STOCHASTIC INTEGRATION OF FUNCTIONS WITH VALUES IN A BANACH SPACE

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Abstract. Let H be a separable real Hilbert space and let E be a real Banach space. In this paper we construct a stochastic integral for certain operator-valued functions $\Phi : (0,T) \to \mathscr{L}(H,E)$ with respect to a cylindrical Wiener process $\{W_H(t)\}_{t\in[0,T]}$. The construction of the integral is given by a series expansion in terms of the stochastic integrals for certain E-valued functions. As a substitute for the Itô isometry we show that the square expectation of the integral equals the radonifying norm of an operator which is canonically associated with the integrand. We obtain characterizations for the class of stochastically integrable functions and prove various convergence theorems. The results are applied to the study of linear evolution equations with additive cylindrical noise in general Banach spaces. An example is presented of a linear evolution equation driven by a one-dimensional Brownian motion which has no weak solution.

In this paper we construct a theory of stochastic integration of operator-valued functions with respect to a cylindrical Wiener process. The range space of the operators is allowed to be an arbitrary real Banach space E. A stochastic integral of this type can be used for solving the linear stochastic Cauchy problem

$$\begin{aligned} dU(t) &= AU(t)\,dt + B\,dW_H(t), \qquad t \in [0,T], \\ U(0) &= u_0. \end{aligned} \tag{0.1}$$

Here, A is the infinitesimal generator of a strongly continuous semigroup $\{S(t)\}_{t\geqslant 0}$ of bounded linear operators on E, the operator B is a bounded linear operator from a separable real Hilbert space H into E, and $\{W_H(t)\}_{t\in[0,T]}$ is a cylindrical H-Wiener process. Formally, equation (0.1) is solved by the stochastic convolution process

$$U(t) = S(t)u_0 + \int_0^t S(t-s)B\,dW_H(s). \tag{0.2}$$

It is well known that the integral on the right hand side can be interpreted as an Itô stochastic integral if E is a Hilbert space. A comprehensive theory of abstract stochastic differential equations in Hilbert spaces is presented in the monograph by Da Prato and Zabczyk [5]. More generally the integral can be defined for spaces E with martingale type 2. This has been worked out by Brzeźniak [2]. Examples of martingale type 2 spaces are Hilbert spaces and the Lebesgue spaces $L^p(\mu)$ with $p \in [2,\infty)$.

In both settings, the integral is defined for step functions first, and for general functions the integral is obtained by a limiting argument. Such a limiting argument depends on

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the availability of *a priori* estimates for the integrals of the approximating step functions, and the martingale type 2 property is precisely designed to provide such estimates.

Without special assumptions on the geometry of the underlying Banach space E there seems to be no general method to give a rigorous interpretation for the integral in (0.2) as a limit of integrals of step functions. Despite this fact, for separable Banach spaces E, necessary and sufficient conditions for the existence and uniqueness of weak solutions of the problem (0.1) have been obtained recently in [4]. The main idea of this paper was to embed the equation (0.1) into some Hilbert space $\tilde{E}$ containing E as a dense subspace. In $\tilde{E}$, the equation can be solved by Hilbert space methods, and the main point is then to show that the resulting $\tilde{E}$-valued process takes its values in the original space E almost surely. This approach has the obvious disadvantage that the construction of solutions is not intrinsic but depends on an *ad hoc* extension argument.

In the present paper we set up a general theory of stochastic integration for $\mathscr{L}(H,E)$-valued functions that does not have these defects. The construction of the stochastic integral, which can be interpreted as a stochastic version of the Pettis integral but which turns out to possess many properties of the Lebesgue integral as well, is intrinsic and relies on a series expansion in terms of the stochastic Pettis integrals for certain E-valued functions. We identify the square expectation of the stochastic integral with the radonifying norm of a certain operator canonically associated with the integrand. The resulting isometry (cf. Theorems 2.3 and 4.2 below) serves as a substitute for the Itô isometry. The idea to use the radonifying norm to extend Hilbert space results to the Banach space setting was introduced in [12], where it was applied to the H^∞-calculus of unbounded operators and square function estimates in harmonic analysis.

Upon identifying $\mathscr{L}(\mathbb{R},E)$ with E, our integral coincides with the one introduced by Rosiński and Suchanewski [18] in the case of strongly measurable E-valued functions. Even here, the connection with operator theory provides simplified proofs and new insights.

The organization of the paper is as follows. After presenting some preliminaries in Section 1, in Section 2 we start with the construction of a stochastic integral for E-valued functions with respect to a real-valued Brownian motion and give an operator-theoretic characterization of the class of integrable functions.

We discuss cylindrical Wiener processes in Section 3, and the construction of a stochastic integral for $\mathscr{L}(H,E)$-valued functions with respect to a cylindrical H-Wiener process is taken up in Section 4. Again we give an operator-theoretic characterization of the class of integrable functions.

The problem of integrating $\mathscr{L}(E)$-valued functions with respect to E-valued Brownian motions is discussed in Section 5.

In Section 6 we prove a dominated convergence theorem and obtain, as applications, various approximation and continuity results for the integral. We further show that a monotone convergence theorem holds if and only if E does not contain an isomorphic copy of c_0.

In the final Section 7 we use our results to obtain natural direct proofs of the main results of [3] and [4] concerning the existence, uniqueness, and mean square continuity of weak solutions for the stochastic Cauchy problem (0.1). An example is presented of a linear evolution equation driven by a one-dimensional Brownian motion which has no weak solution. Our approach also allows us to give conditions directly in terms of the resolvent of A, which in many problems may be more accessible than the semigroup itself.

In order to keep this paper at a reasonable length, we decided to postpone further developments of the theory that require additional geometric properties of the Banach space to a future paper.

It is possible to extend the results of this paper to progressively measurable processes with values in a UMD Banach space. This topic will be taken up in another paper, where we will also discuss applications to stochastic evolution equations driven by multiplicative noise.

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1. Preliminaries

In this preliminary section we fix notations and recall some well known facts about Gaussian covariance operators and the Pettis integral. Throughout this paper, E is a real Banach space with dual E^*. The pairing between E and E^* is denoted by $\langle \cdot, \cdot \rangle$.

1.1. **Reproducing kernel Hilbert spaces.** Let X and X' be real Banach spaces and let $\langle \cdot, \cdot \rangle : X \times X' \to \mathbb{R}$ be a bounded bilinear form:

$$|\langle x, x' \rangle| \leqslant M \|x\| \, \|x'\| \qquad \forall x \in X, \; x' \in X'.$$

A bounded linear operator $Q \in \mathscr{L}(X', X)$ is called $\langle \cdot, \cdot \rangle$-*positive* if

$$\langle Qx', x' \rangle \geqslant 0 \qquad \text{for all } x' \in X'$$

and $\langle \cdot, \cdot \rangle$-*symmetric* if

$$\langle Qx', y' \rangle = \langle Qy', x' \rangle \qquad \text{for all } x', y' \in X'.$$

It is easily checked that on the range of Q, the formula

$$[Qx', Qy']_{H_Q} := \langle Qx', y' \rangle \tag{1.1}$$

defines an inner product $[\,\cdot\,,\cdot\,]_{H_Q}$. Indeed, if either $Qx' = 0$ or $Qy' = 0$, then

$$[Qx', Qy']_{H_Q} = \langle Qx', y' \rangle = \langle Qy', x' \rangle = 0,$$

which shows that $[\,\cdot\,,\cdot\,]_{H_Q}$ is well defined. We denote by H_Q the real Hilbert space obtained by completing the range of Q with respect to $[\,\cdot\,,\cdot\,]_{H_Q}$. This space will be called the *reproducing kernel Hilbert space* associated with Q and $\langle \cdot, \cdot \rangle$. The inclusion mapping from the range of Q into X is easily seen to be continuous with respect to the inner product $[\,\cdot\,,\cdot\,]_{H_Q}$ and extends uniquely to a bounded linear injection i_Q from H_Q into X. Using the Riesz representation theorem, we can define an adjoint operator $i_Q' \in \mathscr{L}(X', H_Q)$ in the natural way and we have the operator identity

$$Q = i_Q \circ i_Q'.$$

For all $x', y' \in X'$ we have $\langle Qx', y' \rangle = [i_Q' x', i_Q' y']_{H_Q}$. Moreover, the range of i_Q' is dense in H_Q. It is an application of the Riesz representation theorem that for two positive symmetric operators $Q, R \in \mathscr{L}(X', X)$ the following assertions are equivalent; cf. [5, Proposition B.1], [15, Proposition 1.1], [8, Proposition 2.2]:

1. $H_Q \subseteq H_R$ and the inclusion mapping $H_Q \hookrightarrow H_R$ is contractive;
2. For all $x' \in X'$ we have $\langle Qx', x' \rangle \leqslant \langle Rx', x' \rangle$.

In particular it follows that if R takes its values in some closed subspace X_0 of X, then so does Q. In the proof of Theorem 2.3 this will be applied to the particular case where $X_0 = E$, $X = E^{**}$, $X' = E^*$, and $\langle \cdot, \cdot \rangle$ is the duality between X' and X.

For more information on this topic we refer to [19, 21].

1.2. **γ-Radonifying operators.** Positive symmetric operators occur naturally as the covariance operators of Gaussian measures. Recall that a Radon measure μ on E is called *Gaussian* if its image under every linear functional $x^* \in E^*$ is a Gaussian measure on $\mathbb{R}$. In this paper, all Gaussian measures will be *centred*, meaning that all image measures are centred as measures on $\mathbb{R}$. An E-valued random variable is called *Gaussian* if its distribution is a Gaussian measure on E.

If μ is a Gaussian measure on E, there exists a unique positive symmetric operator $Q \in \mathscr{L}(E^*, E)$, the *covariance operator* of μ, such that

$$\int_E \langle x, x^* \rangle^2 \, d\mu(x) = \langle Qx^*, x^* \rangle, \qquad x^* \in E^*.$$

The converse is not true: not every positive symmetric $Q \in \mathscr{L}(E^*, E)$ is a Gaussian covariance.

Let H be a separable real Hilbert space and let $T \in \mathscr{L}(H, E)$ be a bounded operator. The operator $T \circ T^* \in \mathscr{L}(E^*, E)$ is positive and symmetric. The following well known result gives a necessary and sufficient condition for $T \circ T^*$ to be a Gaussian covariance.

Proposition 1.1. *Let $(\gamma_n)_{n=1}^\infty$ be a sequence of independent standard normal random variables on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. The following assertions are equivalent:*

(1) *$T \circ T^*$ is the covariance of a Gaussian measure μ on E;*
(2) *There exists an orthonormal basis $(h_n)_{n=1}^\infty$ of H such that the Gaussian series $\sum_{n=1}^\infty \gamma_n T h_n$ converges in $L^2(\Omega; E)$.*

In this situation, for every orthonormal basis $(h_n)_{n=1}^\infty$ of H, the series $\sum_{n=1}^\infty \gamma_n T h_n$ converges unconditionally in $L^p(\Omega; E)$ for all $p \in [1, \infty)$ and almost surely, and we have

$$\mathbb{E} \Big\| \sum_{n=1}^\infty \gamma_n T h_n \Big\|^p = \int_E \|x\|^p \, d\mu(x).$$

An operator $T \in \mathscr{L}(H, E)$ satisfying the equivalent assumptions of the proposition is called *γ-radonifying*. We define its *γ-norm* $\|T\|_\gamma$ by

$$\|T\|_\gamma^2 := \mathbb{E} \Big\| \sum_{n=1}^\infty \gamma_n T h_n \Big\|^2.$$

This number does not depend on the choice of the orthonormal basis $(h_n)_{n=1}^\infty$. If E has type 2, then

$$\|T\|_\gamma^2 \leqslant C_2^2 \sum_{n=1}^\infty \|T h_n\|^2,$$

where C_2 is the (Gaussian) type 2 constant of E.

Notice that a positive symmetric operator $Q \in \mathscr{L}(E^*, E)$ is a Gaussian covariance operator if and only if the embedding $i_Q : H_Q \hookrightarrow E$ is γ-radonifying. In this case H_Q is always separable.

The space of γ-radonifying operators has the following ideal property: if $R \in \mathscr{L}(\tilde{H}, H)$ and $S \in \mathscr{L}(E, \tilde{E})$ are bounded operators and $T \in \mathscr{L}(H, E)$ is γ-radonifying, then

$S \circ T \circ R \in \mathscr{L}(\tilde{H}, \tilde{E})$ is γ-radonifying and

$$\|S \circ T \circ R\|_\gamma \leqslant \|S\| \, \|T\|_\gamma \, \|R\|.$$

Let $M(E)$ denote the set of all Radon probability measures on E. A family $\mathscr{M} \subseteq M(E)$ is called *uniformly tight* if for every $\varepsilon > 0$ there exists a compact set $K \subseteq E$ such that $\mu(K) \geqslant 1 - \varepsilon$ for all $\mu \in \mathscr{M}$. By Prokhorov's theorem, uniform tightness is equivalent to relative compactness in the weak topology of $M(E)$ generated by $C_b(E)$, the space of bounded continuous real-valued functions on E.

The following tightness result follows from the results in [1, Chapter 3].

