.** Let E be a real locally convex topological vector space. A Borel probability measure μ on E is called a *Radon measure* if for every Borel set B in E and every $\varepsilon > 0$ there exists a compact set $K \subseteq B$ such that $\mu(B \setminus K) < \varepsilon$. A Borel measurable random variable $X : (\Omega, \mathbb{P}) \to E$ is called *Radon* if its distribution is a Radon probability measure on E. We refer to [2, Appendix A] whose terminology we follow.

A Radon measure μ on E is called *Gaussian* if its image under every continuous linear functional $x' \in E'$ is a Gaussian measure on $\mathbb{R}$. By [2, Theorem 3.2.3] there exists a unique element $m_\mu \in E$, the *mean* of E, such that for all $x' \in E'$ we have

$$\langle m_\mu, x' \rangle = \int_E \langle \xi, x' \rangle \, d\mu(\xi).$$

In this paper, all Gaussian Radon measures μ on E will be *centred*, meaning that $m_\mu = 0$, or equivalently, that all image measures $\langle \mu, x' \rangle$ are centred as Gaussian measures on $\mathbb{R}$.

Let μ be a (centred) Gaussian Radon measure on E. For an element $x \in E$ we define

$$|x|_\mu := \sup\Big\{ |\langle x, x' \rangle| : \ x' \in E', \ \int_E \langle \xi, x' \rangle^2 \, d\mu(\xi) \le 1 \Big\}.$$

This supremum may be infinite. The *Cameron-Martin space* associated with μ is the space

$$H_\mu := \{ h \in E : \ |h|_\mu < \infty \}.$$

This space has the structure of a separable real Hilbert space [2, Section 3.2]. Moreover, the inclusion mapping $i_\mu : H_\mu \hookrightarrow E$ is continuous. To see this, note that by [2, Corollary 3.2.4], i_μ maps bounded set of H_μ into relatively compact sets of E. Since compact sets in topological vector spaces are bounded [13, Theorem 1.15], the continuity of i_μ now follows from [13, Theorem 1.32]. Let $i'_\mu : E' \to H_\mu$ denote the adjoint mapping. Then,

$$\langle i_\mu i'_\mu x', y'\rangle = [i'_\mu x', i'_\mu y']_{H_\mu} = \int_E \langle \xi, x'\rangle \langle \xi, y'\rangle \, d\mu(\xi).$$

If $(h_n)_{n\geq 1}$ is an orthonormal basis for H_μ and $(\gamma_n)_{n\geq 1}$ is a *Gaussian sequence*, i.e. a sequence of independent standard Gaussian random variables, then the E-valued Gaussian sum $\sum_{n\geq 1} \gamma_n \, i_\mu h_n$ converges in E almost surely, and its sum is an E-valued random variable with distribution μ [2, Theorem 3.5.1]. More generally, if H is a Hilbert space and $T \in \mathscr{L}(H, E)$ is a continuous linear operator, then by using the fact that the mapping $U : H \to H_\mu$ by $U : T'x' \mapsto i'_\mu x'$ satisfies $\|T'x'\|_H^2 = \|i'_\mu x'\|_{H_\mu}^2$ and therefore extends to a unitary mapping from $\overline{\mathrm{ran}(T')} = (\ker(T))^\perp$ onto H_μ, we have:

Proposition 2.1. *Let H be a separable real Hilbert space with orthonormal basis $(h_n)_{n\geq 1}$. For a continuous linear operator $T : H \to E$ the following assertions are equivalent:*

(1) *There exists a Gaussian Radon measure μ on E such that $T \circ T' = i_\mu \circ i'_\mu$;*
(2) *The E-valued Gaussian sum $\sum_{n\geq 1} \gamma_n \, Th_n$ converges almost surely to an E-valued Radon random variable X.*

In this situation, the sum X has distribution μ.

