Uniform anti-maximum principles

Ph. Clément and G. Sweers
Dept. of Mathematics, I.T.S.
Delft University of Technology
PObox 5031
2600 GA Delft
The Netherlands

Abstract

Consider a second or higher order elliptic partial differential equation $\mathcal{A}u = \lambda u + f$ on an open bounded domain Ω of $\mathbb{R}^n$ with homogeneous boundary conditions $\mathcal{B}u = 0$. If there exists a simple eigenvalue for which the corresponding eigenfunction is positive and satisfies appropriate boundary estimates, then an anti-maximum principle holds. For positive $f \in L^p(\Omega)$ with p large enough there exists $\delta_f > 0$ such that for $\lambda \in (\lambda_1, \lambda_1 + \delta_f)$ the solution is negative and for $\lambda \in (\lambda_1 - \delta_f, \lambda_1)$ the solution is positive. We give conditions such that this sign reversing property is uniform: there is $\delta > 0$ such that for all positive f the solution u is negative for $\lambda \in (\lambda_1, \lambda_1 + \delta)$ and positive for $\lambda \in (\lambda_1 - \delta, \lambda_1)$. Two classes of higher order boundary value problems that satisfy these conditions will be given.

1 Introduction

The aim of this paper is to establish some uniform *anti-maximum principles* for classes of higher order elliptic boundary value problems.

Let us briefly recall the situation in the second order case with Dirichlet boundary conditions. Consider for $\Omega \subset \mathbb{R}^n$ a bounded domain with smooth boundary $\partial\Omega$ and $f \in L^p(\Omega)$, $p > 1$, the boundary value problem

$$\begin{cases} -\Delta u = \lambda u + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

and let λ_1 denote the first eigenvalue. Assume that the function f is nonnegative and positive on a set of positive measure. As is well-known, the maximum principle implies that if $\lambda < \lambda_1$, then the solution u is positive. It was observed in [8] that if $p > n$ then there exists $\lambda_f > \lambda_1$ such that if $\lambda \in (\lambda_1, \lambda_f)$, then the solution u of (1.1) is negative in Ω. Such a restriction on p might look surprising for a sign result. However, in [27] it is shown that the condition $p > n$ is sharp and that one cannot have λ_f to be bounded away from λ_1 uniformly for all positive f. On the other hand it was shown in [8] that, if $n = 1$ and the boundary conditions are of Neumann or Robin type, then a uniform result does hold, that is, the corresponding Green function is negative for $\lambda \in (\lambda_1, \lambda_1 + \delta)$ with $\delta > 0$.

In this paper we obtain uniform results for higher order elliptic boundary value problems also in dimensions higher than 1. Let us start with some examples.

Consider the fourth-order problem with Navier boundary conditions

$$\begin{cases} \Delta^2 u = \lambda u + f & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

and let $\lambda_{1,\mathrm{N}}$ be the first eigenvalue (notice that $\lambda_{1,\mathrm{N}} = \lambda_1^2$ with λ_1 as above). It is known that for $0 \leq \lambda < \lambda_{1,\mathrm{N}}$ the solution u is positive whenever f is positive. It appears that if f is positive and $f \in L^p(\Omega)$, with $p > \max\left(1, \frac{1}{3}n\right)$, then there exists $\delta_f > 0$ such that u is negative for $\lambda \in (\lambda_{1,\mathrm{N}}, \lambda_{1,\mathrm{N}} + \delta_f)$. As in the second order Neumann case one can show that the result is uniform when $n = 1$.

If we replace the boundary conditions in (1.2) by Robin type boundary conditions:

$$\left\{ \begin{array}{rcll} \Delta^2 u = \lambda u + f & & & \text{in } \Omega, \\ \left(1 + \theta_1 \frac{\partial}{\partial n}\right) u & = & 0 & \text{on } \partial\Omega, \\ \left(1 + \theta_2 \frac{\partial}{\partial n}\right) \Delta u & = & 0 & \end{array} \right. \tag{1.3}$$

with $\theta_1, \theta_2 \geq c > 0$, then again there exists a first eigenvalue $\lambda_{1,\mathrm{R}} > 0$ and for $0 \leq \lambda < \lambda_{1,\mathrm{R}}$ the solution u is positive whenever f is positive. Similarly there is $\tilde{\delta}_f > 0$ such that u is negative when $\lambda \in \left(\lambda_{1,\mathrm{R}}, \lambda_{1,\mathrm{R}} + \tilde{\delta}_f\right)$. However, in contrast to the case with Navier boundary conditions (1.2), we are able to show that this result is uniform for $n \in \{1, 2, 3\}$. See the example following Corollary 6.

We will consider general elliptic boundary value problems. The results with respect to those systems are twofold.

- We will show that if there exists a positive eigenfunction with appropriate boundary behaviour, and a relation between the dimension and the boundary condition holds, then a uniform anti-maximum principle holds for λ in a right neighbourhood of the eigenvalue λ_1.
- Since higher order elliptic boundary value problems in general do not satisfy a maximum principle it is not obvious that the conditions just mentioned can be satisfied. However, we will give some examples of such boundary value problems for which these conditions are met.

The basic idea for anti-maximum type results is to split f in $f_1 + f_2$ where

$$f_1 = P_{\phi_1} f \text{ and } f_2 = (I - P_{\phi_1}) f, \tag{1.4}$$

P_{ϕ_1} being an appropriate projection on the first eigenfunction ϕ_1, which has to have a fixed sign. From $f > 0$ it follows that $f_1 = c\phi_1 > 0$. Solving (1.1) one finds

$$u = \frac{c}{\lambda_1 - \lambda} \phi_1 + (-\Delta - \lambda)_0^{-1} f_2, \tag{1.5}$$

and using that $\lambda \mapsto (-\Delta - \lambda)_0^{-1} (I - P_{\phi_1})$ is in $C\left((-\infty, \lambda_2); L\left(L^p(\Omega); C_{\phi_1}\left(\bar{\Omega}\right)\right)\right)$ for $p > n$, where $C_{\phi_1}\left(\bar{\Omega}\right)$ is the subspace of $C\left(\bar{\Omega}\right)$ equipped with the norm defined by

$$\|u\|_{\phi_1} = \sup\left\{ \frac{|u(x)|}{\phi_1(x)}; x \in \Omega \right\},$$

one finds that for $|\lambda - \lambda_1|$ small enough the sign of u in (1.5) equals the sign of $\lambda - \lambda_1$. Since the sign of f in general does not imply a relation between $P_{\phi_1} f$ and the $L^p(\Omega)$-norm of $(I - P_{\phi_1}) f$ one does not obtain a uniform result for positive f in $L^p(\Omega)$.

We need a positive first eigenfunction. In general one cannot expect a maximum principle or even a positive first eigenfunction for higher order elliptic equations except in two cases. The first type of examples appears when the higher order equation can be written as a system of second order elliptic equations with appropriate boundary conditions. The second family of examples occurs when operator and domain are close to a polyharmonic equation on a ball. Positivity results for polyharmonic equations on a ball are established by Boggio [7]. We will restrict ourselves in the second case to $\Omega = B$, a ball in $\mathbb{R}^n$. Using recent results of Grunau and coauthor ([16],[17]) one expects a similar situation for small perturbations of domain and operator.

Anti-maximum principles from an abstract point of view are considered in a paper by Takáč [28]. Extensions of the anti-maximum principle for second order operators, respectively including more general domains and nonlinear elliptic operators, are established independently by Birindelli in [6] and by Fleckinger, Gossez, Takáč en de Thélin in [13].

The paper is organized as follows:

- Section 2 contains our main results for general elliptic boundary value problems assuming that appropriate positive eigenfunctions both for the system and its adjoint exist, that is $\phi_1(x) \geq c_1\, d(x,\partial\Omega)^{m_{\mathcal{B}}}$ and $\phi_1^*(x) \geq c_1^*\, d(x,\partial\Omega)^{m_{\mathcal{B}^*}}$. The numbers $m_{\mathcal{B}}$ and $m_{\mathcal{B}^*}$ will be defined in section 2. Secondly we give examples for which indeed such eigenfunctions exist. The combination leads us to a uniform anti-maximum principle (Corollary 6).

- In Section 3 we consider the functional analytic framework of the system and its adjoint in order to prepare the elliptic estimates in a weak setting.

- In Section 4 these regularity results are used for the contribution of f_2. We obtain a Agmon-Douglis-Nirenberg type estimate for λ near λ_1 of the form

  $$\|u_{2,\lambda}\|_{X^{s,p}} \leq c_2\, \|f_2\|_{X^{s-2m,p}},$$

  for $s \in (0, 2m)$, where $X^{\sigma,p}$ is a Sobolev-type space. An imbedding Theorem implies for $s > m_{\mathcal{B}} + \frac{n}{p}$ that

  $$|u_{2,\lambda}(x)| \leq c_3\, d(x,\partial\Omega)^{m_{\mathcal{B}}}\, \|u_{2,\lambda}\|_{X^{s,p}}.$$

- The crucial theorem which makes our result uniform is established in Section 5. Namely, for $2m - s > m_{\mathcal{B}^*} + \frac{n}{q}$ with $\frac{1}{p} + \frac{1}{q} = 1$, it follows that if $f \geq 0$ then

  $$\|f_2\|_{X^{s-2m,p}} \leq c_4\, \|P_{\phi_1} f\|_\infty,$$

  where $P_{\phi_1} f$ is the appropriate projection on ϕ_1. We may choose $s \in (0, 2m)$ satisfying both restrictions whenever $m_{\mathcal{B}^*} + m_{\mathcal{B}} + n < 2m$. Combining the previous estimates we then find

  $$u(x) = \frac{1}{\lambda_1 - \lambda} P_{\phi_1} f(x) + u_{2,\lambda}(x) \begin{cases} \geq & \left(\frac{1}{\lambda_1-\lambda} - \frac{c_2\, c_3\, c_4}{c_1}\right) P_{\phi_1} f(x), \\ \leq & \left(\frac{1}{\lambda_1-\lambda} + \frac{c_2\, c_3\, c_4}{c_1}\right) P_{\phi_1} f(x), \end{cases} \tag{1.6}$$

  implying the sign of u for $|\lambda_1 - \lambda|$ sufficiently small.

- In the last section we will prove that indeed the conditions are satisfied for some classes of boundary value problems.

- We conclude the paper with several appendices.

2 The results

2.1 General systems

In what follows we will assume that $\Omega \subset \mathbb{R}^n$ is bounded and the boundary satisfies $\partial\Omega \in C^\infty$.

We consider

$$\left\{ \begin{array}{rcll} \mathcal{A}u &=& \lambda u + f & \text{in } \Omega, \\ \mathcal{B}u &=& 0 & \text{on } \partial\Omega, \end{array} \right. \tag{2.1}$$

with $\mathcal{A}$ an elliptic operator of order $2m$ and $\mathcal{B} = \{\mathcal{B}_1, \ldots, \mathcal{B}_m\}$ a system of boundary operators and $\lambda \in \mathbb{R}$. The operators $\mathcal{A}$ and $\mathcal{B}$ are defined by

$$\mathcal{A}u = \sum_{|\alpha| \le 2m} a_\alpha(x) \left(\frac{\partial}{\partial x} \right)^\alpha u, \tag{2.2}$$

$$\mathcal{B}_j u = \sum_{|\alpha| \le m_j} b_{j,\alpha}(x) \left(\frac{\partial}{\partial x} \right)^\alpha u \text{ for } j \in \{1, \ldots, m\}, \tag{2.3}$$

where $a_\alpha, b_{j,\alpha}$ are real-valued functions satisfying $a_\alpha \in C^\infty(\bar{\Omega})$, $b_{j,\alpha} \in C^\infty(\partial\Omega)$, $m_j < 2m$; for α a multi-index in $\mathbb{N}^n$ one defines $\left(\frac{\partial}{\partial x}\right)^\alpha = \prod_{i=1}^n \left(\frac{\partial}{\partial x_i}\right)^{\alpha_i}$.