Proposition 1.2. *Let $R \in \mathscr{L}(E^*, E)$ be the covariance of a Gaussian measure ν on E and let $\mathscr{Q} \subseteq \mathscr{L}(E^*, E)$ be a family of positive symmetric operators such that for all $x^* \in E^*$ and all $Q \in \mathscr{Q}$ we have*

$$\langle Qx^*, x^* \rangle \leqslant \langle Rx^*, x^* \rangle.$$

Then every $Q \in \mathscr{Q}$ is the covariance of a Gaussian measure μ_Q on E and for all $p \in [1, \infty)$ we have

$$\int_E \|x\|^p \, d\mu_Q(x) \leqslant \int_E \|x\|^p \, d\nu(x).$$

Moreover, the family $\{\mu_Q\}_{Q \in \mathscr{Q}}$ is uniformly tight.

More generally, the above estimate holds when we replace $\| \cdot \|^p$ by any nonnegative convex symmetric function $g \in L^1(E, \nu)$.

In Section 6 we will need the following result from [16]:

Proposition 1.3. *Let E be a real Banach space not containing a closed subspace isomorphic to c_0. Let (μ_n) be a sequence of Gaussian measures on E and let (Q_n) be their sequence of covariance operators. Let $Q \in \mathscr{L}(E^*, E)$ be a positive symmetric operator such that*

$$\lim_{n \to \infty} \langle Q_n x^*, x^* \rangle = \langle Qx^*, x^* \rangle, \qquad \forall x^* \in E^*. \tag{1.2}$$

If

$$\sup_n \int_E \|x\|^2 \, d\mu_n(x) < \infty, \tag{1.3}$$

then Q is the covariance of a centred Gaussian measure μ on E and

$$\int_E \|x\|^2 \, d\mu(x) \leqslant \liminf_{n \to \infty} \int_E \|x\|^2 \, d\mu_n(x).$$

1.3. **Pettis integration.** Let (S, Σ, ν) be a finite measure space. A function $\phi : S \to E$ is called *weakly measurable* if the function $\langle \phi, x^* \rangle(\cdot) := \langle \phi(\cdot), x^* \rangle$ is measurable for all $x^* \in E^*$, and *Pettis integrable* if it is weakly measurable and for all measurable subsets $A \in \Sigma$ there exists an element $y_A \in E$ such that for all $x^* \in E^*$ we have

$$\langle y_A, x^* \rangle = \int_A \langle \phi, x^* \rangle \, d\nu.$$

In this situation we write

$$y_A = \int_A \phi \, d\nu.$$

Let $1 \leqslant p \leqslant \infty$. We call $\phi : S \to E$ *weakly L^p* if it is weakly measurable and the function $\langle \phi, x^* \rangle$ belongs to $L^p(S)$ for all $x^* \in E^*$. By the closed graph theorem, for such a function the associated operator $x^* \mapsto \langle \phi, x^* \rangle$ is bounded from E^* into $L^p(S)$.

We list some sufficient conditions for Pettis integrability:

- If $\phi : S \to E$ is Bochner integrable, then ϕ is Pettis integrable and the two integrals coincide.
- If $\phi : S \to E$ is strongly measurable and weakly L^p for some $p > 1$, then ϕ is Pettis integrable [17, Corollary 5.31]. As a particular case, if E is separable and $\phi : S \to E$ is weakly L^p for some $p > 1$, then by the Pettis measurability theorem [7, Theorem II.1.2], ϕ is strongly measurable and hence Pettis integrable.
- If E does not contain a closed subspace isomorphic to c_0 and $\phi : S \to E$ is strongly measurable and weakly L^1, then ϕ is Pettis integrable [7, Theorem II.3.7].
- Let $1 \leqslant p, q \leqslant \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. If $\phi : S \to E$ is Pettis integrable and weakly L^p, then for all $f \in L^q(S)$ the function $f\phi$ is Pettis integrable, and the induced operator $I_\phi : L^q(S) \to E$,
$$I_\phi f := \int_S f\phi \, d\nu, \qquad f \in L^q(S),$$
is bounded [17, Theorem 3.4].

2. Stochastic integration of E-valued functions

Let $\mathbb{W} = \{W(t)\}_{t\in[0,T]}$ be a standard Brownian motion over a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, adapted to some given filtration $\{\mathscr{F}_t\}_{t\in[0,T]}$.

Definition 2.1. *We call a function $\phi : (0, T) \to E$* stochastically integrable *with respect to $\mathbb{W}$ if it is weakly L^2 and for all measurable $A \subseteq (0, T)$ there exists an E-valued random variable Y_A such that for all $x^* \in E^*$ we have*

$$\langle Y_A, x^* \rangle = \int_0^T \mathbf{1}_A(t) \langle \phi(t), x^* \rangle \, dW(t) \tag{2.1}$$

almost surely. In this situation we write
$$Y_A = \int_A \phi(t) \, dW(t).$$

For strongly measurable functions ϕ this notion of stochastic integrability has been introduced by Rosiński and Suchanecki; cf. also the discussion at the end of this section.

The random variables Y_A are uniquely determined almost everywhere and Gaussian. Indeed, the right hand side in (2.1) is a real-valued Gaussian variable for all $x^* \in E^*$. Thus, by the Fernique theorem, $Y_A \in L^p(\Omega; E)$ for all $p \in [1, \infty)$.

Remark 2.2. As in [18], the underlying measure space $(0, T)$ (with the Lebesgue measure) may be replaced by an arbitrary finite measure space (S, Σ, ν). The Brownian motion should then be replaced by a *Gaussian random measure* $W : \Sigma \to L^2(\Omega)$ with the following properties:

(1) For all $A \in \Sigma$, the random variable $W(A)$ is centred Gaussian and
$$\mathbb{E}(W(A))^2 = \nu(A);$$
(2) For all disjoint $A_1, \dots, A_n \in \Sigma$, the random variables $W(A_1), \dots, W(A_n)$ are independent and we have
$$W(A_1 \cup \dots \cup A_n) = W(A_1) + \dots + W(A_n).$$

All results of this paper can be generalized without difficulty to this more general setting.

We collect some elementary properties of the stochastic integral that are immediate consequences of Definition 2.1. Let $\phi : (0,T) \to E$ and $\psi : (0,T) \to E$ be stochastically integrable with respect to $\mathbb{W}$.

- For all measurable subsets $B \subseteq (0,T)$ the function $\mathbf{1}_B\phi$ is stochastically integrable with respect to $\mathbb{W}$ and
$$\int_0^T \mathbf{1}_B(t)\phi(t)\,dW(t) = \int_B \phi(t)\,dW(t)$$
almost surely;
- For all $a, b \in \mathbb{R}$ the function $a\phi + b\psi$ is stochastically integrable with respect to $\mathbb{W}$ and
$$\int_0^T a\phi(t) + b\psi(t)\,dW(t) = a\int_0^T \phi(t)\,dW(t) + b\int_0^T \psi(t)\,dW(t)$$
almost surely;
- For all real Banach spaces F and all bounded operators $S \in \mathscr{L}(E,F)$ the function $S\phi : (0,T) \to F$ is stochastically integrable with respect to $\mathbb{W}$ and
$$\int_0^T S\phi(t)\,dW(t) = S\int_0^T \phi(t)\,dW(t)$$
almost surely.

For a weakly L^2 function $\phi : (0,T) \to E$ we define an operator $I_\phi : L^2(0,T) \to E^{**}$ by
$$\langle x^*, I_\phi f\rangle = \int_0^T \langle \phi(t), x^*\rangle f(t)\,dt, \qquad f \in L^2(0,T),\ x^* \in E^*.$$
Note that I_ϕ is the adjoint of the operator $x^* \mapsto \langle \phi(t), x^*\rangle$. If ϕ is strongly measurable, then I_ϕ maps $L^2(0,T)$ into E, as may be seen by approximating ϕ by step functions.

The following theorem gives a characterization of the class of stochastically integrable functions.

Theorem 2.3. *For a weakly L^2 function $\phi : (0,T) \to E$ the following assertions are equivalent:*

(1) *ϕ is stochastically integrable with respect to $\mathbb{W}$;*

(2) *There exists an E-valued random variable Y and a weak*-sequentially dense linear subspace F of E^* such that for all $x^* \in F$ we have*
$$\langle Y, x^*\rangle = \int_0^T \langle \phi(t), x^*\rangle\,dW(t) \quad \textit{almost surely;}$$

(3) *There exists a Gaussian measure μ on E with covariance operator $Q \in \mathscr{L}(E^*,E)$ and a weak*-sequentially dense linear subspace F of E^* such that for all $x^* \in F$ we have*
$$\int_0^T \langle \phi(t), x^*\rangle^2\,dt = \langle Qx^*, x^*\rangle;$$

(4) *There exists a separable real Hilbert space $\mathscr{H}$, a γ-radonifying operator $T \in \mathscr{L}(\mathscr{H},E)$, and a weak*-sequentially dense linear subspace F of E^* such that for all $x^* \in F$ we have*
$$\int_0^T \langle \phi(t), x^*\rangle^2\,dt \leqslant \|T^*x^*\|^2_{\mathscr{H}};$$

(5) *We have $I_\phi \in \mathscr{L}(L^2(0,T),E)$ and this operator is γ-radonifying.*

In this situation, (2), (3), *and* (4) *hold with* $F = E^*$, *the function* ϕ *is Pettis integrable, the measure* μ *is the distribution of the random variable* $\int_0^T \phi(t)\,dW(t)$, *and we have an isometry*

$$\mathbb{E}\Big\|\int_0^T \phi(t)\,dW(t)\Big\|^2 = \|I_\phi\|_\gamma^2. \tag{2.2}$$

Proof. We prove (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2): This is trivial.

(2) $\Rightarrow$ (3): For all $x^* \in F$ the random variable $\langle Y, x^*\rangle$ is Gaussian. This implies that Y is Gaussian (this follows, e.g., from the fact that Y is essentially separably valued and [4, Corollary 1.3]). Let Q denote the covariance operator of its distribution. By the Itô isometry, for all $x^* \in E^*$ we have

$$\langle Qx^*, x^*\rangle = \mathbb{E}\Big(\int_0^T \langle \phi(t), x^*\rangle\,dW(t)\Big)^2 = \int_0^T \langle \phi(t), x^*\rangle^2\,dt.$$

(3) $\Rightarrow$ (4): Take $\mathscr{H} = H_Q$, the RKHS associated with Q, and $T = i_Q$.

(4) $\Rightarrow$ (5): Define a positive symmetric operator $Q \in \mathscr{L}(E^*, E^{**})$ by

$$\langle y^*, Qx^*\rangle := \int_0^T \langle \phi(t), x^*\rangle\langle \phi(t), y^*\rangle\,dt, \qquad x^*, y^* \in E^*.$$

By assumption we have $\langle x^*, Qx^*\rangle \leqslant \langle TT^*x^*, x^*\rangle$ for all $x^* \in F$. We claim that in fact this holds for all $x^* \in E^*$.

For the proof of the claim we consider the RKHS H_R associated with $R := T \circ T^*$ and note that the operator $i_R' : E^* \to H_R$ defined in Section 1.1 equals the adjoint i_R^* of i_R provided we identify H_R and its dual in the usual way.

Fix an arbitrary $x^* \in E^*$ and let (x_n^*) be a sequence in F converging to x^* weak* in E^*. Then $\lim_{n\to\infty} i_R^* x_n^* = i_R^* x^*$ weakly in H_R. Indeed, i_R^* is an adjoint operator and is therefore weak*-continuous; since H_R is a Hilbert space the weak topology and the weak*-topology of H_R coincide. By a corollary to the Hahn-Banach theorem, for suitably chosen convex combinations y_n^* of the x_n^* we may arrange that $\lim_{n\to\infty} i_R^* y_n^* = i_R^* x^*$ strongly in H_R. By choosing each y_n^* in the span of $x_n^*, x_{n+1}^*, \dots$ we may furthermore arrange that $\lim_{n\to\infty} y_n^* = x^*$ weak*. From $y_n^* - y_m^* \in F$ we have

$$\int_0^T \langle \phi(t), y_n^* - y_m^*\rangle^2\,dt \leqslant \langle R(y_n^* - y_m^*), (y_n^* - y_m^*)\rangle = \|i_R^* y_n^* - i_R^* y_m^*\|_{H_R}^2$$

and therefore the sequence $(\langle \phi, y_n^*\rangle)$ is Cauchy in $L^2(0,T)$. Let h denote the limit. Upon passing to a pointwise almost everywhere convergent subsequence and recalling that $\lim_{n\to\infty} y_n^* = x^*$ weak*, we see that $h = \langle \phi, x^*\rangle$ almost everywhere. Then,

$$\begin{aligned}\langle x^*, Qx^*\rangle = \int_0^T \langle \phi(t), x^*\rangle^2\,dt &= \lim_{n\to\infty}\int_0^T \langle \phi(t), y_n^*\rangle^2\,dt \leqslant \lim_{n\to\infty}\langle Ry_n^*, y_n^*\rangle \\ &= \lim_{n\to\infty}\|i_R^* y_n^*\|_{H_R}^2 = \|i_R^* x^*\|_{H_R}^2 = \langle Rx^*, x^*\rangle = \langle TT^*x^*, x^*\rangle.\end{aligned}$$

This proves the claim. From the observations in Section 1.1 it now follows that Q takes its values in E. We will use this to deduce that also I_ϕ takes its values in E.

