Norm discontinuity of Ornstein-Uhlenbeck semigroups

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0. Introduction

Let μ be a centered Gaussian measure on an infinite-dimensional, separable, real Banach space E. For $t \geq 0$ and bounded Borel functions $f : E \to \mathbb{R}$ we define

$$P(t)f(x) = \int_E f(x+y)\, d\mu_t(y), \qquad x \in E, \tag{0.1}$$

where $\mu_t(B) := \mu(B/\sqrt{t})$ for Borel sets $B \subset E$; for $t = 0$ we set $\mu_0 := \delta_0$, the Dirac measure concentrated at 0. Clearly, each $P(t)$ is a contraction which maps $BUC(E)$, the Banach space of all bounded real-valued, uniformly continuous functions on E, into itself. In fact, it is easy to prove that the family $\mathbf{P} = \{P(t)\}_{t\geq 0}$ defines a strongly continuous semigroup of linear operators on $BUC(E)$. We will refer to this semigroup as the *Wiener semigroup* over E associated with μ. On the other hand let $W(t), t \geq 0$ be an $E-$valued Wiener process, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that the distribution $\mathcal{L}(W(1))$ of $W(1)$ is the measure μ. Then $\mathbf{P}$ is the transition semigroup of $\mathbf{W}$:

$$P(t)f(x) = \mathbb{E}(f(x + W(t))), \qquad x \in E. \tag{0.2}$$

It was proved by Guiotto [12] that, if E is a Hilbert space, $\mathbf{P}$ is not eventually differentiable, and Desch and Rhandi [9] that $\mathbf{P}$ is not even eventually norm continuous. Recall that $\mathbf{P}$ is *eventually differentiable*, respectively *eventually norm continuous*, if there exists $t_0 \geq 0$ such that the map $t \mapsto P(t)$ is differentiable, respectively continuous, with respect to the uniform operator topology for $t > t_0$. If $\mathbf{P}$ is differentiable for $t > t_0$, then $\mathbf{P}$ is norm continuous for $t > t_0$ (see, e.g., [16]). Since analyticity implies differentiability, Guiotto's result implies that $\mathbf{P}$ is not eventually analytic, and since compactness implies norm continuity, the result of Desch and Rhandi shows that $\mathbf{P}$ also fails to be eventually compact.

The proof in [9] is based on elaborate estimates which use the fact that the covariance operator of a Gaussian measure on a Hilbert space is a trace class operator and that $\mathbf{P}$ can be approximated by 'finite-dimensional' heat semigroups. This approach does not work for more general processes and if E is a Banach space. Using a completely different method, we extend the result of [9] to arbitrary separable real Banach spaces E by showing that

$$\|P(t_0 + h) - P(t_0)\| = 2, \qquad \forall t_0 \geq 0 \text{ and } h > 0.$$

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It turns out that this is only a reformulation of the following well-known result about Gaussian measures on Banach spaces: if $t \neq s$, then $\mu_t \perp \mu_s$, see [13, Theorem II.5.2]). Once this has been realized, the door is open to prove discontinuity results for the more general class of Ornstein-Uhlenbeck semigroups.

The paper is organized as follows: In Section 1 we present our proof of the Desch-Rhandi result for the Wiener semigroup on $BUC(E)$. This case is treated separately in order to show how the general idea works in the simplest possible situation. In Section 2 we proceed to more general class of *Ornstein-Uhlenbeck semigroups* on Hilbert spaces $E = H$. Before turning to discontinuity results for the operator norm, we first establish some strong continuity results. Then in Sections 3 we specialize to the self-adjoint commuting case and in Section 4 we present a specific application to a generalization of stochastic wave equations. In the final Section 5 we discuss some extensions to transition semigroups corresponding to Ornstein-Uhlenbeck processes stopped on the boundary of an open subset of H.

1. Norm discontinuity of the Wiener semigroup

In this section we present our proof of the Desch-Rhandi result for the Wiener semigroup acting on $BUC(E)$, where E is a separable real Banach space. We need the following simple fact relating the total variation of a Borel measure to its norm when considered as a bounded linear functional on the space $BUC(E)$.

Lemma 1.1. *Let ν be finite signed Borel measure on a complete separable metric space E. Then $\|\nu\|_{(BUC(E))^*} = var(\nu)$.*

This follows, e.g., from [1, Section 1.1].

Theorem 1.2. *Let μ be a non-degenerate centered Gaussian measure on an infinite-dimensional separable real Banach space E and let $\mathbf{P} = \{P(t)\}_{t\geq 0}$ be the associated Wiener semigroup defined by (0.1) on the space $BUC(E)$. For all $t_0 \geq 0$ and $h > 0$ we have*

$$\|P(t_0 + h) - P(t_0)\| = 2.$$

Proof. Since $\|P(t)\| \leq 1$ for all $t \geq 0$, the inequality $\|P(t_0+h)-P(t_0)\| \leq 2$ is trivial. By the disjointness result for Gaussian measures cited in the introduction, for any two $t, s \geq 0$ with $t \neq s$ the measures μ_t and μ_s are mutually singular. Hence, $\mathrm{var}(\mu_t - \mu_s) = 2$. For all $t, s \geq 0$ we have $\mu_t * \mu_s = \mu_{t+s}$. Hence if $f \in BUC(E)$, then

$$\begin{aligned}\langle P^*(t)\mu_s, f\rangle = \langle \mu_s, P(t)f\rangle &= \int_E \int_E f(x+y)\, d\mu_t(y)\, d\mu_s(x) \\ &= \int_E f(z)\, d(\mu_t * \mu_s)(z) \\ &= \int_E f(z)\, d\mu_{t+s}(z) \\ &= \langle \mu_{t+s}, f\rangle,\end{aligned}$$

the brackets $\langle \cdot, \cdot \rangle$ denoting the duality pairing between $BUC(E)$ and its dual. It follows that $P^*(t)\mu_s = \mu_{t+s}$ in $(BUC(E))^*$. Then by Lemma 1.1, for any

$s \geq 0$ fixed we have

$$\begin{aligned} \|P(t_0+h) - P(t_0)\| &= \|P^*(t_0+h) - P^*(t_0)\| \\ &\geq \|P^*(t_0+h)\mu_s - P^*(t_0)\mu_s\| \\ &= \|\mu_{t_0+h+s} - \mu_{t_0+s}\| \\ &= \mathrm{var}\,(\mu_{t_0+h+s} - \mu_{t_0+s}) = 2. \end{aligned} \tag{1.1}$$

