STOCHASTIC CONVOLUTION IN SEPARABLE BANACH SPACES AND THE STOCHASTIC LINEAR CAUCHY PROBLEM

Zdzisław Brzeźniak[1]
Department of Mathematics
The University of Hull
Hull HU6 7RX, England
Z.Brzezniak@maths.hull.ac.uk

Jan van Neerven[2]
Department of Mathematics
Delft University of Technology
PO Box 5031, 2600 GA Delft
The Netherlands
J.vanNeerven@its.tudelft.nl

Abstract - Let H be a separable real Hilbert space and let E be a separable real Banach space. In this paper we develop a general theory of stochastic convolution of $\mathcal{L}(H,E)-$valued functions with respect to a cylindrical Wiener process $\{W_t^H\}_{t\in[0,T]}$ with Cameron-Martin space H. This theory is applied to obtain necessary and sufficient conditions for the existence of a weak solution of the stochastic abstract Cauchy problem

$$(ACP) \qquad \begin{aligned} dX_t &= AX_t\,dt + B\,dW_t^H \qquad (t\in[0,T]),\\ X_0 &= 0 \quad \text{almost surely,} \end{aligned}$$

where A is the generator of a C_0-semigroup $\{S(t)\}_{t\geq 0}$ of bounded linear operators on E and $B\in\mathcal{L}(H,E)$ is a bounded linear operator. We further show that whenever a weak solution exists, it is unique, and given by a stochastic convolution

$$X_t = \int_0^t S(t-s)B\;dW_s^H.$$

1991 AMS Subject Classification: 60G15, 60H15, 60B05, 47D03
Running title: Stochastic convolution

[1] Part of this work was supported by the Deutsche Forschungs Gemeinschaft (DFG)
[2] Part of this research has been made possible by a fellowship of the Royal Netherlands Academy of Arts and Sciences

0. Introduction

Let H be a separable real Hilbert space and let E be a separable real Banach space. In this paper we set up a theory of stochastic convolution for $\mathcal{L}(H,E)-$valued functions which enables us to study existence and uniqueness of solutions to the stochastic abstract Cauchy problem

$$(ACP) \qquad \begin{aligned} dX_t &= AX_t\,dt + B\,dW_t^H \qquad (t \in [0,T]), \\ X_0 &= 0 \quad \text{almost surely.} \end{aligned}$$

Here A is the generator of a C_0-semigroup $\{S(t)\}_{t\geq 0}$ of bounded linear operators on E, $B \in \mathcal{L}(H,E)$ is a bounded linear operator, and $\{W_t^H\}_{t\in[0,T]}$ is a cylindrical Wiener process with Cameron-Martin space H.

If E is a separable *Hilbert space*, it is well known that a weak solution of (ACP) exists if and only if the positive self-adjoint operator operator $Q_T \in \mathcal{L}(E^*,E)$ defined by

$$Q_T x^* = \int_0^T S(t)BB^*S^*(t)x^*\,dt \qquad (x^* \in E^*)$$

is trace class (we do not identify E and its dual E^* here). In this case the weak solution is unique, and given by the Itô-type convolution integral

$$X_t = \int_0^t S(t-s)B\ dW_s^H \qquad (t \in [0,T]).$$

A detailed account of the theory of the problem (ACP) in *Hilbert* spaces E is presented in the recent book by Da Prato and Zabczyk [DZ].

Due to the lack of a satisfactory theory of stochastic integration in Banach spaces, it seems impossible to give a straightforward extension of this theory to the case where E is a Banach space. For this one needs additional assumptions on E, such as $2-$uniform smoothness (equivalently, martingale type 2). This approach is worked out in [Nh], [Br1], [Br3] and the references therein.

From these works it is well known that the solution of (ACP), if it exists, is an $E-$valued Ornstein-Uhlenbeck process associated with $\mathbf{S}$ and B, i.e. a centred Gaussian $E-$valued process $\{X_t\}_{t\in[0,T]}$ with covariance given by

$$\mathbb{E}\left(\langle X_t, x^*\rangle\langle X_s, y^*\rangle\right) = \int_0^{t\wedge s} [B^*S^*(t-u)x^*, B^*S^*(s-u)y^*]_H\,du.$$

In certain special situations, vector-valued Ornstein-Uhlenbeck processes have been studied by various methods by various authors; we mention Antoniadis and Carmona [AC], Millet and Smolenski [MS] and Röckle [Rö]. However, the problem of giving necessary and sufficient conditions in terms of $\mathbf{S}$ and B for the existence of such a process in the general case has not been addressed there.

In this paper we show that it is possible to set up a theory of stochastic *convolution* in arbitrary separable real Banach spaces E. Let us briefly outline its main features. Suppose H is a separable real Hilbert space and $\Phi : (0, T] \to \mathcal{L}(H, E)$ is an operator-valued function satisfying

$$\int_0^T \|\Phi^*(t)x^*\|_H^2 \, dt < \infty, \qquad \forall x^* \in E^*.$$

We show that the formula

$$\langle Q_T x^*, y^* \rangle = \int_0^T [\Phi^*(t)x^*, \Phi^*(t)y^*]_H \, dt \qquad (x^*, y^* \in E^*)$$

defines a positive symmetric operator $Q_T \in \mathcal{L}(E^*, E)$. Knowing this, we can consider the reproducing kernel Hilbert space (RKHS) H_T associated with Q_T; this is a Hilbert subspace of E. Denoting the inclusion operator $H_T \hookrightarrow E$ by i_T, we have $Q_T = i_T \circ i_T^*$. We prove the following result (Theorem 2.6 and Proposition 2.8):

Theorem 0.1. *The following assertions are equivalent:*

(i) There exists an $E-$valued centred Gaussian process $\{\xi_t\}_{t\in[0,T]}$ with covariance given by

$$\mathbb{E}\left(\langle \xi_t, x^* \rangle \langle \xi_s, y^* \rangle\right) = \int_0^{t\wedge s} [\Phi^*(t-u)x^*, \Phi^*(s-u)y^*]_H \, du; \tag{0.1}$$

(ii) The inclusion $i_T : H_T \hookrightarrow E$ is $\gamma-$radonifying.

An $E-$valued centred Gaussian process with covariance given by (0.1) will be called an *Ornstein-Uhlenbeck process* associated with Φ. Note that the second condition is equivalent to Q_T being the covariance operator of a centred Gaussian Borel measure on E.

Our second main result (Theorem 3.3) shows that it is possible to obtain Ornstein-Uhlenbeck processes by convolution with a cylindrical Wiener process $\{W_t^H\}_{t\in[0,T]}$:

Theorem 0.2. *Let $\{W_t^H\}_{t\in[0,T]}$ be a cylindrical Wiener process with Cameron-Martin space H. If the inclusion $i_T : H_T \hookrightarrow E$ is $\gamma-$radonifying, then there exists a predictable $E-$valued Ornstein-Uhlenbeck process $\{X_t\}_{t\in[0,T]}$ which satisfies*

$$\langle X_t, x^* \rangle = \int_0^t \langle \Phi(t-s)\, dW_s^H, x^* \rangle \quad \text{a.s.} \qquad (t \in [0,T],\ x^* \in E^*).$$

Up to a modification, this process is unique.

This justifies the notation

$$X_t = \int_0^t \Phi(t-s)\, dW_s^H.$$

The weak stochastic convolution on the right hand side is defined in an obvious way (cf. Section 3).

If A is the generator of a C_0-semigroup $\{S(t)\}_{t\geq 0}$ on E and B is a bounded linear operator from H into E, we can apply these results to the operator-valued function $\Phi(t) = S(t) \circ B \in \mathcal{L}(H, E)$. This enables us to derive necessary and sufficient conditions for the existence of weak solutions for the problem (ACP) and study some of their properties. The results can be summarized as follows.

Theorem 0.3. *The following assertions are equivalent:*

(i) The problem (ACP) has a weak solution $\{X_t\}_{t\in[0,T]}$ *on* $[0,T]$*;*

(ii) The inclusion $i_T : H_T \hookrightarrow E$ *is* $\gamma-$*radonifying.*

In this situation, the solution is unique, and given by the stochastic convolution

$$X_t = \int_0^t S(t-s)B\, dW_s^H, \qquad t \in [0,T].$$

The process $\{X_t\}_{t\in[0,T]}$ *has a version with almost surely square integrable trajectories. If the semigroup generated by* A *is analytic, then* $\{X_t\}_{t\in[0,T]}$ *has a version with continuous trajectories.*

Recalling that a positive symmetric operator on a Hilbert space E is trace class if and only if it is the covariance of a centred Gaussian measure on E, we see that our results extend the known existence and uniqueness results for Hilbert spaces mentioned above.

In the final section we apply our theory to the stochastic heat equation driven by a homogenous space-time Wiener process,

$$\begin{aligned}
\frac{\partial X}{\partial t}(t,x) &= \Delta X(t,x) + \frac{\partial w}{\partial t}(t,x), \quad t \in [0,T], \\
X(0,x) &= 0, \\
X(t,0) &= X(t,1) = 0.
\end{aligned}$$

Some of the questions that led to our research were motivated by the theory of Feynman path integrals and their close relationship to the theory of integrals with respect to 'Ornstein-Uhlenbeck measures', i.e. Gaussian measures on spaces of vector-valued functions arising as image measures corresponding to the Cameron-Martin spaces of vector-valued Gaussian processes. It is known that certain equivalent norms on the Cameron-Martin space lead to equivalent image measures, cf. [ABB]. In [BN] we apply the results obtained in the present paper to study equivalence of this type of Gaussian measures in the abstract framework considered here.

1. Preliminaries

In this section we briefly recall some well known facts concerning (cylindrical) Gaussian measures. For more details we refer to [VTC], [Schw1], [Schw2], [Kuo].

Let $\mathcal{E}$ be a real locally convex topological vector space, with topological dual $\mathcal{E}'$. A subset C of $\mathcal{E}$ is said to be a *cylindrical set* if it is of the form $C = \{x \in \mathcal{E} : (\langle x, x_1'\rangle, \dots, \langle x, x_n'\rangle) \in B\}$ for some $n \geq 1$, $x_1', \dots, x_n' \in \mathcal{E}'$, and a Borel set $B \subset \mathbb{R}^n$. The set of all cylindrical subsets of $\mathcal{E}$ is an algebra of sets and is denoted by $\mathcal{C}(\mathcal{E})$. A *centred cylindrical Gaussian measure on* $\mathcal{E}$ is a finitely additive set function μ on $\mathcal{C}(\mathcal{E})$, whose images under the maps $x \mapsto (\langle x, x_1'\rangle, \dots, \langle x, x_n'\rangle)$ are $\sigma-$additive Gaussian measures on $\mathbb{R}^n$, or equivalently, whose images under the maps $x \mapsto \langle x, x'\rangle$ are $\sigma-$additive Gaussian measures on $\mathbb{R}$.

If $\mathcal{F}$ is another locally convex space, and if $T : \mathcal{E} \to \mathcal{F}$ is a continuous linear transformation, then the image $T(\mu) := \mu \circ T^{-1}$ of a centred cylindrical Gaussian measure on $\mathcal{E}$ is a centred cylindrical Gaussian measure on $\mathcal{F}$.

Let H be a real Hilbert space. By γ_H we denote the *standard centred cylindrical Gaussian measure* on H, i.e. the centred cylindrical Gaussian measure on H whose image under any of the maps $g \mapsto ([g, h_1]_H, \dots, [g, h_n]_H)$, with $\{h_1, \dots, h_n\}$ orthonormal in H, is the standard Gaussian measure on $\mathbb{R}^n$.