An operator $T : H \to E$ satisfying the equivalent assumptions of the proposition is called *radonifying.*

A function $\phi : [0, T] \to E$ is called *weakly L^2* if $t \mapsto \langle \phi, x'\rangle(t) := \langle \phi(t), x'\rangle$ defines an element of $L^2(0, T)$ for all $x' \in E'$. A function $\phi : [0, T] \to E$ is called *stochastically integrable* with respect to a Brownian motion $W = (W(t))_{t\in[0,T]}$ defined on a probability space $(\Omega, \mathbb{P})$ if it is weakly L^2 and there exists a Radon random variable $X : \Omega \to E$ such that for all $x' \in E'$ we have

$$\langle X, x'\rangle = \int_0^T \langle \phi(t), x'\rangle \, dW(t)$$

almost surely. In this situation we write

$$X = \int_0^T \phi(t) \, dW(t).$$

The random variable X is Gaussian and is uniquely determined almost everywhere. Indeed, suppose X_1 and X_2 are E-valued Radon random variables satisfying $\langle X_1, x'\rangle = \langle X_2, x'\rangle$ for all $x' \in E'$. To prove that $X_1 = X_2$ almost surely it suffices to show that the distributions μ_Y of $Y := X_1 - X_2$ equals the Dirac measure δ_0.

Since μ_Y and δ_0 are Radon measures on E, they are determined by the cylindrical σ-field $\mathscr{E}$ (this follows by noting that $K \in \mathscr{E}$ for all compact sets $K \subseteq E$). Thus it suffices to show that $\mu_Y = \delta_0$ on $\mathscr{E}$. Let $\mathscr{C}$ the field of cylindrical subsets of E. Then $\mathscr{C}$ is closed under taking finite intersections and we have $\sigma(\mathscr{C}) = \mathscr{E}$. Thus it suffices to show that $\mu_Y = \delta_0$ on $\mathscr{C}$. But if $C \in \mathscr{C}$, there exists a Borel set $B \subseteq \mathbb{R}^n$ and elements $x'_1, \dots, x'_n \in E'$ such that $C = \{x \in E : \ (\langle x, x'_1\rangle, \dots, \langle x, x'_n\rangle) \in B\}$, and therefore

$$\mu_Y(C) = \mathbb{P}\{(\langle Y, x'_1\rangle, \dots, \langle Y, x'_n\rangle) \in B\} = \mathbb{P}\{0 \in B\} = \delta_0(C).$$

The following result extends [11, Theorem 2.3] to locally convex spaces and can be proved in a similar way.

Theorem 2.2. *For a weakly L^2 function $\phi : [0,T] \to E$ the following assertions are equivalent:*

(1) *ϕ is stochastically integrable with respect to W;*
(2) *There exists a Gaussian Radon measure μ on E such that for all $x' \in E'$ we have*
$$\int_E \langle \xi, x'\rangle^2 \, d\mu(\xi) = \int_0^T \langle \phi(t), x'\rangle^2 \, dt;$$
(3) *There exists a radonifying operator $I : L^2(0,T) \to E$ such that for all $x' \in E'$ we have*
$$\langle If, x'\rangle = \int_0^T f(t)\langle \phi(t), x'\rangle \, dt.$$

In this situation, μ is the distribution of $\int_0^T \phi(t)\, dW(t)$.

Now let E be a real Banach space with Banach space dual E^*. The theory developed so far can be applied to E^*, considered as a locally convex topological vector space in its weak*-topology. By general results from the theory of locally convex spaces its topological dual is given by
$$(E^*, \text{weak}^*)' = E.$$
Accordingly we say that a function $\phi : [0,T] \to E^*$ *is weak* L^2* if $\langle x, \phi\rangle$ defines an element of $L^2(0,T)$ for all $x \in E$. We call a function $\phi : [0,T] \to E^*$ *weak*-stochastically integrable* with respect to W if it weak* L^2 and there exists a weak*-Radon random variable $X : \Omega \to E^*$ such that for all $x \in E$ we have
$$\langle x, X\rangle = \int_0^T \langle x, \phi(t)\rangle \, dW(t)$$
almost surely. In this situation we write $X = \text{weak}^*\text{-}\int_0^T \phi(t)\, dW(t)$.

Of particular interest is the special case where E itself is a dual space, say $E = F^*$ for some real Banach space F. If $\phi : [0,T] \to F^{**}$ is weak*-integrable, one may ask under which conditions the weak*-integral is an F-valued random variable. In order to make this question precise, let μ_ϕ denote the distribution of weak*-$\int_0^T \phi\, dW$ and let S_ϕ be the *topological support* of μ_ϕ, i.e., the smallest weak*-closed linear subspace of F^{**} with the property that $\mu_\phi(S_\phi) = 1$ [2, Appendix A].