■

In particular, $\mathbf{P}$ is not eventually norm continuous on $BUC(E)$.
There exists an interesting relationship between these results and the theory of adjoint semigroups. If $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is a strongly continuous semigroup of operators on a Banach space F, then $F^{\odot}$ is defined as the largest closed subspace of the dual F^* on which the adjoint semigroup $\mathbf{T}^*$ acts in a strongly continuous way:

$$F^{\odot} := \{x^* \in F^* : \lim_{h\downarrow 0} \|T^*(h)x^* - x^*\| = 0\}.$$

For more information about adjoint semigroups we refer to [14]. If F is a Banach lattice such that F^* has order continuous norm, and if each operator $T(t)$ is positive, then $F^{\odot}$ is a projection band in F^*, i.e. there exists a positive contractive projection Q of F^* onto $F^{\odot}$ with $Qx^* \perp (I-Q)x^*$ for all $x^* \in F^*$ [15, Theorem 2.1]. Applying this fact to the semigroup $\mathbf{P}$ on $F = BUC(E)$, we obtain the following easy consequence of Theorem 1.2:

Corollary 1.3. *For all $t \geq 0$ we have $\mu_t \perp (BUC(E))^{\odot}$ in the lattice sense.*

Proof. Being an abstract L-space, the dual of $F = BUC(E)$ has order continuous norm and we have $\|\nu_1 + \nu_2\| = \|\nu_1\| + \|\nu_2\|$ whenever $\nu_1 \perp \nu_2$ in F^* [17]. Hence, denoting by Q the band projection of F^* onto $F^{\odot}$, it follows from (1.1) that

$$\begin{aligned} 2 &= \limsup_{h\downarrow 0} \|P^*(h)\mu_t - \mu_t\| \\ &\leq \limsup_{h\downarrow 0} \|(P^*(h) - I)Q\mu_t\| + \limsup_{h\downarrow 0} \|(P^*(h) - I)(I-Q)\mu_t\| \\ &= \limsup_{h\downarrow 0} \|(P^*(h) - I)(I-Q)\mu_t\| \\ &\leq 2\|(I-Q)\mu_t\| = 2 - 2\|Q\mu_t\|. \end{aligned}$$

Therefore we must have $\|Q\mu_t\| = 0$, i.e. $\mu_t \perp F^{\odot}$. ■

If $\mathbf{T}$ is a strongly continuous positive semigroup on a Banach lattice F and if $x^* \perp F^{\odot}$, then the following two statements hold [15, Theorem 4.6 and Corollary 3.4]:

(i) $\limsup_{h\downarrow 0} \|T^*(h)x^* - x^*\| \geq 2$;

(ii) If F^* has order continuous norm, then $T^*(h)x^* \perp x^*$ for almost all $h > 0$.

Now assume Corollary 1.3 to be given. Then by (i) and (1.1) applied to $x^* := \mu_{t_0}$ and the semigroup $\mathbf{P}$ on $F = BUC(E)$ we recover most of Theorem 1.2, and from (ii) it follows that $\mu_{t_0+h} \perp \mu_{t_0}$ for almost all $h > 0$ in the lattice sense, hence in the measure theoretic sense by the results quoted from [1]. But this implies that $\mu_{t_0+h} \perp \mu_{t_0}$ for *all* $h > 0$. To see this, choose a sequence $\varepsilon_n \downarrow 0$ such that $\mu_{t_0+h+\varepsilon_n} \perp \mu_{t_0}$ for all n. For each n, choose Borel sets B_n and C_n such that $B_n \cap C_n = \emptyset$ and $\mu_{t_0+h+\varepsilon_n}(B_n) = \mu_{t_0}(C_n) = 1$. Then $B := \cup_n B_n$ is disjoint from $C := \cap_n C_n$, $\mu_{t_0}(C) = 1$, and by the dominated convergence theorem,

$$\mu_{t_0+h}(B) = \mu(B/\sqrt{t_0+h}) = \lim_{n\to\infty} \mu(B/\sqrt{t_0+h+\varepsilon_n}) = \mu_{t_0+h+\varepsilon_n}(B) = 1.$$

Summarizing, we see that Corollary 1.3 implies that $\mu_t \perp \mu_s$ for any two $t, s \geq 0$ with $t \neq s$.

Mutatis mutandis the considerations of this section apply as well to the *(classical) Ornstein-Uhlenbeck semigroup* associated to a non-degenerate centered Gaussian measure μ on E. This is the semigroup $\mathbf{P}$ defined by

$$P(t)f(x) = \int_E f\left(e^{-t/2}x + \sqrt{1-e^{-t}}y\right) d\mu(y), \qquad x \in E,$$

for bounded Borel functions f. This semigroup is strongly continuous on in invariant subspace $BUC(E) \cap C_0(E)$, but by the above arguments it fails to be eventually norm continuous there.

2. Ornstein-Uhlenbeck semigroups

Let H be a separable real Hilbert space equipped with the $\sigma-$field $\mathcal{B} = \mathcal{B}(\mathcal{H})$ of its Borel subsets, and let Q be a self-adjoint nonnegative bounded linear operator on H. Let $\mathbf{S} = \{S(t)\}_{t\geq 0}$ be a C_0-semigroup of bounded linear operators on H with infinitesimal generator A. For all $t > 0$, the operator Q_t defined by

$$Q_t x = \int_0^t S(s)QS^*(s)x\,ds, \qquad x \in H,$$

is self-adjoint and nonnegative on H. *Throughout the rest of this paper we will make the following standing*

Assumption: For each $t > 0$, the operator Q_t is trace class.

As is well-known, this implies that Q_t is the covariance operator of a unique centered Gaussian measure μ_t. If we want to stress that Q_t is the covariance of μ_t we will use the notation $\mu_t = N(0, Q_t)$, and more generally we denote by $N(m, Q_t)$ the Gaussian measure on H with mean m and covariance Q_t.

If Q itself is trace class, then the Assumption is fulfilled. Indeed, if (e_n) is an orthonormal basis in H, then by Fubini's theorem

$$\begin{aligned}\sum_{n=1}^{\infty}[Q_t e_n, e_n] &= \int_0^t \sum_{n=1}^{\infty}[S(s)QS^*(s)e_n, e_n]\,ds \\ &\leq \left(\sup_{0\leq s\leq t} \|S(s)\|^2\right) \cdot t\,\|Q\|_1 < \infty,\end{aligned}$$

where $[\cdot,\cdot]$ is the inner product of H and $\|Q\|_1$ is the trace class norm of Q; we used the fact that for any bounded T, the operator TQT^* is trace class whenever Q is, with $\|TQT^*\|_1 \leq \|T\|\,\|Q\|_1\,\|T^*\| = \|T\|^2\,\|Q\|_1$.