A continuous linear operator $Q \in \mathcal{L}(\mathcal{E}', \mathcal{E})$ is called *positive* if $\langle Qx', x'\rangle \geq 0$ for all $x' \in \mathcal{E}'$, and *symmetric* if $\langle Qx', y'\rangle = \langle Qy', x'\rangle$ for all $x' \in \mathcal{E}'$ and $y' \in \mathcal{E}'$. To every positive symmetric operator $Q \in \mathcal{L}(\mathcal{E}', \mathcal{E})$ one can associate a real Hilbert space H_Q in the following way. On the range of Q one has a well-defined inner product $[\cdot, \cdot]_H$ given by

$$[Qx', Qy'] := \langle Qx', y'\rangle \qquad (x', y' \in \mathcal{E}').$$

Denote by H_Q the Hilbert space completion of range Q with respect to this inner product; this Hilbert space is called the *reproducing kernel Hilbert space* (RKHS) associated with Q. If $\mathcal{E}$ is quasi-complete, then the inclusion mapping from range Q into $\mathcal{E}$ has a continuous extension to an injective linear map $i : H_Q \to \mathcal{E}$. In this way, the pair (i, H_Q) becomes a Hilbert subspace of $\mathcal{E}$. Moreover, upon identifying H_Q with its dual in the natural way, we then have the operator identity $Q = i \circ i'$. In Section 2 these results will be applied to the (quasi-complete) product space $\mathcal{E} = E^{[0,T]}$, with E a separable real Banach space.

Conversely, if (i, H) is a real Hilbert subspace of $\mathcal{E}$ (i.e. i is a continuous injective linear map from some real Hilbert space H into $\mathcal{E}$), then $Q := i \circ i' \in \mathcal{L}(\mathcal{E}', \mathcal{E})$ is positive and symmetric, and its RKHS equals H.

The relationship between centred cylindrical Gaussian measures and positive symmetric operators in described in the following well known result [VTC, Chapter III].

Proposition 1.1. *Let $\mathcal{E}$ be a real locally convex topological vector space.*

(i) Let H be a real Hilbert space and let $T \in \mathcal{L}(H, \mathcal{E})$. The image cylindical measure $\mu := T(\gamma_H)$ is a centred cylindrical Gaussian measure on $\mathcal{E}$, whose Fourier transform is given by

$$\int_{\mathcal{E}} \exp(i\langle x, x'\rangle)\, d\mu(x) = \exp\left(-\frac{1}{2}\langle (T \circ T')x', x'\rangle\right) \qquad (x' \in \mathcal{E}').$$

The RKHS H_Q associated with the positive symmetric operator $Q = T \circ T' \in \mathcal{L}(\mathcal{E}', \mathcal{E})$, equals the range of T, which is a Hilbert space under the inner product

$$[Tg, Th]_{H_Q} = [Pg, Ph]_H,$$

with P the orthogonal projection in H onto $(\ker T)^\perp$, the orthogonal complement in H of the kernel of T. Moreover, as a map from $(\ker T)^\perp$ onto H_Q, the operator T is an isometry.

(ii) If $\mathcal{E}$ is quasi-complete and $Q \in \mathcal{L}(\mathcal{E}', \mathcal{E})$ is positive and symmetric, and if μ is a centred cylindrical Gaussian measure on $\mathcal{E}$ with Fourier transform

$$\int_{\mathcal{E}} \exp(i\langle x, x'\rangle)\, d\mu(x) = \exp\left(-\frac{1}{2}\langle Qx', x'\rangle\right) \qquad (x' \in \mathcal{E}'),$$

then $\mu = i(\gamma_H)$, where H is the RKHS of Q and $i : H \hookrightarrow \mathcal{E}$ is the natural embedding.

Let $\mathcal{E}$ be a real locally convex topological vector space. A measure μ on the $\sigma-$algebra $\sigma(\mathcal{C}(\mathcal{E}))$ generated by the algebra $\mathcal{C}(\mathcal{E})$ is called a (centred) *Gaussian measure on* $\mathcal{E}$ if for all $x' \in \mathcal{E}'$ the image measure $\langle \mu, x'\rangle := \mu \circ (x')^{-1}$ is a (centred) Gaussian Borel measure on $\mathbb{R}$. If H is a real Hilbert space, a continuous linear operator $T \in \mathcal{L}(H, \mathcal{E})$ is said to be $\gamma-$*radonifying* if the image cylindrical measure $T(\gamma_H)$ has a (necessarily unique) countably additive extension to a Gaussian measure on $\mathcal{E}$. Note that in general the $\sigma-$algebra $\sigma(\mathcal{C}(\mathcal{E}))$ is much smaller than the Borel $\sigma-$algebra of $\mathcal{E}$.

The following three examples of $\gamma-$radonifying operators will be of importance:

- If μ is a centred Gaussian measure on $\mathcal{E}$ with RKHS H, then the inclusion map $i : H \hookrightarrow \mathcal{E}$ is $\gamma-$radonifying, and we have $i(\gamma_H) = \mu$.
- If H and $\mathcal{E}$ are Hilbert spaces, then $T \in \mathcal{L}(H, \mathcal{E})$ is $\gamma-$radonifying if and only if T is a Hilbert-Schmidt operator.
- If G and H are Hilbert spaces and $S \in \mathcal{L}(G, H)$ and $T \in \mathcal{L}(H, \mathcal{E})$ are continuous linear operators, then $T \circ S$ is $\gamma-$radonifying whenever T is $\gamma-$radonifying [Bax], [Ram].

As is common, the dual of a Banach space E will be denoted by E^* rather than E'. We will frequently use sequential weak*-approximation arguments in dual Banach spaces. One has to be careful with this, because a weak*-dense linear subspace in the dual E^* of a Banach space E need not be weak*-sequentially dense, even if E is separable. A counterexample is given in [Di]. We get around this in the following way.

Proposition 1.2. *Let E be a separable real Banach space and let Y be a linear subspace of E^* which is both weak*-dense and weak*-sequentially closed. Then $Y = E^*$.*

Proof : The closed unit ball B_{E^*} is weak*-compact, hence certainly weak*-sequentially closed. It follows that $B_{E^*} \cap Y$ is weak*-sequentially closed. Because the weak*-topology of B_{E^*} is metrizable, $B_{E^*} \cap Y$ is actually weak*-closed. Hence by the Krein-Smulyan theorem [DS, Theorem V.5.7], Y is weak*-closed. Since by assumption Y is also weak*-dense, we infer that $Y = E^*$. ∎

As a corollary we record:

Corollary 1.3. *Let μ be Borel probability measure on a separable real Banach space E, and suppose there is a weak*-dense linear subspace Y of E^* such that the image measures $\langle \mu, x^*\rangle$ are Gaussian for all $x^* \in Y$. Then μ is a Gaussian measure.*

Proof : By Zorn's Lemma there exists a maximal linear subspace Y' of E^* with the property that $\langle \mu, x^*\rangle$ is Gaussian for all $x^* \in Y'$. Since obviously $Y \subset Y'$ we see that Y' is weak*-dense.

Let Y'' denote the weak*-sequential closure of Y'. Let $x^* \in Y''$ be arbitrary and suppose that weak*-$\lim_{n\to\infty} x_n^* = x^*$ in E^* for some sequence (x_n^*) in Y'. By the dominated convergence theorem, for the Fourier transforms we have

$$\begin{aligned}\lim_{n\to\infty} \langle \mu, x_n^*\rangle\hat{}(\xi) &= \lim_{n\to\infty} \int_E \exp(i\xi\langle y, x_n^*\rangle)\, d\mu(y)\\ &= \int_E \exp(i\xi\langle y, x^*\rangle)\, d\mu(y) = \langle \mu, x^*\rangle\hat{}(\xi), \qquad \forall \xi \in \mathbb{R}\,.\end{aligned}$$

As is well known [Nv, Lemme 1.5], this implies that $\langle \mu, x^*\rangle$ is Gaussian.

We have shown that $\langle \mu, x^*\rangle$ is Gaussian for all $x^* \in Y''$. By the maximality of Y' we must have $Y'' = Y'$, and therefore Y' is weak*-sequentially closed. Proposition 1.2 now finishes the proof. ∎

2. The canonical Ornstein-Uhlenbeck process

Throughout the rest of this paper, *H is a separable real Hilbert space and E is a separable real Banach space.* Suppose $\Phi : (0,T] \to \mathcal{L}(H,E)$ is an operator-valued function on $(0,T]$ with the property that for all $x^* \in E^*$, $t \mapsto \Phi^*(t)x^*$ is a strongly measurable $H-$valued function satisfying

$$\int_0^T \|\Phi^*(t)x^*\|_H^2\, dt < \infty.$$

By a standard argument, the mapping $E^* \to L^2((0,T];H)$ given by $x^* \mapsto \Phi^*(\cdot)x^*$ is closed, hence bounded by the closed graph theorem. The space of all such Φ can be made into a normed linear space, which we denote by $L^2((0,T];H,E)$, by defining the norm of Φ to be the operator norm of Φ^* regarded as an element of $\mathcal{L}(E^*, L^2((0,T];H))$,

$$\|\Phi\|^2_{L^2((0,T];H,E)} := \sup_{\|x^*\|\le 1} \left(\int_0^T \|\Phi^*(t)x^*\|_H^2\, dt \right).$$

For the rest of this section we fix $\Phi \in L^2((0,T];H,E)$.

Lemma 2.1. *For all $x^* \in E^*$ the function $\Phi(\cdot)\Phi^*(\cdot)x^*$ is a strongly measurable $E-$valued function on $(0,T]$.*

Proof : Fix $x^* \in E^*$. Choose an orthonormal basis (h_n) in H. Then, for all $t \in (0,T]$ and $y^* \in E^*$,

$$\langle \Phi(t)\Phi^*(t)x^*, y^*\rangle = [\Phi^*(t)x^*, \Phi^*(t)y^*]_H = \sum_n [\Phi^*(t)x^*, h_n]_H\, [\Phi^*(t)y^*, h_n]_H,$$

which is measurable as a function of t. This shows that $\Phi(\cdot)\Phi^*(\cdot)x^*$ is weakly measurable. Since by assumption E is separable, Pettis's measurability theorem [DU, Chapter 2] implies that this function is actually strongly measurable. ∎

Proposition 2.2. *For all $x^* \in E^*$ and $t \in (0,T]$ there exists a unique element $Q_t x^* \in E$ satisfying*

$$\langle Q_t x^*, y^* \rangle = \int_0^t \langle \Phi(s)\Phi^*(s)x^*, y^* \rangle \, ds, \qquad \forall y^* \in E^*.$$

The linear operators Q_t from E^* to E obtained in this way are bounded, positive and symmetric.

Proof : Fix $t \in (0,T]$. Define $Q_t x^* \in E^{**}$ by

$$\langle y^*, Q_t x^* \rangle := \int_0^t \langle \Phi(s)\Phi^*(s)x^*, y^* \rangle \, ds = \int_0^t [\Phi^*(s)x^*, \Phi^*(s)y^*]_H \, ds \qquad (y^* \in E^*).$$

Note that this integral is finite by Hölder's inequality and the integrability assumption on Φ. By the boundedness of the map $x^* \mapsto \Phi^*(\cdot)x^*$ from E^* into $L^2((0,T];H)$, the resulting linear operator $Q_t : E^* \to E^{**}$ is bounded. We must prove that Q_t is actually $E-$valued.

Fix $x^* \in E^*$ arbitrary. We claim that $Q_t x^*$ acts weak$^*-$continuously on the closed unit ball B_{E^*} of E^*. By the Krein-Smulyan theorem, this implies that $Q_t x^*$ belongs to E, and the proof will be complete.