Theorem 2.3. *Let F be a real Banach space and let $\phi : [0,T] \to F^{**}$ be weak*-stochastically integrable. With the notations as above, the following assertions are equivalent:*

(1) *The topological support S_ϕ is contained in F;*
(2) *There exists a Gaussian Radon measure μ on F such that for all $x^* \in F^*$ we have*
$$\int_E \langle \xi, x^*\rangle^2 \, d\mu(\xi) = \int_0^T \langle x^*, \phi(t)\rangle^2 \, dt;$$
(3) *There exists a radonifying operator $I : L^2(0,T) \to F$ such that for all $x^* \in F^*$ we have*
$$\langle If, x^*\rangle = \int_0^T f(t)\langle x^*, \phi(t)\rangle \, dt.$$

In this situation, μ is the distribution of weak*-$\int_0^T \phi(t)\, dW(t)$.

Proof. (1) $\Rightarrow$ (2): We need to show that μ restricts to a Radon measure on F.

By [2, Theorem 3.6.1] the Cameron-Martin space H_ϕ of μ_ϕ is contained in S_ϕ, and hence in F. Let $i_\phi : H_\phi \to F^{**}$ be the inclusion mapping. If $(h_n)_{n\geq 1}$ is an orthonormal basis for H_ϕ, then by Proposition 2.1 the sum $\sum_{n\geq 1} \gamma_n i_\phi h_n$ converges weak* in F^{**} almost surely. Since i_ϕ takes its values in F, the sum $\sum_{n\geq 1} \gamma_n i_\phi h_n$ converges weakly in F almost surely. Its sum Y is a random variable which takes its values in a weakly separable, hence separable, closed subspace F_0 of F. Thus we see that μ_ϕ is supported on F_0. Since the Borel σ-fields generated by the weak and the strong topologies coincide on F_0, μ_ϕ is a Borel measure on F_0. By a standard result, the separability of F_0 then implies that μ_ϕ is actually a Radon measure on F_0, and hence on F.

(2) $\Rightarrow$ (3): By Theorem 2.2 there exists a radonifying operator $I : L^2(0,T) \to F^{**}$ such that for all $x^* \in F^*$ we have

$$\langle x^*, If\rangle = \int_0^T f(t)\langle x^*, \phi(t)\rangle \, dt.$$

We need to show that I takes its values in F. But $I^*x^* = \langle x^*, \phi\rangle$ and therefore

$$\langle y^*, II^*x^*\rangle = \int_0^T \langle x^*, \phi(t)\rangle\langle y^*, \phi(t)\rangle \, dt = \langle y^*, i_\mu i_\mu^* x^*\rangle$$

for all $y^* \in F^*$, where i_μ is the inclusion operator of the Cameron-Martin space H_μ into F. It follows that $II^*x^* \in F$ for all $x^* \in F^*$. Since the range of I^* is dense in the orthogonal complement of the kernel of I in $L^2(0,T)$, the result follows from this.

(3) $\Rightarrow$ (1): Choose an orthonormal basis $(f_n)_{n\geq 1}$ for $L^2(0,T)$. Denoting by $\text{Itô} : L^2(0,T) \to L^2(\Omega)$ the Itô isometry, the sequence $\gamma_n := \text{Itô} f_n$ consists of independent standard normal random variables. It follows from Proposition 2.1 that the F-valued Gaussian series $\sum_{n\geq 1} \gamma_n I f_n$ converges almost surely to an F-valued Radon random variable X. For all $x^* \in F^*$ we have

$$\begin{aligned}\langle X, x^*\rangle &= \sum_{n\geq 1} \gamma_n \langle If_n, x^*\rangle = \sum_{n\geq 1} \int_0^T \big[\langle \phi, x^*\rangle, f_n\big] f_n(t) \, dW(t) \\ &= \int_0^T \sum_{n\geq 1} \big[\langle \phi, x^*\rangle, f_n\big] f_n(t) \, dW(t) = \int_0^T \langle \phi(t), x^*\rangle \, dW(t)\end{aligned}$$

almost surely. This proves that ϕ is weak*-stochastically integrable in F^{**} with $X = \int_0^T \phi \, dW$ almost surely. Let μ_X and μ_ϕ be the distribution of X and $\int_0^T \phi \, dW$. Then μ_X is a Radon measure on F and μ_ϕ is a weak*-Radon measure on F. Moreover, $j\mu_X = \mu_\phi$, where $j : F \to F^{**}$ is the canonical inclusion operator. It follows that μ_ϕ is a Radon measure on F^{**}. By [2, Lemma 3.2.2 and Theorem 3.6.1] this implies $S_X = \overline{H_X} = \overline{H_\phi} = S_\phi$, where the closures are taken with respect to the strong topologies of F and F^{**}, respectively. In particular, S_ϕ is contained in F. $\square$