Notation *We shall denote by*

i. $M_\mathcal{B}$ *the largest order of the derivatives that appears in* $\mathcal{B}$;

ii. $m_\mathcal{B}$ *the largest integer such that* $\mathcal{B}u = 0$ *implies* $\left(\frac{\partial}{\partial n}\right)^k u = 0$ *for all* $k \in \{0, \ldots, m_\mathcal{B} - 1\}$.

Clearly $M_\mathcal{B} \ge m_\mathcal{B} - 1$ holds, with equality for the Dirichlet problem. In a regular elliptic boundary value problem one finds that $M_\mathcal{B} \in \{m-1, 2m-1\}$ and $m_\mathcal{B} \in \{0, m\}$.

Our general assumption on the pair $(\mathcal{A}, \mathcal{B})$ is:

Assumption 1 *System (2.1) is a* regular *elliptic problem.*

For the definition see [22], [29, Definition 5.2.1.4] or Appendix B.

Before we are able to state our second general assumption we need to introduce the adjoint problem. The operators $\mathcal{A}^*$ and $\mathcal{B}^*$ in the adjoint boundary value problem

$$\left\{ \begin{array}{rcll} \mathcal{A}^*u &=& \lambda u + f & \text{in } \Omega, \\ \mathcal{B}^*u &=& 0 & \text{on } \partial\Omega, \end{array} \right. \tag{2.4}$$

are defined as follows. If $\mathcal{A}$ is as in (2.2) then the differential operator $\mathcal{A}^*$ is given by the formal adjoint operator of $\mathcal{A}$, that is,

$$\mathcal{A}^* v(x) = \sum_{|\alpha| \le 2m} (-\mathcal{D})^\alpha \Big(a_\alpha(x) v(x) \Big). \tag{2.5}$$

The adjoint boundary operators are found in the following way. Since $\mathcal{B}$ is normal one may extend $\mathcal{B}$ by m additional operators $\mathcal{B}_c$ such that $\{\mathcal{B}, \mathcal{B}_c\}$ is a normal system of $2m$ boundary operators, see [22, Chapter 2, Theorem 2.1] or [29, Theorem 5.4.2]. For $u, v \in C^\infty(\bar{\Omega})$ an integration by parts yields in a unique way two sets of m operators $\mathcal{B}^*$, $\mathcal{B}_c^*$ such that the Green formula

$$\int_\Omega \mathcal{A}u\, v\, dx - \int_\Omega u\, \mathcal{A}^* v\, dx = \int_{\partial\Omega} (\mathcal{B}u \cdot \mathcal{B}_c^* v - \mathcal{B}_c u \cdot \mathcal{B}^* v)\, dx, \tag{2.6}$$

holds, where $\cdot$ denotes the inner product in $\mathbb{R}^m$. Although depending on the choice of $\mathcal{B}_c$ the boundary operators $\mathcal{B}^*$ uniquely determine the adjoint system in the following sense. If $\mathcal{B}_{c,1}$ is another extension of $\mathcal{B}$ and $\mathcal{B}_1^*$ and $\mathcal{B}_{c,1}^*$ are the corresponding operators, then for all $v \in C^\infty\left(\bar{\Omega}\right)$

$$\mathcal{B}_1^* v = 0 \text{ on } \partial\Omega \quad \Leftrightarrow \quad \mathcal{B}^* v = 0 \text{ on } \partial\Omega. \tag{2.7}$$

We recall that, see [29, Theorem 5.4.2] or [22, Chapter 2, Thms 2.1 and 2.2], the adjoint system (2.4) is a regular elliptic problem, if and only if system (2.1) is a regular elliptic problem. Moreover, the following relation between $m_{\mathcal{B}}$, $m_{\mathcal{B}^*}$, $M_{\mathcal{B}}$ and $M_{\mathcal{B}^*}$ holds.

Lemma 1 *Let (2.1) be regular. Then it holds that*

$$m_{\mathcal{B}^*} + M_{\mathcal{B}} = m_{\mathcal{B}} + M_{\mathcal{B}^*} = 2m - 1. \tag{2.8}$$

Proof. By using the formula in (2.6) one finds as in [22, Chapter 2, Theorem 2.1] that corresponding terms of $\mathcal{B}_j$ and $\mathcal{B}_{c,j}^*$ respectively $\mathcal{B}_{c,j}$ and $\mathcal{B}_j^*$ have orders that add up to $2m-1$, that is, $\mathcal{B}$ having orders $\{m_j\}_{j=1}^m$ implies that $\mathcal{B}_c^*$ has orders $\{2m - m_j - 1\}_{j=1}^m$ and hence does not contain all orders strictly less then $2m - M_{\mathcal{B}} - 1$. Since $\{\mathcal{B}_c^*, \mathcal{B}^*\}$ is a normal system all boundary operators with these orders appear in $\mathcal{B}^*$ implying that $m_{\mathcal{B}^*} = 2m - M_{\mathcal{B}} - 1$. □

Examples *Let $m \in \mathbb{N}^+$ and set $\mathcal{A} = (-\Delta)^m$. Then (2.1) will be regular with each of the following sets of boundary conditions. We assume that $\theta_1, \dots, \theta_m \in C^\infty(\partial\Omega)$ with each $\theta_1, \dots, \theta_m$ either identical zero or strictly positive. Here $\frac{\partial}{\partial n}$ denotes the outward normal derivative.*

$$\begin{array}{rlcl} i. & \mathcal{B}_{m,Dirichlet}\, u & = & \left\{u, \frac{\partial}{\partial n}u, \left(\frac{\partial}{\partial n}\right)^2 u, \dots, \left(\frac{\partial}{\partial n}\right)^{m-1} u\right\}; \\ ii. & \mathcal{B}_{m,Navier}\, u & = & \left\{u, \Delta u, \Delta^2 u, \dots, \Delta^{m-1} u\right\}; \\ iii. & \mathcal{B}_{m,Robin}\, u & = & \left\{\left(1 + \theta_1 \frac{\partial}{\partial n}\right) u, \left(1 + \theta_2 \frac{\partial}{\partial n}\right) \Delta u, \dots, \left(1 + \theta_m \frac{\partial}{\partial n}\right) \Delta^{m-1} u\right\}; \\ iv. & \mathcal{B}_{m,D,m_0,D}\, u & = & \left\{\mathcal{B}_{m_0,Dirichlet} u, \mathcal{B}_{m-m_0,Dirichlet} \Delta^{m_0} u\right\}. \end{array} \tag{2.9}$$

Notice that $\mathcal{B}_{m,Navier}$ is a subcase of $\mathcal{B}_{m,Robin}$.
We find that $\mathcal{A}^* = (-\Delta)^m$ and for appropriate choices of $\mathcal{B}_c$:

$$\begin{array}{rlcl} i. & \mathcal{B}_{m,Dirichlet}^* & = & \mathcal{B}_{m,Dirichlet}; \\ ii. & \mathcal{B}_{m,Navier}^* & = & \mathcal{B}_{m,Navier}; \\ iii. & \mathcal{B}_{m,Robin}^* & & \text{is of the same type as } \mathcal{B}_{m,Robin}, \text{ with } \theta_i^* = \theta_{m+1-i} \text{ for all } i = 1, \dots, m; \\ iv. & \mathcal{B}_{m,D,m_0,D}^* & = & \mathcal{B}_{m,D,m-m_0,D}. \end{array}$$

The second assumption is concerned with the existence of an algebraically simple (in an appropriate sense) eigenvalue λ_1 for system (2.1) with a corresponding positive eigenfunction φ_1.

A function $\varphi_1 \in C^\infty\left(\bar{\Omega}\right)$ is called an *eigenfunction* with *eigenvalue* λ_1 for the pair $(\mathcal{A}, \mathcal{B})$ if it satisfies

$$\left\{\begin{array}{rcll} \mathcal{A}\varphi & = & \lambda_1 \varphi & \text{in } \Omega, \\ \mathcal{B}\varphi & = & 0 & \text{on } \partial\Omega. \end{array}\right. \tag{2.10}$$

The eigenvalue λ_1 is called *geometrically simple* if every solution φ of (2.10) is a multiple of φ_1.

Our second general assumption reads as follows.

Assumption 2 *There exists $\lambda_1 \in \mathbb{R}$ which is a geometrically simple eigenvalue both for $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{A}^*, \mathcal{B}^*)$, with corresponding eigenfunctions φ_1, respectively φ_1^*, satisfying $\varphi_1(x) > 0$ and $\varphi_1^*(x) > 0$ for all $x \in \Omega$.*

Remark 2.1 Since both eigenfunctions are strictly positive we may and will assume that φ_1 and φ_1^* are normalized such that

$$\int_\Omega \varphi_1(y)\,\varphi_1^*(y)\,dy = 1 \text{ and } \|\varphi_1\|_{L^2(\Omega)} = \|\varphi_1^*\|_{L^2(\Omega)}\,. \tag{2.11}$$

Remark 2.2 According to Lemma C.2 of Appendix C Assumption 2 implies that λ_1 is an algebraically simple eigenvalue for the realization of the boundary value problem in $L^p(\Omega)$, $1 < p < \infty$.

In the theorems the distance function of x to the boundary $\partial\Omega$ will appear. This distance function is defined by:

$$d(x,\partial\Omega) = \inf_{y\in\partial\Omega} |x-y|\,. \tag{2.12}$$

Before stating the main result we will mention a non-uniform result which can be obtained under much weaker assumptions.

Proposition 2 *Let (2.1) be such that Assumptions 1 and 2 are satisfied. Let $m_{\mathcal{B}}$ be as above. Suppose that:*

a. *for the eigenfunction φ_1 in Assumptions 2 there exists $c_1 > 0$ such that*

$$\varphi_1(x) \geq c_1\, d(x,\partial\Omega)^{m_{\mathcal{B}}} \quad \text{for all } x\in\Omega; \tag{2.13}$$

b. $m_{\mathcal{B}} + \frac{n}{p} < 2m$.

Then for each $f \in L^p(\Omega)$ with

$$\int_\Omega f(y)\,\varphi_1^*(y)\,dy > 0 \tag{2.14}$$

there exists $\delta_f > 0$ such that the following holds for the solution u of (2.18).

i. If $\lambda \in (\lambda_1 - \delta_f, \lambda_1)$, then there exists $c > 0$ such that

$$u(x) \geq c\, d(x,\partial\Omega)^{m_{\mathcal{B}}} > 0 \qquad \text{for all } x\in\Omega.$$

ii. If $\lambda \in (\lambda_1, \lambda_1 + \delta_f)$, then there exists $c > 0$ such that

$$u(x) \leq -c\, d(x,\partial\Omega)^{m_{\mathcal{B}}} < 0 \qquad \text{for all } x\in\Omega.$$

(The proof is postponed to section 5.2.)

Remark 2.3 The condition $n < p$ that appears in [8], [27] for the anti-maximum principle in the case of a second order Dirichlet problem coincides with ***b***. Indeed for $m_{\mathcal{B}} = m = 1$ condition ***b*** becomes $\frac{n}{p} < 1$.

Remark 2.4 Observe that (2.14) may hold even if f changes sign. In the second order case Hopf's maximum principle implies that if f is positive conclusion *i.* holds for all $\lambda < \lambda_1$.