The family $\mathbf{P} = \{P(t)\}_{t\geq 0}$ of linear operators, defined on the space $B_b(H)$ of bounded Borel functions f on H, by

$$P(t)f(x) = \int_H f(S(t)x + y)\, d\mu_t(y), \qquad x \in H,$$

is called the *Ornstein-Uhlenbeck semigroup* associated with $\mathbf{S}$ and Q. If $S(t) = I$, $t \geq 0$, then $Q_t = tQ$ and the measures μ_t are given by $\mu_t(B) = \mu(B/\sqrt{t})$. Thus we recover the Wiener semigroup discussed in the previous section, with $\mu = N(0,Q)$ (the Assumption implies that Q_t, hence also Q, is trace class). Similarly, if $S(t) = e^{-t/2}I$, $t \geq 0$, we recover the classical Ornstein-Uhlenbeck semigroup.

The operators $P(t)$, $t \geq 0$, form the transition semigroup corresponding to the *Ornstein-Uhlenbeck process* $\mathbf{X}$ given by the formula

$$X(t,x) := S(t)x + \int_0^t S(t-s)\, dW_Q(s),$$

where $W_Q(t)$ is a Q-Wiener process on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\mathbb{E}\,[W_Q(t),x][W_Q(s),y] = (t \wedge s)\,[Qx,y], \qquad t,s \geq 0;\; x,y \in H.$$

Thus, for all $f \in B_b(H)$ we have

$$P(t)f(x) = \mathbb{E}(f(X(t,x))), \qquad x \in H,\, t \geq 0.$$

It is known that the process $X(t,x)$ is Gaussian and Markov [7, Chapters 5 and 9]. For more details on these concepts see e.g. [7, Chapter 4].

The formula for $P(t)$ easily implies that $\mathbf{P}$ maps $BUC(H)$ into itself. In contrast to the situation for the Wiener semigroup, however, the Ornstein-Uhlenbeck semigroup $\mathbf{P}$ is generally not strongly continuous on all of $BUC(H)$. The closed subspace of all $f \in BUC(H)$ on which $\mathbf{P}$ acts in a strongly continuous way depends on the semigroup $\mathbf{S}$ and is denoted by $BUC_{\mathbf{S}}(H)$. We show first that the strong continuity on $BUC(H)$ actually implies that $S(t) = I$ for all $t \geq 0$.

Theorem 2.1. *The identity $BUC_{\mathbf{S}}(H) = BUC(H)$ holds if and only if $S(t) = I$ for all $t \geq 0$.*

Proof. Define the operators $R(t)$ on $BUC(H)$ by

$$R(t)f(x) := f(S(t)x), \qquad x \in H,\, t \geq 0.$$

The family $\mathbf{R} = \{R(t)\}_{t\geq 0}$ is the transition semigroup of the degenerate process $X(t,x) = S(t)x$, $t \geq 0$, corresponding to $Q = 0$. First we prove the following lemma:

Lemma 2.2. *If for some $t_0 \geq 0$ and all $f \in BUC(H)$ we have*

$$\lim_{h\downarrow 0} \|R(t_0+h)f - R(t_0)f\| = 0,$$

then $S(t_0+s) = S(t_0)$ for all $s \geq 0$.

Proof. Define, for each $y \in H$, the function e_y by

$$e_y(x) := e^{i[y,x]}, \qquad x \in H.$$

The real and imaginary parts of e_y belong to $BUC(H)$ and therefore,

$$\sup_{x\in H} |R(t_0+s)e_y(x) - R(t_0)e_y(x)| \to 0 \quad \text{as } s \downarrow 0.$$

But

$$\begin{aligned} \sup_{x\in H} |R(t_0+s)e_y(x) - R(t_0)e_y(x)| &= \sup_{x\in H} |e^{i[y,S(t_0+s)x-S(t_0)x]} - 1| \\ &= \sup_{x\in H} |e^{i[S^*(t_0+s)y-S^*(t_0)y,x]} - 1|. \end{aligned} \tag{2.1}$$

Taking $x := \gamma(S^*(t_0+s)y - S^*(t_0)y)$, where $\gamma \in \mathbb{R}$ is an arbitrary fixed constant, one obtains

$$\sup_{x\in H} |R(t_0+s)e_y(x) - R(t_0)e_y(x)| \geq \sup_{\gamma\in\mathbb{R}} |e^{i\gamma\|S^*(t_0+s)y-S^*(t_0)y\|^2} - 1| = 2$$

whenever $\|S^*(t_0+s)y - S^*(t_0)y\| \neq 0$. Thus (2.1) implies that for all $y \in H$ there exists $s_0 \geq 0$ such that

$$S^*(t_0+s)y = S^*(t_0)y, \qquad 0 \leq s \leq s_0.$$

But then the semigroup property implies that $S^*(t_0+s)y = S^*(t_0)y$ for all $s \geq 0$, and since this is true for all $y \in H$ the lemma follows. ∎

We now continue the proof of Theorem 2.1. It follows from the lemma that the semigroup $\mathbf{R}$ is strongly continuous for $t > 0$ on $BUC(H)$ if and only if $A = 0$. On the other hand it is well-known, cf. [5], that a function $f \in BUC(H)$ belongs to $BUC_{\mathbf{S}}(H)$ if and only if

$$\lim_{h\downarrow 0} \|R(h)f - f\| = 0,$$

so the subspaces of $BUC(H)$ of strong continuity of $\mathbf{P}$ and $\mathbf{R}$ coincide and by Lemma 2.2 the theorem follows. ∎

The theorem shows that there is no point of studying norm continuity of $\mathbf{P}$ for $t > 0$ in $BUC(H)$, except if $A = 0$ in which case the semigroup $\mathbf{P}$ is the transition semigroup of the Wiener process discussed in Section 1. For this reason, we will restrict our considerations to the closed subspace $BUC_{\mathbf{S}}(H)$ of $BUC(H)$ where $\mathbf{P}$ is strongly continuous.

We need the following generalization of Lemma 1.1.