Assume, for a contradiction, that the claim is not true. Since E is separable, the closed unit ball of E^* is weak*-sequentially compact, and we can find an $\varepsilon > 0$ and a sequence (y_n^*) in B_{E^*} that weak$^*-$converges to some $y^* \in B_{E^*}$ such that

$$|\langle y_n^*, Q_t x^* \rangle - \langle y^*, Q_t x^* \rangle| \geq \varepsilon \qquad (n \geq 0). \tag{2.1}$$

For each s, the adjoint operator $\Phi^*(s)$ is weak$^*-$continuous from E^* into H, and hence weak$^*-$to$-$weakly continuous. Therefore, $\lim_n \Phi^*(s)y_n^* = \Phi^*(s)y^*$ weakly in H for all $s \in (0,T]$, and

$$\begin{aligned} \lim_{n\to\infty} \langle \Phi(s)\Phi^*(s)x^*, y_n^* \rangle &= \lim_{n\to\infty} [\Phi^*(s)x^*, \Phi^*(s)y_n^*]_H \\ &= [\Phi^*(s)x^*, \Phi^*(s)y^*]_H \\ &= \langle \Phi(s)\Phi^*(s)x^*, y^* \rangle. \end{aligned} \tag{2.2}$$

The boundedness of (y_n^*) in E^* implies that the sequence of functions $(\Phi^*(\cdot)y_n^*)$ is bounded in $L^2((0,t];H)$. Since $L^2((0,t];H)$ is reflexive, upon passing to a subsequence we may assume that $(\Phi^*(\cdot)y_n^*)$ is weakly convergent in $L^2((0,t];H)$ to some limit function f. As $\Phi^*(\cdot)x^* \in L^2((0,t];H)$, we then have

$$\lim_{n\to\infty} \int_0^t \langle \Phi(s)\Phi^*(s)x^*, y_n^* \rangle \, ds = \int_0^t [\Phi^*(s)x^*, f(s)]_H \, ds. \tag{2.3}$$

The weak convergence $\Phi^*(\cdot)y_n^* \to f$ implies further that there exist convex combinations

$$z_n^* = \sum_{k=n}^{K_n} \lambda_{k,n} y_k^*$$

such that $\Phi^*(\cdot)z_n^* \to f$ *strongly* in $L^2((0,t],H)$. Passing, if necessary, to a further subsequence of (z_n^*), we even have $\Phi^*(\sigma)z_n^* \to f(\sigma)$ strongly in H for almost all $\sigma \in (0,t]$. For any σ with this property,

$$\lim_{n\to\infty} \langle \Phi(\sigma)\Phi^*(\sigma)x^*, z_n^*\rangle = [\Phi^*(\sigma)x^*, f(\sigma)]_H. \tag{2.4}$$

On the other hand, because we take z_n^* in the convex hull of $\{y_k^* : \ k \ge n\}$, by (2.2) we have

$$\lim_{n\to\infty} \langle \Phi(s)\Phi^*(s)x^*, z_n^*\rangle = \langle \Phi(s)\Phi^*(s)x^*, y^*\rangle$$

for all $s \in (0,t]$. From this and (2.4) it follows that

$$[\Phi^*(\sigma)x^*, f(\sigma)]_H = \langle \Phi(\sigma)\Phi^*(\sigma)x^*, y^*\rangle$$

for almost all $\sigma \in (0,t]$. Combining this with (2.3) we obtain

$$\begin{aligned}
\lim_{n\to\infty} \langle y_n^*, Q_t x^*\rangle &= \lim_{n\to\infty} \int_0^t \langle \Phi(\sigma)\Phi^*(\sigma)x^*, y_n^*\rangle \, d\sigma \\
&= \int_0^t [\Phi^*(\sigma)x^*, f(\sigma)]_H \, d\sigma \\
&= \int_0^t \langle \Phi(\sigma)\Phi^*(\sigma)x^*, y^*\rangle \, d\sigma \\
&= \langle y^*, Q_t x^*\rangle.
\end{aligned}$$

But this contradicts (2.1). ∎

This result shows that for all $t \in (0,T]$ we have a well defined bounded linear operator $Q_t \in \mathcal{L}(E^*, E)$, which can be represented as a Pettis integral by

$$Q_t x^* = \int_0^t \Phi(s)\Phi^*(s)x^* \, ds \qquad (x^* \in E^*).$$

Clearly Q_t is positive and symmetric; by (i_t, H_t) we denote its RKHS (cf. Section 1 for the definition). If $0 < s \le t \le T$, then for all $x^* \in E^*$ we have $\|Q_s x^*\|_{H_s} \le \|Q_t x^*\|_{H_t}$, which implies that there is a natural inclusion $H_s \hookrightarrow H_t$ (cf. [Ne1], where it is shown that in this inclusion is in fact a contraction).

Just as in Lemma 2.1 one proves:

Lemma 2.3. *For all $t \in (0,T]$ and $x^* \in E^*$ the function $s \mapsto \Phi(t-s)\Phi^*(s)x^*$ is a strongly measurable $E-$valued function on $(0,t]$.*

For each $t \in (0,T]$ we let $\mathcal{H}_t = \mathcal{H}_t^\Phi$ denote the closure in $L^2((0,T];H)$ of its linear subspace $\{\chi_{(0,t]}\Phi^*(\cdot)x^* : \ x^* \in E^*\}$.

Lemma 2.4. *For each $t \in (0,T]$ there exists a unique bounded linear operator $I_{\Phi,t} : \mathcal{H}_t \to H_t$ which satisfies*

$$[I_{\Phi,t}(\chi_{(0,t]}\Phi^*(\cdot)x^*), i_t^* y^*]_{H_t} = \int_0^t \langle \Phi(t-s)\Phi^*(s)x^*, y^*\rangle\, ds, \qquad \forall x^*, y^* \in E^*.$$

Proof : By the Cauchy-Schwartz inequality and the identity

$$\|i_t^* y^*\|_{H_t} = \|\chi_{(0,t]}\Phi^*(\cdot)y^*\|_{L^2((0,t];H)}$$

we have

$$\begin{aligned}\Big|\int_0^t \langle \Phi(t-s)\Phi^*(s)x^*, y^*\rangle\, ds\Big| &= \Big|\int_0^t [\Phi^*(s)x^*, \Phi^*(t-s)y^*]_H\, ds\Big| \\ &\le \|\chi_{(0,t]}\Phi^*(\cdot)x^*\|_{L^2((0,T];H)} \cdot \|\chi_{(0,t]}\Phi^*(\cdot)y^*\|_{L^2((0,T];H)} \\ &= \|\chi_{(0,t]}\Phi^*(\cdot)x^*\|_{\mathcal{H}_t} \cdot \|i_t^* y^*\|_{H_t}.\end{aligned}$$

It follows that the map

$$i_t^* y^* \mapsto \int_0^t \langle \Phi(t-s)\Phi^*(s)x^*, y^*\rangle\, ds$$

defines a bounded linear functional on H_t of norm $\le \|\chi_{(0,t]}(\cdot)\Phi^*(\cdot)x^*\|_{\mathcal{H}_t}$. By the Riesz representation theorem, this functional can be identified with an element of H_t; we will denote it by $I_{\Phi,t}(\chi_{(0,t]}\Phi^*(\cdot)x^*)$. In this way we obtain a bounded linear operator $I_{\Phi,t}$ of norm ≤ 1 from the linear span of $\{\chi_{(0,t]}\Phi^*(\cdot)x^* : x^* \in E^*\}$ into H_t. Since this span is dense in $\mathcal{H}_t$, this proves the result. ∎

From the identity

$$\begin{aligned}\langle (i_t \circ I_{\Phi,t})(\chi_{(0,t]}\Phi^*(\cdot)x^*), y^*\rangle &= [I_{\Phi,t}(\chi_{(0,t]}\Phi^*(\cdot)x^*), i_t^* y^*]_{H_t} \\ &= \int_0^t \langle \Phi(t-s)\left(\chi_{(0,t]}\Phi^*(\cdot)x^*\right)(s), y^*\rangle\, ds\end{aligned}$$

and a continuity argument we see that $i_t \circ I_{\Phi,t}$ can be represented as a Pettis integral by

$$(i_t \circ I_{\Phi,t})g = \int_0^t \Phi(t-s)g(s)\, ds \qquad (g \in \mathcal{H}_t).$$

Noting that $f \mapsto \chi_{(0,t]}f$ defines a contraction from $\mathcal{H}_T$ onto $\mathcal{H}_t$, we can define a continuous linear operator $I_\Phi : \mathcal{H}_T \to E^{[0,T]}$ by

$$(I_\Phi f)(t) := \begin{cases} 0, & t = 0 \\ (i_t \circ I_{\Phi,t})(\chi_{(0,t]}f), & t \in (0,T] \end{cases} \qquad (f \in \mathcal{H}_T).$$

Theorem 2.5. *If the embedding $i_T : H_T \hookrightarrow E$ is $\gamma-$radonifying, then the operator $I_\Phi : \mathcal{H}_T \to E^{[0,T]}$ is $\gamma-$radonifying.*

Proof : We noted earlier that for each $0 < t \le T$ there is a natural inclusion $i_{t,T} : H_t \hookrightarrow H_T$. Composing this with the inclusion $i_T : H_T \hookrightarrow E$ we obtain a factorization $i_t = i_T \circ i_{t,T}$. Since i_T is $\gamma-$radonifying by assumption, it follows that each of the inclusions i_t is $\gamma-$radonifying.

Let $\nu = \nu_\Phi := I_\Phi(\gamma_{\mathcal{H}_T})$ denote the image cylindrical measure on $E^{[0,T]}$ under I_Φ of the standard cylindrical Gaussian measure $\gamma_{\mathcal{H}_T}$ of $\mathcal{H}_T$. Let $\delta_t : E^{[0,T]} \to E$ denote the point evaluation at t, and let $\nu_t := \delta_t(\nu)$ be the corresponding image cylindrical measure on E. By Proposition 1.1 the covariance operator $R_t \in \mathcal{L}(E^*, E)$ of ν_t is given by $R_t = \delta_t \circ I_\Phi \circ I'_\Phi \circ \delta'_t$. For $y^* \in E^*$ and $f = \Phi^*(\cdot)x^* \in \mathcal{H}_T$ we have

$$[(I'_\Phi \circ \delta'_t)y^*, f]_{\mathcal{H}_T} = \langle (\delta_t \circ I_\Phi)f, y^* \rangle = \int_0^t \langle \Phi(t-s)\Phi^*(s)x^*, y^* \rangle \, ds = [\chi_{(0,t]}\Phi^*(t-\cdot)y^*, f]_{\mathcal{H}_T}.$$

Therefore,

$$(I'_\Phi \circ \delta'_t)y^* = \chi_{(0,t]}\Phi^*(t-\cdot)y^* \tag{2.5}$$

and for all $x^*, y^* \in E^*$ we obtain

$$\begin{aligned}
\langle R_t x^*, y^* \rangle &= [(I'_\Phi \circ \delta'_t)x^*, (I'_\Phi \circ \delta'_t)y^*]_{\mathcal{H}_T} \\
&= [\chi_{(0,t]}\Phi^*(t-\cdot)x^*, \chi_{(0,t]}\Phi^*(t-\cdot)y^*]_{\mathcal{H}_T} \\
&= \int_0^t \langle \Phi(s)\Phi^*(s)x^*, y^* \rangle \, ds \\
&= \langle Q_t x^*, y^* \rangle.
\end{aligned}$$

But Q_t is the covariance operator of the Gaussian Borel measure $\mu_t := i_t(\gamma_{H_t})$, and it thus follows that $\nu_t = \mu_t$ as cylindrical measures on E. We conclude that ν_t extends to a centred Gaussian Borel measure on E.