Next we consider the uniform case:

Theorem 3 *Let (2.1) be such that Assumptions 1 and 2 are satisfied. Let $m_{\mathcal{B}}$ and $m_{\mathcal{B}^*}$ be as above. Suppose that:*

aa. *for the eigenfunction φ_1 there exists $c_1 > 0$ such that*

$$\varphi_1(x) \geq c_1\, d(x, \partial\Omega)^{m_{\mathcal{B}}} \quad \text{for all } x \in \Omega; \tag{2.15}$$

the corresponding eigenfunction for the adjoint system is such that for some $c_1^* > 0$

$$\varphi_1^*(x) \geq c_1^*\, d(x, \partial\Omega)^{m_{\mathcal{B}^*}} \quad \text{for all } x \in \Omega; \tag{2.16}$$

bb. $m_{\mathcal{B}} + m_{\mathcal{B}^*} + n < 2m$.

Then there exists $\delta > 0$ such that for all $f \in C(\bar{\Omega})$ with $0 \neq f \geq 0$ the following holds for the solution u of (2.18).

i. If $\lambda \in (\lambda_1 - \delta, \lambda_1)$, then there exists $c > 0$ such that

$$u(x) \geq c\, d(x, \partial\Omega)^{m_{\mathcal{B}}} > 0 \qquad \text{for all } x \in \Omega.$$

ii. If $\lambda \in (\lambda_1, \lambda_1 + \delta)$, then there exists $c > 0$ such that

$$u(x) \leq -c\, d(x, \partial\Omega)^{m_{\mathcal{B}}} < 0 \qquad \text{for all } x \in \Omega.$$

(The proof is postponed to section 5.2.)

Remark 2.5 A necessary condition for a uniform anti-maximum principle to hold is that the Green function for some λ be bounded. For the Green function to be bounded $n < 2m$ is a necessary condition. Notice that this is guaranteed by condition ***bb***. It will be necessary that the Green function $G(x, y)$ is bounded by a multiple of $\varphi_{1,(\mathcal{A},\mathcal{B})}(x)\, \varphi^*_{1,(\mathcal{A}^*,\mathcal{B}^*)}(y)$ with φ_1, φ_1^* respectively the first eigenfunction of (2.1) and its adjoint (2.4). For known explicit Green functions ([17]) one may show that such a bound is equivalent with $m_{\mathcal{B}} + m_{\mathcal{B}^*} + n < 2m$. Note that this is exactly condition ***bb***. If ***bb*** is not satisfied then we expect that a counterexample for the uniform anti-maximum principle can be constructed by using similar arguments as in [27]. Takáč in [28] studied anti-maximum principles proceeding by the Green function. He uses that the resolvent is positive for all $\lambda \ll 0$ and satisfies the estimate $\left\| (A - \lambda I)^{-1} \right\| \leq \frac{M}{1+|\lambda|}$ which is the notion of positivity also used by Triebel (see [29, Definition 1.14.1]). Such bounds on the resolvent are implied by Assumption 1. The 'pointwise' notion of positivity for the resolvent operator for all $\lambda \ll 0$ is satisfied only for second order equations or cooperative systems of second order equations.

Remark 2.6 For the examples above we have

		$m_{\mathcal{B}}$	$M_{\mathcal{B}}$	***bb***
$\mathcal{B}_{m,\text{Dirichlet}}$		m	$m-1$	false
$\mathcal{B}_{m,\text{Navier}}$		1	$2m-2$	$n < 2m-2$
$\mathcal{B}_{m,\text{Robin}}$	$\theta_1 = 0$ and $\theta_m = 0$	1	$2m-2$	$n < 2m-2$
	$\theta_1 > 0$ and $\theta_m = 0$	0	$2m-2$	$n < 2m-1$
	$\theta_1 = 0$ and $\theta_m > 0$	1	$2m-1$	$n < 2m-1$
	$\theta_1 > 0$ and $\theta_m > 0$	0	$2m-1$	$n < 2m$
$\mathcal{B}_{m,\text{D},m_0,\text{D}}$		m_0	$m+m_0-1$	$n < m$

(2.17)

2.2 Systems that have an appropriate first eigenfunction

We will show that condition ***aa*** is satisfied for a large class of boundary value problems. We consider $\mathcal{A} = (-\Delta)^m$:

$$\begin{cases} (-\Delta)^m u &= \lambda u + f & \text{in } \Omega, \\ \mathcal{B}u &= 0 & \text{on } \partial\Omega. \end{cases} \tag{2.18}$$

The first type of boundary conditions are those for which the system can be written as a system of second order equations. $\mathcal{B}_{m,\text{Robin}}$ (and hence $\mathcal{B}_{m,\text{Navier}}$) is of this type. We shall refer to these by Case **I**.

Proposition 4 *Let $\mathcal{A} = (-\Delta)^m$ and assume that $\mathcal{B}$ is as in (2.9, iii) with $\theta_i \in C^\infty(\partial\Omega)$ and each θ_i either identical zero or strictly positive. Then Assumptions 1 and 2 hold and the eigenfunction φ_1 satisfies:*

i. if $\theta_1 = 0$ then there exists some constants $c_1, c_2 > 0$ such that

$$c_1\, d(x, \partial\Omega) \leq \varphi_1(x) \leq c_2\, d(x, \partial\Omega) \qquad \textit{for all } x \in \Omega; \tag{2.19}$$

ii. if $\theta_1(x) \geq c > 0$ then there exists some constants $c_1, c_2 > 0$ such that

$$c_1 \leq \varphi_1(x) \leq c_2 \qquad \textit{for all } x \in \Omega. \tag{2.20}$$

(The proof is postponed to section 6.3.)

The second type of boundary conditions that we can handle can be described as poly-Dirichlet:

$$\mathcal{B}u = \left\{\mathcal{B}_{m_1,\text{Dirichlet}}u, \mathcal{B}_{m_2,\text{Dirichlet}}\Delta^{m_1}u, \ldots, \mathcal{B}_{m_k,\text{Dirichlet}}\Delta^{m_1+m_2+\ldots+m_{k-1}}u\right\}. \tag{2.21}$$

Systems of this type will be referred to by Case **II**.

In order to have an appropriate eigenfunction the domain should be equal or close to a ball. Indeed, for general domains the system in (2.18) with $\mathcal{B} = \mathcal{B}_{m,\text{D},m_0,\text{D}}$ does not have a first eigenfunction which is positive. See the results for $\mathcal{B}_{2,\text{Dirichlet}}$ of Coffman and Duffin in [10], [9] and [11]. For Ω a ball a result of Boggio in [7] implies that the first eigenfunction does satisfy the estimate in (2.22). Recently Grunau and coauthor ([18]) obtained results that showed that the first eigenfunction remained positive under some small perturbations of the domain. It does not state the estimate (2.22), but a careful observation of the proof will yield this estimate.

Proposition 5 *Suppose that $\Omega = B$, a ball in $\mathbb{R}^n$, and that $\mathcal{B}$ is as in (2.21) with $m_1 + \ldots + m_k = m$. Then Assumptions 1 and 2 hold and the corresponding eigenfunction φ_1 satisfies for some $c_1, c_2 > 0$*

$$c_1\, d(x, \partial\Omega)^{m_1} \leq \varphi_1(x) \leq c_1\, d(x, \partial\Omega)^{m_1} \textit{ for all } x \in \Omega. \tag{2.22}$$

(The proof is postponed to section 6.3.)

In the examples which are considered in Propositions 4 and 5 inverse positivity results hold for $0 \leq \lambda < \lambda_1$, that is, if f is positive then u is positive. Such results are often referred to as 'maximum principle' type results.

2.3 Uniform anti-maximum principle

The combination of the theorem and the propositions above leads to the following corollary.

Corollary 6 (uniform anti-maximum principle) *Suppose that the conditions of either Proposition 4 or 5 are satisfied and that condition **bb** of Theorem 3 holds.*

Then there exists $\delta > 0$ such that for all $f \in C\left(\bar{\Omega}\right)$ with $0 \leq f \neq 0$ and $\lambda_1 < \lambda < \lambda_1 + \delta$ the solution u of (2.18) satisfies for some $c_u > 0$ the estimate $u(x) < -c_u\, d(x, \partial\Omega)^{m_{\mathcal{B}}}$ holds for all $x \in \Omega$.

Remark 2.7 It follows that the corresponding Green's function for $\lambda \in (\lambda_1, \lambda_1 + \delta)$ is nonpositive. In particular for any nonnegative f (say in $L^p(\Omega)$ for some $p > 0$) the corresponding solution u is nonpositive.

Examples *Recalling the results from the table in (2.17) we find a uniform anti-maximum principle for $\mathcal{A} = (-\Delta)^m$ and*

i. $\mathcal{B} = \mathcal{B}_{m,Navier}$, $\Omega \subset \mathbb{R}^n$ *bounded and smooth, if* $n < 2m - 2$;

ii. $\mathcal{B} = \mathcal{B}_{m,Robin}$, *with* $\theta_1 > 0$ *and* $\theta_n > 0$, $\Omega \subset \mathbb{R}^n$ *bounded and smooth, if* $n < 2m$;

iii. $\mathcal{B} = \mathcal{B}_{m,D,m_0,D}$, $\Omega = B \subset \mathbb{R}^n$ *a ball, if* $n < m$.

3 Solving the system in strong and weak sense

The aim of this section is to recall and establish some regularity results concerning the strong and the weak formulations of the regular elliptic problem (2.1). By using extrapolation and interpolation techniques, (see [22, Chapter 2] for general elliptic operators in the Hilbert space case; see [3] for the second order case in L^p-setting; the general case in L^p-setting is announced for [4, Vol. 2]) we obtain an intermediate result which is needed to establish a uniform anti-maximum principle. In particular we shall use a formulation where the boundary conditions are satisfied in a strong sense although the right hand side of the differential equation will be a distribution (Theorem 13).

We start be recalling a solvability result in the C^∞-framework which will be used later on.

3.1 Solvability in $C^\infty\left(\bar{\Omega}\right)$.

Let $(\mathcal{A}, \mathcal{B})$ satisfy Assumption 1, let $(\mathcal{A}^*, \mathcal{B}^*)$ be the adjoint system and let $f \in C^\infty\left(\bar{\Omega}\right)$. We are interested in the solvability of problem (2.1). The following holds (see [22, Chapter 2, Proposition 5.3]).

Theorem 7 *Let $f \in C^\infty\left(\bar{\Omega}\right)$. Problem (2.1) has at least one solution $u \in C^\infty\left(\bar{\Omega}\right)$ if and only if the condition*

$$\int_\Omega f\, v\, dx = 0$$

holds for all $v \in C^\infty\left(\bar{\Omega}\right)$ such that $(\mathcal{A}^* - \lambda)\, v = 0$, $\mathcal{B}^* v = 0$.

Before considering the solvability of (2.1) in appropriate L^p-type spaces, the so-called Bessel potential spaces, we recall some definitions and properties.

3.2 Bessel potential spaces

For convenience we recall the definitions of standard Bessel potential spaces and corresponding spaces associated with boundary conditions.

We assume that $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. The Bessel potential spaces on $\mathbb{R}^n$ are defined by

$$H^{s,p}(\mathbb{R}^n) = \left\{ u \in S'(\mathbb{R}^n) ; \left\| \mathcal{F}^{-1} \left(1 + |\xi|^2\right)^{\frac{s}{2}} \mathcal{F}u \right\|_{L^p} < \infty \right\}, \tag{3.1}$$

where $S'(\mathbb{R}^n)$ is the set of tempered distributions and $\mathcal{F}$ the Fourier transform, see [29].