Lemma 2.3. *If ν is a finite signed Borel measure on H, then $\|\nu\|_{(BUC_{\mathbf{S}}(H))^*} = var(\nu)$.*

Proof. The inequality $\|\nu\|_{(BUC_{\mathbf{S}}(H))^*} \leq \mathrm{var}(\nu)$ is obvious. To prove the opposite inequality we can assume, without any loss of generality, that $\mathrm{var}(\nu) = 1$. By Lemma 1.1, for arbitrary $\varepsilon > 0$ there exists $f_0 \in BUC(H)$ with $\|f_0\| \leq 1$ such that $\langle \nu, f_0 \rangle \geq 1 - \varepsilon$. For $\delta > 0$ define

$$f_\delta(x) = \frac{1}{\delta} \int_0^\delta P(s) f_0(x)\, ds, \qquad x \in H.$$

Then $f_\delta \in BUC(H)$, $\|f_\delta\| \le 1$, and for all $x \in H$ and $t \in (0, \delta)$ we have

$$|P(t)f_\delta(x) - f_\delta(x)| = \frac{1}{\delta}\left|\int_t^{\delta+t} P(s)f_0(x)\,ds - \int_0^\delta P(s)f_0(x)\,ds\right| \le \frac{2t}{\delta}.$$

Hence $\|P(t)f_\delta - f_\delta\| \le 2t/\delta$, which shows that $f_\delta \in BUC_{\mathbf{S}}(H)$.

By inner regularity of the measure ν, the supremum of $\nu(K)$, with K ranging over all compact subsets of H, equals 1. To prove the lemma it is therefore enough to show that f_δ tends to f_0, uniformly on compact sets, as δ tends to 0. Define

$$\xi(t) = \int_0^t S(t-s)\,dW_Q(s),$$

and note that

$$|f_\delta(x) - f_0(x)| = \frac{1}{\delta}\left|\int_0^\delta \mathbb{E}\big(f_0(S(s)x + \xi(s)) - f_0(x)\big)\,ds\right|.$$

Moreover, for arbitrary $\gamma > 0$,

$$\mathbb{E}|f_0(S(s)x + \xi(s)) - f_0(x)| = \mathbb{E}\big(|f_0(S(s)x + \xi(s)) - f_0(x)|\,|\chi_{A_{s,\gamma}} + \chi_{A^c_{s,\gamma}}\big),$$

where

$$A_{s,\gamma} = \{\omega;\ |\xi(s,\omega)| \le \gamma\}, \quad A^c_{s,\gamma} = \{\omega;\ |\xi(s,\omega)| > \gamma\}.$$

Let us fix a compact set K and note that $S(s)x \to x$ as $s \downarrow 0$, uniformly on K. Taking into account that f_0 is uniformly continuous, for arbitrary $\varepsilon > 0$ one can find $\gamma > 0$ and $s_0 > 0$ such that for all $s \in (0, s_0)$ and all $x \in K$,

$$\mathbb{E}\big(|f_0(S(s)x + \xi(s)) - f_0(x)|\,|\chi_{A_{s,\gamma}}\big) < \varepsilon.$$

On the other hand, by Chebyshev's inequality and [7, Theorem 5.2],

$$\mathbb{P}(A^c_{s,\gamma}) \le \frac{1}{\gamma^2}\mathbb{E}(|\xi(s)|^2) = \frac{1}{\gamma^2}\int_0^s \mathrm{Trace}\,(S(r)QS^*(r))\,dr.$$

It is therefore clear that $\mathbb{P}(A^c_{s,\gamma})$ can be made small by taking s small enough. Thus there exists $s_1 \in (0, s_0)$ such that for all $x \in K$ and for all $s \in (0, s_1)$:

$$\mathbb{E}\big(|f_0(S(s)x + \xi(s)) - f_0(x)|\,|\chi_{A^c_{s,\gamma}}\big) < \varepsilon.$$

This proves the required uniform convergence and completes the proof of the lemma. ■

The main result of the present section is the following theorem listing sufficient conditions for the norm discontinuity of Ornstein-Uhlenbeck semigroups. Recall that the *pseudo-inverse*, notation B^{-1}, of a bounded linear operator B from a Hilbert space H into another Hilbert space H' is defined by $\mathcal{D}(B^{-1}) = \mathrm{Im}\,B$ and $B^{-1}h'$ is the element in $\{h \in H : Bh = h'\}$ of minimal norm.

Theorem 2.4. *Let $t_0 > 0$ and $h > 0$ be fixed. We have*

$$\|P(t_0+h) - P(t_0)\|_{BUC_{\mathbf{S}}(H)} = 2$$

provided at least one of the following three conditions is satisfied:

(i) $Im\, Q_{t_0}^{1/2} \neq Im\, Q_{t_0+h}^{1/2}$;

(ii) *For some* $x \in H$, $S(t_0+h)x - S(t_0)x \notin Im\, Q_{t_0}^{1/2}$;

(iii) $Im\, Q_{t_0+h}^{1/2} = Im\, Q_{t_0}^{1/2}$, *but the operator* $\left(Q_{t_0}^{-1/2} Q_{t_0+h}^{1/2}\right)\left(Q_{t_0}^{-1/2} Q_{t_0+h}^{1/2}\right)^* - I$ *is not Hilbert-Schmidt on* $\overline{Im\, Q_{t_0}^{1/2}}$.

For the proof we need the following result on Gaussian measures due to Feldman and Hajek (cf. [7, Theorem 2.23]).

Lemma 2.5. *Let $\mu = N(m_\mu, Q_\mu)$ and $\nu = N(m_\nu, Q_\nu)$ be Gaussian measures on a Hilbert space H. Then either $\mu \perp \nu$ or μ and ν are absolutely continuous with respect to each other. The second of these possibilities happens if and only if the following two conditions are satisfied:*

(i) $Im\, Q_\mu^{1/2} = Im\, Q_\nu^{1/2}$, *and the operator* $(Q_\mu^{-1/2} Q_\nu^{1/2})(Q_\mu^{-1/2} Q_\nu^{1/2})^* - I$ *is Hilbert-Schmidt on* $\overline{Im\, Q_\mu^{1/2}}$;

(ii) $m_\mu - m_\nu \in Im\, Q_\mu^{1/2}$.