Now suppose $0 \le t_1 < ... < t_n \le T$ are fixed and consider the canonical projection $\delta_{\{t_1,...,t_n\}} : E^{[0,T]} \to E^n$, $f \mapsto (f(t_1), ..., f(t_n))$. Let $\nu_{\{t_1,...,t_n\}} := \delta_{\{t_1,...,t_n\}}(\nu)$. By a result of Dudley, Feldman and Le Cam [DFL, Lemma 5], the fact that each ν_{t_k} extends to a centred Gaussian Borel measure on E implies that $\nu_{\{t_1,...,t_n\}}$ extends to a centred Gaussian Borel measure on E^n. By the Kolmogorov consistency theorem the projective limit of these measures exists and defines a probability measure $\tilde{\nu}$ on the product $\sigma-$algebra $\mathcal{B}(E^{[0,T]})$ of $E^{[0,T]}$. But since this measure is completely determined by its finite marginals $\nu_{\{t_1,...,t_n\}}$ it follows that $\tilde{\nu} = \nu$. This proves that ν extends to a Gaussian measure on $(E^{[0,T]}, \mathcal{B}(E^{[0,T]}))$. ∎

Suppose the embedding $i_T : H_T \hookrightarrow E$ is $\gamma-$radonifying and let $\nu_\Phi := I_\Phi(\gamma_{\mathcal{H}_T})$. By Theorem 2.5, this is a Gaussian measure on $(E^{[0,T]}, \mathcal{B}(E^{[0,T]}))$. On the resulting probability space $(\Omega, \mathcal{F}, \mathbb{P}) = (E^{[0,T]}, \mathcal{B}(E^{[0,T]}), \nu_\Phi)$ we consider the canonical process $\xi = \{\xi_t\}_{t\in[0,T]}$ defined by point evaluation:

$$\xi_t(\omega) := \omega(t), \qquad t \in [0,T].$$

Theorem 2.6. *Suppose the embedding $i_T : H_T \hookrightarrow E$ is $\gamma-$radonifying. The canonical process $\{\xi_t\}_{t\in[0,T]}$ is an $E-$valued Gaussian process with covariance*

$$\mathbb{E}\left(\langle \xi_t, x^*\rangle\langle \xi_s, y^*\rangle\right) = \int_0^{t\wedge s} [\Phi^*(t-u)x^*, \Phi^*(s-u)y^*]_H\, du.$$

Proof : We compute, using (2.5) and Proposition 1.1,

$$\begin{aligned}\mathbb{E}\left(\langle \xi_t, x^*\rangle\langle \xi_s, y^*\rangle\right) &= [I'_\Phi(x^*\otimes\delta_t), I'_\Phi(y^*\otimes\delta_s)]_{\mathcal{H}_T} \\ &= [(I'_\Phi\circ\delta'_t)x^*, (I'_\Phi\circ\delta'_s)y^*]_{\mathcal{H}_T} \\ &= \int_0^{t\wedge s} [\Phi^*(t-u)x^*, \Phi^*(s-u)y^*]_H\, du.\end{aligned}$$

■

Definition 2.7. An $E-$valued stochastic process $\{X_t\}_{t\in[0,T]}$ will be called an *Ornstein-Uhlenbeck process associated with the operator-valued function* $\Phi \in L^2((0,T];H,E)$ if it is centred Gaussian with covariance given by

$$\mathbb{E}\left(\langle X_t, x^*\rangle\langle X_s, y^*\rangle\right) = \int_0^{t\wedge s} [\Phi^*(t-u)x^*, \Phi^*(s-u)y^*]_H\, du.$$

The canonical process $\{\xi_t\}_{t\in[0,T]}$ of Theorem 2.6 will be called the *canonical Ornstein-Uhlenbeck process associated with* Φ.

We close this section with the following converse of Theorem 2.6:

Proposition 2.8. *Suppose $\{X_t\}_{t\in[0,T]}$ is an Ornstein-Uhlenbeck process with respect to a function $\Phi \in L^2((0,T];H,E)$. Then the inclusion mapping $i_T : H_T \hookrightarrow E$ is $\gamma-$radonifying.*

Proof : Let μ_T denote the distribution of the $E-$valued random variable X_T. Then μ_T is a centred Gaussian Borel measure on E, whose covariance operator $R_T \in \mathcal{L}(E^*, E)$ satisfies

$$\langle R_T x^*, x^*\rangle = \mathbb{E}\left(\langle X_T, x^*\rangle^2\right) = \int_0^T [\Phi^*(T-u)x^*, \Phi^*(T-u)x^*]_H\, du = \langle Q_T x^*, x^*\rangle.$$

This implies that $Q_T = R_T$, from which we infer that Q_T is the covariance operator of μ_T. On the other hand, $Q_T = i_T \circ i_T^*$ is the covariance operator of image cylindical measure $i_T(\gamma_{H_T})$. Since a cylindical measure is uniquely determined by its covariance operator, it follows that $i_T(\gamma_{H_T}) = \mu_T$ as cylindical measures. This implies that $i_T(\gamma_{H_T})$ has a $\sigma-$additive extension to a Borel measure on E, and thus i_T is $\gamma-$radonifying. ■

3. Stochastic convolution

As before we let E be a separable real Banach space and H a separable real Hilbert space.

In this section we shall investigate under which conditions it is possible to define a stochastic convolution of an operator-valued function $\Phi : (0,T] \to \mathcal{L}(H,E)$ with respect to a cylindrical Wiener process $\{W_t^H\}_{t\in[0,T]}$ with Cameron-Martin space H. We start with a definition.

Definition 3.1. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space. A *cylindrical Wiener process with Cameron-Martin space* H is a family $\{W_t^H\}_{t\in[0,T]}$ of bounded linear operators from H into $L^2(\mathbb{P})$ with the following properties:

(i) For all $h \in H$, $\{W_t^H h\}_{t\in[0,T]}$ is a real-valued Brownian motion adapted to $\{\mathcal{F}_t\}_{t\in[0,T]}$;
(ii) For all $t, s \in [0,T]$ and $h, g \in H$ we have

$$\mathbb{E}\,(W_t^H h \cdot W_s^H g) = (t \wedge s)[h,g]_H.$$

Instead of $W_t^H h$ we will usually write $[W_t^H, h]$.

Consider an operator-valued function $\Phi \in L^2((0,T];H,E)$ (we recall that this space has been defined at the beginning of Section 2) and let $\{W_t^H\}_{t\in[0,T]}$ be a cylindrical Wiener process with Cameron-Martin space H. We briefly outline how to define, for all $x^* \in E^*$, a stochastic Itô type integral

$$\int_0^T \langle \Phi(s)\, dW_s^H, x^* \rangle.$$

If $\Phi(s) = \chi_{(t_0,t_1]}(s) U$ for some fixed $U \in \mathcal{L}(H,E)$, we put

$$\int_0^T \langle \Phi(s)\, dW_s^H, x^* \rangle := [W_{t_1}^H, U^* x^*] - [W_{t_0}^H, U^* x^*].$$

Extending this definition by linearity, we obtain a stochastic integral for $\mathcal{L}(H,E)-$valued step functions. For such a step function Φ it is straightforward to verify that

$$\mathbb{E}\left\{\left(\int_0^T \langle \Phi(s)\, dW_s^H, x^* \rangle\right)^2\right\} = \|\Phi^*(\cdot)x^*\|^2_{L^2((0,T];H)}. \tag{3.1}$$

The construction is completed by the following observation:

Lemma 3.2. *For each $\Phi \in L^2((0,T];H,E)$ and $x^* \in E^*$ there exists a sequence of step functions (Φ_n) in $L^2((0,T];H,E)$ such that*

$$\lim_{n\to\infty} \|\Phi^*(\cdot)x^* - \Phi_n^*(\cdot)x^*\|_{L^2((0,T];H)} = 0.$$

Proof : Let H' be the closed linear subspace in H generated by the set $\{\Phi^*(t)x^* : t \in (0,T]\}$. Choose a sequence (ϕ_n) in $L^2((0,T];H')$ consisting of step functions of the form

$$\phi_n(\cdot) = \sum_{j=1}^{N_n} \chi_{(t_{n,j},t_{n,j+1}]}(\cdot) \otimes h'_{n,j}$$

such that $\lim_{n\to\infty} \phi_n(\cdot) = \Phi^*(\cdot)x^*$ almost surely and in $L^2((0,T];H')$. There is no loss in generality to assume that each $h'_{n,j}$ is in the linear span of $\{\Phi^*(t)x^* : t \in (0,T]\}$, say $h'_{n,j} = \sum_{k=1}^{N_{n,j}} \Phi^*(t_{n,j,k})x^*$. Defining $U_{n,j} := \sum_{k=1}^{N_{n,j}} \Phi(t_{n,j,k})$, and

$$\Phi_n(\cdot) = \sum_{j=1}^{N_n} \chi_{(t_{n,j},t_{n,j+1}]}(\cdot) \otimes U_{n,j},$$

we have $\Phi_n^*(\cdot)x^* = \phi_n(\cdot)$ and the lemma follows. ∎

For all $t \in (0,T]$ and $\Phi \in L^2((0,T];H,E)$ we have $\chi_{(0,t]}\Phi \in L^2((0,T];H,E)$. This allows us to define

$$\int_0^t \langle \Phi(s)\, dW_s^H, x^* \rangle := \int_0^T \langle \chi_{(0,t]}(s)\Phi(s)\, dW_s^H, x^* \rangle.$$

For the rest of this section we fix $\Phi \in L^2((0,T];H,E)$ and a cylindrical Wiener process $\{W_t^H\}_{t\in[0,T]}$ with Cameron-Martin space H. As before we let

$$Q_T x^* = \int_0^T \Phi(s)\Phi^*(s)x^*\, ds$$

and denote by H_T the RKHS associated with Q_T; for the natural embedding map $i_T : H_T \hookrightarrow E$ we then have $Q_T = i_T \circ i_T^*$.

Theorem 3.3. *If the inclusion $i_T : H_T \hookrightarrow E$ is $\gamma-$radonifying, then there exists a predictable $E-$valued process $\{X_t\}_{t\in[0,T]}$, adapted to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$, such that for all $x^* \in E^*$ and $t \in [0,T]$ we have*

$$\langle X_t, x^* \rangle = \int_0^t \langle \Phi(t-s)\, dW_s^H, x^* \rangle \qquad \text{a.s.} \tag{3.2}$$

Up to a modification this process is unique. For all $x^*, y^* \in E^*$ and $0 \le s,t \le T$ we have

$$\mathbb{E}\left(\langle X_t, x^* \rangle \langle X_s, y^* \rangle\right) = \int_0^{t\wedge s} [\Phi^*(t-u)x^*, \Phi^*(s-u)y^*]_H\, du, \tag{3.3}$$

i.e., the process $\{X_t\}_{t\in[0,T]}$ is an Ornstein-Uhlenbeck process associated with Φ.

Proof : Uniquess up to a modification is obvious from the Hahn-Banach theorem and the separability of E.

Let $j : E \hookrightarrow \tilde{E}$ be a continuous dense embedding of E into a separable real Hilbert space $\tilde{E}$. As is well known, such a pair $(j, \tilde{E})$ always exists (for instance, let (x_n^*) be a weak*-dense sequence in the dual unit ball B_{E^*}, let (λ_n) be a summable sequence of strictly positive real numbers and define $\tilde{E}$ to be the completion of E with respect to the inner product $[x, y]_{\tilde{E}} := \sum_{n=1}^{\infty} \lambda_n \langle x, x_n^* \rangle \langle y, x_n^* \rangle$; cf. [Kuo, p. 154]).

For $t \in (0, T]$ define $\tilde{\Phi}(t) \in \mathcal{L}(H, \tilde{E})$ by

$$\tilde{\Phi}(t) := j \circ \Phi(t).$$

It is immediate that $\tilde{\Phi} \in L^2((0, T]; H, \tilde{E})$. For $t \in (0, T]$ let $\tilde{Q}_t \in \mathcal{L}(\tilde{E}^*, \tilde{E})$ be defined by

$$\tilde{Q}_t \tilde{x}^* := \int_0^t \tilde{\Phi}(s) \tilde{\Phi}^*(s) \tilde{x}^* \, ds.$$

We have $\tilde{Q}_t = j \circ Q_t \circ j^*$. Let $(\tilde{i}_T, \tilde{H}_T)$ denote the RKHS associated with $\tilde{Q}_T$. The map $k_T : \tilde{Q}_T \tilde{x}^* \mapsto Q_T x^*$ extends to an isometry from $\tilde{H}_T$ onto H_T, and we have $\tilde{i}_T = j \circ i_T \circ k_T$. It follows that $\tilde{i}_T$ is $\gamma-$radonifying (cf. Section 1).

The space $\tilde{E}$ being a Hilbert space, we may define an $\tilde{E}-$valued process $\{\tilde{X}_t\}_{t \in [0,T]}$ by the Hilbert space-valued stochastic Itô convolution integral

$$\tilde{X}_t = \int_0^t \tilde{\Phi}(t - s) \, dW_s^H$$

(cf. [DZ, Chapter 4]); this process is predictable and adapted to the filtration $\{\mathcal{F}_t\}_{t \in [0,T]}$.

We denote by $\tilde{\mu}_t$ the distribution of the $\tilde{E}-$valued random variable $\tilde{X}_t$. This is a centred Gaussian Borel measure on $\tilde{E}$. By the theory of stochastic convolutions in Hilbert spaces, $\{\tilde{X}_t\}_{t \in [0,T]}$ is an Ornstein-Uhlenbeck process associated with the function $\tilde{\Phi}$; in particular the covariance operator of $\tilde{\mu}_t$ equals $\tilde{Q}_t$.