Remark 2.4. We have formulated condition (1) in terms of the topological support in order to avoid the following subtle issue. For general Banach space E, it is not clear whether E is always a μ_ϕ-measurable subset of (E^{**}, weak^*) (at least we could not find a reference for this problem). Thus one has to be careful when using the phrase

(2.1) "the weak*-stochastic integral of ϕ is almost surely E-valued"

If E is separable, then E is a Borel subset of (E^{**}, weak^*) by [2, Theorem A.3.15(ii)] and (2.1) becomes meaningful. Also, the proof of the theorem shows that if (3) holds, then

μ_ϕ is actually Radon on $(E^{**}, \|\cdot\|)$ and (2.1) becomes meaningful since E is norm closed as a subspace of E^{**}.

3. Delay differential equations

In this section we apply our results on weak*-stochastic integration to represent the solution of a real-valued stochastic delay differential equation as a C-valued weak*-stochastic integral in the bidual of C, where $C = C[-h, 0]$ is the space of history functions. Before turning our attention to stochastic equations, we summarize some results on deterministic delay differential equations. Proofs may be found in [6, 9].

Let $h > 0$ be fixed and consider the deterministic linear delay differential equation

$$\begin{aligned} \dot{x}(t) &= \int_{[-h,0]} x(t+s)\, d\mu(s) \quad \text{for } t \geq 0, \\ x(t) &= f(t) \quad \text{for } t \in [-h, 0], \end{aligned} \tag{3.1}$$

where $\mu \in M = M[-h, 0]$, the space of signed Borel measure on $[-h, 0]$ with the total variation norm $\|\cdot\|_{TV}$. The initial function $f : [-h, 0] \to \mathbb{R}$ is assumed to be Borel measurable. A function $x : [-h, \infty) \to \mathbb{R}$ is called *classical solution* of (3.1) if x is continuous on $[-h, \infty)$, its restriction to $[0, \infty)$ is continuously differentiable, and x satisfies the first and second identity of (3.1) for all $t \geq 0$ and $t \in [-h, 0]$, respectively. It is well known that for every $f \in C = C[-h, 0]$ the problem (3.1) admits a unique classical solution $x = x(\cdot, f)$.

For a continuous function $x : [-h, \infty) \to \mathbb{R}$ and $t \geq 0$ we define the *segment* $x_t \in C$ by

$$x_t(u) := x(t+u), \quad u \in [-h, 0].$$

The solution operators $T(t) : C \to C$,

$$T(t)f = x_t(\cdot, f), \qquad t \geq 0,$$

where $x(\cdot, f)$ is the solution of (3.1), form a strongly continuous semigroup $T = (T(t))_{t\geq 0}$ on C.

The *fundamental solution* or *resolvent* of (3.1) is the unique locally absolutely continuous function $r : [0, \infty) \to \mathbb{R}$ which satisfies

$$r(t) = 1 + \int_0^t \int_{[\max\{-h,-s\},0]} r(s+u)\, d\mu(u)\, ds \quad \text{for } t \geq 0.$$

It plays a role which is analogous to the fundamental system in linear ordinary differential equations and the Green function in partial differential equations. Formally, it is the solution of (3.1) corresponding to the initial function $f = \mathbf{1}_{\{0\}}$.

From [9, Theorem 6.3.2] and [9, Equation (6.3.13)] we deduce:

Proposition 3.1. *The adjoint $T^*(t)$ of the solution operator $T(t)$ satisfies*

$$\langle T^*(s)\nu, B\rangle = \int_{[\max\{-h,-s\},0]} r(s+u)\, d\nu(u) \qquad \text{for all } s \geq 0,\ \nu \in M, \tag{3.2}$$

where $B \in C^{**} = M^*$ is defined by $\langle \nu, B\rangle := \nu(\{0\})$.