Let $G := \{\langle \phi, x^*\rangle : x^* \in E^*\}$; this is a linear subspace of $L^2(0,T)$. We have $f \in \ker I_\phi$ if and only if $\langle x^*, I_\phi f\rangle = 0$ for all $x^* \in E^*$, and by the very definition of I_ϕ this happens if and only if $f \perp \langle \phi, x^*\rangle$ for all $x^* \in E^*$, i.e., if and only $f \perp G$. Therefore,

$$L^2(0,T) = \overline{G} \oplus \ker I_\phi. \tag{2.3}$$

Since for all $x^*, y^* \in E^*$,

$$\langle I_\phi \langle \phi, x^* \rangle, y^* \rangle = \int_0^T \langle \phi(t), x^* \rangle \langle \phi(t), y^* \rangle \, dt = \langle Qx^*, y^* \rangle$$

it follows that $I_\phi \langle \phi, x^* \rangle = Qx^*$ for all $x^* \in E^*$. Since Q takes its values in E, it follows that $I_\phi \, g \in E$ for all $g \in G$. Then (2.3) shows that I_ϕ takes values in E.

Now the identity $I_\phi \circ I_\phi^* = Q$ and the fact that Q is a Gaussian covariance operator imply that $I_\phi : L^2(0,T) \to E$ is γ-radonifying.

(5) $\Rightarrow$ (1): Fix $A \subseteq (0,T)$ measurable and put $\phi_A(t) := \mathbf{1}_A(t)\phi(t)$. Defining $M_A : L^2(0,T) \to L^2(0,T)$ by $M_A f := \mathbf{1}_A f$, we have $I_{\phi_A} = I_\phi \circ M_A$. Hence by the right ideal property of γ-radonifying operators, I_{ϕ_A} is γ-radonifying.

Choose an orthonormal basis (f_n) for $L^2(0,T)$. Denoting by $J : L^2(0,T) \to L^2(\Omega)$ the Itô isometry, the sequence (Jf_n) consists of independent standard normal random variables. It follows that the E-valued Gaussian series $Y_A := \sum_n Jf_n \, I_{\phi_A} f_n$ converges in $L^2(\Omega; E)$. Then for all $x^* \in E^*$,

$$\begin{aligned}
\langle Y_A, x^* \rangle &= \sum_n \langle I_{\phi_A} f_n, x^* \rangle \, Jf_n \\
&= \int_0^T \sum_n [\langle \phi_A, x^* \rangle, f_n]_{L^2(0,T)} \, f_n(t) \, dW(t) = \int_0^T \langle \phi_A(t), x^* \rangle \, dW(t)
\end{aligned}$$

almost surely. This proves that ϕ is stochastically integrable.

It remains to prove the final assertions. The Pettis integrability of ϕ follows from (5) by observing that for all measurable $A \subseteq (0,T)$ and all $x^* \in E^*$ we have

$$\langle I_\phi \mathbf{1}_A, x^* \rangle = \int_A \langle \phi(t), x^* \rangle \, dt.$$

Finally, (2.2) is an immediate consequence of Proposition 1.1 and the fact that μ is the distribution of $\int_0^T \phi(t) \, dW(t)$. $\square$

Remark 2.4. If ϕ is strongly measurable (in particular, by the Pettis measurability theorem, if E separable), then in (2), (3), and (4) it suffices to assume that F is weak*-dense. Indeed, by Zorn's lemma there exist maximal weak*-dense linear subspaces $F^{(2)}$, $F^{(3)}$, and $F^{(4)}$ of E^* for which these conditions hold. By arguing as in the proof of (4) $\Rightarrow$ (5) one shows that these subspaces are weak*-sequentially closed. By an easy application of the Krein-Smulyan theorem it then follows that they equal E^*.

The above notion of stochastic integrability was introduced, for the class of strongly measurable functions, by Rosiński and Suchanecki [18, Section 4], who also noted the equivalence (1) $\Leftrightarrow$ (3) of Theorem 2.3 for such functions (with $F = E^*$). [18, Corollary 4.2]. They also showed [18, Theorem 4.1] that, for a strongly measurable function ϕ, condition (3) (with $F = E^*$) is satisfied if and only if ϕ is stochastically integrable in probability, i.e., there exists a sequence of strongly measurable step functions (ϕ_n) and a random variable Y such that $\lim_{n\to\infty} \phi_n = \phi$ in measure and

$$Y = \lim_{n\to\infty} \int_0^T \phi_n(t) \, dW(t) \quad \text{in probability.}$$

In this case we have $Y = \int_0^T \phi(t) \, dW(t)$. In Section 4 below, we shall extend this characterization as follows:

Theorem 2.5 (Approximation with step functions)**.** *For a weakly L^2 function $\phi : (0,T) \to E$ the following assertions are equivalent:*

1. *ϕ is stochastically integrable with respect to $\mathbb{W}$;*
2. *There exists a sequence of step functions $\phi_n : (0,T) \to E$ with the following properties:*
   (a) *For all $x^* \in E^*$ we have*

$$\lim_{n\to\infty} \langle \phi_n, x^*\rangle = \langle \phi, x^*\rangle \quad \textit{in measure}; \tag{2.4}$$

   (b) *There exists a random variable $Y : \Omega \to E$ such that*

$$Y = \lim_{n\to\infty} \int_0^T \phi_n(t)\, dW(t) \quad \textit{in probability.} \tag{2.5}$$

In this situation we have $Y = \int_0^T \phi(t)\, dW(t)$, the convergence in (2.4) *is in $L^2(0,T)$, and the convergence in* (2.5) *is in $L^p(\Omega; E)$ for every $p \in [1,\infty)$.*

By results in [10, 18], a Banach space has type 2 if and only if every function $\phi \in L^2(0,T;E)$ is stochastically integrable. This also follows readily from Theorem 2.3 and the fact that for a step function

$$\phi = (\tfrac{n}{T})^{\frac12} \sum_{j=1}^n \mathbf{1}_{\left(\frac{(j-1)T}{n}, \frac{jT}{n}\right]} \otimes x_j$$

with $x_j \in E$ we have

$$\mathbb{E}\Big\|\int_0^T \phi(t)\, dW(t)\Big\|^2 = \mathbb{E}\Big\|\sum_{j=1}^n \gamma_j x_j\Big\|^2$$

where $\gamma_j := (\frac{n}{T})^{\frac12}\big(W(\frac{jT}{n}) - W(\frac{(j-1)T}{n})\big)$ are independent standard normal random variables. A similar result with the implications reversed holds for spaces with cotype 2. In Section 6 we shall extend these results to the operator-valued case.

The following example shows that stochastic integrability does not imply strong measurability and that Theorem 2.3 may fail if F is only assumed to be weak*-dense.

Example 2.6. For $p \in [1,\infty)$ let $E_p := l^p(0,1)$, the space of all functions $x : (0,1) \to \mathbb{R}$ for which

$$\|x\|^p := \sup_F \sum_{t\in F} |x(t)|^p < \infty,$$

where the supremum is taken over all finite subsets F of $(0,1)$. Note that for every $x \in E_p$ the set $\{t \in (0,1) :\ x(t) \neq 0\}$ is at most countable. Similarly we define $E_\infty := l^\infty(0,1)$ as the space of all bounded functions $x : (0,1) \to \mathbb{R}$ endowed with the supremum norm.

First take $p \in (1,\infty)$ and define $f : (0,1) \to E_p$ by $\phi(t) = \mathbf{1}_{\{t\}}$. For all $x^* \in E_p^* = E_q$ we have $\langle \phi(t), x^*\rangle = 0$ for all but at most countably many $t \in (0,1)$, and therefore ϕ is Pettis integrable with $\int_A \phi(t)\, dt = 0$ for all measurable $A \subseteq (0,1)$. Similarly we have

$$\int_0^1 \langle \phi(t), x^*\rangle\, dt = \langle Q_0 x^*, x^*\rangle = 0 \qquad \forall x^* \in E_q,$$

where $Q_0 := 0$ is the covariance operator of the Dirac measure on E_p concentrated at the origin. It follows that ϕ is stochastically integrable with

$$\int_A \phi(t)\, dW(t) = 0$$

for all measurable $A \subseteq (0,1)$. On the other hand, since $\|\phi(t)-\phi(s)\|_{E_p}^p = 2$ for all $t,s \in (0,1)$ with $t \neq s$, the function ϕ fails to be essentially separably-valued. Consequently ϕ is not strongly measurable by the Pettis measurability theorem.

Next consider the function $\phi : (0,T) \to E_1 = l^1(0,1)$ defined by $\phi_t := \delta_{\{t\}}$. We will show that ϕ is not stochastically integrable, although (2), (3), and (4) hold for all $x^* \in F := c_0(0,1)$ if we take $Y = 0$, $Q = 0$, and $T = 0$ (with $\mathscr{H}$ arbitrary). Here $c_0(0,1)$ is defined as closure in $l^\infty(0,T)$ of all finitely supported functions. This space is norming for $l^1(0,1)$, hence weak*-dense as a subspace of $l^\infty(0,1)$, but it is not weak*-sequentially dense in $l^\infty(0,1)$.

For all $x^* \in F$ we have

$$\langle Y, x^*\rangle = \int_0^1 \langle \phi(t), x^*\rangle \, dW(t) = 0$$

almost surely, because $\langle \phi(t), x^*\rangle = 0$ for all but at most countably many $t \in (0,1)$. Similarly,

$$\int_0^1 \langle \phi(t), x^*\rangle \, dt = \langle Qx^*, x^*\rangle = \langle TT^*x^*, x^*\rangle = 0$$

for all $x^* \in F$.

To show that ϕ is not stochastically integrable, we argue by contradiction. If ϕ were stochastically integrable could consider the random variable

$$Y := \int_0^1 \phi(t)\, dW(t).$$

As before, for all $x^* \in F$ we have $\langle Y, x^*\rangle = 0$ almost surely. Since F is norming it follows that $Y = 0$ almost surely. For the constant one function $\mathbf{1} \in l^\infty(0,1)$ we then have

$$0 = \langle Y, \mathbf{1}\rangle = \int_0^1 \langle \phi(t), \mathbf{1}\rangle \, dW(t) = \int_0^1 1 \, dW(t) = W(1)$$

almost surely, a contradiction.

We proceed with some consequences of Theorem 2.3. As a first observation we note that the stochastic integrability of a function $\phi : (0,T) \to E$ does not depend on the particular choice of the Brownian motion $\mathbb{W}$ (its integral does depend on $\mathbb{W}$, of course). Moreover, it follows from Theorem 2.3 that a function $\phi : (0,T) \to E$ is stochastically integrable with respect to $\mathbb{W}$ if and only if $\check{\phi} : (0,T) \to E$ is, where $\check{\phi}(t) := \phi(T-t)$.

Functions that are dominated by some stochastically integrable function are stochastically integrable:

Corollary 2.7. *Let F be a weak*-sequentially dense linear subspace of E^*. If $\phi : (0,T) \to E$ is stochastically integrable with respect to $\mathbb{W}$, and if $\psi : (0,T) \to E$ is a weakly measurable function which satisfies*

$$\int_0^T \langle \psi(t), x^*\rangle^2 \, dt \leqslant \int_0^T \langle \phi(t), x^*\rangle^2 \, dt \qquad \forall x^* \in F,$$

then ψ is stochastically integrable with respect to $\mathbb{W}$ and for all $p \in [1,\infty)$ we have

$$\mathbb{E}\Big\|\int_0^T \psi(t)\, dW(t)\Big\|^p \leqslant \mathbb{E}\Big\|\int_0^T \phi(t)\, dW(t)\Big\|^p.$$

If E is separable, it is enough to assume that F is weak*-dense in E^*.

Proof. This follows from Proposition 1.2 and condition (4) in Theorem 2.3, applied to $\mathscr{H} = L^2(0,T)$ and $T = I_\phi$. The final assertion follows from Remark 2.4. $\square$

As a second corollary to Theorem 2.3 we show that the stochastic integral defines a martingale that is continuous in p-th mean for all $p \in [1,\infty)$:

Corollary 2.8. *Let $\phi : (0,T) \to E$ be stochastically integrable with respect to $\mathbb{W}$. Then the E-valued process*

$$M(t) := \int_0^t \phi(r)\, dW(r), \qquad t \in [0,T],$$

is a martingale adapted to the filtration $\{\mathscr{F}_t\}_{t\in[0,T]}$ that is continuous in p-th moment for all $p \in [1,\infty)$.

Proof. For all $0 \leqslant s \leqslant t \leqslant T$ and all $x^* \in E^*$,

$$\begin{aligned}\Big\langle \mathbb{E}\Big(\int_0^t \phi(r)\, dW(r) \,\Big|\, \mathscr{F}_s\Big), x^* \Big\rangle &= \mathbb{E}\Big(\int_0^t \langle \phi(r), x^*\rangle\, dW(r) \,\Big|\, \mathscr{F}_s\Big) \\ &= \int_0^s \langle \phi(r), x^*\rangle\, dW(r) = \Big\langle \int_0^s \phi(r)\, dW(r), x^* \Big\rangle\end{aligned}$$

almost surely. This implies that

$$\mathbb{E}\Big(\int_0^t \phi(r)\, dW(r) \,\Big|\, \mathscr{F}_s\Big) = \int_0^s \phi(r)\, dW(r)$$

almost surely. This proves the martingale property.