Let Ω be a bounded domain in $\mathbb{R}^n$ with $\partial\Omega \in C^\infty$. For $s \in \mathbb{R}$ the Bessel-potential spaces on Ω are defined by (see [29, Definition 4.2.1])

$$\begin{aligned} H^{s,p}(\Omega) &= \left\{ u = g_{|\Omega} ; g \in H^{s,p}(\mathbb{R}^n) \right\}, \text{ with} \\ \|u\|_{H^{s,p}(\Omega)} &= \inf\nolimits_{g \in H^{s,p}(\mathbb{R}^n)} \|g\|_{H^{s,p}(\mathbb{R}^n)} . \end{aligned} \tag{3.2}$$

For $k \in \mathbb{N}$ we have $H^{k,p}(\Omega) = W^{k,p}(\Omega)$, where $W^{k,p}(\Omega)$ is the usual Sobolev space equipped with $\|u\|_{W^{k,p}} = \sum_{|\alpha| \le k} \|\mathcal{D}^\alpha u\|_{W^{s,p}}$.

Two types of subspaces of functions that vanish on $\partial\Omega$ in an appropriate sense are

$$H_0^{s,p}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{H^{s,p}(\Omega)}} \text{ (closure in } H^{s,p}(\Omega)\text{)} \tag{3.3}$$

and

$$\tilde{H}^{s,p}(\Omega) = \left\{ u \in H^{s,p}(\mathbb{R}^n) ; \operatorname{supp} u \subset \bar{\Omega} \right\}. \tag{3.4}$$

For sake of completeness let us recall the definition of support. For continuous functions, say $f \in C(\mathbb{R}^n)$, the support is defined by

$$\operatorname{supp}(f) = \overline{\{x \in \mathbb{R}^n ; f(x) \neq 0\}}. \tag{3.5}$$

The support of a generalized function is defined as follows (see e.g. [31, page 62]). A distribution $\delta \in C_0^\infty(\mathbb{R}^n)'$ is said to vanish on an open set $\mathcal{O} \subset \mathbb{R}^n$ if $\delta(f) = 0$ for all $f \in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp}(f) \subset \mathcal{O}$. The support of δ is defined by

$$\operatorname{supp}(\delta) = \text{the smallest closed set } \mathcal{F} \subset \mathbb{R}^n \text{ such that } \delta \text{ vanishes on } \mathbb{R}^n \backslash \mathcal{F}. \tag{3.6}$$

If δ can be represented by a continuous function f one has $\operatorname{supp}(\delta) = \operatorname{supp}(f)$.

In Triebel [29, Theorem 4.8.1] the following duality results are stated:

$$H^{-s,p}(\Omega) = \left(\tilde{H}^{s,q}(\Omega)\right)'.$$

Moreover in [29, Theorem 4.3.2] one finds:

$$\begin{array}{llcl} \text{if} & s - \frac{1}{p} \le 0 & \text{then} & \tilde{H}^{s,p}(\Omega) = H^{s,p}(\Omega), \\ \text{if} & -1 < s - \frac{1}{p} \notin \mathbb{N} & \text{then} & \tilde{H}^{s,p}(\Omega) = H_0^{s,p}(\Omega). \end{array}$$

Related spaces corresponding to other boundary conditions $\mathcal{B}$ are defined in [29, Definition 4.3.3.2]. The space

$$H_{\mathcal{B}}^{s,p}(\Omega) = \{ u \in H^{s,p}(\Omega) ; \mathcal{B}u = 0 \text{ on } \partial\Omega \} \tag{3.7}$$

is well defined if $M_{\mathcal{B}}$, the largest order derivative that appears in $\mathcal{B}$, is strictly less then $s - \frac{1}{p}$. Indeed, if $s > M_{\mathcal{B}} + \frac{1}{p}$ then $\mathcal{B}_i : H^{s,p}(\Omega) \to L^p(\partial\Omega)$, $i = 1, \dots, m$ are bounded linear operators, see [29, Theorem 4.7.1]. Moreover, $H^{s,p}_{\mathcal{B}}(\Omega)$ is a Banach-space as a closed subspace of $H^{s,p}(\Omega)$.

Finally we mention that for $p \in (1,\infty)$ and $s \in \left(m - 1 + \frac{1}{p}, m + \frac{1}{p}\right)$

$$H^{s,p}_{\mathcal{B}_{m,\text{Dirichlet}}}(\Omega) = H^{s,p}_0(\Omega), \tag{3.8}$$

where $\mathcal{B}_{m,\text{Dirichlet}}$ is defined in (2.9). Again we refer to [29, Theorem 4.7.1].

3.3 Some notations

In the sequel clarity will greatly benefit from an accurate notation:

pairing	$\langle\cdot,\cdot\rangle : L^p \times L^q \to \mathbb{R}$	defined by	$\langle u,v\rangle = \int_\Omega u(x)\, v(x)\, dx$	for $u \in L^p$, $v \in L^q$;
imbedding	$j_p : H^{2m,p}_{\mathcal{B}} \to L^p$	.. ..	$(j_p u)(x) = u(x)$	for $u \in H^{2m,p}_{\mathcal{B}}$, $x \in \Omega$;
isometry	$i_p : L^p \to (L^q)'$	.. ..	$(i_p u)(v) = \langle u,v\rangle$	for $u \in L^p$, $v \in L^q$;
imbedding	$j'_q : (L^q)' \to \left(H^{2m,q}_{\mathcal{B}^*}\right)'$	.. ..	$(j'_q U)(v) = U(j_q v)$	for $U \in (L^q)'$, $v \in H^{2m,q}_{\mathcal{B}^*}$.

Here we used $\frac{1}{q} + \frac{1}{p} = 1$ and $H^{s,p}_{\mathcal{B}} = H^{s,p}_{\mathcal{B}}(\Omega)$. The operator i_p is an isometric isomorphism. For a linear operator $T : X \to Y$ we denote by T' the dual operator $T' : Y' \to X'$ which is defined by

$$(T'U)(v) = U(Tv) \text{ for } U \in Y',\ v \in X.$$

An element u of $H^{2m,p}_{\mathcal{B}}$ is a function and if there is no ambivalence we will denote the function $j_p u \in L^p$ also by u.

3.4 Realization in $L^p(\Omega)$

In this subsection we shall consider the realization A of the operator $(\mathcal{A},\mathcal{B})$ in $L^p(\Omega)$ and of its adjoint in $L^q(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$ and $p, q > 1$.

We define the unbounded operator $A : D(A) \subset L^p \to L^p$ by

$$D(A) = H^{2m,p}_{\mathcal{B}} \text{ and } Au = \mathcal{A}u \text{ for } u \in D(A). \tag{3.9}$$

It appears that under Assumption 1 the realization of the operator $(\mathcal{A}^*,\mathcal{B}^*)$ in $L^q(\Omega)$ is the adjoint of A. In the following we summarize the properties of the operator A which will be used later on. We will also use the bounded operator $A_0 = A \circ j_p : H^{2m,p}_{\mathcal{B}} \to L^p$.

Theorem 8 *Let the pair $(\mathcal{A},\mathcal{B})$ satisfy Assumption 1 and let A be as above. We have:*

i. *For all $r > 0$ there exists $c > 0$ such that for all $u \in H^{2m,p}_{\mathcal{B}}$ the following holds*

$$\|u\|_{H^{2m+r,p}} \le c\left(\|Au\|_{H^{r,p}} + \|u\|_{H^{r,p}}\right). \tag{3.10}$$

ii. *If $u \in D(A)$ and $Au \in H^{\ell,p}$ for some $\ell \in \mathbb{N}^+$, then $u \in H^{2m+\ell,p}$.*

iii. *The operator A is densely defined, closed, and has closed range $R(A)$.*

iv. *The null space $N(A)$ is finite dimensional, contained in $C^\infty(\bar{\Omega})$ and independent of p.*

v. *The adjoint operator* A^* *in* $L^q(\Omega)$ *satisfies* $D(A^*) = H^{2m,p}_{\mathcal{B}^*}$ *and* $A^*v = \mathcal{A}^*v$ *for* $v \in D(A^*)$.

If moreover Assumption 2 is satisfied, then

vi. $\dim N(A - \lambda_1) = \dim N(A^* - \lambda_1) = 1$.

vii. *There exists* $\delta > 0$ *such that for* $0 < |\lambda - \lambda_1| < \delta$ *we have* $\lambda \in \rho(A)$.

Proof. Parts i.-v. can be found in [22] for $p = 2$ or e.g. in [29] for other values of p. For a sketch of the proof see Appendix C.
Part vi.: Since $N(A - \lambda_1) \subset C^\infty(\bar{\Omega})$ and the only $C^\infty(\bar{\Omega})$-solutions are multiples of φ_1 we have $\dim N(A - \lambda_1) = 1$. Since A^* is also the realization of $(\mathcal{A}^*, \mathcal{B}^*)$ in L^q it follows from iv.-v. that $\dim N(A^* - \lambda_1) \subset C^\infty(\bar{\Omega})$ and hence by Assumption 2 that $\dim N(A^* - \lambda_1) = 1$.
Part vii.: Let $A_0 = A \circ j_p$. We have $(A_0 - \lambda_1 j_p)\varphi_1 = 0$, $\dim N(A_0 - \lambda_1 j_p) = 1$ and since $R(A - \lambda_1)^\perp = N(A^* - \lambda_1)$ it follows that

$$\operatorname{codim} R(A_0 - \lambda_1 j_p) = \operatorname{codim} R(A - \lambda_1) = 1.$$

Since $\langle \varphi_1, \varphi_1^* \rangle \neq 0$ we have $\varphi_1 \notin R(A_0 - \lambda_1 j_p)$. Hence, see Definition A.1, φ_1 is a j_p-simple eigenvalue of A_0. The result follows from Theorem A.3. □

3.5 Extrapolation

For the extrapolation procedure we will use the adjoint operator A^*. Let A_0^* denote the bounded operator

$$A_0^* := (A^*)_0 = A^* \circ j_q : H^{2m,q}_{\mathcal{B}^*} \to L^q.$$

The weak formulation of the differential equation becomes: let $F \in \left(H^{2m,q}_{\mathcal{B}^*}\right)'$ and find $U \in (L^q)'$ such that

$$U\left((A_0^* - \lambda j_q)(v)\right) = F(v) \text{ for all } v \in H^{2m,q}_{\mathcal{B}^*} \tag{3.11}$$

which we may rewrite as $\left((A_0^*)' - \lambda j_q'\right)U = F$. The extension of $A_0 = A \circ j_p$ is defined by

$$A_{-1} := (A_0^*)' : (L^q)' \to \left(H^{2m,q}_{\mathcal{B}^*}\right)'.$$

For $u \in L^p$, the image $A_{-1} \circ i_p(u)$ lies in $\left(H^{2m,q}_{\mathcal{B}^*}\right)'$ and for all $v \in H^{2m,q}_{\mathcal{B}^*}$ we have

$$(A_{-1} \circ i_p(u))(v) = (i_p u)(A_0^*(v)) = \int_\Omega u\, \mathcal{A}^* v\, dx. \tag{3.12}$$

Hence (3.11) can be rewritten as

$$\int_\Omega u\,(\mathcal{A}^* - \lambda)\, v\, dx = F(v) \text{ for all } v \in H^{2m,q}_{\mathcal{B}^*}. \tag{3.13}$$

We have:

Theorem 9 *Assume that* $(\mathcal{A}, \mathcal{B})$ *are such that (2.1) forms a regular system and let* A *be defined as in (3.9). Assume that* $\lambda \in \rho(A)$.