Now let us fix a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with a filtration $(\mathscr{F}_t)_{t\geq 0}$. We will study the stochastic linear delay differential equation

$$\begin{aligned} dX(t) &= \Big(\int_{[-h,0]} X(t+s)\, d\mu(s) \Big)\, dt + dW(t) \quad \text{for } t \geq 0, \\ X(t) &= f(t) \quad \text{for } t \in [-h, 0], \end{aligned} \tag{3.3}$$

where μ is a finite signed Borel measure on $[-h, 0]$ and $W = (W(t))_{t\geq 0}$ is a standard Brownian motion on $(\Omega, \mathscr{F}, \mathbb{P})$. As before the initial function f is taken from C. A *solution* of (3.3) is an adapted process $(X(t, f))_{t\geq -h}$ with continuous paths such that almost surely we have

$$X(t, f) = f(0) + \int_0^t \Big(\int_{[-h,0]} X(s+u)\, d\mu(u) \Big)\, ds + W(t) \quad \text{for } t \geq 0, \tag{3.4}$$

with $X(u, f) = f(u)$ for $u \in [-h, 0]$ almost surely.

For $t \geq 0$ and $u \in [-h, 0]$ we define $(I(t))(u) := 0$ if $t + u < 0$ and

$$(I(t))(u) := W(t+u) - \int_0^{t+u} W(t-s+u)\, dr(s) \quad \text{if } t+u \geq 0.$$

Clearly, $u \mapsto (I(t))(u)$ is continuous for all $t \geq 0$. By the Pettis measurability theorem, $I(t)$ is strongly measurable as a C-valued random variable. Since for $t + u \geq 0$ we have, almost surely,

$$W(t+u) - \int_0^{t+u} W(t-s+u)\, dr(s) = \int_0^{t+u} r(t-s+u)\, dW(s)$$

we may think of $I(t)$ as a continuous version of the convolution process $r * W$.

The following existence and uniqueness result is proved in [10]:

Proposition 3.2. *For every $f \in C$ the problem (3.3) admits a solution $(X(t, f))_{t\geq -h}$. This solution is unique up to indistinguishability and almost surely, for all $t \geq 0$ we have*

$$X_t(\cdot, f) = T(t)f + I(t) \ \text{ in } C.$$

The first main result of this section identifies the segment process $(X_t(\cdot, f))_{t\geq 0}$ as a weak*-stochastic integral in C^{**} which actually takes its values in C.

Theorem 3.3. *Let $f \in C$ and denote by $(X(t, f))_{t\geq -h}$ the solution of (3.3). The function $s \mapsto T^{**}(t-s)B$ is* weak**-stochastically integrable in C^{**} on the interval $[0, t]$ and the segment process $(X_t(\cdot, f))_{t\geq 0}$ can be represented in C^{**} by*

$$X_t(\cdot, f) = T(t)f + \text{weak}^*\text{-}\int_0^t T^{**}(t-s)B\, dW(s).$$

Proof. For $t \geq 0$ we define $\phi : [0,t] \to C^{**}$ by $\phi(s) := T^{**}(t-s)B$. It is immediate from (3.2) that ϕ is weak* L^2. By the stochastic Fubini theorem, for all $t \geq 0$ we have

$$\begin{aligned}
\langle I(t), \nu\rangle &= \int_{[-h,0]} (I(t))(u)\, d\nu(u) \\
&= \int_{[-h,0]} \Big(\int_0^{\max\{0,t+u\}} r(t-s+u)\, dW(s)\Big) d\nu(u) \\
&= \int_0^t \Big(\int_{[\max\{-h,s-t\},0]} r(t-s+u)\, d\nu(u)\Big)\, dW(s) \\
&= \int_0^t \langle \nu, T^{**}(t-s)B\rangle\, dW(s)
\end{aligned}$$

almost surely. The distribution of $I(t)$ is a Radon probability measure on C. As the inclusion $C \subseteq C^{**}$ is strongly-to-weak*-continuous, $I(t)$ is weak*-Radon as a C^{**}-valued random variable. Consequently the function ϕ is weak*-stochastically integrable and in C^{**} we have

$$I(t) = \text{weak*-}\int_0^t T^{**}(t-s)B\, dW(s)$$

almost surely. $\square$

The representation of the solution $(X(t,f))_{t\geq -h}$ in C given by Theorem 3.3 identifies the segment process

$$U(t,f) := X_t(\cdot, f)$$

as the *mild* weak*-*solution* of the following Cauchy problem in C^{**}:

$$\begin{aligned}
dU(t) &= AU(t)\, dt + B\, dW(t) \qquad \text{for } t \geq 0, \\
U(0) &= f,
\end{aligned} \tag{3.5}$$

where A denotes the generator of the solution semigroup $(T(t))_{t\geq 0}$. As in [11, Theorem 7.1] one checks that for all $t \in [0,T]$ and $\mu \in D(A^*)$ we have, almost surely,

$$\langle U(t,f), \mu\rangle = \langle f, \mu\rangle + \int_0^t \langle U(s,f), A^*\mu\rangle\, ds + \mu(\{0\})W(t).$$