Next let $t_n \downarrow t$ in the interval $[0,T]$; the case $t_n \uparrow t$ is handled similarly. Let R_n denote the covariance operator of the distribution ν_n of the Gaussian random variable $\int_t^{t_n} \phi(r)\, dW(r)$. For all $x^* \in E^*$ we have

$$\lim_{n\to\infty} \langle R_{t_n} x^*, x^*\rangle = \lim_{n\to\infty} \int_t^{t_n} \langle \phi(r), x^*\rangle^2\, dr = 0.$$

Moreover, for all n and all $x^* \in E^*$ we have $\langle R_{t_n} x^*, x^*\rangle \leqslant \langle Qx^*, x^*\rangle$, where Q is as in (3) of Theorem 2.3. Hence by Proposition 1.2 and a standard argument, for the associated distributions we have $\lim_{n\to\infty} \nu_{t_n} = \delta_0$ (the Dirac measure at 0) weakly. By [1, Lemma 3.8.7], this implies

$$\lim_{n\to\infty} \mathbb{E}\Big\| \int_t^{t_n} \phi(r)\, dW(r) \Big\|^p = \lim_{n\to\infty} \int_E \|x\|^p\, d\nu_n(x) = 0.$$

$\square$

Remark 2.9. By the same argument one proves that for every $p \in [1,\infty)$,

$$V(A) := \int_A \phi(t)\, dW(t), \qquad A \subseteq (0,T) \ \text{ measurable},$$

defines a countably additive $L^p(\Omega;E)$-valued vector measure V which is absolutely continuous with respect to the Lebesgue measure.

We conclude with some remarks on stochastic integrability in certain concrete classes of Banach spaces.

For L^p-spaces, a precise characterization of Gaussian covariance operators is known, cf. [21, Theorem V.5.5], and this can be used in conjunction with condition (3) in Theorem 2.3 to give a precise description the class of stochastically integrable functions;

cf. also [18, Corollary 4.3]. The following result extends this description to a wider class of function spaces and does not depend on such a characterization. Rather, it exploits condition (4) of Theorem 2.3 in a direct way.

Corollary 2.10. *Let E be Banach function space with finite cotype over a σ-finite measure space (S, Σ, ν). A strongly measurable function $\phi : (0,T) \to E$ is stochastically integrable with respect to $\mathbb{W}$ if and only if*

$$\Big\| \Big(\int_0^T |\phi(t,\cdot)|^2 \, dt \Big)^{\frac12} \Big\|_E < \infty.$$

In this case we have

$$\Big(\mathbb{E} \Big\| \int_0^T \phi(t) \, dW(t) \Big\|_E^2 \Big)^{\frac12} \sim \Big\| \Big(\int_0^T |\phi(t,\cdot)|^2 \, dt \Big)^{\frac12} \Big\|_E,$$

where '$\sim$' means that both quantities are proportional to each other with constants depending on E only.

Here we write $\phi(t,\xi) := (\phi(t))(\xi)$. Since ϕ can be approximated pointwise a.e. by a sequence of E-valued step functions, and since every convergent sequence in E contains a subsequence which converges pointwise μ-a.e., the function $(t,\xi) \mapsto \phi(t,\xi)$ is easily seen to be jointly measurable. Hence by Fubini's theorem, for μ-almost $\xi \in S$ the function $t \mapsto \phi(t,\xi)$ is measurable, and the expressions in the statement of the theorem are well defined.

Proof. Let (f_n) be an orthonormal basis of $L^2(0,T)$ and let (r_n) be a sequence of independent Rademacher variables.

If ϕ is stochastically integrable, then by the finite cotype assumption, [14, Theorem 1.d.6, Corollary 1.f.9] and [6, Proposition 12.11, Theorem 12.27] we obtain:

$$\begin{aligned}
\Big(\mathbb{E} \Big\| \int_0^T \phi(t) \, dW(t) \Big\|_E^2 \Big)^{\frac12} &= \|I_\phi\|_\gamma = \Big(\mathbb{E} \Big\| \sum_n \gamma_n \int_0^T \phi(t,\cdot) f_n(t) \, dt \Big\|_E^2 \Big)^{\frac12} \\
&\sim \Big(\mathbb{E} \Big\| \sum_n r_n \int_0^T \phi(t,\cdot) f_n(t) \, dt \Big\|_E^2 \Big)^{\frac12} \sim \Big\| \Big(\sum_n \Big| \int_0^T \phi(t,\cdot) f_n(t) \, dt \Big|^2 \Big)^{\frac12} \Big\|_E \\
&= \Big\| \Big(\int_0^T |\phi(t,\cdot)|^2 \, dt \Big)^{\frac12} \Big\|_E.
\end{aligned}$$

Conversely, if $\Big\| \Big(\int_0^T |\phi(t,\cdot)|^2 \, dt \Big)^{\frac12} \Big\|_E < \infty$, we can read these estimates backwards, but we have to check that ϕ is weakly L^2 and that I_ϕ takes values in E, since otherwise I_ϕ is not well defined as an operator from $L^2(0,T)$ into E. We estimate the middle expression by applying an element $x^* \in E^*$ of norm one:

$$\begin{aligned}
&\Big(\mathbb{E} \Big\| \sum_n r_n \int_0^T \phi(t,\cdot) f_n(t) \, dt \Big\|_E^2 \Big)^{\frac12} \\
&\qquad \geqslant \Big(\mathbb{E} \Big| \sum_n r_n \int_0^T \langle \phi(t), x^* \rangle f_n(t) \, dt \Big|^2 \Big)^{\frac12} \\
&\qquad = \Big(\sum_n \Big| \int_0^T \langle \phi(t), x^* \rangle f_n(t) \, dt \Big|^2 \Big)^{\frac12} = \Big(\int_0^T |\langle \phi(t), x^* \rangle|^2 \, dt \Big)^{\frac12}.
\end{aligned}$$

This shows that ϕ is weakly L^2. Since ϕ is also strongly measurable, it follows that ϕ is Pettis integrable. Then for any $f \in L^2(0,T)$, $f\phi$ is Pettis integrable as well and I_ϕ takes values in E. □

3. Cylindrical Wiener processes

Let H be a real Hilbert space.

Definition 3.1. *A* cylindrical H-Wiener process $\mathbb{W}_H = \{W_H(t)\}_{t\in[0,T]}$ *is a family of bounded linear operators from H into $L^2(\Omega)$ with the following properties:*

(1) *For all $h \in H$, $\{W_H(t)h\}_{t\in[0,T]}$ is a standard Brownian motion;*
(2) *For all $s,t \in [0,T]$ and $g,h \in H$ we have*

$$\mathbb{E}(W_H(s)g \cdot W_H(t)h) = (s \wedge t)[g,h]_H.$$

We shall always assume that $\mathbb{W}_H$ is adapted to a given filtration $\{\mathscr{F}_t\}_{t\in[0,T]}$; by this we mean that the Brownian motions $\{W_H(t)h\}_{t\in[0,T]}$ are adapted to $\{\mathscr{F}_t\}_{t\in[0,T]}$.

Example 3.2. If $\mathbb{W}$ is a standard Brownian motion, then $W_{\mathbb{R}}(t) : \mathbb{R} \to L^2(\Omega)$,

$$W_{\mathbb{R}}(t)h := hW(t), \qquad h \in \mathbb{R},$$

is a cylindrical $\mathbb{R}$-Wiener process. Conversely, every cylindrical $\mathbb{R}$-Wiener process arises in this way.

More generally, suppose $C \in \mathscr{L}(E^*, E)$ is a Gaussian covariance operator and $\mathbb{W}^C$ is an E-valued Brownian motion satisfying

$$\mathbb{E}\langle W^C(t), x^*\rangle^2 = t\,\langle Cx^*, x^*\rangle \qquad \forall t \geqslant 0,\ x^* \in E^*,$$

i.e., C is the covariance operator of the Gaussian random variable $W^C(1)$. Let (H_C, i_C) be the reproducing kernel Hilbert space associated with C. Then the mapping

$$W_{H_C}(t) i_C^* x^* := \langle W^C(t), x^*\rangle \qquad t \geqslant 0,\ x^* \in E^*,$$

uniquely extends to a cylindrical H_C-Wiener process $\mathbb{W}_{H_C}$. We return to this example in Section 5.

In the next section we shall be concerned with setting up a stochastic integral for certain $\mathscr{L}(H,E)$-valued functions with respect to a cylindrical H-Wiener process $\mathbb{W}_H$. Before we can do so, we need to observe first that we can integrate certain H-valued functions with respect to $\mathbb{W}_H$. In contrast to the other integrals studied in this paper, the construction is completely straightforward. Indeed, for a step function of the form $\mathbf{1}_{(a,b]} \otimes h$ we define

$$\int_0^T \mathbf{1}_{(a,b)}(t) \otimes h\, dW_H(t) := W_H(b)h - W_H(a)h. \tag{3.1}$$

This is extended to arbitrary step functions ψ by linearity, and a standard computation shows that

$$\mathbb{E}\Big(\int_0^T \psi(t)\, dW_H(t)\Big)^2 = \int_0^T \|\psi(t)\|_H^2\, dt.$$

Since the step functions are dense in $L^2(0,T;H)$, the map

$$J_H : \psi \mapsto \int_0^T \psi(t)\, dW_H(t)$$

extends uniquely to an isometry from $L^2(0,T;H)$ into $L^2(\Omega)$. For all $f \in L^2(0,T)$ and all $h \in H$ we have

$$\int_0^T f(t) \otimes h \, dW_H(t) = \int_0^T f(t) \, dW_H(t)h \tag{3.2}$$

as can be seen by approximating f with step functions and using (3.1). In Section 7 the following integration by parts formula will be useful: for all $f \in C^1[0,T]$, $B \in \mathscr{L}(H,E)$, and $x^* \in E^*$,

$$\int_0^T f'(t)W_H(t)B^*x^* \, dt = f(T)W_H(T)B^*x^* - \int_0^T f(t) \otimes B^*x^* \, dW_H(t). \tag{3.3}$$

This formula follows from (3.2) and the corresponding integration by parts formula for standard Brownian motions. We will also need the following version of Fubini's theorem. See also [5], where an analogous version of this result for processes is given in a slightly different formulation.

Theorem 3.3 (Fubini). *Let $\psi : (0,T) \times (0,T) \to H$ be jointly measurable and assume that*

$$\int_0^T \|\psi(s,\cdot)\|_{L^2(0,T;H)} ds < \infty.$$

(1) *The $L^2(\Omega)$-valued function*

$$s \mapsto \int_0^T \psi(s,t) \, dW_H(t)$$

is Bochner integrable;

(2) *For almost all $t \in (0,T)$ we have $s \mapsto \psi(s,t) \in L^1(0,T;H)$, and the H-valued function*

$$t \mapsto \int_0^T \psi(s,t) \, ds$$

is square Bochner integrable;

(3) *In $L^2(\Omega)$ we have*

$$\int_0^T \Big(\int_0^T \psi(s,t) \, dW_H(t) \Big) \, ds = \int_0^T \Big(\int_0^T \psi(s,t) \, ds \Big) \, dW_H(t).$$

4. Stochastic integration of $\mathscr{L}(H,E)$-valued functions

From this point onwards we shall assume that H is a separable real Hilbert space. As before, E is a real Banach space. In this section we define a stochastic integral for certain $\mathscr{L}(H,E)$-valued functions with respect to a cylindrical H-Wiener process $\mathbb{W}_H$. In the case $H = \mathbb{R}$ we may identify $\mathscr{L}(\mathbb{R},E)$ with E, in which case the integral reduces to the integral for E-valued functions of Section 2.

We say that a function $\Phi : (0,T) \to \mathscr{L}(H,E)$ is *H-weakly L^2* if for all $x^* \in E^*$ the map $t \mapsto \Phi^*(t)x^*$ is strongly measurable and satisfies

$$\int_0^T \|\Phi^*(t)x^*\|_H^2 \, dt < \infty.$$

By the separability of H and the Pettis measurability theorem, the strong measurability of $t \mapsto \Phi^*(t)x^*$ is equivalent to its weak measurability.

Definition 4.1. *We call a function $\Phi : (0,T) \to \mathscr{L}(H,E)$* stochastically integrable *with respect to $\mathbb{W}_H$ if it is H-weakly L^2 and for all measurable $A \subseteq (0,T)$ there exists an E-valued random variable Y_A such that for all $x^* \in E^*$ we have*

$$\langle Y_A, x^* \rangle = \int_0^T \mathbf{1}_A(t)\, \Phi^*(t) x^* \, dW_H(t)$$

almost surely. In this situation we write

$$Y_A = \int_A \Phi(t)\, dW_H(t).$$

The random variables Y_A are uniquely determined almost everywhere and Gaussian; this is proved in the same way as in Section 2. In particular, $Y_A \in L^p(\Omega; E)$ for all $p \in [1,\infty)$.

This definition agrees with the one in [20], where the case of an E-valued Brownian motion was considered; cf. Section 5 below.