By a theorem of Kuratowski [VTC, Chapter 1], jE is a Borel subset of $\tilde{E}$. We are going to show that $\tilde{\mu}_t(jE) = 1$.

Since by assumption the inclusion map $i_T : H_T \hookrightarrow E$ is $\gamma-$radonifying, the remark preceding Lemma 2.3 and the results mentioned in Section 1 show that for each $t \in (0, T]$ the inclusion map $i_t : H_t \hookrightarrow E$ is $\gamma-$radonifying as well. Let $\nu_t := i_t(\gamma_{H_t})$ and let $\tilde{\nu}_t := j(\nu_t)$; these are centred Gaussian Borel measures on E and $\tilde{E}$, respectively. The covariance operator $\tilde{R}_t$ of $\tilde{\nu}_t$ is given by

$$\langle \tilde{R}_t \tilde{x}^*, \tilde{x}^* \rangle = \int_0^t [\tilde{\Phi}^*(s) \tilde{x}^*, \tilde{\Phi}^*(s) \tilde{x}^*]_H \, ds = \langle \tilde{Q}_t \tilde{x}^*, \tilde{x}^* \rangle.$$

It follows that $\tilde{R}_t = \tilde{Q}_t$. Since a centred Gaussian Borel measure is completely determined by its covariance, we conclude that $\tilde{\nu}_t = \tilde{\mu}_t$. But from $\tilde{\nu}_t = j(\nu_t)$ it follows that $\tilde{\nu}_t(jE) = \nu_t(E) = 1$. This proves that $\tilde{\mu}_t(jE) = 1$.

As a consequence we have $\tilde{X}_t \in jE$ almost surely. This allows us to define an $\mathcal{F}_t$-measurable E-valued random variable X_t by insisting that $jX_t = \tilde{X}_t$. The resulting adapted process $\{X_t\}_{t\in[0,T]}$ is predictable.

The distribution μ_t of X_t is a probability Borel measure on E which satisfies $j(\mu_t) = \tilde{\mu}_t$. For all $x^* \in E^*$ of the form $x^* = j^*\tilde{x}^*$ for some $\tilde{x}^* \in \tilde{E}^*$ we have $\langle \mu_t, x^*\rangle = \langle \tilde{\mu}_t, \tilde{x}^*\rangle$, the right hand side being a centred Gaussian Borel measure on $\mathbb{R}$. Because the subspace $j^*\tilde{E}^*$ is weak*-dense in E^*, the measure μ_t is centred Gaussian by Corollary 1.3.

Next we prove (3.2). First note that for all $x^* = j^*\tilde{x}^*$ with $\tilde{x}^* \in \tilde{E}^*$ we have

$$\langle X_t, x^*\rangle = \langle \tilde{X}_t, \tilde{x}^*\rangle = \int_0^t \langle \tilde{\Phi}(t-s)\, dW_s^H, \tilde{x}^*\rangle = \int_0^t \langle \Phi(t-s)\, dW_s^H, x^*\rangle.$$

Therefore the subspace Y consisting of all $x^* \in E^*$ for which (3.2) holds is weak*-dense. In order to prove that $Y = E^*$, by Proposition 1.2 it suffices to check that Y is weak*-sequentially closed.

Let (x_n^*) be a sequence in Y converging to some $x^* \in E^*$ in the weak*-topology. We will show that $x^* \in Y$.

First we note that for all $t \in (0,T]$ we have $\lim_{n\to\infty} \Phi^*(t)x_n^* = \Phi^*(t)x^*$ weakly in H. The sequence (x_n^*) being bounded, the sequence $(\Phi^*(\cdot)x_n^*)$ is bounded in $L^2((0,T];H)$. Upon passing to a weakly convergent subsequence we may assume that $\lim_{n\to\infty} \Phi^*(\cdot)x_n^* = f$ weakly for some $f \in L^2((0,T];H)$. By a convex combination argument as in the proof of Proposition 2.2, we find a sequence (y_n^*) in Y such that $\lim_{n\to\infty} y_n^* = x^*$ weak* and $\lim_{n\to\infty} \Phi^*(\cdot)y_n^* = f$ strongly in $L^2((0,T];H)$. Upon passing to a pointwise a.e. convergent subsequence we conclude that

$$f = \lim_{n\to\infty} \Phi^*(\cdot)y_n^* = \Phi^*(\cdot)x^* \quad \text{a.e.}$$

Next we note that

$$\langle X_t, x^*\rangle = \lim_{n\to\infty} \langle X_t, y_n^*\rangle = \lim_{n\to\infty} \int_0^t \langle \Phi(t-s)\, dW_s^H, y_n^*\rangle \quad \text{a.e.} \tag{3.4}$$

But by (3.1), which in view of Lemma 3.2 extends to arbitrary $\Phi \in L^2((0,T];H,E)$,

$$\lim_{n\to\infty} \mathbb{E}\left(\int_0^t \langle \Phi(t-s)\, dW_s^H, y_n^* - x^*\rangle\right)^2 = \lim_{n\to\infty} \|\Phi^*(\cdot)(y_n^* - x^*)\|^2_{L^2((0,T];H)} = 0.$$

Therefore,

$$\lim_{n\to\infty} \int_0^t \langle \Phi(t-s)\, dW_s^H, y_n^*\rangle = \int_0^t \langle \Phi(t-s)\, dW_s^H, x^*\rangle \quad \text{in} \quad L^2(\mathbb{P}). \tag{3.5}$$

Upon passing once more to pointwise a.e. convergent subsequence if necessary, we conclude that (3.2) follows from (3.4) and (3.5).

It remains to show that (3.3) holds, i.e. that $\{X_t\}_{t\in[0,T]}$ is an Ornstein-Uhlenbeck process associated with Φ. Let $j^*\tilde{y}^* \in j^*\tilde{E}^*$ be fixed and let Y denote the set of all $x^* \in E^*$ such that

$$\mathbb{E}\left(\langle X_t, x^*\rangle\langle X_s, j^*\tilde{y}^*\rangle\right) = \int_0^{t\wedge s} [\Phi^*(t-u)x^*, \Phi^*(s-u)j^*\tilde{y}^*]_H \, du \tag{3.6}$$

holds for all $t, s \in [0,T]$. Since $\{\tilde{X}_t\}_{t\in[0,T]}$ is an Ornstein-Uhlenbeck process with values in $\tilde{E}$ we have $j^*\tilde{E}^* \subset Y$ and therefore Y is a weak*-dense linear subspace of E^*. By the dominated convergence theorem it is also weak*-sequentially closed. Hence by Proposition 1.2, $Y = E^*$.

Let Z denote the set of all $y^* \in E^*$ such that (3.3) holds for all $x^* \in E^*$ and all $t, s \in [0,T]$. By what we already know, $j^*\tilde{E}^* \subset Z$ and therefore Z is a weak*-dense linear subspace of E^*. Once more the dominated convergence theorem shows that Z is also weak*-sequentially closed, and therefore $Z = E^*$. This proves the $\{X_t\}_{t\geq 0}$ is an Ornstein-Uhlenbeck process with covariance given by (3.3).

∎

Remark. By the Kolmogorov scheme, to the process $\{X_t\}_{t\in[0,T]}$ one can associate a *canonical* process on the probability space $(E^{[0,T]}, \nu)$, where ν is the measure obtained as the projective limit of the finite-dimensional distributions of $\{X_t\}_{t\in[0,T]}$. In this way we just obtain the canonical process $\{\xi_t\}_{t\in[0,T]}$ of Section 2.

Definition 3.4. The predictable $E-$valued process $\{X_t\}_{t\in[0,T]}$ constructed in Theorem 3.3 will be called the *stochastic convolution of* Φ *with respect to* $\{W_t^H\}_{t\in[0,T]}$; notation:

$$X_t = \int_0^t \Phi(t-s)\, dW_s^H.$$

4. Path regularity

In this section we discuss path regularity of the stochastic convolution process

$$X_t = \int_0^t \Phi(t-s)\, dW_s^H$$

under the assumptions that $\{W_t^H\}_{t\in[0,T]}$ is a cylindrical Wiener process with Cameron-Martin space H and $\Phi \in L^2((0,T];H,E)$ is such that the embedding $i_T : H_T \hookrightarrow E$ is $\gamma-$radonifying.

We begin with some preparations. As before μ_t denotes the distribution of X_t; this is the centred Gaussian Borel measure on E whose covarariance operator is Q_t. The following inequality is a direct consequence of [Nh, Lemma 28] and the observation that $\|Q_t x^*\|_{H_t} \leq \|Q_T x^*\|_{H_T}$ whenever $0 < t \leq T$:

Proposition 4.1. *If $0 < t \le T$, then*

$$\int_E \|x\|^2 \, d\mu_t(x) \le \int_E \|x\|^2 \, d\mu_T(x).$$

Proposition 4.2. *The process $\{X_t\}_{t\in[0,T]}$ has a strongly measurable modification such that for almost all $\omega \in \Omega$,*

$$\int_0^T \|X_t(\omega)\|^2 \, dt < \infty.$$

Proof : The process $\{X_t\}_{t\in[0,T]}$ has a predictable, and therefore a progressively measurable, modification. Hence by Fubini's theorem, the paths of this modification are strongly measurable almost surely, and by Proposition 4.1 we have for almost all $\omega \in \Omega$,

$$\begin{aligned}
\int_\Omega \int_0^T \|X_t(\omega)\|^2 \, dt \, d\mathbb{P}(\omega) &= \int_0^T \int_\Omega \|X_t(\omega)\|^2 \, d\mathbb{P}(\omega) \, dt \\
&= \int_0^T \int_E \|x\|^2 \, d\mu_t(x) \, dt \\
&\le T \int_E \|x\|^2 \, d\mu_T(x) < \infty.
\end{aligned}$$

This shows that the non-negative extended-real valued function $\omega \mapsto \int_0^T \|X_t(\omega)\|^2 \, dt$ is integrable, and therefore almost surely finitely-valued. ∎

It is well-known that if E is a Hilbert space, the stochastic convolution processes $\{X_t\}_{t\ge0}$ is mean square continuous (cf. [DZ, Theorem 5.2]). In the Banach space case, $\{X_t\}_{t\in[0,T]}$ is mean square continuous as well; for the proof we refer to [BGN].

We shall now give a sufficient condition for the existence of a continuous version for $\{X_t\}_{t\in[0,T]}$.

Proposition 4.3. *Assume there exist $\theta \in (0,1]$ and $L \ge 0$ such that for all $0 \le s < t \le T$ and $x^* \in E^*$ we have:*

(i) $\displaystyle\int_0^{t-s} \|\Phi^*(u)x^*\|_H^2 \, du \le L(t-s)^\theta \|x^*\|^2$;

(ii) $\displaystyle\int_0^{s} \|\Phi^*(t-s+u)x^* - \Phi^*(u)x^*\|_H^2 \, du \le L(t-s)^\theta \|x^*\|^2$.

Then the process $\{X_t\}_{t\in[0,T]}$ has a continuous modification.