We now prove Theorem 2.4. Assume that condition 2.4 (i) holds. Then by the Feldman-Hajek lemma the measures $\mu_{t_0+h} = N(0, Q_{t_0+h})$ and $\mu_{t_0} = N(0, Q_{t_0})$ are singular. But for arbitrary $f \in BUC_{\mathbf{S}}(H)$,

$$P(t_0+h)f(0) = \langle \mu_{t_0+h}, f\rangle \quad \text{and} \quad P(t_0)f(0) = \langle \mu_{t_0}, f\rangle.$$

Consequently, for arbitrary $f \in BUC_{\mathbf{S}}(H)$ of norm not exceeding 1,

$$\|P(t_0+h) - P(t_0)\|_{BUC_{\mathbf{S}}(H)} \geq |\langle \mu_{t_0+h} - \mu_{t_0}, f\rangle|.$$

Since the total variation of $\mu_{t_0+h} - \mu_{t_0}$ equals 2, the result follows by Lemma 2.3.

If 2.4 (ii) holds then measures $\mu_{t_0+h}^x = N(S(t_0+h)x, Q_{t_0+h})$ and $\mu_{t_0}^x = N(S(t_0)x, Q_{t_0})$, are singular and the argument of 2.4 (i) applies.

Finally if 2.4 (iii) holds, then Lemma 2.5 (i) is violated. ■

Applications of the theorem are postponed to the next sections.

Let us comment on the conditions 2.4 (i), (ii), and (iii). If one assumes that

$$\sup_{t>0} \int_0^t \operatorname{Trace} S(s)QS^*(s)\, ds < \infty,$$

then

$$Q_\infty x := \int_0^\infty S(s)QS^*(s)x\, ds, \qquad x \in H, \tag{2.2}$$

defines a positive, self-adjoint trace class operator Q_∞ on H and therefore it is the covariance of the centered Gaussian measure $\mu_\infty = N(0, Q_\infty)$ on H [7, Theorem 11.7]. Moreover, μ_∞ is an invariant measure for $\mathbf{P}$. If one further assumes the *null controllability* condition

$$\operatorname{Im} S(t) \subset \operatorname{Im} Q_t^{1/2} \qquad \forall t > 0, \tag{2.3}$$

then it can be shown that

$$\operatorname{Im} Q_t^{1/2} = \operatorname{Im} Q_\infty^{1/2}, \qquad \forall t > 0,$$

and that the operators $\big(Q_t^{-1/2} Q_{t+h}^{1/2}\big)\big(Q_t^{-1/2} Q_{t+h}^{1/2}\big)^* - I$ are Hilbert-Schmidt [6], [7, Theorem 11.13]. It follows that if (2.2) and (2.3) are satisfied, then none of the conditions (i), (ii), or (iii) of Theorem 2.4 are satisfied and the theorem is not applicable. We do not know whether in this situation $\mathbf{P}$ is norm continuous for $t > 0$. In this connection it is instructive to recall the situation when the space $BUC_{\mathbf{S}}(H)$ is replaced by an L^p-space. It is known (cf. [2]) that under conditions (2.2) and (2.3), for each $p \in [1, \infty)$ the semigroup $\mathbf{P}_A$ has a unique extension to a C_0-contraction semigroup on $L^p(H, \mu_\infty)$, and if $p \in (1, \infty)$ this semigroup is compact, hence norm continuous for $t > 0$.

3. Self-adjoint commutative systems

As a first application of the general theorem from the previous section we will consider Ornstein-Uhlenbeck processes with a self-adjoint semigroup $\mathbf{S}$ which commutes with Q. Our result covers, for instance, the classical Ornstein-Uhlenbeck semigroup corresponding to $S(t) = e^{-t} I$, $t \geq 0$.

Theorem 3.1. *Suppose that $\mathbf{S}$ is self-adjoint and commutes with Q. For a fixed $t_0 > 0$, each of the following conditions implies that*

$$\| P_A(t_0 + h) - P_A(t_0) \|_{BUC_{\mathbf{S}}(H)} = 2$$

for all $h > 0$:

- *(i) The operator $S(2t_0)$ does not map $\operatorname{Im} Q_{t_0}^{1/2}$ into itself;*
- *(ii) The operator $S(2t_0)$ maps $\operatorname{Im} Q_{t_0}^{1/2}$ into itself, but with respect to the norm of $\operatorname{Im} Q_{t_0}^{1/2}$ the restriction $S(2t_0)|_{\operatorname{Im} Q_{t_0}^{1/2}}$ is not Hilbert-Schmidt.*

Proof. First observe that $S(t)Q_s = Q_s S(t)$ and $S(t) Q_s^{1/2} = Q_s^{1/2} S(t)$ for all $s > 0$ and $t > 0$. Let us fix an arbitrary $h > 0$.

Step 1 - If $\operatorname{Im} Q_{t_0+h}^{1/2} \neq \operatorname{Im} Q_{t_0}^{1/2}$ we have $\| P(t_0 + h) - P(t_0) \|_{BUC_{\mathbf{S}}(H)} = 2$ by Theorem 2.4 (i).

Step 2 - In the rest of the proof we assume that $\operatorname{Im} Q_{t_0+h}^{1/2} = \operatorname{Im} Q_{t_0}^{1/2}$.

We claim that $\operatorname{Im} Q_h^{1/2} = \operatorname{Im} Q_{t_0}^{1/2}$. To start the proof of this claim, we first recall the elementary fact [7, Proposition B1] that $\operatorname{Im} Q_s^{1/2} \subset \operatorname{Im} Q_t^{1/2}$ if and only if there exists a constant $k > 0$ such that

$$[Q_s x, x] \leq k [Q_t x, x], \qquad \forall x \in H.$$

In particular, from the definition of the operators Q_t it is clear that $\operatorname{Im} Q_s^{1/2} \subset \operatorname{Im} Q_t^{1/2}$ whenever $t \geq s$. Hence if $h \in [t_0, t_0 + h]$ the claim obviously follows from the assumption $\operatorname{Im} Q_{t_0+h}^{1/2} = \operatorname{Im} Q_{t_0}^{1/2}$. Therefore we will assume that $0 < h \leq t_0$.