We collect some elementary properties of the stochastic integral that are immediate consequences of Definition 4.1. Let $\Phi : (0,T) \to \mathscr{L}(H,E)$ and $\Psi : (0,T) \to \mathscr{L}(H,E)$ be stochastically integrable with respect to $\mathbb{W}_H$.

- For all measurable subsets $B \subseteq (0,T)$ the function $\mathbf{1}_B\Phi$ is stochastically integrable with respect to $\mathbb{W}_H$ and
$$\int_0^T \mathbf{1}_B(t)\Phi(t)\, dW_H(t) = \int_B \Phi(t)\, dW_H(t)$$
almost surely;
- For all $a, b \in \mathbb{R}$ the function $a\Phi + b\Psi$ is stochastically integrable with respect to $\mathbb{W}_H$ and
$$\int_0^T a\Phi(t) + b\Psi(t)\, dW_H(t) = a\int_0^T \Phi(t)\, dW_H(t) + b\int_0^T \Psi(t)\, dW_H(t)$$
almost surely;
- For all real Banach spaces F and all bounded operators $S \in \mathscr{L}(E,F)$ the function $S\Phi : (0,T) \to \mathscr{L}(H,F)$ is stochastically integrable with respect to $\mathbb{W}_H$ and
$$\int_0^T S\Phi(t)\, dW_H(t) = S\int_0^T \Phi(t)\, dW_H(t)$$
almost surely.

For an H-weakly L^2 function $\Phi : (0,T) \to \mathscr{L}(H,E)$ we define an operator $I_\Phi : L^2(0,T;H) \to E^{**}$ by

$$\langle x^*, I_\Phi f \rangle := \int_0^T [\Phi^*(t)x^*, f(t)]_H \, dt, \qquad f \in L^2(0,T;H),\ x^* \in E^*.$$

Note that I_Φ is the adjoint of the operator $x^* \mapsto \Phi^*(\cdot)x^*$ from E^* into $L^2(0,T;H)$.

If the functions $t \mapsto \Phi(t)h$ are strongly measurable for all $h \in H$, then I_Φ maps $L^2(0,T;H)$ into E. Indeed, this is clear for step functions Φ of the form $\sum_{n=1}^N \mathbf{1}_{A_n} \otimes h_n$ with the property that $t \mapsto \Phi(t)h_n$ is bounded on A_n; the general case follows from the fact that these step functions are dense in $L^2(0,T;H)$. The following theorem characterizes the class of stochastically integrable functions.

Theorem 4.2. *For an H-weakly L^2 function $\Phi : (0,T) \to \mathscr{L}(H,E)$ the following assertions are equivalent:*

(1) *Φ is stochastically integrable with respect to $\mathbb{W}_H$;*
(2) *There exists an E-valued random variable Y and a weak*-sequentially dense linear subspace F of E^* such that for all $x^* \in F$ we have*
$$\langle Y, x^* \rangle = \int_0^T \Phi^*(t)x^* \, dW_H(t) \quad \textit{almost surely;}$$
(3) *There exists a Gaussian measure μ on E with covariance operator Q and a weak*-sequentially dense linear subspace F of E^* such that for all $x^* \in F$ we have*
$$\int_0^T \|\Phi^*(t)x^*\|_H^2 \, dt = \langle Qx^*, x^* \rangle;$$
(4) *There exists a separable real Hilbert space $\mathscr{H}$, a γ-radonifying operator $T \in \mathscr{L}(\mathscr{H}, E)$, and a weak*-sequentially dense linear subspace F of E^* such that for all $x^* \in F$ we have*
$$\int_0^T \|\Phi^*(t)x^*\|_H^2 \, dt \leqslant \|T^*x^*\|_{\mathscr{H}}^2;$$
(5) *I_Φ maps $L^2(0,T;H)$ into E and $I_\Phi \in \mathscr{L}(L^2(0,T;H), E)$ is γ-radonifying.*

If these equivalent conditions hold, then in (2), (3), and (4) we may take $F = E^*$, for all $h \in H$ the function $\Phi(\cdot)h$ is both Pettis integrable and stochastically integrable with respect to $\mathbb{W}_H(\cdot)h$, and we have the representation

$$\int_0^T \Phi(t) \, dW_H(t) = \sum_n \int_0^T \Phi(t)h_n \, dW_H(t)h_n, \tag{4.1}$$

where (h_n) is any orthonormal basis for H; the series converges unconditionally in $L^p(\Omega; E)$ for all $p \in [1, \infty)$ and almost surely. The measure μ is the distribution of $\int_0^T \Phi(t) \, dW_H(t)$ and we have an isometry

$$\mathbb{E}\Big\| \int_0^T \Phi(t) \, dW_H(t) \Big\|^2 = \|I_\Phi\|_\gamma^2. \tag{4.2}$$

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) are proved in the same way as the corresponding ones in Theorem 2.3.

(5) $\Rightarrow$ (1): Fix $A \subseteq (0,T)$ measurable and put $\Phi_A(t) := \mathbf{1}_A(t)\Phi(t)$. Defining $M_A : L^2(0,T;H) \to L^2(0,T;H)$ by $M_A f := \mathbf{1}_A f$, we have $I_{\Phi_A} = I_\Phi \circ M_A$. Hence by the right ideal property of γ-radonifying operators, I_{Φ_A} is γ-radonifying.

Choose an orthonormal basis (f_n) for $L^2(0,T)$ and an orthonormal basis (h_m) for H. Let $k \mapsto (n(k), m(k))$ be a bijection from $\mathbb{N}$ onto $\mathbb{N} \times \mathbb{N}$. Denoting by $J_H : L^2(0,T;H) \to L^2(\Omega)$ the Itô isometry, the sequence $(J_H(f_{n(k)} \otimes h_{m(k)}))$ consists of independent standard normal random variables. It follows that the E-valued Gaussian series
$$Y_A := \sum_k J_H(f_{n(k)} \otimes h_{m(k)} \, I_{\Phi_A}(f_{n(k)} \otimes h_{m(k)})$$
converges unconditionally in $L^2(\Omega; E)$. For all $x^* \in E^*$ we have
$$\begin{aligned}\langle Y_A, x^* \rangle &= \sum_k \langle I_{\Phi_A}(f_{n(k)} \otimes h_{m(k)}), x^* \rangle J_H(f_{n(k)} \otimes h_{m(k)}) \\ &= \sum_k [\langle \Phi_A(\cdot)h_{m(k)}, x^* \rangle, f_{n(k)}]_{L^2(0,T)} \int_0^T f_{n(k)}(t) \otimes h_{m(k)} \, dW_H(t)\end{aligned}$$

$$
\begin{aligned}
&\overset{\text{(i)}}{=} \sum_k [\langle \Phi_A(\cdot)h_{m(k)}, x^*\rangle, f_{n(k)}]_{L^2(0,T)} \int_0^T f_{n(k)}(t)\, dW_H(t)h_{m(k)} \\
&\overset{\text{(ii)}}{=} \sum_m \int_0^T \sum_n [\langle \Phi_A(\cdot)h_m, x^*\rangle, f_n]_{L^2(0,T)}\ f_n(t)\, dW_H(t)h_m \\
&= \sum_m \int_0^T \langle \Phi_A(t)h_m, x^*\rangle\, dW_H(t)h_m \\
&\overset{\text{(iii)}}{=} \Big\langle \sum_m \int_0^T \Phi_A(t)h_m\, dW_H(t)h_m, x^*\Big\rangle.
\end{aligned}
\tag{4.3}
$$

In (i) we used the identity (3.2), in (ii) we used unconditionality, and the convergence of the sum at the right hand side of (iii), both in $L^p(\Omega; E)$ for all $p \in [1,\infty)$ and in the almost sure sense, follows from theorems of Itô-Nisio theorem [11] and Hoffmann-Jørgensen [9]. Moreover,

$$
\begin{aligned}
&\Big\langle \sum_m \int_0^T \Phi_A(t)h_m\, dW_H(t)h_m, x^*\Big\rangle \\
&\quad = \sum_m \int_0^T [h_m, \Phi_A^*(t)x^*]_H\, dW_H(t)h_m = \sum_m \int_0^T [h_m, \Phi_A^*(t)x^*]_H h_m\, dW_H(t) \\
&\quad = \int_0^T \sum_m [h_m, \Phi_A^*(t)x^*]_H h_m\, dW_H(t) = \int_0^T \mathbf{1}_A(t)\Phi^*(t)x^*\, dW_H(t).
\end{aligned}
$$

This proves that Φ is stochastically integrable. The series representation (4.1) for the stochastic integral of Φ has already been obtained in (4.3). The sum converges unconditionally since $(h_{\pi(n)})$ is an orthonormal basis for H for every permutation π of the indices.

It remains to prove the final assertions. The Pettis integrability of $\Phi(\cdot)h$ follows by observing that for all $A \subseteq (0,T)$ measurable and all $x^* \in E^*$ we have

$$\langle I_\Phi \mathbf{1}_A \otimes h, x^*\rangle = \int_A \langle \Phi(t)h, x^*\rangle\, dt.$$

Finally, (4.2) is an immediate consequence of Proposition 1.1 and the fact that μ is the distribution of $\int_0^T \Phi(t)\, dW_H(t)$. □

Remark 4.3. If Φ is strongly measurable (in particular, if E is separable), then in (2), (3), and (4) it suffices to assume that F is weak*-dense.

The equivalence (2) $\Leftrightarrow$ (3) (with $F = E^*$) was proved, in the case of separable Banach spaces E, in [4]. The proof presented above is not only more general, but also considerably simpler.

As in the case of E-valued functions, Theorem 4.2 shows that the stochastic integrability of a function Φ does not depend on the particular choice of the cylindrical Wiener process $\mathbb{W}_H$ and that a function Φ is stochastically integrable with respect to $\mathbb{W}_H$ if and only if its time reflection $\check{\Phi}$ is.

Functions that are dominated by some stochastically integrable function are stochastically integrable:

Corollary 4.4. *Let F be a weak*-sequentially dense linear subspace of E^*. If $\Phi : (0,T) \to \mathscr{L}(H,E)$ is stochastically integrable with respect to $\mathbb{W}_H$, and if $\Psi : (0,T) \to \mathscr{L}(H,E)$ is an H-weakly L^2 function which satisfies*

$$\int_0^T \|\Psi^*(t)x^*\|_H^2 \, dt \leqslant \int_0^T \|\Phi^*(t)x^*\|_H^2 \, dt \qquad \forall x^* \in F,$$

then Ψ is stochastically integrable with respect to $\mathbb{W}_H$ and for all $p \in [1,\infty)$ we have

$$\mathbb{E}\Big\|\int_0^T \Psi(t)\, dW_H(t)\Big\|^p \, dt \leqslant \mathbb{E}\Big\|\int_0^T \Phi(t)\, dW_H(t)\Big\|^p \, dt.$$

If E is separable, it is enough to assume that F is weak*-dense in E^*.

Parallel to Corollary 2.8 we have:

Corollary 4.5. *Let $\Phi : (0,T) \to \mathscr{L}(H,E)$ be stochastically integrable with respect to $\mathbb{W}_H$. Then the E-valued process*

$$t \mapsto \int_0^t \Phi(r)\, dW_H(r), \qquad t \in [0,T],$$

is a martingale adapted to the filtration $\{\mathscr{F}_t\}_{t\in[0,T]}$ that is continuous in p-th moment for all $p \in [1,\infty)$.

Proof. The martingale property follows from the first part of Corollary 2.8 and the representation formula (4.1) and the continuity in p-th moment is proved along the lines of the second part of Corollary 2.8. $\square$

By the remarks in Section 1 we obtain the following sufficient condition for stochastic integrability if E has type 2:

Corollary 4.6. *Let E be a separable real Banach space with type 2 and let $\Phi : (0,T) \to \mathscr{L}(H,E)$ be an H-weakly L^2 function. If*

$$\sum_n \Big\|\int_0^T \Phi(t) g_n(t)\, dt\Big\|^2 < \infty$$

for some orthonormal basis (g_n) for $L^2(0,T;H)$, then Φ is stochastically integrable with respect to $\mathbb{W}_H$ and

$$\mathbb{E}\Big\|\int_0^T \Phi(t)\, dW_H(t)\Big\|^2 \leqslant C_2^2 \sum_n \Big\|\int_0^T \Phi(t) g_n(t)\, dt\Big\|^2,$$

where C_2 is the type 2 constant of E.

Proof. The separability of E ensures that the functions $t \mapsto \Phi(t)g_n(t)$ are Pettis integrable; cf. the remarks in Section 1.3. The corollary now follows from

$$\|I_\Phi\|_\gamma^2 = \mathbb{E}\Big\|\sum_n \gamma_n \int_0^T \Phi(t) g_n(t)\, dt\Big\|^2 \leqslant C_2^2 \sum_n \Big\|\int_0^T \Phi(t) g_n(t)\, dt\Big\|^2$$

and an application of Theorem 4.2. $\square$

A similar result with the implications reversed holds for spaces with cotype 2. For separable real Hilbert spaces E we recover the well known fact that Φ is stochastically integrable if and only if I_Φ is a Hilbert-Schmidt operator.

We conclude this section with a reformulation of a result of [16], which gives another sufficient condition for stochastic integrability in space of type 2, this time in terms of a direct integrability assumption on Φ. A related result in a slightly different setting was obtained by Brzeźniak [2].