Proof : Let $x^* \in E^*$ and $r > 0$ be arbitrary and fixed. Let $\{\xi_t\}_{t\in[0,T]}$ denote the canonical Ornstein-Uhlenbeck process associated with Φ. Observing that $X_t - X_s$ and $\xi_t - \xi_s$ have the same distribution, we have

$$\begin{aligned}
\mathbb{E}\,|\langle X_t - X_s, x^*\rangle|^r &= \int_\Omega |\langle X_t(\omega) - X_s(\omega), x^*\rangle|^r \, d\mathbb{P}(\omega) \\
&= \int_\mathbb{R} |\tau|^r \, d\langle (x^* \otimes (\delta_t - \delta_s)), \nu\rangle(\tau) \\
&= \frac{1}{\sqrt{2\pi}} \int_\mathbb{R} \|I'_\Phi(x^* \otimes (\delta_t - \delta_s))\|^r_{\mathcal{H}_T} \, |\tau|^r \, \exp\left(-\tau^2/2\right) \, d\tau.
\end{aligned}$$

Recalling that $I'_\Phi(x^* \otimes \delta_t) = \chi_{(0,t]}\, \Phi^*(t - \ \cdot)x^*$, for $t \geq s$ we have

$$\begin{aligned}
&\|I'_\Phi(x^* \otimes (\delta_t - \delta_s))\|^2_{\mathcal{H}_T} \\
&\qquad = \int_0^s \|\Phi^*(t-u)x^* - \Phi^*(s-u)x^*\|_H^2\, du + \int_s^t \|\Phi^*(t-u)x^*\|_H^2\, du.
\end{aligned}$$

Hence by (i) and (ii),

$$\|I'_\Phi(x^* \otimes (\delta_t - \delta_s))\|^2_{\mathcal{H}_T} \leq 2L|t-s|^\theta \|x^*\|^2$$

for all $t, s \in [0, T]$. It follows that

$$\mathbb{E}\,|\langle X_t - X_s, x^*\rangle|^r \leq M|t-s|^{r\theta/2}\|x^*\|^r \tag{4.1}$$

for some $M \geq 0$ and all $t, s \in [0, T]$. In particular,

$$\mathbb{E}\,|\langle X_t - X_s, x^*\rangle|^2 \leq M|t-s|^\theta$$

for all $t, s \in [0, T]$ and $x^* \in E^*$ with $\|x^*\| \leq 1$.

To finish the proof we proceed as in [MS, Proposition 3.1] and check that the assumptions of [Ca, Proposition 5] are satisfied. The existence of a continuous modification then follows. For the convenience of the reader we give the details.

First we consider the Gaussian process $\mathbf{X}_T = \{\langle X_T, x^*\rangle\}_{x^* \in U}$ indexed by the closed unit ball U of E^*. This process has weak*-continuous paths. Putting

$$\Gamma(t, s; x^*, y^*) := \int_0^{t \wedge s} [\Phi^*(t-u)x^*, \Phi^*(s-u)y^*]_H\, du,$$

for $0 \leq t \leq T$ we have

$$\begin{aligned}
\Gamma(t, t; x^* - y^*, x^* - y^*) &= \int_0^t \|\Phi^*(v)(x^* - y^*)\|_H^2\, dv \\
&\leq \int_0^T \|\Phi^*(v)(x^* - y^*)\|_H^2\, dv \\
&= \langle Q_T(x^* - y^*), (x^* - y^*)\rangle \\
&= \mathbb{E}\,|\langle X_T, x^* - y^*\rangle|^2.
\end{aligned}$$

This verifies the first condition of [Ca, Proposition 5].

Next, noting that the function $(t, s) \mapsto \frac{M}{2}(t^\theta + s^\theta - |t-s|^\theta)$ is symmetric and positive definite, there exists a centred real-valued Gaussian process $\{Y_t\}_{t \in [0,T]}$ with

$$\mathbb{E}\,(Y_t Y_s) = \frac{M}{2}\big(t^\theta + s^\theta - |t-s|^\theta\big).$$

Then,

$$\mathbb{E}\,|Y_t - Y_s|^2 = M|t-s|^\theta.$$

This process being Gaussian, we have

$$\mathbb{E}\,|Y_t - Y_s|^{2p} = C_p|t-s|^{\theta p}$$

and by taking p large enough we see that it has a continuous modification. By the computations above, for $0 \le s \le t \le T$ and $x^* \in U$ we have

$$\begin{aligned}\Gamma(t,t;x^*,x^*) - 2\Gamma(t,s;x^*,x^*) &+ \Gamma(s,s;x^*,x^*) \\ &= \mathbb{E}\,|\langle X_t - X_s, x^*\rangle|^2 \le M|t-s|^\theta = \mathbb{E}\,|Y_t - Y_s|^2.\end{aligned}$$

This verifies the second condition of [Ca, Proposition 5]. ■

In particular, it follows from this proposition that the process $\{X_t\}_{t\in[0,T]}$ has a continuous modification if there exists a constant M such that

$$\|\Phi(t) - \Phi(s)\| \le M|t-s|, \qquad t,s \in (0,T].$$

Remark 4.4. If the conditions (i) and (ii) in Proposition 4.3 hold for a single $x^* \in E^*$, then $\{\langle X_t, x^*\rangle\}_{t\in[0,T]}$ admits a continuous version. This follows upon taking r large in in (4.1) and applying the Kolmogorov-Chentsov theorem. In particular, if there exists a constant M such that

$$\|\Phi(t)x^* - \Phi(s)x^*\| \le M|t-s|, \qquad t,s \in (0,T],$$

then the process $\{\langle X_t, x^*\rangle\}_{t\in[0,T]}$ has a continuous modification.

5. Weak solutions of the stochastic Cauchy problem

In this section we will apply our theory to the study of the following stochastic abstract Cauchy problem:

$$(ACP) \qquad \begin{aligned} dX_t &= AX_t\,dt + B\,dW_t^H \qquad (t \in [0,T]),\\ X_0 &= 0 \quad \text{a.s.}\end{aligned}$$

Here A is the generator of a C_0-semigroup $\mathbf{S} = \{S(t)\}_{t\ge 0}$ on a separable real Banach space E, B is a bounded linear operator from a separable real Hilbert space H into E, and $\{W_t^H\}_{t\in[0,T]}$ is a cylindrical Wiener process with Cameron-Martin space H.

In this setting we may define an operator-valued function $\Phi : (0,T] \to \mathcal{L}(H,E)$ by

$$\Phi(t) = S(t) \circ B \qquad (t \in (0,T]).$$

Clearly we have $\Phi \in L^2((0,T];H,E)$. The operators $Q_t \in \mathcal{L}(E^*,E)$ are given by

$$Q_t x^* = \int_0^t S(s)QS^*(s)x^*\,ds \qquad (x^* \in E^*,\ t \in (0,T]),$$

where $Q = B \circ B^*$. This integral can be shown to exist in the sense of Bochner [Ne1], but this will not play a role in what follows. As before we let (i_T, H_T) denote the RKHS associated with Q_T.

Definition 5.1. A *weak solution* of (ACP) is a predictable $E-$valued stochastic process $\{X_t\}_{t\in[0,T]}$ such that for all $x^* \in D(A^*)$ the function $s \mapsto \langle X_s, A^* x^*\rangle$ is almost surely integrable on $[0,T]$ and

$$\langle X_t, x^*\rangle = \int_0^t \langle X_s, A^* x^*\rangle\, ds + [W_t^H, B^* x^*] \qquad (t \in [0,T]). \tag{5.1}$$

Remark. Although we do not assume that a weak solution $\{X_t\}_{t\in[0,T]}$ has a (weakly) continuous version, it is an immediate consequence of our definition and Definition 3.1 that for every $x^* \in D(A^*)$ the process $\{\langle X_t, x^*\rangle\}_{t\in[0,T]}$ does have a continuous version.

The proof of our main result depends on the following extension result for C_0-semigroups [Ne2]:

Proposition 5.2. *There exists a separable real Hilbert space $\tilde{E}$, a continuous and dense embedding $j : E \hookrightarrow \tilde{E}$, and a C_0-semigroup $\tilde{\mathbf{S}}$ on $\tilde{E}$ such that $j \circ S(t) = \tilde{S}(t) \circ j$ for all $t \geq 0$.*

Theorem 5.3. *If the embedding $i_T : H_T \hookrightarrow E$ is $\gamma-$radonifying, then the process $\{X_t\}_{t\in[0,T]}$ defined by stochastic convolution,*

$$X_t = \int_0^t S(t-s)B\; dW_s^H \qquad (t \in [0,T])$$

is a weak solution of (ACP). This process has a strongly measurable modification that satisfies

$$\int_0^T \|X_t\|^2\, dt < \infty$$

almost surely.

Proof : By Proposition 4.2, with $\Phi(t) = S(t) \circ B$, the process $\{X_t\}_{t\in[0,T]}$ has a strongly measurable modification which satisfies $\int_0^T \|X_t\|^2\, dt < \infty$ almost surely.

Let $j : E \hookrightarrow \tilde{E}$ denote the embedding of Proposition 5.2 and let $\tilde{\mathbf{S}}$ denote the C_0-extension of $\mathbf{S}$ to $\tilde{E}$. By the theory of (ACP) in Hilbert spaces [DZ, Chapter 5], the $\tilde{E}-$valued process $\{\tilde{X}_t\}_{t\in[0,T]}$ defined by the Hilbert space stochastic Itô convolution integral

$$\tilde{X}_t = \int_0^t \tilde{S}(t-s)\tilde{B}\, dW_s^H,$$

where $\tilde{B} = j \circ B$, is a weak solution of the problem

$$\begin{aligned} d\tilde{X}_t &= \tilde{A}\tilde{X}_t\, dt + \tilde{B}\, dW_t^H \qquad (t \in [0,T]) \\ \tilde{X}_0 &= 0 \quad \text{a.s.} \end{aligned}$$

in $\tilde{E}$, where $\tilde{A}$ is the generator of $\tilde{\mathbf{S}}$. As we have seen in the proof of Theorem 3.3, for all $t \in [0,T]$ we have $\tilde{X}_t = jX_t$.

For all $\tilde{v}^* \in D(\tilde{A}^*)$ we have $j^*\tilde{v}^* \in D(A^*)$ and $A^*(j^*\tilde{v}^*) = j^*(\tilde{A}^*\tilde{v}^*)$. This implies that for all elements in $v^* \in D(A^*)$ of the form $v^* = j^*\tilde{v}^*$ for some $\tilde{v}^* \in D(\tilde{A}^*)$ we have

$$\begin{aligned} \langle X_t, v^* \rangle = \langle \tilde{X}_t, \tilde{v}^* \rangle &= \int_0^t \langle \tilde{X}_s, \tilde{A}^*\tilde{v}^* \rangle\, ds + [W_t^H, \tilde{B}^*\tilde{v}^*] \\ &= \int_0^t \langle X_s, A^*v^* \rangle\, ds + [W_t^H, B^*v^*] \qquad (t \in [0,T]). \end{aligned} \tag{5.2}$$

Fix $\lambda \in \varrho(A)$. Let Y denote the set of all $v^* \in E^*$ such that (5.1) holds for the element $x^* := (\lambda - A^*)^{-1}v^* \in D(A^*)$. By the above, Y is a linear subspace of E^* containing the weak*-dense subspace $j^*\tilde{E}^*$.

We will show next that Y is weak*-sequentially closed. Let (x_n^*) be a sequence in Y converging weak* to some $x^* \in E^*$. Then $y_n^* := (\lambda - A^*)^{-1}x_n^*$ belongs to $D(A^*)$ and the sequence (y_n^*) converges weak* to $y^* := (\lambda - A^*)^{-1}x^*$. Hence for all ω we have

$$\lim_{n\to\infty} \langle X_t(\omega), y_n^* \rangle = \langle X_t(\omega), y^* \rangle. \tag{5.3}$$

Moreover, from $A^*y_n^* = A^*(\lambda - A^*)^{-1}x_n^* = \lambda(\lambda - A^*)^{-1}x_n^* - x_n^*$ we see that $(A^*y_n^*)$ converges weak* to $\lambda(\lambda - A^*)^{-1}x^* - x^* = A^*y^*$. By dominated convergence, for all $\omega \in \Omega$ we have

$$\lim_{n\to\infty} \int_0^t \langle X_s(\omega), A^*y_n^* \rangle\, ds = \int_0^t \langle X_s(\omega), A^*y^* \rangle\, ds. \tag{5.4}$$

The weak*-to-weak continuity of B^* implies that $B^*y_n^* \to B^*y^*$ weakly in H. Since bounded linear operators are weakly continuous, it follows that

$$\lim_{n\to\infty} [W_t^H, B^*y_n^*] = [W_t^H, B^*y^*] \quad \text{weakly in } L^2(\mathbb{P}). \tag{5.5}$$

On the other hand, combining (5.2) with (5.3) and (5.4), it follows that for all $\omega \in \Omega$ the limit

$$\lim_{n\to\infty} [W_t^H, B^*y_n^*]\,(\omega) =: Y(\omega)$$

exists. With a convex combination argument as in the proof of Proposition 2.2, together with (5.5) this shows that $Y = [W_t^H, B^*y^*]$ a.e. Hence for almost all $\omega \in \Omega$ we have

$$\lim_{n\to\infty} [W_t^H, B^*y_n^*]\,(\omega) = [W_t^H, B^*y^*]\,(\omega). \tag{5.6}$$

By (5.2), (5.3), (5.4), (5.6) and dominated convergence we finally obtain

$$\begin{aligned} \langle X_t, y^* \rangle = \lim_{n\to\infty} \langle X_t, y_n^* \rangle &= \lim_{n\to\infty} \left(\int_0^t \langle X_s, A^*y_n^* \rangle\, ds + [W_t^H, B^*y_n^*] \right) \\ &= \int_0^t \langle X_s, A^*y^* \rangle\, ds + [W_t^H, B^*y^*] \end{aligned}$$

almost everywhere. This shows that $x^* \in Y$, and Y is weak*-sequentially closed as claimed.