Choose a positive integer k such that $kh \ge t_0$. By induction on the identity $Q_t = Q_h + S(h)Q_{t-h}S(h)$ we have $Q_{kh} = \sum_{j=0}^{k-1} S(jh)Q_hS(jh)$. Hence for all $x \in H$,

$$\begin{aligned}
[Q_{kh}x, x] &= \sum_{j=0}^{k-1}[S(jh)Q_hS(jh)x, x] \\
&= \sum_{j=0}^{k-1} \|Q_h^{1/2}S(jh)x\|^2 = \sum_{j=0}^{k-1} \|S(jh)Q_h^{1/2}x\|^2 \\
&\le M\|Q_h^{1/2}x\|^2 = M[Q_hx, x],
\end{aligned}$$

where $M := \sum_{j=0}^{k-1} \|S(jh)\|$. This shows that $\operatorname{Im} Q_{kh}^{1/2} \subset \operatorname{Im} Q_h^{1/2}$. But from $0 < h < kh$ we also have $\operatorname{Im} Q_{kh}^{1/2} \supset \operatorname{Im} Q_h^{1/2}$, and hence $\operatorname{Im} Q_{kh}^{1/2} = \operatorname{Im} Q_h^{1/2}$. But then $t_0 \in [h, kh]$ implies that also $\operatorname{Im} Q_{t_0}^{1/2} = \operatorname{Im} Q_h^{1/2}$ and the claim is proved.

Step 3 - Let us assume that the operator $T := \big(Q_{t_0}^{-1/2}Q_{t_0+h}^{1/2}\big)\big(Q_{t_0}^{-1/2}Q_{t_0+h}^{1/2}\big)^* - I$ from Theorem 2.4 (iii) is Hilbert-Schmidt on $\overline{\operatorname{Im} Q_{t_0}^{1/2}}$. The theorem is proved if we can show that this implies that $S(2t_0)$ restricts to a Hilbert-Schmidt operator on $\operatorname{Im} Q_{t_0}^{1/2}$, i.e. that neither (i) or (ii) holds.

From

$$\begin{aligned}
Tx &= \big(Q_{t_0}^{-1/2}Q_{t_0+h}^{1/2}\big)\big(Q_{t_0}^{-1/2}Q_{t_0+h}^{1/2}\big)^* x - x \\
&= Q_{t_0}^{-1/2}Q_{t_0+h}Q_{t_0}^{-1/2}x - x \\
&= Q_{t_0}^{-1/2}S(t_0)Q_hS(t_0)Q_{t_0}^{-1/2}x, \qquad x \in \operatorname{Im} Q_{t_0}^{1/2},
\end{aligned}$$

we see that T extends to a Hilbert-Schmidt operator on $\overline{\operatorname{Im} Q_{t_0}^{1/2}}$ if and only if the map

$$T_1 : Q_{t_0}x \mapsto S(t_0)Q_hS(t_0)x = Q_hS(2t_0)x, \qquad x \in H,$$

extends to a Hilbert-Schmidt operator on $\operatorname{Im} Q_{t_0}^{1/2}$. In the above identities we used the algebraic relation $Q_{t_0+h} = Q_{t_0} + S(t_0)Q_hS(t_0)$, which in particular implies that $S(t_0)Q_hS(t_0)$ maps H into $\operatorname{Im}(Q_{t_0+h} - Q_{t_0}) \subset \operatorname{Im} Q_{t_0+h}^{1/2} + \operatorname{Im} Q_{t_0}^{1/2} = \operatorname{Im} Q_{t_0}^{1/2}$, showing that the vector $Q_{t_0}^{-1/2}S(t_0)Q_hS(t_0)Q_{t_0}^{-1/2}x$ is well-defined if $x \in \operatorname{Im} Q_{t_0}^{1/2}$.

From Step 2 and [7, Proposition B.1] it follows that $Q_{t_0}Q_h^{-1} : Q_hy \mapsto Q_{t_0}y$, $y \in H$, extends to a (Banach space) isomorphism of $\operatorname{Im} Q_{t_0}^{1/2}$ onto itself. Thus, T_1 extends to a Hilbert-Schmidt operator on $\operatorname{Im} Q_{t_0}^{1/2}$ if and only if

$$T_2 : Q_{t_0}x \mapsto (Q_{t_0}Q_h^{-1})T_1Q_{t_0}x = Q_{t_0}S(2t_0)x = S(2t_0)Q_{t_0}x, \qquad x \in H,$$

extends to a Hilbert-Schmidt operator on $\operatorname{Im} Q_{t_0}^{1/2}$. But since $\operatorname{Im} Q_{t_0}$ is dense in $\operatorname{Im} Q_{t_0}^{1/2}$, this can only happen if $S(2t_0)$ maps $\operatorname{Im} Q_{t_0}^{1/2}$ into itself and defines a Hilbert-Schmidt operator on this space. ∎

Remark 3.2. The above proof essentially shows this: if $\mathbf{S}$ is self-adjoint and commutes with Q, and if $t_0 > 0$ and $h > 0$ are such that the measures μ_{t_0} and μ_{t_0+h} are equivalent, then $\mathbf{S}$ leaves $H_{t_0} := \operatorname{Im} Q_{t_0}^{1/2}$ invariant and the restriction of $S(2t_0)$ to this space is Hilbert-Schmidt. In oder to understand this better, observe the restriction $S(t_0)|_{H_{t_0}}$ is self-adjoint on H_{t_0}, so that

$$S(2t_0)|_{H_{t_0}} = \left(S(t_0)|_{H_{t_0}}\right)\left(S(t_0)|_{H_{t_0}}\right)^*.$$

Thus, in some sense Theorem 3.1 is analogous to the well-known results (cf. [3]) that in the presence of an invariant measure μ_∞, one always has that (i) $\mathbf{S}$ always leaves $H_\infty = \operatorname{Im} Q_\infty^{1/2}$ invariant and (ii) equivalence of the measures μ_{t_0} and μ_∞ implies that the operator $(S(t_0)|_{H_\infty})(S(t_0)|_{H_\infty})^*$ is Hilbert-Schmidt on H_∞.

4. Hyperbolic systems

Let Λ be a non-negative self-adjoint unbounded operator on a Hilbert space H. Define

$$\mathcal{H} = D(\Lambda^{1/2}) \oplus H$$

and

$$S(t) := \begin{pmatrix} \cos t\Lambda^{1/2} & \Lambda^{-1/2}\sin t\Lambda^{\frac{1}{2}} \\ -\Lambda^{1/2}\sin t\Lambda^{1/2} & \cos t\Lambda^{1/2} \end{pmatrix}, \qquad t \geq 0.$$

Then $\mathbf{S} = \{S(t)\}_{t\geq 0}$ extends to a C_0-group on $\mathcal{H}$. If $\Lambda = -\Delta$ then $\mathbf{S}$ is the wave semigroup.