Then for every $F \in \left(H^{2m,q}_{\mathcal{B}^*}\right)'$ *one and only one* $U \in (L^q)'$ *exists for which (3.11) holds. Moreover, there exists* $c_\lambda > 0$, *independent of* F, *such that*

$$\left\| i_p^{-1} U \right\|_{L^p} \le c_\lambda \sup \left\{ F(v) ; v \in H^{2m,q}_{\mathcal{B}^*}, \|v\|_{H^{2m,q}} \le 1. \right\} \tag{3.14}$$

Proof. If $(\mathcal{A},\mathcal{B})$ is regular elliptic then also the formal adjoint $(\mathcal{A}^*,\mathcal{B}^*)$ is regular elliptic. Since the realization A^* of $(\mathcal{A}^*,\mathcal{B}^*)$ is indeed the adjoint of A (see Theorem 8) we find that A and A^* have the same spectrum. We may use Theorem 8 for $(\mathcal{A}^*,\mathcal{B}^*)$ and solve

$$\begin{cases} \mathcal{A}^* v &= \lambda v + g \quad \text{in } \Omega, \\ \mathcal{B}^* v &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3.15}$$

for any $g \in L^q$ to find $v \in H^{2m,q}_{\mathcal{B}^*}$ with $\|v\|_{H^{2m,q}} \leq \tilde{c}_\lambda \|g\|_{L^q}$. The estimate in (3.14) follows from duality. □

In remains to show that A_{-1} is indeed an extension of A_0, that is $A_{-1} \circ i_p^{-1}$ extends $i_p \circ A_0$. It is sufficient to show that for some $\lambda \in \rho(A)$ and every $f \in L^p$ the functions

$$u = (A_0 - \lambda I)^{-1} f \qquad \text{and} \qquad \tilde{u} = i_p^{-1} \circ (A_{-1} - \lambda I)^{-1} \circ j_q' \circ i_p f$$

coincide. Note that $\tilde{u}$ is the unique function in $L^p(\Omega)$ such that

$$\int_\Omega \tilde{u}\, (\mathcal{A}^* - \lambda)\, v\, dx = (i_p f)(j_q v) \text{ for all } v \in H^{2m,q}_{\mathcal{B}^*}(\Omega).$$

Since $u = (A_0 - \lambda I)^{-1} f \in H^{2m,p}_{\mathcal{B}}(\Omega)$ an integration by part shows

$$\int_\Omega u\, (\mathcal{A}^* - \lambda)\, v\, dx = \int_\Omega (\mathcal{A} - \lambda)\, u\, v\, dx = \int_\Omega f\, v\, dx = (i_p f)(j_q v) \text{ for all } v \in H^{2m,q}_{\mathcal{B}^*}(\Omega)$$

and hence $u = \tilde{u}$.

4 The system near the first eigenvalue

4.1 Projection on the first eigenfunction

The eigenfunctions φ_1 of A and φ_1^* of A^* are assumed to be normalized such that

$$\|\varphi_1\|_{L^p} = \|\varphi_1^*\|_{L^q} \text{ and } \int_\Omega \varphi_1(x)\, \varphi_1^*(x)\, dx = 1. \tag{4.16}$$

By Assumption 2 λ_1 is a j_p-simple eigenvalue of A_0 and a j_q-simple eigenvalue of A_0^*. See Appendix A. By duality we have that λ_1 is a j_q'-simple eigenvalue of A_{-1}. The j_p-, j_q- and j_q'-eigenfunctions are respectively $\varphi_1 \in H^{2m,p}_{\mathcal{B}}$, $\varphi_1^* \in H^{2m,q}_{\mathcal{B}^*}$ and $\Phi_1 = i_p \circ j_p \varphi_1 \in (L^q)'$. The corresponding projections for A_0 as in Theorem A.3 are $P_0 : L^p \to L^p$ and $P_1 : H^{2m,p}_{\mathcal{B}} \to H^{2m,p}_{\mathcal{B}}$ defined by

$$P_0 f = \langle f, \varphi_1^* \rangle\, j_p \varphi_1 \quad \text{for all } f \in L^p, \tag{4.17}$$

$$P_1 u = \langle j_p u, \varphi_1^* \rangle\, \varphi_1 \quad \text{for all } u \in H^{2m,p}_{\mathcal{B}}. \tag{4.18}$$

In a similar way we may define projections $\tilde{P}_0 : \left(H^{2m,q}_{\mathcal{B}^*}\right)' \to \left(H^{2m,q}_{\mathcal{B}^*}\right)'$ and $\tilde{P}_1 : (L^q)' \to (L^q)'$ by

$$\tilde{P}_0 F = F(\varphi_1^*)\, j_q' \circ \Phi_1 \quad \text{for all } F \in \left(H^{2m,q}_{\mathcal{B}^*}\right)', \tag{4.19}$$

$$\tilde{P}_1 U = U(j_q \varphi_1^*)\, \Phi_1 \quad \text{for all } U \in (L^q)'. \tag{4.20}$$

Notice that for $F \in \left(H^{2m,q}_{\mathcal{B}^*}\right)'$ and $v \in H^{2m,q}_{\mathcal{B}^*}$

$$\begin{aligned} \left(\tilde{P}_0 F\right)(v) &= F(\varphi_1^*)\,\langle \varphi_1, j_q v\rangle \text{ for } F \in \left(H^{2m,q}_{\mathcal{B}^*}\right)' \text{ and } v \in H^{2m,q}_{\mathcal{B}^*}, \\ \left(\tilde{P}_1 U\right)(g) &= U(j_q \varphi_1^*)\,\langle \varphi_1, g\rangle \text{ for } U \in (L^q)' \text{ and } g \in L^q. \end{aligned}$$

The projections P_0 and $\tilde{P}_1$ are related through $i_p \circ P_0 = \tilde{P}_1 \circ i_p$:

$$\left(\tilde{P}_1 \circ i_p f\right)(g) = \langle f, \varphi_1^*\rangle\,\langle \varphi_1, g\rangle = (i_p \circ P_0 f)(g) \text{ for } f \in L^p \text{ and } g \in L^q.$$

Let us summarize the results in the following scheme.

$$\begin{array}{ccccccc} H^{2m,p}_{\mathcal{B}} & \overset{A_0 - \lambda j_p}{\longrightarrow} & L^p & \overset{i_p}{\simeq} & (L^q)' & \overset{A_{-1} - \lambda j_p'}{\longrightarrow} & \left(H^{2m,q}_{\mathcal{B}^*}\right)' \\ \shortparallel & & \shortparallel & & \shortparallel & & \shortparallel \\ N(P_1) & \overset{\text{isomorphism}}{\longrightarrow} & R(A_0 - \lambda_1 j_p) = N(P_0) & \simeq & N(\tilde{P}_1) & \overset{\text{isomorphism}}{\longrightarrow} & R\left(A_{-1} - \lambda_1 j_q'\right) = N(\tilde{P}_0) \\ \oplus & & \oplus & & \oplus & & \oplus \\ R(P_1) & \longrightarrow & \text{span}\{\varphi_1\} = R(P_0) & \simeq & R(\tilde{P}_1) & \longrightarrow & \text{span}\left\{j_q' \circ i_p(\varphi_1)\right\} = R(\tilde{P}_0) \end{array}$$

As a consequence of Theorem A.3 we find:

Corollary 10 *There exist $c > 0$ and $\delta > 0$ such that for $|\lambda - \lambda_0| < \delta$ the following holds.*

i. For all $f \in L^p$ with $\langle f, \varphi_1^*\rangle = 0$ there exists a unique $u_\lambda \in H^{2m,p}_{\mathcal{B}}$ with $\langle u_\lambda, \varphi_1^*\rangle = 0$ such that $(A_0 - \lambda j_p)\, u_\lambda = f$ and moreover

$$\|u_\lambda\|_{H^{2m,p}} \leq c\,\|f\|_{L^p}.$$

ii. For all $F \in \left(H^{2m,q}_{\mathcal{B}^*}\right)'$ with $F(\varphi_1^*) = 0$ there exists a unique $U_\lambda \in (L^q)'$ with $U_\lambda(\varphi_1^*) = 0$ such that $\left(A_{-1} - \lambda j_q'\right) U_\lambda = F$ and moreover

$$\|U_\lambda\|_{(L^q)'} \leq c\,\|F\|_{\left(H^{2m,q}_{\mathcal{B}^*}\right)'}.$$

Remark 4.1 In a formulation closer to the differential equation the second statement means: for every bounded linear form $F \in \left(H^{2m,q}_{\mathcal{B}^*}\right)'$ with $F(\varphi_1^*) = 0$ there exists a unique $u_\lambda \in L^p$ with $\langle u_\lambda, \varphi_1^*\rangle = 0$ such that

$$\int_\Omega u_\lambda(x)\,(\mathcal{A}^* - \lambda)\, v(x)\, dx = F(v) \text{ for all } v \in H^{2m,q}_{\mathcal{B}^*}.$$

Moreover,

$$\|u_\lambda\|_{L^p} \leq c \sup\left\{F(v)\,;\, v \in H^{2m,q}_{\mathcal{B}^*} \text{ and } \|v\|_{H^{2m,q}} \leq 1\right\}.$$

4.2 Interpolation

We will use interpolation results for Bessel-potential space with boundary conditions. These kind of results are due to Grisvard, [15], for real interpolation and Seeley, [26], for the complex interpolation.

Lemma 11 *Let $s \in (0,2m)$ with $s - \frac{1}{p} \notin \mathbb{N}$. For $\theta = \frac{s}{2m}$ it follows that*

$$\left[L^p, H^{2m,p}_{\mathcal{B}}\right]_{\frac{s}{2m}} = H^{s,p}_{\left\{\mathcal{B}_j \in \mathcal{B}; m_j < s - \frac{1}{p}\right\}}. \tag{4.21}$$

For $s \in \left(M_{\mathcal{B}} + \frac{1}{p}, 2m\right)$ with $s - \frac{1}{p} \notin \mathbb{N}$ we find $[L^p, H^{2m,p}_{\mathcal{B}}]_\theta = H^{s,p}_{\mathcal{B}}$ and $[(H^{2m,q}_{\mathcal{B}^*})', (L^q)']_\theta = H^{s-2m,p}$.

Proof. By results of Seeley (see [25], [26] or [29, Theorem 4.3.3]) one finds if $s - \frac{1}{p} \notin \mathbb{N}$ that (4.21) holds. Hence by our definition of $M_{\mathcal{B}}$ and $m_{\mathcal{B}}$ we have for $0 < \sigma < 1$ that

$$\begin{array}{llll} \text{if} \quad 2m\sigma > M_{\mathcal{B}} + \frac{1}{p}, & \text{then} \quad \left[L^p, H^{2m,p}_{\mathcal{B}}\right]_\sigma & = & H^{2m\sigma,p}_{\mathcal{B}}, \\ \text{if} \quad 2m\sigma < m_{\mathcal{B}} + \frac{1}{p}, & \text{then} \quad \left[L^p, H^{2m,p}_{\mathcal{B}}\right]_\sigma & = & H^{2m\sigma,p}_0, \end{array}$$

where we refer to (3.8) for the second identity.