Theorem 4.7. *Let E be a separable real Banach space with type 2 and let $\Phi : (0,T) \to \mathscr{L}(H,E)$ be H-weakly L^2. If $\Phi(t)$ is γ-radonifying for almost all $t \in (0,T)$ and*

$$\int_0^T \|\Phi(t)\|_\gamma^2 \, dt < \infty,$$

then Φ is stochastically integrable with respect to $\mathbb{W}_H$ and

$$\mathbb{E}\Big\|\int_0^T \Phi(t)\, dW_H(t)\Big\|^2 \leqslant C_2^2 \int_0^T \|\Phi(t)\|_\gamma^2 \, dt,$$

where C_2 denotes the type 2 constant of E.

The special case for $H = \mathbb{R}$ is well known [10, 18]; cf. Section 2. In the same paper [16], an example is given of a stochastically integrable function $\Phi : (0,T) \to \mathscr{L}(l^2, l^p)$ (with $2 < p < \infty$) for which none of the operators $\Phi(t)$ is γ-radonifying.

5. Stochastic integration of $\mathscr{L}(E)$-valued functions

Let $C \in \mathscr{L}(E^*, E)$ be a Gaussian covariance operator and let $\mathbb{W}^C$ be a *C-Wiener process* in E, i.e., an E-valued Brownian motion satisfying

$$\mathbb{E}\langle W^C(t), x^*\rangle^2 = t\,\langle Cx^*, x^*\rangle \qquad \forall t \geqslant 0,\ x^* \in E^*.$$

Thus, C is the covariance operator associated with the Gaussian random variable $W^C(1)$. As was pointed out in Example 3.2, there is a canonical way of associating a cylindrical Wiener process $\mathbb{W}_{H_C}$ to $\mathbb{W}^C$. This construction enables us to integrate certain functions $\Phi : (0,T) \to \mathscr{L}(E)$ with respect to $\mathbb{W}^C$. In practice, it may be rather cumbersome to compute the space H_C explicitly, however. For this reason we will reformulate the definition of stochastic integrability in a way that avoids explicit reference to H_C. The resulting formalism is modelled after the one used in the monograph by Da Prato and Zabczyk [5].

The identity $i_C \circ i_C^* = C$ implies, via the Pettis measurability theorem and the fact that H_C is separable, that $t \mapsto i_C^*\Phi^*(t)x^*$ is strongly measurable if and only if

$$t \mapsto C\Phi^*(t)x^* \ \text{ is weakly measurable.} \tag{5.1}$$

If E is separable, this condition is automatically satisfied if all orbits $t \mapsto \Phi(t)x$ are strongly measurable. If (5.1) holds, every function $t \mapsto i_C^*\Phi^*(t)x^*$ is strongly measurable, and from $\|i_C^*\Phi^*(t)x^*\|_{H_C}^2 = \langle \Phi(t)C\Phi^*(t)x^*, x^*\rangle$ we deduce that $\int_0^T \|i_C^*\Phi^*(t)x^*\|_{H_C}^2\,dt < \infty$ if and only if

$$\int_0^T \langle \Phi(t)C\Phi^*(t)x^*, x^*\rangle\, dt < \infty. \tag{5.2}$$

Motivated by this we say that $\Phi : (0,T) \to \mathscr{L}(E)$ is *C-weakly L^2* if (5.1) and (5.2) hold for all $x^* \in E^*$. For such a function we may define a stochastic integral

$$\int_0^T \langle \Phi(t)\, dW^C(t), x^*\rangle \tag{5.3}$$

as follows. For a step function $\mathbf{1}_{(a,b]} \otimes \Phi_0$ we put

$$\int_0^T \langle \mathbf{1}_{(a,b)}(t) \otimes \Phi_0\, dW^C(t), x^*\rangle := \langle W^C(b), \Phi_0^* x^*\rangle - \langle W^C(a), \Phi_0^* x^*\rangle$$

and we extend this to arbitrary C-weakly L^2 functions by linearity and an approximation argument in the usual way.

We call a function $\Phi : (0,T) \to \mathscr{L}(E)$ *stochastically integrable* with respect to $\mathbb{W}^C$ if it is C-weakly L^2 and for every measurable set $A \subseteq (0,T)$ there exists a random variable Y_A such that for all $x^* \in E^*$ we have

$$\langle Y_A, x^* \rangle = \int_0^T \langle \mathbf{1}_A(t)\Phi(t)\, dW^C(t), x^* \rangle$$

almost surely. The random variable Y_A is uniquely determined and Gaussian, and we write

$$Y_A = \int_A \Phi(t)\, dW^C(t).$$

Proposition 5.1. *A function $\Phi : (0,T) \to \mathscr{L}(E)$ is stochastically integrable with respect to $\mathbb{W}^C$ if and only if $\Phi \circ i_C : (0,T) \to \mathscr{L}(H_C, E)$ is stochastically integrable with respect to $\mathbb{W}_{H_C}$. In this situation the two integrals agree.*

Proof. All we have to observe is that Φ is C-weakly L^2 if and only if $\Phi \circ i_C$ is H_C-weakly L^2, and that in this case we have

$$\int_0^T \langle \mathbf{1}_A(t)\Phi(t)\, dW^C(t), x^* \rangle = \int_0^T \mathbf{1}_A(t) i_C^* \Phi^*(t) x^*\, dW_{H_C}(t) \qquad \forall x^* \in E^*$$

for all measurable $A \subseteq (0,T)$. $\square$

This observation enables us to reformulate Theorem 4.2 for stochastic integrals with respect to $\mathbb{W}^C$. Omitting any statements containing explicit reference H_C, we obtain:

Theorem 5.2. *For a C-weakly L^2 function $\Phi : (0,T) \to \mathscr{L}(E)$ the following assertions are equivalent:*

1. *Φ is stochastically integrable with respect to $\mathbb{W}^C$;*
2. *There exists an E-valued random variable Y and a weak*-sequentially dense linear subspace F of E^* such that for all $x^* \in F$ we have*
$$\langle Y, x^* \rangle = \int_0^T \langle \Phi(t)\, dW^C(t), x^* \rangle \quad \textit{almost surely;}$$
3. *there exists a Gaussian measure μ on E with covariance operator Q and a weak*-sequentially dense linear subspace F of E^* such that for all $x^* \in E^*$ we have*
$$\int_0^T \langle \Phi(t) C \Phi^*(t) x^*, x^* \rangle\, dt = \langle Q x^*, x^* \rangle;$$
4. *There exists a separable real Hilbert space $\mathscr{H}$, a γ-radonifying operator $T \in \mathscr{L}(\mathscr{H}, E)$, and and a weak*-sequentially dense linear subspace F of E^* such that for all $x^* \in F$ we have*
$$\int_0^T \langle \Phi(t) C \Phi^*(t) x^*, x^* \rangle^2\, dt \leqslant \|T^* x^*\|_{\mathscr{H}}^2.$$

The measure μ is the distribution of $\int_0^T \Phi(t)\, dW^C(t)$.

If E is separable, then in (2), (3), and (4) it suffices to assume that F is weak*-dense in E^*.

Notice that the Itô isometry cannot be conveniently formulated in this setting; it refers to H_C in an explicit way.

Remark 5.3. In the definition of the stochastic integral (5.3) we only used the scalar Brownian motions $\langle \mathbb{W}^C, x^* \rangle$ rather than $\mathbb{W}^C$ itself. For this reason, all that has been said in this section generalizes to the case of arbitrary positive symmetric operators $C \in \mathscr{L}(E^*, E)$, i.e., to the case of *cylindrical C-Wiener processes* in the terminology of [5]; these can be defined analogously to Definition 3.1.

6. Approximation and convergence theorems

In this section we shall be concerned with convergence theorems for the stochastic integral for $\mathscr{L}(H,E)$-valued functions. If we apply the result to the case $H = \mathbb{R}$ we obtain corresponding convergence theorems for the stochastic integral for E-valued functions with respect to a Brownian motion.

We start with a simple finite-dimensional approximation result:

Proposition 6.1 (Finite-dimensional approximation). *Let $\Phi : (0,T) \to \mathscr{L}(H,E)$ be stochastically integrable with respect to $\mathbb{W}_H$. Let (h_n) be an orthonormal basis in H and denote by P_n the orthogonal projection onto the linear span of $(h_k)_{k\leqslant n}$. Then the functions $\Phi \circ P_n : (0,T) \to \mathscr{L}(H,E)$ are stochastically integrable with respect to $\mathbb{W}_H$ and*

$$\lim_{n\to\infty} \mathbb{E}\Big\| \int_0^T \Phi(t) \circ P_n - \Phi(t)\, dW_H(t) \Big\|^p = 0 \qquad \forall p \in [1,\infty).$$

Proof. Noting that $\lim_{n\to\infty} \|I_{\Phi\circ P_n} - I_\Phi\|_\gamma = 0$, this is immediate from Theorem 4.2 and Proposition 1.1. $\square$

Next we turn to a dominated convergence theorem.

Theorem 6.2 (Dominated Convergence). *Let $\Phi_n : (0,T) \to \mathscr{L}(H,E)$ be a sequence of stochastically integrable functions with respect to $\mathbb{W}_H$. Assume that there exists an H-weakly L^2 function $\Phi : (0,T) \to \mathscr{L}(H,E)$ such that*

$$\lim_{n\to\infty} \int_0^T \|\Phi_n^*(t)x^* - \Phi^*(t)x^*\|_H^2\, dt = 0 \qquad \forall x^* \in E^*. \tag{6.1}$$

Assume further that there exists a stochastically integrable function $\Psi : (0,T) \to \mathscr{L}(H,E)$ such that for all $x^* \in E^*$ and all n we have

$$\int_0^T \|\Phi_n^*(t)x^*\|_H^2\, dt \leqslant \int_0^T \|\Psi^*(t)x^*\|_H^2\, dt. \tag{6.2}$$

Then Φ is stochastically integrable with respect to $\mathbb{W}_H$ and

$$\lim_{n\to\infty} \mathbb{E}\Big\| \int_0^T \Phi_n(t) - \Phi(t)\, dW_H(t) \Big\|^p \qquad \forall p \in [1,\infty). \tag{6.3}$$

Proof. It follows from (6.1) and (6.2) that

$$\int_0^T \|\Phi^*(t)x^*\|_H^2\, dt \leqslant \int_0^T \|\Psi^*(t)x^*\|_H^2\, dt \qquad \forall x^* \in E^*.$$

Hence by (4) of Theorem 4.2 (with $\mathscr{H} = H_R$ and $T = i_R$, where (H_R, i_R) is the reproducing kernel Hilbert space associated with the covariance operator R of the random variable $\int_0^T \Psi(t)\, dW_H(t)$), Φ is stochastically integrable with respect to $\mathbb{W}_H$. Let Q_n, μ_n, and R_n, ν_n, denote the covariance operators and the distributions of $\int_0^T \Phi_n(t)\, dW_H(t)$ and $\int_0^T \Phi_n(t) - \Phi(t)\, dW_H(t)$, respectively. By (6.2) and Proposition 1.2, the measures μ_n are uniformly tight. By a standard argument, cf. [11], this implies that the measures

ν_n are uniformly tight as well. Thanks to condition (6.1) we have $\lim_{n\to\infty}\langle R_n x^*, x^*\rangle = 0$ for all $x^* \in E^*$. It follows that $\lim_{n\to\infty} \nu_n = \delta_0$ weakly. Hence, by [1, Lemma 3.8.7],

$$\lim_{n\to\infty} \mathbb{E}\Big\| \int_0^T \Phi_n(t) - \Phi(t)\, dW_H(t)\Big\|^p = \lim_{n\to\infty} \int_E \|x\|^p \, d\nu_n(x) = 0 \tag{6.4}$$

for all $p \in [1, \infty)$. □

Notice that Proposition 6.1 is contained in Theorem 6.2 as a special case.

Remark 6.3. Noting that $\langle R_n x^*, x^*\rangle \leqslant 4\langle Rx^*, x^*\rangle$ for all $x \in E^*$, we see that this result remains true if we replace $\|\cdot\|^p$ by any nonnegative convex symmetric function $g : E \to \mathbb{R}$ satisfying

$$\int_E g(2x)\, d\nu(x) < \infty.$$

As a first application, we show that stochastically integrable functions can be approximated by step functions in a suitable sense.

Theorem 6.4 (Approximation with step functions)**.** *For an H-weakly L^2 function $\Phi : (0,T) \to \mathscr{L}(H,E)$ the following assertions are equivalent:*

(1) *Φ is stochastically integrable with respect to $\mathbb{W}_H$;*
(2) *There exists a sequence of step functions $\Phi_n : (0,T) \to \mathscr{L}(H,E)$ with the following properties:*
   (a) *For all $x^* \in E^*$ we have*

$$\lim_{n\to\infty} \Phi_n^*(\cdot)x^* = \Phi^*(\cdot)x^* \quad \textit{in measure}; \tag{6.5}$$

   (b) *There exists an E-valued random variable Y such that*

$$Y = \lim_{n\to\infty} \int_0^T \Phi_n(t)\, dW_H(t) \quad \textit{in probability.} \tag{6.6}$$

In this situation we have $Y = \int_0^T \Phi(t)\, dW_H(t)$, the convergence in (6.5) *is in $L^2(0,T;H)$, and the convergence in* (6.6) *is in $L^p(\Omega;E)$ for every $p \in [1,\infty)$.*

Proof. (1) ⇒ (2): By Theorem 4.2 the functions $\Phi(\cdot)h$ are Pettis integrable. Therefore we may define, for $n \geqslant 1$ and $j = 1, \dots, 2^n$, bounded operators $\Phi_{j,n} \in \mathscr{L}(H,E)$ by

$$\Phi_{j,n} h := \frac{2^n}{T} \int_{\frac{(j-1)T}{2^n}}^{\frac{jT}{2^n}} \Phi(t)h\, dt.$$

Now define $\Phi_n : (0,T) \to \mathscr{L}(H,E)$ by $\Phi_n(t) := \Phi_{j,n}$ for $t \in \big(\frac{(j-1)T}{2^n}, \frac{jT}{2^n}\big]$.