Further properties of the segment process $(X_t(\cdot, f))_{t\geq 0}$ are investigated in [12].

It is shown in [8] that the problem (3.3) admits an invariant measure if and only if $r \in L^2(0,\infty)$. The second main result of this section shows that this condition is in fact necessary and sufficient for the existence of an invariant measure for the problem (3.5).

Theorem 3.4. *The problem* (3.5) *admits an invariant measure if and only if* $r \in L^2(0,\infty)$. *In this situation the invariant measure is unique.*

Proof. First assume that $r \in L^2(0,\infty)$. Noting that by Proposition 3.1 and the Cauchy-Schwarz inequality we have

$$\begin{aligned}
\int_0^\infty |\langle T^*(s)\nu, B\rangle|^2\, ds &= \int_0^\infty \Big|\int_{[\max\{-h,-s\},0]} r(s+u)\, d\nu(u)\Big|^2 ds \\
&\leq \|\nu\|_{TV} \int_0^\infty \int_{[\max\{-h,-s\},0]} |r(s+u)|^2\, d|\nu|(u)\, ds \\
&\leq \|\nu\|_{TV}^2\, \|r\|_{L^2(0,\infty)}^2,
\end{aligned}$$

we may define $Q_\infty : M \to C^{**}$ by

$$\langle \nu, Q_\infty \xi\rangle = \int_0^\infty \langle \nu, T^{**}(s)B\rangle\langle \xi, T^{**}(s)B\rangle\,ds, \quad \nu,\xi\in M.$$

We claim that the mapping $\nu \mapsto \langle \nu, T^{**}(\cdot)B\rangle$ is weak*-to-weakly continuous from M to $L^2(0,T)$. Indeed, if $\lim_\alpha \nu_\alpha = \nu$ weak* in M, then for all $f\in L^2(0,T)$ we have

$$\begin{aligned}
[\langle \nu_\alpha, T^{**}(\cdot)B\rangle, f]_{L^2(0,T)} &= \int_0^T f(t)\langle T^*(t)\nu_\alpha, B\rangle\,dt\\
&= \int_0^T f(t)\Big(\int_{[\max\{-h,-t\},0]} r(u+t)\,d\nu_\alpha(u)\Big)\,dt\\
&= \int_{[-h,0]}\Big(\int_{-u}^T f(t)r(u+t)\,dt\Big)d\nu_\alpha(u).
\end{aligned}$$

Since $u\mapsto \int_{-u}^T f(t)r(u+t)\,dt$ belongs to C, the right hand side tends to

$$\int_{[-h,0]}\Big(\int_{-u}^T f(t)r(u+t)\,dt\Big)d\nu(u) = [\langle \nu, T^{**}(\cdot)B\rangle, f]_{L^2(0,T)}$$

and the claim is proved.

It follows from the claim that $Q_\infty\xi\in C$ for all $\xi\in M$. Indeed, the claim shows that for all $\xi\in M$, $Q_\infty\xi$ is weak*-continuous on M. As a consequence, Q_∞ is the adjoint of some operator acting from M to C, and by symmetry this operator can only be Q_∞.

Let μ_t denote the distribution of the random variable $U(t) := U(t,0)$, the solution of (3.5) with initial condition 0. Next we show that the family $\{\mu_t : t\ge h\}$ is uniformly tight on C. According to [1, Theorem 8.2], we have to show that for every $\eta>0$ there exists an $a\ge 0$ such that

$$\mathbb{P}\left(|U(t)(-h)|>a\right)\le \eta \qquad \text{for every } t\ge h, \tag{3.6}$$

and that for every $\varepsilon>0$ and $\kappa>0$ there exists a $\delta>0$ such that

$$\mathbb{P}\Big(\sup_{\substack{u,v\in[-h,0]\\ |u-v|\le\delta}} |U(t)(u)-U(t)(v)|\ge\varepsilon\Big)\le\kappa \tag{3.7}$$

for every $t\ge h$.