The first identity implies with $\theta = \sigma = \frac{s}{2m}$ and $s - \frac{1}{p} \notin \mathbb{N}$ that

$$\text{if } s > M_{\mathcal{B}} + \frac{1}{p} \text{ then } \left[L^p, H^{2m,p}_{\mathcal{B}}\right]_\theta = H^{s,p}_{\mathcal{B}}. \tag{4.22}$$

For the last claim we proceed as follows. Since $[X', Y']_\theta = [X, Y]'_\theta$ (see [29, Lemma 1.11.3]) it is sufficient to identify $[H^{2m,q}_{\mathcal{B}^*}, L^q]_\theta$ for $\theta = \frac{s}{2m}$. By (4.21) we find, assuming $2m\theta + \frac{1}{q} \notin \mathbb{N}$, that

$$\left[H^{2m,q}_{\mathcal{B}^*}, L^q\right]_\theta = H^{2m(1-\theta),q}_{\{\mathcal{B}^*_j \in \mathcal{B}^*; m_j < 2m(1-\theta) - \frac{1}{q}\}}.$$

We have $2m\theta = s$ and since $s > M_{\mathcal{B}} + \frac{1}{p}$ it follows that $2m(1-\theta) - \frac{1}{q} < 2m - M_{\mathcal{B}} - \frac{1}{p} - \frac{1}{q} = m_{\mathcal{B}^*}$ and moreover that $2m(1-\theta) - \frac{1}{q} = 2m - 1 - \left(s - \frac{1}{p}\right) \notin \mathbb{N}$, implying

$$H^{2m(1-\theta),q}_{\{\mathcal{B}^*_j \in \mathcal{B}^*; m_j < 2m(1-\theta) - \frac{1}{q}\}} = H^{2m(1-\theta),q}_0.$$

Since $H^{2m(1-\theta),q}_0 = \tilde{H}^{2m(1-\theta),q}$ for $2m(1-\theta) - \frac{1}{q} \notin \mathbb{N}$ (see [29, Theorem 4.3.2.1]) the proof of Lemma 11 is complete. □

Proposition 12 *Let X, X_1 and X_2 be Banach spaces such that X_1, X_2 are continuously imbedded in X. Let $P : X \to X$ be a bounded linear operator satisfying $P = P^2$. Moreover, assume $P(X_i) \subset X_i$ and $P_{|X_i} : X_i \to X_i$ is continuous ($i = 1, 2$). Let $\theta \in (0,1)$. Then*

$$[X_1 \cap P(X), X_2 \cap P(X)]_\theta = [X_1, X_2]_\theta \cap P(X).$$

Remark 4.2 The Banach space $X_i \cap P(X)$ is considered as a subspace of X_i, that is, equipped with the X_i-norm. Similarly, $[X_1, X_2]_\theta \cap P(X)$ is considered to be equipped with the $[X_1, X_2]_\theta$-norm.

Proof. For general interpolation functors the result is due to [5] and can also be found in [29, Theorem 1.17.1.1]. Indeed, following the notation of [29],

$$\begin{array}{rcl} X_1 + X_2 & = & \{x \in X; \exists x_i \in X_i \text{ s. t. } x = x_1 + x_2\}, \\ \|x\|_{X_1+X_2} & = & \inf\left\{\|x_1\|_{X_1} + \|x_2\|_{X_2}; x = x_1 + x_2, x_i \in X_i\right\}, \end{array}$$

is a Banach space. Take $B = P(X_1 + X_2)$, equipped with the $\|\cdot\|_{X_1+X_2}$-norm. Since $P(X_1 + X_2) \subset X_1 + X_2$ we have $X_i \cap P(X) = X_i \cap P(X_1 + X_2)$ and may apply [29, Theorem 1.17.1.1] using complex interpolation. □

We conclude this section by interpolating the two results in Corollary 10.

Theorem 13 *Let $s \in (0, 2m)$ and $s - \frac{1}{p} \notin \mathbb{N}$, then there exists $c > 0$ and $\delta > 0$ such that for all $\lambda \in \mathbb{R}$ with $|\lambda - \lambda_1| < \delta$ and $F \in (H^{2m-s,q}_{\{\mathcal{B}^*_j \in \mathcal{B}^*; m_j < 2m-s-\frac{1}{q}\}})'$ with $F(\varphi_1^*) = 0$ the following holds.*

i. There exists a unique $u_\lambda \in H^{s,p}_{\{\mathcal{B}_j \in \mathcal{B}; m_j < s-\frac{1}{p}\}}$ with $\langle u_\lambda, \varphi_1^* \rangle = 0$ for which (3.13) holds.

ii. Moreover,

$$\begin{aligned} \|u_\lambda\|_{H^{s,p}} &\le c\,\|F\|_{H^{s-2m}} && \text{if} \quad s \ge 2m - m_{\mathcal{B}^*}, \\ \|u_\lambda\|_{H^{s,p}} &\le c\,\|F\|_{\left(H^{2m-s,q} \cap H_0^{m_{\mathcal{B}^*},q}\right)'} && \text{if} \quad s \le 2m - m_{\mathcal{B}^*}. \end{aligned} \tag{4.23}$$

Remark 4.3 For $s \le 2m - m_{\mathcal{B}^*}$ the norm of F is defined by

$$\|F\|_{\left(H^{2m-s,q} \cap H_0^{m_{\mathcal{B}^*},q}\right)'} = \sup\left\{F(v)\,; v \in H^{2m-s,q} \cap H_0^{m_{\mathcal{B}^*},q} \text{ with } \|v\|_{H^{2m-s,q}} \le 1\right\}.$$

Proof. The operator $T_{-1,\lambda} := \left(A_{-1} - \lambda j_q'\right) \circ i_p : L^p \to \left(H^{2m,q}_{\mathcal{B}^*}\right)'$ is an extension of $T_{0,\lambda} := i_p \circ (A_0 - \lambda j_p) : H^{2m,p}_{\mathcal{B}} \to (L^q)'$ in the sense that $j_q' \circ T_{0,\lambda} = T_{-1,\lambda} \circ j_p$. By interpolation we define for $\theta \in (0, 1)$ the intermediate operator

$$T_{-\theta,\lambda} : \left[L^p, H^{2m,p}_{\mathcal{B}}\right]_\theta \to \left[(H^{2m,q}_{\mathcal{B}^*})', (L^q)'\right]_\theta.$$

Let $P_0, P_1, \tilde{P}_0$ and $\tilde{P}_1$ be as in the previous section and we interpolate between

$$\begin{aligned} S_{-1,\lambda} &:= \left(\left(A_{-1} - \lambda j_q'\right) \circ i_p\right)_{|N(P_0)} &: \ N(P_0) &\to N(\tilde{P}_0), \\ S_{0,\lambda} &:= \left(i_p \circ (A_0 - \lambda j_p)\right)_{|N(P_1)} &: \ N(P_1) &\to N(\tilde{P}_1), \end{aligned}$$

and define the intermediate operator

$$S_{-\theta,\lambda} : [N(P_0), N(P_1)]_\theta \to \left[N(\tilde{P}_0), N(\tilde{P}_1)\right]_\theta. \tag{4.24}$$

Since

$$\begin{aligned} N(P_0) &= L^p \cap P_0(L_p) \quad \text{and} \quad N(\tilde{P}_0) = (H^{2m,q}_{\mathcal{B}^*})' \cap \tilde{P}_0\left((H^{2m,q}_{\mathcal{B}^*})'\right), \\ N(P_1) &= H^{2m,p}_{\mathcal{B}} \cap P_0(L_p) \quad \text{and} \quad N(\tilde{P}_1) = (L^q)' \cap \tilde{P}_0\left((H^{2m,q}_{\mathcal{B}^*})'\right), \end{aligned}$$

we may use Proposition 12 to find that

$$[N(P_0), N(P_1)]_\theta = \left[L^p, H^{2m,p}_{\mathcal{B}}\right]_\theta \cap P_0(L_p) \quad \text{and} \tag{4.25}$$

$$\left[N(\tilde{P}_0), N(\tilde{P}_1)\right]_\theta = \left[(H^{2m,q}_{\mathcal{B}^*})', (L^q)'\right]_\theta \cap \tilde{P}_0\left((H^{2m,q}_{\mathcal{B}^*})'\right). \tag{4.26}$$

Using Lemma 11 we have for $\theta = \frac{s}{2m}$ that

$$\left[L^p, H^{2m,p}_{\mathcal{B}}\right]_\theta = H^{s,p}_{\{\mathcal{B}_j \in \mathcal{B}; m_j < s-\frac{1}{p}\}} \quad \text{and} \quad \left[(H^{2m,q}_{\mathcal{B}^*})', (L^q)'\right]_\theta = \left(H^{2m-s,q}_{\{\mathcal{B}^*_j \in \mathcal{B}^*; m_j < 2m-s-\frac{1}{q}\}}\right)'.$$

For all $|\lambda-\lambda_0|<\delta$ Corollary 10 implies that $\lambda\to S_{-\theta,\lambda}$ is analytic and hence shows

$$\|u_\lambda\|_{H^{s,p}}=\|u_\lambda\|_{H^{s,p}_{\{\mathcal{B}_j\in\mathcal{B};m_j<s-\frac{1}{p}\}}}\leq c\,\|F\|_{(H^{2m-s,q}_{\{\mathcal{B}^*_j\in\mathcal{B}^*;m_j<2m-s-\frac{1}{q}\}})'}.$$

Since for $2m-s\geq m_{\mathcal{B}^*}$ one has $H^{2m-s,q}_{\{\mathcal{B}^*_j\in\mathcal{B}^*;m_j<2m-s-\frac{1}{q}\}}\subset H^{2m-s,q}\cap H_0^{m_{\mathcal{B}^*},q}$ it follows that

$$\|F\|_{(H^{2m-s,q}_{\{\mathcal{B}^*_j\in\mathcal{B}^*;m_j<2m-s-\frac{1}{q}\}})'}\leq\|F\|_{\left(H^{2m-s,q}\cap H_0^{m_{\mathcal{B}^*},q}\right)'}.$$

For $2m-s\leq m_{\mathcal{B}^*}$ the claim results from $H^{2m-s,q}_{\{\mathcal{B}^*_j\in\mathcal{B}^*;m_j<2m-s-\frac{1}{q}\}}=H_0^{2m-s,q}$. □

5 Proof of the main results

5.1 The link between positivity and a Sobolev type estimate

In this section we will show that for positive f we may estimate a norm, in a negative Bessel-potential space, of $(f-P_0f)$ by the projection of f on the first eigenvalue.

Theorem 14 *Let $p,q\in(1,\infty)$ with $\frac{1}{p}+\frac{1}{q}=1$, $f\in L^p$ and suppose that $s\in\mathbb{R}^+$ satisfies*

$$s-\frac{n}{q}>m_{\mathcal{B}^*}. \tag{5.1}$$

Also assume that (2.16) holds: $\varphi_1^*(x)\geq c_1\,d(x,\partial\Omega)^{m_{\mathcal{B}^*}}$.