Let $\mathscr{G}_n$ denote the finite σ-algebra in $(0,T)$ generated by the 2^n-th equipartition of $(0,T)$. Then $\Phi_n^*(\cdot)x^* = \mathbb{E}(\Phi^*(\cdot)x^*|\mathscr{G}_n)$ and (6.5) follows follows from the vector-valued martingale convergence theorem.

By Jensen's inequality we have

$$\int_0^T \|\Phi_n^*(t)x^*\|_H^2\, dt \leqslant \int_0^T \|\Phi^*(t)x^*\|_H^2\, dt.$$

Hence by Corollary 4.4, each Φ_n is stochastically integrable. Assertion (6.6) now follows from Theorem 6.2, with convergence in every $L^p(\Omega;E)$.

(2) ⇒ (1): Let $x^* \in E^*$ be arbitrary and fixed. Then, by (6.6),

$$\langle Y, x^*\rangle = \lim_{n\to\infty} \int_0^T \Phi_n^*(t)x^*\, dW_H(t) \quad \text{in probability.}$$

The variables on the right hand side being Gaussian, it follows that Y is Gaussian and by general results on convergence of Gaussian variables the convergence takes place in $L^2(\Omega)$. Hence by the Itô isometry, the functions $\Phi_n^*(\cdot)x^*$ define a Cauchy sequence in $L^2(0,T;H)$. By (6.5), the limit equals $\Phi^*(\cdot)x^*$. Thus, the convergence in (6.5) takes place in $L^2(0,T;H)$, and by another application of the Itô isometry it follows that

$$\langle Y, x^*\rangle = \int_0^T \Phi^*(t)x^*\,dW_H(t) \quad \text{almost surely.}$$

By Theorem 4.2(2), this implies that Φ is stochastically integrable, with integral Y. $\square$

As a further corollary to Theorem 6.2 we prove a continuity result for stochastic convolutions, which generalizes and simplifies the main result of [3].

Corollary 6.5. *Let $\Phi : (0,T) \to \mathscr{L}(H,E)$ be stochastically integrable with respect to $\mathbb{W}_H$. Then the E-valued process*

$$t \mapsto \int_0^t \Phi(t-s)\,dW_H(s) \qquad (t \in [0,T])$$

is continuous in p-th moment for all $p \in [1,\infty)$.

Proof. Apply Theorem 6.2 to the functions $\Psi(s) := \mathbf{1}_{(0,t)}(s)\Phi(t-s)$ and $\Psi_n(s) := \mathbf{1}_{(0,t_n)}(s)\Phi(t_n - s)$, where $t_n \to t$. $\square$

We continue with a monotone convergence theorem:

Theorem 6.6 (Monotone Convergence)**.** *Suppose E is separable and does not contain a closed subspace isomorphic to c_0. Let $\Phi_n : (0,T) \to \mathscr{L}(H,E)$ be a sequence of functions that are stochastically integrable with respect to $\mathbb{W}_H$. Assume that there exists an H-weakly L^2 function $\Phi : (0,T) \to \mathscr{L}(H,E)$ such that for all $x^* \in E^*$ we have*

$$\lim_{n\to\infty} \int_0^T \|\Phi_n^*(t)x^* - \Phi^*(t)x^*\|_H^2\,dt = 0 \tag{6.7}$$

and

$$\int_0^T \|\Phi_n^*(t)x^*\|_H^2\,dt \ \uparrow \ \int_0^T \|\Phi^*(t)x^*\|_H^2\,dt \quad \text{as } n \to \infty \tag{6.8}$$

monotonically. If

$$\sup_{n\geqslant 1} \mathbb{E}\Big\|\int_0^T \Phi_n(t)\,dW_H(t)\Big\|^2 < \infty, \tag{6.9}$$

then Φ is stochastically integrable with respect to $\mathbb{W}_H$ and for all $p \in [1,\infty)$ we have

$$\mathbb{E}\Big\|\int_0^T \Phi_n(t) - \Phi(t)\,dW_H(t)\Big\|^p = 0. \tag{6.10}$$

Proof. First note that for all $x^* \in E^*$ the function $t \mapsto \Phi(t)\Phi^*(t)x^*$ is weakly L^1. Since E does not contain a isomorphic copy of c_0, these functions are in fact Pettis integrable by the remarks in Section 1.3. Thus we may define a positive symmetric operator $Q \in \mathscr{L}(E^*,E)$ by

$$Qx^* := \int_0^T \Phi(t)\Phi^*(t)x^*\,dt, \qquad x^* \in E^*.$$

Using notations as before, from (6.8) we have $\lim_{n\to\infty}\langle Q_n x^*, x^*\rangle = \langle Qx^*, x^*\rangle$ for all $x^* \in E^*$. Hence the assumptions of Proposition 1.3 are satisfied and we obtain that Q is a Gaussian covariance operator. The stochastic integrability of Φ and (6.10) now follow from the dominated convergence theorem. $\square$

Thus we see that the stochastic integral admits versions of the two classical convergence theorems of Integration Theory. This is possible because of the underlying tightness conditions, which provide the compactness needed to arrive at convergence of the integrals. Let us point out that Proposition 1.3, when specialized to the covariances of stochastic integrals, also gives a Fatou lemma for the stochastic integral.

The following example, which is essentially a reformulation of an example by Linde and Pietsch [13], shows that the theorem fails in every Banach space E containing an isomorphic copy of c_0. Thus, the validity of the monotone convergence theorem actually characterizes the Banach spaces without an copy of c_0.

Example 6.7. Let $H := \mathbb{R}$, $E := c_0$, and identify $\mathscr{L}(\mathbb{R}, c_0)$ with c_0. We fix an $\mathbb{R}$-cylindrical Wiener process, i.e., a standard Brownian motion, and denote it by $\mathbb{W}$.

Define $I_n := (t_{n-1}, t_n)$, where $t_0 := 0$ and $t_n := \sum_{j=1}^n \frac{1}{j^2}$ for $n \geqslant 1$. Let $T := \sum_{j=1}^\infty \frac{1}{j^2}$ and define $\phi : (0,T) \to c_0$ by

$$\phi(t) := \big(0, \dots, 0, n/\sqrt{\ln(n+1)}, 0, \dots\big) \quad \text{for} \ \ t \in I_n.$$

The function ϕ is not stochastically integrable with respect to $\mathbb{W}$. To see this, note that

$$\int_0^T \langle \phi(t), e_n^* \rangle^2 \, dt = 1/\ln(n+1),$$

where $e_n^* = (0, \dots, 0, 1, 0 \dots)$ is the n-th unit vector of $c_0^* = l^1$. Hence,

$$\int_0^T \langle \phi(t), x^* \rangle^2 \, dt = \langle Q x^*, x^* \rangle \qquad \forall x^* \in l^1,$$

where $Q \in \mathscr{L}(l^1, c_0)$ is given by $Q\big((\alpha_n)\big) := \big(\alpha_n / \ln(n+1)\big)$. It is shown in [13, Theorem 11] that this operator is not a Gaussian covariance. Theorem 4.2 therefore implies that ϕ is not stochastically integrable.

On the other hand, the functions $\phi_n := \mathbf{1}_{(0,t_n)}\phi$ are stochastically integrable with respect to $\mathbb{W}$ and

$$\int_0^T \phi_n(t) \, dW(t) = \sum_{j=1}^n \big(W(t_j) - W(t_{j-1})\big) \otimes \big(0, \dots, 0, j/\sqrt{\ln(j+1)}, 0, \dots\big).$$

It is easy to check that (6.7) and (6.8) hold, and (6.9) follows from [13, Theorem 11]. Thus, the monotone convergence theorem fails for this example.

7. Application to the stochastic linear Cauchy problem

In this section we will apply our results to study linear abstract Cauchy problems with additive noise. As before, H is a separable real Hilbert space and E is a real Banach space.

We consider the stochastic Cauchy problem

$$\begin{aligned} dU(t) &= AU(t)\, dt + B\, dW_H(t), \qquad t \in [0,T], \\ U(0) &= u_0. \end{aligned} \tag{7.1}$$

Here A is the generator of a C_0-semigroup $\{S(t)\}_{t \geqslant 0}$ on E and $B \in \mathscr{L}(H,E)$ is a given bounded operator, and the cylindrical Wiener process $\mathbb{W}_H = \{W_H(t)\}_{t \in [0,T]}$ is adapted to some given filtration $\{\mathscr{F}_t\}_{t \in [0,T]}$.

An E-valued process $\{U(t,u_0)\}_{t\in[0,T]}$ is called a *weak solution* if it is weakly progressively measurable and for all $x^* \in D(A^*)$, the domain of the adjoint operator A^*, the following two conditions are satisfied:

(1) Almost surely, the paths $t \mapsto \langle U(t,u_0), A^*x^*\rangle$ are integrable;
(2) For all $t \in [0,T]$ we have, almost surely,

$$\langle U(t,u_0), x^*\rangle = \langle u_0, x^*\rangle + \int_0^t \langle U(s,u_0), A^*x^*\rangle\,ds + W_H(t)B^*x^* \tag{7.2}$$

Notice that this notion of weak solution is slightly more general than the one used in [5]. It will follow from the theorem below that both notions are in fact equivalent.

For separable Hilbert spaces E, necessary and sufficient conditions for the existence and uniqueness of weak solutions are presented in [5, Chapter 5]. These were extended to separable Banach spaces E in [4] with an indirect method of proof; see also the remarks in the introduction. Continuity in p-th moment of weak solutions in the Banach space setting was proved in [3]. With the tools developed in this paper a direct proof of the results of [4] becomes possible, which generalizes the Hilbert space arguments in [5]. In view of the many subtle differences in the execution we present it in full detail.

As a preliminary observation we remark that, notwithstanding the fact that the adjoint semigroup generally fails to be strongly continuous, the function $t \mapsto S(t)BB^*S^*(t)x^*$ is always Bochner integrable; cf. [15, Proposition 1.2].

Theorem 7.1. *The following assertions are equivalent:*

(1) *The problem* (7.1) *has a weak solution* $\{U(t,u_0)\}_{t\in[0,T]}$*;*
(2) *The function* $t \mapsto S(t)B$ *is stochastically integrable on* $(0,T)$ *with respect to* $\mathbb{W}_H$*;*
(3) *The operator* $R \in \mathscr{L}(E^*,E)$ *defined by*
$$Rx^* := \int_0^T S(t)BB^*S^*(t)x^*\,dt, \qquad x^* \in E^*,$$
is a Gaussian covariance operator;
(4) *The operator* $I \in \mathscr{L}(L^2(0,T;H),E)$ *defined by*
$$If := \int_0^T S(t)Bf(t)\,dt$$
is γ*-radonifying from* $L^2(0,T;H)$ *into* E*.*

In this situation, for every $t \in [0,T]$ *the function* $s \mapsto S(t-s)B$ *is stochastically integrable on* $(0,t)$ *with respect to* $\mathbb{W}_H$ *and we have*

$$U(t,u_0) = S(t)u_0 + \int_0^t S(t-s)B\,dW_H(s) \tag{7.3}$$

almost surely. In particular, up to a modification the problem (7.1) *has a unique weak solution. For all* $p \in [1,\infty)$ *the paths* $t \mapsto U(t,u_0)$ *belong to* $L^p(0,T;E)$ *almost surely, the process* $\{U(t)\}_{t\in[0,T]}$ *is continuous in* p*-th moment and it has a predictable version.*

Proof. We start with noting that $U(\cdot,u_0)$ is a weak solution corresponding to the initial value u_0 if and only if $U(\cdot,u_0) - S(\cdot)u_0$ is a weak solution corresponding to the initial value 0. Without loss of generality we shall therefore assume that $u_0 = 0$.

We will prove the equivalence (1) $\Leftrightarrow$ (2); the equivalences (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) are consequences of Theorem 4.2.

(1) ⇒ (2): We write $U(t) := U(t,0)$ for convenience. Let $A^{\odot}$ denote the part of the adjoint generator A^* in $E^{\odot} := \overline{D(A^*)}$. We will show first that for all $x^* \in D(A^{\odot})$ and $t \in [0,T]$, almost surely we have

$$\langle U(t), x^*\rangle = \int_0^t B^*S^*(t-s)x^*\, dW_H(s). \tag{7.4}$$

Once this is established, condition (2) of Theorem 4.2 (for F we take the weak*-sequentially dense subspace $D(A^{\odot})$) shows that for all $t \in [0,T]$ the function $s \mapsto S(t-s)B$ is stochastically integrable on $(0,t)$ with respect to $\mathbb{W}_H$. Assertion (2) and the representation (7.3) then follow.