By Proposition 1.2, $Y = E^*$ and the proof is complete. ∎

Remark. As we noted above, the fact that $\{X_t\}_{t\in[0,T]}$ is a weak solution implies that for each $x^*\in D(A^*)$, the scalar process $\{\langle X_t, x^*\rangle\}_{t\in[0,T]}$ has a continuous modification. This can also be seen more directly from the observation in Remark 4.4. Indeed, if $x^*\in D(A^*)$, the identity

$$S^*(t)x^* - x^* = \int_0^t S^*(s)A^*x^*\,ds$$

shows that the orbit $t\mapsto S^*(t)x^*$ is Lipschitz continuous on the bounded interval $[0,T]$.

Theorem 5.3 admits the following converse:

Theorem 5.4. *Suppose (ACP) admits a weak solution $\{X_t\}_{t\in[0,T]}$. Then the embedding $i_T : H_T \hookrightarrow E$ is $\gamma-$radonifying and $\{X_t\}_{t\in[0,T]}$ is an Ornstein-Uhlenbeck process.*

Proof : Let $j : E \hookrightarrow \tilde{E}$ and $\tilde{\mathbf{S}}$ be as in Proposition 5.2. By the results of [BRS], $\tilde{E}$ may be densely embedded into another separable real Hilbert space $\overline{E}$ in such a way that $\tilde{\mathbf{S}}$ extends to a C_0-semigroup $\overline{\mathbf{S}}$ on $\overline{E}$ and the embedding $\tilde{j} : \tilde{E} \hookrightarrow \overline{E}$ is Hilbert-Schmidt. Let $\overline{j} := \tilde{j}\circ j$.

The operator $\overline{B} := \overline{j}\circ B = \tilde{j}\circ\tilde{B} : H \to \overline{E}$ is Hilbert-Schmidt, being the composition of the bounded operator $\tilde{B} = j\circ B$ and the Hilbert-Schmidt operator $\tilde{j}$. It follows that $\overline{Q} := \overline{B}\circ\overline{B}^*$ is trace class. Define the positive selfadjoint operator $\overline{Q}_T$ on $\overline{E}$ by

$$\overline{Q}_T\overline{h} = \int_0^T \overline{S}(s)\overline{Q}\,\overline{S}^*(s)\overline{h}\,ds \qquad (\overline{h}\in\overline{E}).$$

Then it easy to check (cf. [Ne1]) that $\overline{Q}_T$ is trace class as well.

It now follows from the general theory of stochastic equations in Hilbert spaces [DZ, Chapter 5] that the stochastic convolution process $\overline{X}_t = \int_0^t \overline{S}(t-s)\overline{B}\;dW_s^H$ is the unique weak solution to the problem

$$\begin{aligned} d\overline{X}_t &= \overline{A}\,\overline{X}_t\,dt + \overline{B}\,dW_t^H \qquad (t\in[0,T]),\\ \overline{X}_0 &= 0 \quad \text{a.s.} \end{aligned}$$

But the process $\{\overline{j}X_t\}_{t\in[0,T]}$ is a weak solution of this problem as well, and hence by uniqueness it follows that $\overline{X}_t = \overline{j}X_t$ for all $t\in[0,T]$. We conclude that $\{\overline{j}X_t\}_{t\in[0,T]}$ is an $\overline{E}-$valued Ornstein-Uhlenbeck process, this being true for $\{\overline{X}_t\}_{t\in[0,T]}$. This implies that for all $\overline{x}^*, \overline{y}^* \in \overline{E}^*$ and $t,s\in[0,T]$ we have

$$\begin{aligned} \mathbb{E}\left(\langle X_t, \overline{j}^*\overline{x}^*\rangle, \langle X_s, \overline{j}^*\overline{y}^*\rangle\right) &= \mathbb{E}\left(\langle \overline{X}_t, \overline{x}^*\rangle, \langle \overline{X}_s, \overline{y}^*\rangle\right)\\ &= \int_0^{t\wedge s} [\overline{B}^*\overline{S}^*(t-u)\overline{x}^*, \overline{B}^*\overline{S}^*(s-u)\overline{y}^*]_H\,du\\ &= \int_0^{t\wedge s} [B^*S^*(t-u)(\overline{j}^*\overline{x}^*), B^*S^*(s-u)(\overline{j}^*\overline{y}^*)]_H\,du. \end{aligned} \tag{5.7}$$

The linear subspace $Y = \{\overline{j}^*\overline{x}^* : \overline{x}^* \in \overline{E}^*\}$ is weak*-dense in E^*, as $\overline{j}$ is a dense embedding.

We claim that $\{X_t\}_{t\in[0,T]}$ is a Gaussian process. To see this, fix $t \in [0,T]$ and let μ_t and $\overline{\mu}_t$ be the distributions of X_t and $\overline{X}_t$, respectively. These are Borel probability measures on E and $\overline{E}$, respectively, and we have $\overline{\mu}_t = \overline{j}(\mu_t)$. Moreover, because $\{\overline{X}_t\}_{t\in[0,T]}$ is an Ornstein-Uhlenbeck process, hence a Gaussian process, the measure $\overline{\mu}_t$ is a Gaussian measure. Hence for all $y^* = \overline{j}^*\overline{x}^*$ in the weak*-dense subspace Y of E^*, the image measures $\langle \mu_t, y^*\rangle = \langle \overline{\mu}_t, \overline{x}^*\rangle$ are Gaussian on $\mathbb{R}$. By Corollary 1.3, this implies that μ_t is Gaussian, and the claim is proved.

The process $\{X_t\}_{t\in[0,T]}$ being Gaussian, the weak second moments $\mathbb{E}\,(\langle X_t, x^*\rangle^2)$ are finite for all $t \in [0,T]$ and $x^* \in E^*$. Departing from (5.7), the proof that $\{X_t\}_{t\in[0,T]}$ is an Ornstein-Uhlenbeck process now proceeds along the lines of the proof of Theorem 3.3. ∎

Concerning uniqueness of weak solutions, we have the following result:

Theorem 5.5. *Let $X^{(0)} = \{X_t^{(0)}\}_{t\in[0,T]}$ and $X^{(1)} = \{X_t^{(1)}\}_{t\in[0,T]}$ be two weak solutions of the problem (ACP). Then $X^{(0)}$ and $X^{(1)}$ are versions of each other.*

Proof : This follows immediately by embedding E into a Hilbert space $\overline{E}$ in the way described in the proof of Theorem 5.4 and the fact that the corresponding uniqueness result for weak solutions holds in the Hilbert space setting. ∎

So far, we were concerned only with solutions on a finite time interval $[0,T]$. By obvious modifications, the theory extends to the interval $[0,\infty)$. In particular, a weak *global* solution of (ACP) exists if and only if for *all* $T > 0$ the associated inclusion mapping $i_T : H_T \hookrightarrow E$ is $\gamma-$radonifying; in this case the solution is unique, and given by stochastic convolution.

Under this assumption, for each $t > 0$ we let $\mu_t = i_t(\gamma_{H_t})$ denote the corresponding centred Gaussian measure on E; we futher set $\mu_0 = \delta_0$, the Dirac measure concentrated at 0. For each $t \geq 0$ we define a linear contraction $P(t)$ on the space $B_b(E)$ of bounded real-valued Borel functions on E by the formula

$$P(t)f(x) = \int_E f(S(t)x + y)\, d\mu_t(y) \qquad (x \in E,\ f \in B_b(E)).$$

From the identity

$$Q_{t+s} = Q_t + S(t)Q_sS^*(t)$$

we see that

$$\mu_{t+s} = \mu_t * S(t)\mu_s,$$

from which it easily follows that $P(t+s) = P(t)\circ P(s)$ for all $t, s \geq 0$. Thus the family $\{P(t)\}_{t\geq 0}$ is a semigroup of contractions on $B_b(E)$. This semigroup has been studied in some detail in [Ne1] from a functional-analytic point of view. We conclude this section by showing that it arises as the transition semigroup of the weak solution of the stochastic Cauchy problem (ACP):

Proposition 5.6. *Let $\{X_t\}_{t\geq 0}$ be a weak solution of the problem (ACP). For all $t \in [0,T]$ we have, for almost all $x \in E$,*

$$P(t)f(x) = \mathbb{E}\,(f(X_{t,x})),$$

where $X_{t,x} := S(t)x + X_t$.

Proof : Fix $t \in [0, T]$. Recalling that μ_t is the distribution of X_t, for almost all $x \in E$ we have

$$\begin{aligned}\mathbb{E}\left(f(X_{t,x})\right) &= \int_\Omega f(X_{t,x}(\omega))\, d\mathbb{P}(\omega) \\ &= \int_\Omega f(S(t)x + X_t(\omega))\, d\mathbb{P}(\omega) \\ &= \int_E f(S(t)x + y)\, d\mu_t(y) \\ &= P(t)f(x).\end{aligned}$$

■

We point out that the weak solution is always a Markov process. This can be seen directly as in the proof of Proposition 5.6 or by using the fact that this is true for the Hilbert space case and using the extension argument of Theorem 5.5.

6. The analytic case

The results of the previous section do note take into account possible regularization effects of the semigroup $\mathbf{S}$. We will present now a result in this direction for the case where $\mathbf{S}$ is an analytic semigroup. Roughly speaking it turns out that if $\mathbf{S}$ maps E into some smaller space F, then under some natural assumptions the weak solution of (ACP) is also $F-$valued.

Theorem 6.1. *Suppose that F and E are separable real Banach spaces, with F continuously embedded in E. Let $\mathbf{S}_E = \{S_E(t)\}_{t\geq 0}$ be a C_0 semigroup on E, with generator A_E such that for all $t > 0$, $S_E(t)E \subset F$. Denote by $S_{EF}(t)$ the operator $S_E(t)$, regarded as a bounded linear operator from E into F, and let $S_F(t)$ denote the restriction of $S_{EF}(t)$ to F.*

Let B be a bounded linear operator from a separable real Hilbert space H into E, and let $Q = B \circ B^*$. Let $Q_T \in \mathcal{L}(F^*, F)$ be the positive symmetric operator defined by the Pettis integral

$$Q_T x^* = \int_0^T S_{EF}(t) Q S_{EF}^*(t) x^*\, dt \qquad (x^* \in F^*).$$

Let (i_T, H_T) be the RKHS associated with Q_T. We assume:

(i) For each $x^* \in F^*$, the function $t \mapsto B^* S_{EF}^*(t) x^*$ is strongly measurable and

$$\int_0^T \|B^* S_{EF}^*(t) x^*\|_H^2\, dt < \infty; \tag{6.1}$$

(ii) The semigroup $\mathbf{S}_F = \{S_F(t)\}_{t\geq 0}$ is an analytic C_0-semigroup on F, with generator A_F, and there exist $\lambda \in \varrho(A_F)$ and $\theta \in (0,1]$ such that

$$\int_0^T \|(\lambda - A_F)^\theta S_{EF}(t)\|^2_{\mathcal{L}(E,F)}\, dt < \infty; \tag{6.2}$$

(iii) The embedding $i_T : H_T \hookrightarrow F$ is γ−radonifying.