The infinitesimal generator A of $\mathbf{S}$ is of the form

$$A = \begin{pmatrix} 0 & I \\ -\Lambda & 0 \end{pmatrix}$$

with domain $D(A) = D(\Lambda) \oplus D(\Lambda^{1/2})$.

Ornstein-Uhlenbeck processes with the operator A of the introduced form are called *hyperbolic systems.* The first coordinate of the process satisfies a stochastic equation which is second order in time, see e.g. [7], [8]. Since $\mathbf{S}$ is in fact a C_0-group, it is easy to check (cf. [8, Theorem 9.2.1] that the corresponding linear equation has a solution if and only if the covariance operator Q of the noise is trace class.

Theorem 4.1. *Transition semigroups corresponding to hyperbolic systems are not eventually norm continuous in* $BUC_{\mathbf{S}}(H)$.

Proof. We are going to check that $\mathbf{S}$ fulfills condition (ii) of Theorem 2.4 for all $t_0 > 0$ and $h > 0$. We claim that it is enough to show that the operator $S(h) - I$ is not compact. Indeed, if condition (ii) of Theorem 2.4 fails for some $t_0 > 0$ and $h > 0$, then $\operatorname{Im}(S(t_0+h) - S(t_0)) \subset \operatorname{Im} Q_{t_0}^{1/2}$. By general results on abstract Wiener spaces [13, Section 1.4], the inclusion map $i_{t_0} : \operatorname{Im} Q_{t_0}^{1/2} \subset H$ is compact. By composition, it then follows that $S(t_0+h) - S(t_0)$ is compact as an operator on H. But since $S(t_0+h) - S(t_0) = S(t_0)(S(h) - I)$ and the operator $S(t_0)$ is invertible, it follows that $S(h) - I$ is compact.

In order to prove that $S(h) - I$ is not compact we need then the following simple fact.

Lemma 4.2. *Let $(\lambda_n) \subset \mathbb{R}_{>0}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} \lambda_n = \infty$, and let $I \subset [0,1)$ be Borel subset with non-zero Lebesgue measure. Then the set of all $s > 0$ for which*

$$\#\{n \geq 1 :\ s\lambda_n \ mod\ 1 \in I\} = \infty \tag{4.1}$$

is dense in $(0,\infty)$.

Proof. Fix $0 < a < b < \infty$; we check the existence of an $s \in (a,b)$ for which (4.1) holds.

For a given $\lambda > 0$ define

$$J_\lambda := \{s \in (a,b) :\ s\lambda \bmod 1 \in I\}.$$

Denoting by $\lfloor \gamma \rfloor$ the integer part of a real number γ and by $|J|$ the Lebesgue measure of a set J, we clearly have

$$|J_\lambda| \geq \frac{\lfloor (b-a)\lambda \rfloor}{\lambda}\,|I|.$$

Hence by dominated convergence and Fubini's theorem,

$$\begin{aligned}
&\int_a^b \Big(\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \chi_I(s\lambda_n \bmod 1)\Big)\, ds \\
&\qquad\qquad \geq \liminf_{N\to\infty} \int_a^b \Big(\frac{1}{N}\sum_{n=1}^{N} \chi_I(s\lambda_n \bmod 1)\Big)\, ds \\
&\qquad\qquad = \liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \int_a^b \chi_I(s\lambda_n \bmod 1)\, ds \\
&\qquad\qquad \geq \liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \frac{\lfloor (b-a)\lambda_n \rfloor}{\lambda_n}\,|I| = (b-a)\,|I|
\end{aligned}$$

It follows that the set of all $s \in (a,b)$ such that $\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \chi_I(s\lambda_n \bmod 1) > 0$ has positive measure. But in order that this limes superior should be positive for some s we certainly need that

$$\#\{n \geq 1 :\ \chi_I(s\lambda_n \bmod 1) = 1\} = \infty,$$

which is the same as (4.1). ■

The conclusion of the theorem is the direct consequence of the next lemma:

Lemma 4.3. *The set of all $t > 0$ such that $S(t) - I$ is not compact is dense in $(0,\infty)$.*

Proof. Let $\pi(d\lambda)$ denote the spectral measure on $\mathcal{H}$ corresponding to Λ.

Then for arbitrary $x \in H$ we have, with $z := \Lambda^{1/2}x$,

$$\begin{aligned}
\left\| S(t)\begin{pmatrix} x \\ 0 \end{pmatrix} - \begin{pmatrix} x \\ 0 \end{pmatrix}\right\|_{\mathcal{H}}^2 &= \|\Lambda^{1/2}(\cos t\Lambda^{1/2}x - x)\|^2 + \|\sin t\Lambda^{1/2}x\|^2 \\
&= \int_0^\infty (\cos t\sqrt{\lambda} - 1)^2 \,\langle \pi(d\lambda)z, z\rangle \\
&\qquad + \int_0^\infty (\sin t\sqrt{\lambda})^2 \,\langle \pi(d\lambda)z, z\rangle \\
&= 2\int_0^\infty (1 - \cos t\sqrt{\lambda})\,\langle \pi(d\lambda)z, z\rangle \\
&= 2\int_0^\infty (1 - \cos t\lambda)\,\langle \tilde{\pi}(d\lambda)z, z\rangle \\
&=: \|B_t z\|^2,
\end{aligned}$$

where $\tilde{\pi}(d\lambda)$ is the spectral measure corresponding to $\Lambda^{1/2}$.