We follow the argument in [5, Chapter 5.2]. Fix $x^* \in D(A^*)$ and $t \in [0,T]$ and let $f : [0,t] \to \mathbb{R}$ be an arbitrary C^1 function. Since the process $\{U(t)\}_{t\in[0,T]}$ is weakly progressively measurable, Fubini's theorem implies that almost surely the identity

$$\langle U(s), x^*\rangle = \int_0^s \langle U(r), A^*x^*\rangle\, dr + W_H(s)B^*x^*$$

holds for almost all $s \in (0,t)$. Using this in combination with (3.3), an integration by parts gives

$$\begin{aligned}
&\int_0^t f'(s)\langle U(s), x^*\rangle\, ds \\
&\qquad = \int_0^t f'(s)\Big[\int_0^s \langle U(r), A^*x^*\rangle\, dr\Big]\, ds + \int_0^t f'(s)W_H(s)B^*x^*\, ds \\
&\qquad = f(t)\int_0^t \langle U(s), A^*x^*\rangle\, ds - \int_0^t f(s)\langle U(s), A^*x^*\rangle\, ds \\
&\qquad\qquad + f(t)W_H(t)B^*x^* - \int_0^t f(s)\otimes B^*x^*\, dW_H(s)
\end{aligned} \tag{7.5}$$

almost surely. Multiplying both sides of (7.2) with $f(t)$, putting $F := f\otimes x^*$, using (7.5) and rewriting, we obtain

$$\langle U(t), F(t)\rangle = \int_0^t \langle U(s), F'(s) + A^*F(s)\rangle\, ds + \int_0^t B^*F(s)\, dW_H(s) \tag{7.6}$$

almost surely. Since linear combinations of the functions $f\otimes x^*$ with $f \in C^1[0,t]$ and $x^* \in D(A^*)$ are dense in $C^1([0,t];E^*)\cap C([0,t];D(A^*))$, by approximation this identity extends to arbitrary functions $F \in C^1([0,t];E^*)\cap C([0,t];D(A^*))$. In particular we may take $F(s) = S^*(t-s)x^*$, with $x^* \in D(A^{\odot})$. For this choice of F, the identity (7.6) reduces to (7.4).

(2) ⇒ (1): Suppose now that the function $t \mapsto S(t)B$ is stochastically integrable on $(0,T)$. This implies the stochastic integrability of $s \mapsto S(t-s)B$ on $(0,t)$ for all $t \in (0,T)$. We will check that the process defined by

$$U(t) := \int_0^t S(t-s)B\, dW_H(s)$$

is a weak solution of the problem (7.1) with initial value 0.

Fix $x^* \in D(A^*)$ and $t \in [0,T]$. Then,

$$\langle U(t), A^*x^*\rangle = \int_0^t B^*S^*(t-s)A^*x^*\, dW_H(s)$$

almost surely. By Theorem 3.3, the $L^2(\Omega)$-valued function $t \mapsto \langle U(t), A^*x^*\rangle$ is integrable on $(0,T)$ and

$$\begin{aligned}
\int_0^t \langle U(s), A^*x^*\rangle\, ds &= \int_0^t \Big(\int_0^t \mathbf{1}_{(0,s)}(r) B^*S^*(s-r)A^*x^*\, dW_H(r)\Big)\, ds \\
&= \int_0^t \Big(\int_0^t \mathbf{1}_{(0,s)}(r) B^*S^*(s-r)A^*x^*\, ds\Big)\, dW_H(r) \\
&= \int_0^t \Big(\int_r^t B^*S^*(s-r)A^*x^*\, ds\Big)\, dW_H(r) \\
&= \int_0^t B^*S^*(t-r)x^* - B^*x^*\, dW_H(r) \\
&= \int_0^t B^*S^*(t-r)x^*\, dW_H(r) - W_H(t)B^*x^* \\
&= \langle U(t), x^*\rangle - W_H(t)B^*x^*,
\end{aligned}$$

where all identities are understood in the sense of $L^2(\Omega)$. In particular the identities hold almost surely.

The continuity in p-th moment of the process $\{U(t)\}_{t\in[0,T]}$ follows from Corollary 6.5. Since this process is also adapted, its is progressively measurable. By general results on stochastic processes this implies the existence of a predictable version.

It remains to show that for all $p \in [1,\infty)$ we have $t \mapsto U(t)$ in $L^p(0,T;E)$ almost surely. Following an argument of [4], we let μ_t denote the distribution of $U(t)$ and note that its covariance operator R_t satisfies

$$\langle R_t\, x^*, x^*\rangle = \int_0^t \|B^*S^*(s)x^*\|_H^2\, ds \leqslant \langle R_T x^*, x^*\rangle = \langle Rx^*, x^*\rangle.$$

By Fubini's theorem,

$$\mathbb{E}\int_0^T \|U(t)\|^p\, dt = \int_0^T \int_E \|x\|^p\, d\mu_t(x)\, dt \leqslant T \int_E \|x\|^p\, d\mu_T(x) < \infty,$$

and the claim follows. □

Corollary 7.2. *If the problem* (7.1) *has a weak solution on the interval* $[0,T]$*, it has a weak solution on* $[0,\infty)$.

Proof. By Theorem 7.1, the operator $R_T \in \mathscr{L}(E^*,E)$ defined in condition (2) is a Gaussian covariance. To prove that (7.1) has a weak solution on the interval $[0,T']$ we need to show that the operator $R_{T'} \in \mathscr{L}(E^*,E)$ is a Gaussian covariance as well.

Given any $T' \geqslant 0$, choose an integer $N \geqslant 1$ such that $T'/N \leqslant T$ and notice that

$$R_{T'} = \sum_{n=0}^{N-1} S(nT'/N)R_{T'/N}S^*(nT'/N).$$

Since R_T is a Gaussian covariance, so is $R_{T'/N}$. Let μ be the centred Gaussian measure with covariance $R_{T'/N}$. Then the image measures $\mu_n := S(nT'/N)\mu$ have covariances $S(nT'/N)R_{T'/N}S^*(nT'/N)$ and their convolution $\mu_0 * \cdots * \mu_{N-1}$ has covariance $R_{T'}$. □

The conditions in Theorem 7.1 allow us to exhibit examples of stochastic evolution equations driven by one-dimensional Brownian motions that have no weak solution.

Example 7.3. Let $E = L^p(\Gamma)$, where Γ denotes the unit circle in the complex plane with its normalized Lebesgue measure. We let $A = d/d\theta$ denote the generator of the rotation (semi)group S on $L^p(\Gamma)$, i.e., $S(t)f(\theta) = f(\theta + t \bmod 2\pi)$.

We define an $\mathbb{R}$-cylindrical Wiener process $\mathbb{W}_{\mathbb{R}}$ by

$$W_{\mathbb{R}}(t)h := W(t)h, \qquad h \in \mathbb{R},$$

where $\mathbb{W}$ is a standard real Brownian motion, and for a fixed element $\phi \in L^p(\Gamma)$ we define $B_\phi \in \mathscr{L}(\mathbb{R}, L^p(\Gamma))$ by

$$B_\phi h := h\phi, \qquad h \in \mathbb{R}.$$

Let (f_n) denote an orthonormal basis for $L^2((0, 2\pi); \mathbb{R})$. The corresponding stochastic initial problem has a weak solution on $[0, 2\pi]$ (and hence on $[0, \infty)$) if and only the operator I defined in condition (3) of Theorem 7.1 (with $T = 2\pi$) is γ-radonifying. By definition this happens if and only if $\sum_{n=1}^\infty \gamma_n I f_n$ converges in $L^2(\Omega; L^p(\Gamma))$, where (γ_n) is a sequence of independent standard normal random variables. For all integers $N \geqslant M \geqslant 1$, by Fubini's theorem and the Khinchine inequalities we have

$$\begin{aligned}
\mathbb{E}\Big\| \sum_{n=M}^{N} \gamma_n I f_n \Big\|_{L^p(\Gamma)}^p &= \int_0^{2\pi} \mathbb{E}\Big| \sum_{n=M}^{N} \gamma_n I f_n(\theta) \Big|^p \, d\theta \\
&\sim \int_0^{2\pi} \Big[\sum_{n=M}^{N} |I f_n(\theta)|^2 \Big]^{\frac{p}{2}} d\theta = \Big\| \Big[\sum_{n=M}^{N} |I f_n|^2 \Big]^{\frac{1}{2}} \Big\|_{L^p(\Gamma)}^p.
\end{aligned}$$

Now,

$$\sum_{n=M}^{N} |I f_n(\theta)|^2 = \sum_{n=M}^{N} \Big| \int_0^{2\pi} f_n(t)\phi(\theta + t \bmod 2\pi)\, dt \Big|^2 = \sum_{n=M}^{N} \big| [f_n, \phi_\theta]_{L^2(0,2\pi)} \big|^2$$

where $\phi_\theta(t) := \phi(\theta + t \bmod 2\pi)$. From this we deduce that a weak solution exists if and only if $\phi \in L^2(\Gamma)$. In particular, for $p \in [1, 2)$ and $\phi \in L^p(\Gamma) \setminus L^2(\Gamma)$ the resulting initial value problem has no weak solution.

It is not a coincidence that a nonexistence is obtained in the range $p \in [1, 2)$ only. Indeed, for $p \in [2, \infty)$ the space $L^p(\Gamma)$ has type 2 and Theorem 4.7 implies that the operator I is γ-radonifying if B has this property. In the above situation, $B = B_\phi$ is a rank one operator, and such operators are trivially γ-radonifying.

In many applications the resolvent of A is more accessible than the semigroup generated by A. We can use Theorem 7.1 to give an existence criterium directly in terms of the resolvent of A:

Corollary 7.4. *Let $M \geqslant 1$ and $\omega_0 \in \mathbb{R}$ be chosen in such a way that $\|S(t)\| \leqslant M e^{\omega_0 t}$ for all $t \geqslant 0$. The following assertions are equivalent:*

1. *The problem (7.1) has a weak solution $\{U(t, u_0)\}_{t \in [0,T]}$;*
2. *There exist $a > \omega_0$ and an orthonormal basis $(h_j)_{j=1}^\infty$ of H such that*

$$\mathbb{E}\Big\| \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}} \gamma_{jk}\, R\big(a + \tfrac{2\pi i k}{T}, A\big) B h_j \Big\|^2 < \infty.$$

Here, for $\lambda \in \varrho(A)$ (the resolvent set of A) we write $R(\lambda, A) := (\lambda - A)^{-1}$ and $(\gamma_{jk})_{j \geqslant 1,\, k \in \mathbb{Z}}$ is a doubly indexed sequence of independent standard normal random variables.

Notice that in assertion (2) we consider a complexification of E. We leave it to the reader to supply the complexification arguments to make the proof below rigorous.

Proof. Without loss of generality we may assume that $T = 2\pi$. We have, for all $a > \omega_0$, $k \in \mathbb{Z}$, and $x \in E$,

$$\int_0^{2\pi} e^{-(a+ik)t} S(t)x\,dt = UR(a+ik, A)x,$$

where $U := I - e^{-2\pi a}S(2\pi)$ is invertible since by elementary semigroup theory the spectral radius of $e^{-2\pi a}S(2\pi)$ is less than one. Choose $(e^{-ik(\cdot)})_{k\in\mathbb{Z}}$ as an orthonormal basis of $L^2(0, 2\pi)$. Then $(e^{-ik(\cdot)} \otimes h_j)_{j\geqslant 1,\, k\in\mathbb{Z}}$ is an orthonormal basis of $L^2((0, 2\pi); H)$.

Let $\Phi(t) := S(t)B$. The function $\Phi_a(t) := e^{-at}\Phi(t)$ is H-weakly L^2 and we have

$$\begin{aligned}
\|I_{\Phi_a}\|_\gamma^2 &= \mathbb{E}\Big\|\sum_{j=1}^\infty \sum_{k\in\mathbb{Z}} \gamma_{jk} \int_0^{2\pi} e^{-ikt}\Phi_a(t)h_j\,dt\Big\|^2 \\
&= \mathbb{E}\Big\|\sum_{j=1}^\infty \sum_{k\in\mathbb{Z}} \gamma_{jk} \int_0^{2\pi} e^{-(a+ik)t}S(t)Bh_j\,dt\Big\|^2 \\
&= \mathbb{E}\Big\|U\sum_{j=1}^\infty \sum_{k\in\mathbb{Z}} \gamma_{jk}\, R(a+ik, A)Bh_j\Big\|^2.
\end{aligned}$$

Taking into account the invertibility of U, it follows that (2) holds if and only if Φ_a is stochastically integrable. By Corollary 4.4 this happens if and only if Φ is stochastically integrable, and by Theorem 7.1 this happens if and only if (1) holds. $\square$

Remark 7.5. Under certain geometric assumptions on E, which are satisfied by the L^p-spaces and $W^{\alpha,p}$-spaces for $1 < p < \infty$, the conditions in Theorem 7.1 and its corollary can be further simplified and effectively verified for concrete classes of equations. This point will be taken up in a forthcoming paper.

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