Under these assumptions there exists an F−valued stochastic process $\{X_t\}_{t\in[0,T]}$ with covariance

$$\mathbb{E}\left(\langle X_t, x^*\rangle\langle X_s, y^*\rangle\right) = \int_0^{t\wedge s} [B^* S^*_{EF}(t-u)x^*, B^* S^*_{EF}(s-u)y^*]_H\, du \qquad (x^*, y^* \in F^*). \tag{6.3}$$

This process has a continuous modification. As an E−valued process, it is a weak solution to the stochastic abstract Cauchy problem

$$\begin{aligned} dX_t &= A_E X_t\, dt + B\, dW^H_t, \quad t \in [0,T], \\ X_0 &= 0. \end{aligned} \tag{6.4}$$

Proof : By (i), (iii), and Theorem 3.3 applied to the $\mathcal{L}(H,F)$−valued function $t \mapsto S_{EF}(t)\circ B$, there exists an F−valued process Ornstein-Uhlenbeck process $\{X_t\}_{t\in[0,T]}$ with covariance given by (6.3) and we have

$$\langle X_t, x^*\rangle = \int_0^t \langle S_{EF}(t-s) B\, dW^H_s, x^*\rangle \qquad (t\in[0,T],\ x^* \in F^*).$$

We shall prove that the process $\{X_t\}_{t\in[0,T]}$ has a continuous version. We argue as in [MS, Remark 3.2]. Fix $\lambda \in \varrho(A_F)$ and $\theta \in (0,1]$ as in assumption (ii). For all $x^* \in F^*$ we have

$$\begin{aligned} &\|B^* S^*_{EF}(t-s+u)x^* - B^* S^*_{EF}(u)x^*\|_H \\ &\quad \leq \|B^*\|_{\mathcal{L}(E^*,H)}\, \|x^*\|\, \|S_F(t-s)S_{EF}(u) - S_{EF}(u)\|_{\mathcal{L}(E,F)} \\ &\quad \leq \|B^*\|_{\mathcal{L}(E^*,H)}\, \|x^*\|\, \|\left(S_F(t-s) - I\right)(\lambda - A_F)^{-\theta}\|_{\mathcal{L}(F,F)}\, \|(\lambda - A_F)^\theta S_{EF}(u)\|_{\mathcal{L}(E,F)} \\ &\quad \leq C\, \|x^*\|(t-s)^\theta\, \|(\lambda - A_F)^\theta S_{EF}(u)\|_{\mathcal{L}(E,F)}. \end{aligned}$$

Hence by (6.2),

$$\begin{aligned} &\int_0^s \|B^* S^*_{EF}(t-s+u)x^* - B^* S^*_{EF}(u)x^*\|^2_H\, du \\ &\qquad\qquad \leq C^2\, \|x^*\|^2 (t-s)^{2\theta} \int_0^T \|(\lambda - A_F)^\theta S_{EF}(u)\|^2_{\mathcal{L}(E,F)}\, du < \infty. \end{aligned}$$

By Proposition 4.3, it follows that the process $\{X_t\}_{t\geq 0}$ has a continuous modification.

Let $j : F \hookrightarrow E$ denote the inclusion mapping. It remains to check that the E−valued process defined by $\tilde{X}_t = jX_t$ $(t \in [0,T])$ is a weak solution of (6.4). Let $\tilde{x}^* \in E^*$ be given and let $x^* := j^*\tilde{x}^*$. Recalling that $j \circ S_{EF}(t) = S_E(t)$ we have

$$\langle \tilde{X}_t, \tilde{x}^*\rangle = \langle X_t, x^*\rangle = \int_0^t \langle S_{EF}(t-s)B\, dW^H_s, x^*\rangle = \int_0^t \langle S_E(t-s)B\, dW^H_s, \tilde{x}^*\rangle.$$

Hence, by Theorem 5.3 and the uniqueness part of Theorem 3.3, $\{\tilde{X}_t\}_{t\in[0,T]}$ is a weak solution of (6.4). ∎

For $F = E$ this reduces to:

Corollary 6.2. *If A generates an analytic semigroup, then the weak solution $\{X_t\}_{t\in[0,T]}$ admits a continous version.*

Consider the stochastic heat equation driven by spatio-temporal white noise:

$$\begin{aligned} \frac{\partial X}{\partial t}(t,x) &= \Delta X(t,x) + \frac{\partial w}{\partial t}(t,x), \quad t \geq 0, \\ X(0,x) &= 0, \\ X(t,0) &= X(t,1) = 0. \end{aligned} \tag{6.5}$$

As an application of Theorem 6.1 we will show that for any $\beta \in [0, \frac{1}{2})$, this problem has a unique weak solution with a continuous modification taking values in the space of Hölder continuous functions of exponent β. This result has been obtained by entirely different methods in [Br3]; see [Wa]. Extensions of this equation with more general types of noise have been discussed in, e.g., [DaS], [PZ] and [BP].

By a *weak solution* of (6.5) we understand a weak solution to the problem

$$\begin{aligned} dX_t &= \Delta X_t + dW_t, \quad t \geq 0, \\ X_0 &= 0, \end{aligned} \tag{6.6}$$

where Δ is the Dirichlet Laplacian in $E = L^2[0,1]$ and $\{W_t\}_{t\in[0,T]}$ is a cylindrical Wiener process with Cameron-Martin space $H = E = L^2[0,1]$.

For $\beta \in [0,1]$ let

$$c_0^\beta[0,1] = \{u \in c^\beta[0,1] : u(0) = u(1) = 0\},$$

where $c^\beta[0,1]$ is the little Hölder space of all continuous functions f on $[0,1]$ for which

$$\|f\|_{c^\beta[0,1]} := \sup_{t\in[0,1]} |f(t)| + \sup_{0\leq s<t\leq 1} \frac{|f(t)-f(s)|}{(t-s)^\beta} < \infty$$

and

$$\lim_{\delta\downarrow 0} \sup_{|t-s|\leq\delta} \frac{|f(t)-f(s)|}{(t-s)^\beta} = 0.$$

Theorem 6.3. *The problem (6.5) has a unique global weak solution $\{X_t\}_{t\geq 0}$. For each $t \geq 0$ the random variable X_t takes values in $c_0^\beta[0,1]$ almost surely. As a $c_0^\beta[0,1]$−valued process, $\{X_t\}_{t\geq 0}$ has a continuous modification.*

Proof : For $p \in [1,\infty)$, let $A_p = \Delta$ be the Laplacian on $L^p[0,1]$ with Dirichlet boundary conditions, i.e. $D(A_p) = H_0^{1,p}[0,1] \cap H^{2,p}[0,1]$, and let $\mathbf{S}_p$ denote the heat semigroup on $L^p[0,1]$, i.e. the analytic C_0-semigroup on $L^p[0,1]$ generated by A_p.

Let $T > 0$ be arbitrary. As is well-known, see e.g. [DZ], the RKHS corresponding to the selfadjoint operator $R_T \in \mathcal{L}(L^2[0,1])$ defined by

$$R_T f = \int_0^T S_2^*(t) S_2(t) f \, dt \qquad (f \in L^2[0,1]),$$

equals $H_0^{1,2}[0,1]$. The inclusion $H_0^{1,2}[0,1] \hookrightarrow L^2[0,1]$ is Hilbert-Schmidt and hence $\gamma-$radonifying. Hence (6.5), and therefore (6.6), has a unique global weak solution $\{X_t\}_{t\geq 0}$.

Fix $\alpha \in (0, \frac{1}{4})$ and $2 < p < \infty$ such that $2\alpha > \frac{1}{p}$. We are going to check first that Theorem 6.1 applies, with $H = E = L^2[0,1]$, $B : H \to E$ the identity operator, $F = H_0^{2\alpha,p}[0,1]$, and $\mathbf{S}_E = \mathbf{S}_2$.

The restriction $\mathbf{S}_{2\alpha,p}$ of $\mathbf{S}_p$ to $H_0^{2\alpha,p}[0,1]$ is strongly continuous and analytic on $H_0^{2\alpha,p}[0,1]$. Notice that, with the notation of Theorem 6.1, $\mathbf{S}_{2\alpha,p}$ equals the semigroup $\mathbf{S}_F$. Let $A_{2\alpha,p}$ be its generator. Put $2\delta := \frac{1}{2} - \frac{1}{p}$ and note that $2(\alpha + \delta) < 1$ since we assume that $\alpha \in (0, \frac{1}{4})$. Choose $\theta \in (0,1]$ so small that $2(\alpha + \delta + 2\theta) < 1$. Suppressing subscripts we then have, with a suitable choise of $0 < \eta < \delta + \theta$,

$$\begin{aligned} \|S(t)\|_{\mathcal{L}(L^2[0,1],L^p[0,1])} &\leq Ct^{-\eta} \qquad (t \in (0,1]), \\ \|S(t)\|_{\mathcal{L}(L^p[0,1],H_0^{2\alpha,p}[0,1])} &\leq Ct^{-\alpha} \qquad (t \in (0,1]), \\ \|(-A_{2\alpha,p})^\theta S(t)\|_{\mathcal{L}(L^p[0,1],H_0^{2\alpha,p}[0,1])} &\leq Ct^{-\alpha-\theta} \qquad (t \in (0,1]). \end{aligned}$$

The first of these estimates follows from

$$\|S(t)f\|_{L^\infty[0,1]} \leq C\|S(t)f\|_{H_0^{2(\delta+\theta),2}[0,1]} \leq C' t^{-\delta-\theta} \|f\|_{L^2[0,1]}, \qquad t > 0,$$

and interpolation; here we use that by assumption $\delta + \theta > \frac{1}{4}$, so that $H_0^{2(\delta+\theta),2}[0,1] \hookrightarrow L^\infty[0,1]$ by the Sobolev embedding theorem. The second and third estimate follow from general results about analytic semigroups.

The first two estimates show that assumption (i) of Theorem 6.1 holds. From the first and third estimate we infer that

$$\int_0^1 \|(-A_{2\alpha,p})^\theta S(t)\|^2_{\mathcal{L}(L^2[0,1],H_0^{2\alpha,p}[0,1])} \, dt \leq C \int_0^1 t^{-2(\alpha+\eta+\theta)} \, dt < \infty,$$

which shows that assumption (ii) of Theorem 6.1 is satisfied (cf. the remark following the formulation of the theorem). By [Br1] and a closed subspace argument, the inclusion $i_{2\alpha,p} : H_0^{1,2}[0,1] \hookrightarrow H_0^{2\alpha,p}[0,1]$ is $\gamma-$radonifying; this verifies assumption (iii) of Theorem 6.1. Hence by Theorem 6.1 the weak solution $\{X_t\}_{t\geq 0}$ of (6.5) has a modification that is a continuous $H_0^{2\alpha,p}[0,1]-$valued process with covariance

$$\mathbb{E}\left(\langle X_t, \varphi\rangle \langle X_s, \psi\rangle\right) = \int_0^{t\wedge s} [i_{2\alpha,p}^* S_{2\alpha,p}^*(t-u)\varphi, i_{2\alpha,p}^* S_{2\alpha,p}^*(s-u)\psi]_{H_0^{1,2}[0,1]} \, du$$

for all $t, s \geq 0$ and $\varphi, \psi \in (H_0^{2\alpha,p}[0,1])^*$.

Now fix $\beta \in [0, \frac{1}{2})$. Choose $\alpha \in (0, \frac{1}{4})$ and $p > 2$ in such a way that $2\alpha > \beta + \frac{1}{p}$. By the Sobolev embedding theorem we then have a continuous inclusion $H_0^{2\alpha,p}[0,1] \hookrightarrow c_0^\beta[0,1]$. Combining this with the above, it follows that $\{X_t\}_{t\geq 0}$ takes values in $c_0^\beta[0,1]$, and that is continuous as a $c_0^\beta[0,1]-$valued process. ∎

Acknowledgement - The authors thank the referee for his valuable remarks and for pointing out some inaccuracies in the original draft of the paper. An enlightening discussion with Ben Goldys led to an improvement of Theorem 3.3, and Ben de Pagter pointed out to us the proof of Corollary 1.3.

7. References

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