Let $\lambda_n \to \infty$ be a strictly increasing sequence of elements in $\sigma(\Lambda^{1/2})$. For $t > 0$ and $\gamma \in (0,2)$ let

$$I_{t,\gamma} := \{\lambda > 0 :\, 1 - \cos t\lambda > \gamma\}.$$

Denoting $\lambda_{t,\gamma} := \min\{\lambda \geq 0 :\, 1 - \cos t\lambda = \gamma\}$ we see that

$$\lambda \in I_{t,\gamma} \quad \text{if and only if} \quad \frac{t\lambda}{2\pi} \bmod 1 \in \left(\frac{t\lambda_{t,\gamma}}{2\pi}, 1 - \frac{t\lambda_{t,\gamma}}{2\pi}\right) = \left(\frac{\lambda_{1,\gamma}}{2\pi}, 1 - \frac{\lambda_{1,\gamma}}{2\pi}\right).$$

Hence by Lemma 4.2, for $\gamma \in (0,2)$ fixed we can always find a dense set $J \subset (0,\infty)$ (depending on γ) such that $I_{t,\gamma} \cap \sigma(\Lambda^{1/2})$ is an infinite set for all $t \in J$. Fix $t \in J$ and let $I_{t,\gamma} \cap \sigma(\Lambda^{1/2}) = \{\lambda_{n_j} :\, j = 1, 2, ...\}$; of course the index set (n_j) depends on γ and t. Choose closed intervals I_j such that for each j we have $\lambda_{n_j} \in I_j \subset I_{t,\gamma}$. These intervals are disjoint. Let π_j denote the spectral projection corresponding to $I_j \cap \sigma(\Lambda^{1/2})$ and let $z_j \in H$ be a norm one vector contained in the range $H_j := \pi_j H$. Since $1 - \cos t\lambda > \gamma$ for $\lambda \in I_j \cup I_l$ it follows that

$$\frac{1}{2}\|B_t(z_j - z_l)\|^2 \geq 2\gamma, \qquad j \neq l.$$

Noting that the part of $\Lambda^{1/2}$ in H_j is invertible, we can find $x_j \in D(\Lambda^{1/2})$ such that $\Lambda^{1/2}x_j = z_j$. Then $\|(x_j, 0)\|_{\mathcal{H}} = 1$ and for all $j \neq l$ we have

$$\left\| S(t)\begin{pmatrix} x_j - x_l \\ 0 \end{pmatrix} - \begin{pmatrix} x_j - x_l \\ 0 \end{pmatrix}\right\|_{\mathcal{H}}^2 \geq 4\gamma.$$

This proves that $S(t) - I$ fails to be compact in $\mathcal{H}$ for each $t \in J$. ∎

The proof of the lemma shows a little more: noting that $\|z_j - z_l\| = \sqrt{2}$ by orthogonality of H_j and H_l (or by replacing $z_j - z_l$ by z_j) we see that

$$\|S(t) - I\|_{\mathcal{H}}^2 \geq 2\gamma,$$

and since $\gamma \in (0,2)$ was arbitrary this gives, for all t in the dense set $J \subset (0,\infty)$,

$$\|S(t) - I\|_{\mathcal{H}} \geq 2.$$

Also, inspection of the proof of the lemma shows that we can quantify the dense set of t, but this will not be needed.

We cannot expect $S(t) - I$ to be non-compact for all $t > 0$: Let $H = l^2$ and $\Lambda e_n := n^2 e_n$ (where e_n is the nth unit vector) with maximal domain. Then on $\mathcal{H}$ we have

$$S(t)\begin{pmatrix} e_n \\ e_m \end{pmatrix} = \begin{pmatrix} \cos nt\, e_n + m^{-1}\sin mt\, e_m \\ -n\sin nt\, e_n + \cos mt\, e_m \end{pmatrix} \qquad \forall n, m = 1, 2, ...;$$

in particular,

$$S(2\pi k)\begin{pmatrix} e_n \\ e_m \end{pmatrix} = \begin{pmatrix} e_n \\ e_m \end{pmatrix}, \qquad \forall k \in \mathbb{N},\ n, m = 1, 2, ...$$

Hence, $S(2\pi k) - I = 0$ for all $k \in \mathbb{N}$.

5. Norm discontinuity of the stopped semigroup

Let $\mathbf{X}$ be an Ornstein-Uhlenbeck process on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in H. Fix an open set $V \subset H$ and denote by τ_x the *first exit time* of V,

$$\tau_x^V(\omega) := \inf\{t \geq 0 :\ X(t, x)(\omega) \notin V\}.$$

In order to avoid measurability problems, *in this section we impose the following*

Assumption: The Ornstein-Uhlenbeck process $\mathbf{X}$ has continuous trajectories.

This Assumption guarantees that τ_x is a random variable and in fact a stopping time. It is well-known that $\mathbf{X}$ always has a continuous modification if Q is trace class, and more generally if there exists a real number $\alpha > 0$ such that for all $t > 0$,

$$\int_0^t s^{-\alpha}\,\mathrm{Trace}\, S(s)QS^*(s)\, ds < \infty;$$

cf. [7, Section 5.3].

The *stopped transition semigroup* $\mathbf{P}^V$ associated with $\mathbf{X}$ and V is defined by

$$P^V(t)f(x) = \mathbb{E}(f(X(t, x))|\tau_x^V > t).$$

Clearly $P^V(t)$ is a linear contraction on $B_b(V)$, $P^V(0) = I$ and we have $P^V(t)P^V(s) = P^V(t+s)$. Thus $\mathbf{P}^V$ is a contraction semigroup on $B_b(V)$; cf. [4], [10], [11]. It is not clear however under what conditions it is a semigroup on $BUC(V)$. We denote by $BUC_{\mathbf{S}}(V)$ the smallest closed subspace of $B_b(V)$ which is invariant under $\mathbf{P}^V$ and contains all functions g of the form

$$g(x) = \int_0^t P^V(s)f(x)\, ds, \qquad f \in BUC(V),\ t > 0,\ x \in V.$$

We have the following generalization of Theorem 2.4.

Theorem 5.1. *Assume that one of the conditions of Theorem 2.4 is satisfied. Then*

$$\|P^V(t_0+h) - P^V(t_0)\|_{BUC_{\mathbf{S}}(V)} = \mathbb{P}(\tau_x^V > t_0+h) \ + \mathbb{P}(\tau_x^V > t_0).$$

Proof. With our definition of the space $BUC_{\mathbf{S}}(V)$, Lemma 2.3 has a straightforward extension to finite, signed measures ν on V with the same proof. Thus if ν is a finite, signed Borel measure on V, then $\|\nu\|_{(BUC_{\mathbf{S}}(V))^*} = \mathrm{var}\,(\nu)$. Letting

$$\mu_t^{x,V}(\Gamma) = P^V(t)\chi_\Gamma, \quad \Gamma \in \mathcal{B}(V),$$

then

$$\mu_t^{x,V}(\Gamma) = \mathbb{P}(X(t,x) \in \Gamma \ \text{ and } \tau_x^V > t).$$

Consequently, if the measures $\mu_{t_0+h}^x$ and $\mu_{t_0}^x$ are singular and therefore supported by disjoint sets, the same is true for the measures $\mu_{t_0+h}^{x,V}$ and $\mu_{t_0}^{x,V}$. ■

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