Then there exists $c_2>0$ such that for all $f\in L^p(\Omega)$ with $f\geq0$:

$$\|f-P_0f\|_{H^{-s,p}}\leq c_2\,\langle f,\varphi_1^*\rangle. \tag{5.2}$$

Proof. Writing $f=P_0f-f_e$, it follows from $f\geq0$ that $f_e\leq P_0f$. By a Sobolev imbedding, see [1], we find using (5.1) that $H^{s,q}(\Omega)\hookrightarrow C^{m_{\mathcal{B}^*}}\left(\bar{\Omega}\right)$. Moreover, by Lemma D.1 one finds that there exists a uniform constant $c^\star$ such that for all $x\in\Omega$ and all $\psi\in H^{s,q}(\Omega)\cap H_0^{m_{\mathcal{B}^*},q}(\Omega)$ with $\|\psi\|_{H^{s,q}}=1$ one has

$$|\psi(x)|\leq c^\star\,d(x,\partial\Omega)^{m_{\mathcal{B}^*}}\leq c^*\,\varphi_1^*(x).$$

In the last inequality we used (2.16) and denoted $c^*=c^\star\,c_1$. Consequently we find for such ψ that

$$\langle f_e,\psi\rangle=\langle f_e,\psi+c^*\,\varphi_1^*\rangle=\int_\Omega f_e\,(\psi+c^*\,\varphi_1^*)\,dx\leq$$

(since $\psi+c^*\,\varphi_1^*\geq0$ we are able to use $f_e\leq P_0f$)

$$\leq\int_\Omega P_0f\,(\psi+c^*\,\varphi_1^*)\,dx=\langle f,\varphi_1^*\rangle\int_\Omega\varphi_1\,(\psi+c^*\,\varphi_1^*)\,dx\leq$$

(since $\varphi_1>0$ we may use $\psi\leq c^*\,\varphi_1^*$)

$$\leq\langle f,\varphi_1^*\rangle\int_\Omega\varphi_1\,(2c^*\,\varphi_1^*)\,dx=2c^*\,\langle f,\varphi_1^*\rangle.$$

Collecting the above it follows that

$$\|f_e\|_{\left(H^{s,q}\cap H_0^{m_{\mathcal{B}^*},q}\right)'}=\sup\left\{\int_\Omega f_e\,\psi\,dx;\ \psi\in H^{s,q}\cap H_0^{m_{\mathcal{B}^*},q},\ \|\psi\|_{H^{s,q}}=1\right\}\leq2c^*\,\langle f,\varphi_1^*\rangle,$$

which completes the proof of the theorem. □

5.2 Conclusion for Proposition 2 and Theorem 3

Proof of Proposition 2. For $f \in L^p(\Omega)$ we have $f = \langle f, \varphi_1^* \rangle \varphi_1 - f_e$ and the corresponding solution is, as stated in (1.5):

$$u = \frac{\langle f, \varphi_1^* \rangle}{\lambda_1 - \lambda} \varphi_1 - (A_0 - \lambda)^{-1} f_e.$$

By the previously stated result $u_{e,\lambda} = (A_0 - \lambda)^{-1} f_e \in H^{2m,p}_{\mathcal{B}}$ for $|\lambda - \lambda_0| < \delta$. Condition **b** states that $2m - m_{\mathcal{B}} - \frac{n}{p} > 0$. Since $2m - \frac{1}{p} > m_{\mathcal{B}}$ we have $u_{e,\lambda} = \frac{\partial}{\partial n} u_{e,\lambda} = \dots = \left(\frac{\partial}{\partial n}\right)^{m_{\mathcal{B}}-1} u_{e,\lambda} = 0$ on $\partial\Omega$ in $L^p(\partial\Omega)$. Moreover, Lemma D.1 and Corollary 10 yields with $c_{f_e} = c \|u_{e,\lambda}\|_{H^{2m,p}} \leq \tilde{c} \|f_e\|_{L^p}$ that

$$\left| \left((A_0 - \lambda)^{-1} f_e \right)(x) \right| \leq c_{f_e} \, d(x, \partial\Omega)^{m_{\mathcal{B}}}$$

implying (1.6) and hence the result stated in Theorem 2. □

Proof of Theorem 3. By Theorem 14 we find that if

$$2m - s - \frac{n}{q} > m_{\mathcal{B}^*} \tag{5.3}$$

holds, then $\|f_e\|_{H^{s-2m,p}} \leq c_1 \, c_2 \, \langle \varphi_1^*, f \rangle$.

Denote again $u_{e,\lambda} = ((-\Delta)^m - \lambda)^{-1}_{\mathcal{B}} f_e$. Assume $s \leq 2m - m_{\mathcal{B}^*}$ and $s - \frac{1}{p} \notin \mathbb{N}$. By Theorem 13 we have if for all λ with $|\lambda - \lambda_1|$ small enough that

$$\|u_{e,\lambda}\|_{H^{s,p}} \leq c_3 \|f_e\|_{\left(H^{2m-s,q} \cap H_0^{m_{\mathcal{B}^*},q}\right)'}.$$

By Lemma D.1 we have for

$$s - \frac{n}{p} > m_{\mathcal{B}} \tag{5.4}$$

that there exists $c > 0$ independent of $u_{e,\lambda}$ such that

$$|u_{e,\lambda}(x)| \leq c \|u_{e,\lambda}\|_{H^{s,p}} \, d(x, \partial\Omega)^{m_{\mathcal{B}}}.$$

Collecting the above we find that if

$$m_{\mathcal{B}} + \frac{n}{p} < s < 2m - m_{\mathcal{B}^*} - \frac{n}{q} \tag{5.5}$$

then

$$|u_{e,\lambda}(x)| \leq c \, c_3 \, \|f_e\|_{\left(H^{2m-s,q} \cap H_0^{m_{\mathcal{B}^*},q}\right)'} \, d(x, \partial\Omega)^{m_{\mathcal{B}}}.$$

Hence a necessary and sufficient condition for the existence of $s \in [0, 2m]$ satisfying (5.5) is $m_{\mathcal{B}} + m_{\mathcal{B}^*} + n < 2m$. This condition is satisfied by assumption. Note that we may choose s such that both (5.5) and $s - \frac{1}{p} \notin \mathbb{N}$ holds. □

6 Proof of the existence of an appropriate eigenfunction.

In this section we will prove the Propositions 4 and 5. First we verify Assumption 1 for Case **I** and **II**.

6.1 Regular boundary value problem

Lemma 15 (Case **I**) *The system in (2.18) with $\mathcal{B}$ as in (2.9, iii) and $\theta_i \in C^\infty(\partial\Omega)$ such that each θ_i is either identical zero or strictly positive, is a regular elliptic problem.*

Proof. The first two conditions of Definition B.1 (properly elliptic and normal) are immediate. It remains to show that $((-\Delta)^m, \mathcal{B})$ satisfies the complementary condition. The symbol of $\mathcal{A} = (-\Delta)^m$ used with $\xi + \tau\eta$, with $\xi, \eta \in \mathbb{R}^n$ and $\tau \in \mathbb{R}$, is decomposed as follows:

$$|\xi + \tau\eta|^{2m} = a^+(x; \xi, \eta, \tau)\ a^-(x; \xi, \eta, \tau)$$

where

$$a^\pm(x; \xi, \eta, \tau) = \left(|\eta|\,\tau + \frac{\langle \eta, \xi \rangle}{|\eta|} \mp i \frac{\sqrt{|\xi|^2 |\eta|^2 - \langle \eta, \xi \rangle^2}}{|\eta|} \right)^m .$$

The top-order symbol for the boundary condition $\mathcal{B}_j u = 0$ used with $\xi_x + \tau\nu_x$, where ξ_x tangential and ν_x the outside normal direction at $x \in \partial\Omega$, is

$$p_j(x; \xi_x, \nu_x, \tau) = \begin{cases} |\xi_x + \tau\nu_x|^{2j-2} & \text{if } \theta_j = 0, \\ \theta_j(x)\, |\xi_x + \tau\nu_x|^{2j-2}\, \tau\, |\nu_x| & \text{if } \theta_j > 0. \end{cases}$$

Since $|\xi_x + \tau\nu_x|^{2j-2} = (\tau + i|\xi_x|)^{j-1}(\tau - i|\xi_x|)^{j-1}$ and $a^+(x; \xi_x, \nu_x, \tau) = (\tau - i|\xi_x|)^m$ it follows by $m > j-1$ that the set $\{p_j(x; \xi_x, \nu_x, \cdot)\}_{j=1}^m$ is independent modulo $a^+(x; \xi_x, \nu_x, \tau)$. Hence the complementary condition is satisfied. □

Lemma 16 (Case **II**) *The system in (2.18) with $\mathcal{B}$ as in (2.21) is a regular elliptic problem.*

Proof. Again the first two conditions of Definition B.1 are immediate. The (top-order) symbol for the boundary conditions $\mathcal{B}_{m_j,\text{Dirichlet}} u = 0$ applied to $\xi_x + \tau\nu_x$ with ξ_x tangential and ν_x the outside normal direction, are

$$\begin{aligned} p_{m_1+\ldots+m_{j-1}+1}(x; \xi_x, \nu_x, \tau) &= \left(\tau^2 + |\xi_x|^2\right)^{m_1+\ldots+m_{j-1}}, \\ p_{m_1+\ldots+m_{j-1}+2}(x; \xi_x, \nu_x, \tau) &= \tau\left(\tau^2 + |\xi_x|^2\right)^{m_1+\ldots+m_{j-1}}, \\ &\vdots \\ p_{m_1+\ldots+m_{j-1}+m_j}(x; \xi_x, \nu_x, \tau) &= \tau^{m_j-1}\left(\tau^2 + |\xi_x|^2\right)^{m_1+\ldots+m_{j-1}}, \end{aligned}$$

and $\tau^2 + |\xi_x|^2 = (\tau + i|\xi_x|)(\tau - i|\xi_x|)$. The highest order symbol is

$$p_m(x; \xi_x, \nu_x, \tau) = \tau^{m_k-1}(\tau + i|\xi_x|)^{m_1+\ldots+m_{k-1}}(\tau - i|\xi_x|)^{m_1+\ldots+m_{k-1}} .$$

Since $a^+(x; \xi_x, \nu_x, \tau) = (\tau - i|\xi_x|)^{m_1+\ldots+m_k}$ it follows that the set $\{p_j(x; \xi_x, \nu_x, \cdot)\}_{j=1}^m$ is independent modulo $a^+(x; \xi_x, \nu_x, \tau)$. □

6.2 Reformulation of the problem

Let $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. In order to verify Assumption 2 it will be convenient to rewrite the operator A as a product of (unbounded) operators A_i, $i = 1, \ldots, \kappa$, which are realizations in L^p of lower order operators $(\mathcal{A}_i, \mathcal{B}_i)$, $i = 1, \ldots, \kappa$.

Case **I**. Let A be the realization in L^p of $((-\Delta)^m, \mathcal{B}_{m,\text{Robin}})$ defined in 2.9. Set A_i the realization in L^p of $(\mathcal{A}_i, \mathcal{B}_i)$, where $\mathcal{A}_i = -\Delta$ and $\mathcal{B}_i = \left(1 + \theta_i \frac{\partial}{\partial n}\right)$, $i = 1, \ldots, \kappa = m$. Notice that $m_i = 1$ and $m_{\mathcal{B}_i} = 0$ if $\theta_i > 0$. If $\theta_i = 0$ we have $m_{\mathcal{B}_i} = 1$.

Case **II**. Let A be the realization in L^p of $((-\Delta)^m, \mathcal{B})$ with $\mathcal{B}$ defined in 2.21. Set A_i the realization in L^p of $(\mathcal{A}_i, \mathcal{B}_i)$, where $\mathcal{A}_i = (-\Delta)^{m_i}$ and $\mathcal{B}_i = \mathcal{B}_{m_i,\text{Dirichlet}}$, $i = 1, \ldots, \kappa = k$. Here $m_{\mathcal{B}_i} = m_i$.

Notice that in both cases the pairs $(\mathcal{A}_i, \mathcal{B}_i)$, $i = 1, \ldots, \kappa$, satisfy Assumption 1. Moreover we have $\mathcal{A}_i = \mathcal{A}_i^*$ and we may choose $\mathcal{B}_i^* = \mathcal{B}_i$, $i = 1, \ldots, \kappa$.