The first condition (3.6) coincides with the tightness of the laws of $\{X(t) : t\ge h\}$ in $\mathbb{R}$, where $X(t) := X(t,0)$ is the solution of (3.3) with initial condition 0. The latter are tight since equation (3.3) admits an invariant measure by the result of [8] mentioned above.

Towards (3.7), for $-h\le v\le u\le 0$ and $t\ge h$ we have, by (3.4),

$$U(t)(u)-U(t)(v) = \int_{t+v}^{t+u}\int_{[-h,0]} X(s+m)\,d\mu(m)\,ds + W(t+u)-W(t+v).$$

The Burkholder-Davis-Gundy inequality yields, for all $\delta>0$,

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{0\le\rho\le\delta}|W(t)-W(t+\rho)|\ge\varepsilon\Big)\\
&\qquad\le \varepsilon^{-2m}\mathbb{E}\Big(\sup_{0\le\rho\le\delta}|W(t)-W(t+\rho)|^{2m}\Big)\le C_m\varepsilon^{-2m}\delta^m
\end{aligned}$$

for every $m \geq 1$, with a constant C_m depending on m only. By using this inequality and a sufficiently small partition of the interval $[-h, 0]$ one obtains that for all $\varepsilon, \kappa > 0$ there exists a $\delta > 0$ such that, for all $t \geq h$,

$$\mathbb{P}\Big(\sup_{\substack{u,v\in[-h,0] \\ |u-v|\leq\delta}} |W(t+u) - W(t+v)| \geq \varepsilon \Big) \leq \kappa. \tag{3.8}$$

Furthermore, Proposition 3.2 and Itô's isometry imply that

$$\mathbb{E}\, |X(t)|^2 = \int_0^t r^2(t-s)\, ds \leq \|r\|^2_{L^2(0,\infty)} \qquad \text{for } t \geq 0.$$

Using the Cauchy-Schwartz inequality twice we compute, for $t \geq h$,

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{\substack{u,v\in[-h,0] \\ |u-v|\leq\delta}} \Big| \int_{t+u}^{t+v} \int_{[-h,0]} X(s+m)\, d\mu(m)\, ds \Big|^2 \Big] \\
&\quad \leq \mathbb{E}\Big[\sup_{\substack{u,v\in[-h,0] \\ |u-v|\leq\delta}} |u-v| \int_{t+u}^{t+v} \Big| \int_{[-h,0]} X(s+m)\, d\mu(m) \Big|^2 ds \Big] \\
&\quad \leq \delta\, \mathbb{E}\Big[\int_{t-h}^{t} \Big| \int_{[-h,0]} X(s+m)\, d\mu(m) \Big|^2 ds \Big] \\
&\quad \leq \delta\, \|\mu\|_{TV} \int_{t-h}^{t} \int_{[-h,0]} \mathbb{E}\, |X(s+m)|^2 \; d|\mu|(m)\, ds \\
&\quad \leq \delta h\, \|\mu\|^2_{TV}\, \|r\|^2_{L^2(0,\infty)}\, .
\end{aligned}$$

Applying Chebyshev's inequality and (3.8), we obtain (3.7) and thus the tightness of $\{\mu_t : t \geq 0\}$. Therefore, Q_∞ is the covariance operator of a centred Gaussian Radon measure μ_∞ on C which satisfies $\mu_\infty = \lim_{t\to\infty} \mu_t$ weakly. A standard argument shows that μ_∞ is invariant.

Conversely, if there exists an invariant measure for (3.5), then the same holds true for (3.3). By the result in [8], the latter is equivalent to $r \in L^2(0, \infty)$.

Finally, the uniqueness of the invariant measure follows from the fact, proved in [8], that the invariant measure for (3.3) is unique. □

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Delft Institute of Applied Mathematics, Technical University of Delft, P.O. Box 5031, 2600 GA Delft, The Netherlands
E-mail address: J.vanNeerven@math.tudelft.nl

Department of Mathematics, Humboldt University of Berlin, Unter den Linden 6, 10099 Berlin, Germany
E-mail address: riedle@mathematik.hu-berlin.de