We have:

Lemma 17 *Let p and A_i, $i = 1, \ldots, \kappa$, be as above. Then $0 \in \rho(A_i)$ and A_i^{-1} is compact. Moreover, if $p > n$ then for all $f \in L^p$ with $0 \neq f \geq 0$ there exist $b_f, c_f > 0$ such that*

$$b_f \, d(x, \partial\Omega)^{m_{\mathcal{B}_i}} \leq \left(A_i^{-1} f\right)(x) \leq c_f \, d(x, \partial\Omega)^{m_{\mathcal{B}_i}} \quad \text{in } \Omega. \tag{6.1}$$

Proof. In view of Theorem 8 iv., v. and $(\mathcal{A}_i, \mathcal{B}_i) = (\mathcal{A}_i^*, \mathcal{B}_i^*)$ we have $N(A_i) = N(A_i^*) \subset C^\infty(\bar{\Omega})$. An integration by parts shows that $u \in N(A_i)$ implies $u = 0$. From Theorem 8 iii. and v. it follows that $0 \in \rho(A_i)$. The compactness of A_i^{-1} follows from Theorem 8 i. and the compact imbedding of $H^{2m_i,p}$ in L^p.

Let $p > n$ and $f \in L^p$ with $0 \neq f \geq 0$. Set $u(x) = \left(A_i^{-1} f\right)(x)$. By standard regularity results we find that $u \in H^{2m_i,p}$ and hence by Sobolev imbedding $u \in C^{2m_i - 1}(\bar{\Omega})$. We distinguish three cases.

i. $\mathcal{A}_i = -\Delta$ and $\mathcal{B}_i u = \left(1 + \theta_i \frac{\partial}{\partial n}\right) u$, then $m_i = 1$, $m_{\mathcal{B}_i} = 0$. We claim that $u(x) = \left(A_i^{-1} f\right)(x) > 0$ in $\bar{\Omega}$ and proceed by contradiction. Indeed suppose that $u(x_0) \leq 0$ for some $x_0 \in \bar{\Omega}$. Since $p > n$ we have that $u \in C^1(\bar{\Omega})$. Lemma E.1 implies that $\inf_{x \in \Omega} u(x) = \min_{x \in \partial\Omega} u(x)$ and hence we may assume that $x_0 \in \partial\Omega$ and $u(x_0) = \min_{x \in \partial\Omega} u(x)$. From $\frac{\partial}{\partial n} u(x_0) \leq 0$ it follows that

$$\left(1 + \theta_i \frac{\partial}{\partial n}\right) u(x_0) \leq 0. \tag{6.2}$$

If $u(x_0) < 0$ then (6.2) is strict and we have a contradiction. If $u(x_0) = 0$ the version of Hopf's boundary point lemma in Lemma E.1 implies $\frac{\partial}{\partial n} u(x_0) > 0$ and hence again a contradiction.

The bound from above follows form $u \in C(\bar{\Omega})$.

ii. $\mathcal{A}_i = -\Delta$ and $\mathcal{B}_i u = u$, then $m_i = 1$, $m_{\mathcal{B}_i} = 1$. By Lemma E.1 one finds that since $u \in C^1(\bar{\Omega})$ it satisfies $\frac{\partial}{\partial n} u(x) > 0$ for all $x \in \partial\Omega$. Hence there is $c > 0$ such that $u(x) \geq c \, d(x, \partial\Omega)$. The estimate from above follows from $u_{|\partial\Omega} = 0$ and $u \in C^1(\bar{\Omega})$.

iii. $\mathcal{A}_i = (-\Delta)^{m_i}$ and $\mathcal{B}_i u = \left\{u, \frac{\partial}{\partial n} u, \left(\frac{\partial}{\partial n}\right)^2 u, \ldots, \left(\frac{\partial}{\partial n}\right)^{m_i - 1} u\right\}$, then $m_{\mathcal{B}_i} = m_i$. By the explicit integral formula of Boggio (see [7] or [17]) for $\Omega = B$ one finds that there are $c_{m_i,n}$ such that the Green function satisfies $G(x, y) > 0$ for $x, y \in \Omega$ and even

$$G(x, y) \geq c_{m_i,n} \begin{cases} |x - y|^{2m_i - n} \min\left(1, \left(\frac{d(x)d(y)}{|x-y|^2}\right)^{m_i}\right) & \text{if } 2m_i < n, \\ \log\left(1 + \left(\frac{d(x)d(y)}{|x-y|^2}\right)^{m_i}\right) & \text{if } 2m_i = n, \\ (d(x)\ d(y))^{m_i - n/2} \min\left(1, \left(\frac{d(x)d(y)}{|x-y|^2}\right)^{n/2}\right) & \text{if } 2m_i > n, \end{cases}$$

where $d(x) = d(x, \partial\Omega)$. Hence there exists $c = c(\operatorname{diam}\Omega, m_i, n)$ such that

$$G(x, y) \geq c \, d(x, \partial\Omega)^{m_i} \ d(y, \partial\Omega)^{m_i} \text{ for all } x, y \in \Omega,$$

and $u(x) = \int_B G(x,y)\, f(y)\, dy$ with $0 \neq f \geq 0$ implies $u(x) \geq c_f\, d(x,\partial\Omega)^{m_i}$ for some $c_f > 0$. The estimate from above follows from $\mathcal{B}_i u = 0$ and $u \in C^{m_i}(\bar{\Omega})$. □

Proposition 18 *Let $p \in (1,\infty)$ and let A be as in Case* **I** *or Case* **II**. *Then $0 \in \rho(A)$ and A^{-1} is compact. Moreover, if $p > n$ then for all $f \in L^p$ with $0 \neq f \geq 0$ there exists $b_f, c_f > 0$ such that*

$$b_f\, d(x,\partial\Omega)^{m_{\mathcal{B}}} \leq \left(A^{-1}f\right)(x) \leq c_f\, d(x,\partial\Omega)^{m_{\mathcal{B}}} \quad \text{in } \Omega. \tag{6.3}$$

Proof. Let A_i, $i = 1,\ldots,\kappa$ be as above. Let $f \in L^p$ and set

$$\tilde{u} = A_1^{-1} \circ A_2^{-1} \circ A_3^{-1} \circ \ldots \circ A_\kappa^{-1} f,$$

which is well defined by the previous lemma. By a repeated use of Theorem 8 we find that $\tilde{u} \in H^{2m,p}$. Since $A_1\tilde{u} = A_2^{-1} \circ A_3^{-1} \circ \ldots \circ A_\kappa^{-1} f$ it follows that $\mathcal{B}_1\tilde{u} = 0$. Similarly $A_2 \circ A_1\tilde{u} = A_3^{-1} \circ \ldots \circ A_\kappa^{-1} f$ implies $\mathcal{B}_2\mathcal{A}_1\tilde{u} = 0$ etc. and it follows that $\mathcal{A}\tilde{u} = f$ and $\mathcal{B}\tilde{u} = \{\mathcal{B}_1\tilde{u}, \mathcal{B}_2\mathcal{A}_1\tilde{u}, \ldots, \mathcal{B}_\kappa\mathcal{A}_{\kappa-1}\cdots\mathcal{A}_1\tilde{u}\} = 0$. We have $\tilde{u} \in H_{\mathcal{B}}^{2m,p}$ and $A\tilde{u} = f$, meaning $f \in R(A)$. Since $R(A) = L^p$ we find $N(A^*) = \{0\}$.

Since $(\mathcal{A}^*, \mathcal{B}^*)$ is of similar type we also find $R(A^*) = L^q$ implying $N(A) = \{0\}$. Together $R(A) = L^p$ and $N(A) = \{0\}$ imply that $0 \in \rho(A)$. Similarly $0 \in \rho(A^*)$.

Since $0 \in \rho(A)$ and $A\tilde{u} = f$ it follows that $\tilde{u} = A^{-1}f$, that is

$$A^{-1} = A_1^{-1} \circ A_2^{-1} \circ A_3^{-1} \circ \ldots \circ A_\kappa^{-1}.$$

The compactness of A^{-1} follows from the compactness of A_i^{-1}.

Let $p > n$ and $f \in L^p$ with $0 \neq f \geq 0$. Since $m_{\mathcal{B}} = m_{\mathcal{B}_1}$ the estimate in (6.3) follows from the strong positivity of A_i^{-1}, $i = 2,\ldots,\kappa$ and the estimate in (6.5) for A_1^{-1}. □

Remark 6.1 It follows from the proof that

$$A^{-1} = (A_\kappa^*)^{-1} \circ \left(A_{\kappa-1}^*\right)^{-1} \circ \left(A_{\kappa-2}^*\right)^{-1} \circ \ldots \circ (A_1^*)^{-1}. \tag{6.4}$$

6.3 Conclusion

Proof of Proposition 4 respectively 5: By Lemma 15 and 16 Assumption 1 is satisfied.

By Proposition 18 we have $0 \in \rho(A)$ and A^{-1} compact. Now let $p > n$. Then the operator A^{-1} is $d(\cdot,\partial\Omega)^{m_{\mathcal{B}}}$-bounded and hence Theorem F.2 implies that its spectral radius $\nu\left(A^{-1}\right)$ is a geometrically simple eigenvalue and that the corresponding eigenfunction satisfies for some $c_1, c_2 > 0$

$$c_1\, d(x,\partial\Omega)^{m_{\mathcal{B}}} \leq \varphi_{\mathcal{A},\mathcal{B}}(x) \leq c_2\, d(x,\partial\Omega)^{m_{\mathcal{B}}}. \tag{6.5}$$

We assumed that $p > n$. However, due to Theorem 8 iv., $N(A - \lambda)$ does not depend on p and we find that $\nu\left(A^{-1}\right)$ is a geometrically simple eigenvalue for any $p \in (1,\infty)$. Since by Theorem 8 v. A^* is the realization of $(\mathcal{A}^*, \mathcal{B}^*)$ it follows from (6.4) and Proposition 18 applied to A^* that $\nu\left((A^*)^{-1}\right)$ is also a geometrically simple eigenvalue of $(A^*)^{-1}$. Moreover, since $\nu\left(A^{-1}\right) = \nu\left((A^*)^{-1}\right)$ we find that $\varphi_{\mathcal{A},\mathcal{B}}$ and $\varphi_{\mathcal{A}^*,\mathcal{B}^*}$ are the unique eigenfunctions for A, respectively A^*, with eigenvalue $\lambda_1 = \nu\left(A^{-1}\right)^{-1} = \nu\left((A^*)^{-1}\right)^{-1}$. Since the eigenfunctions $\varphi_{\mathcal{A},\mathcal{B}}$ and $\varphi_{\mathcal{A}^*,\mathcal{B}^*}$ are strictly positive, even satisfy (6.5) with $m_{\mathcal{B}}$ respectively $m_{\mathcal{B}^*}$, we find $\int_\Omega \varphi_{\mathcal{A},\mathcal{B}}\, \varphi_{\mathcal{A}^*,\mathcal{B}^*}\, dx > 0$. The eigenvalue λ_1 is algebraically simple according to Lemma C.2. Hence Assumption 2 and the estimates (2.19), (2.20) and (2.22) are satisfied. □

Appendix A - Simple eigenvalues

The aim of this section is to recall the notion of K-simple eigenvalue introduced by Crandall and Rabinowitz in [12] and to extend some of their results for our purposes.

Let X and Y be Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Definition A.1 *Let $T, K \in L(X;Y)$. Then $\lambda_0 \in \mathbb{K}$ is called a K-simple eigenvalue of T if:*

i. *there exists $\varphi \in X$ with $T\varphi = \lambda_0 K\varphi$ and $\varphi \neq 0$;*

ii. $\dim N(T - \lambda_0 K) = 1$ *and* $\operatorname{codim} R(T - \lambda_0 K) = 1$

iii. $K\varphi \notin R(T - \lambda_0 K)$.

By ii. $T - \lambda_0 K$ is Fredholm of index 0. In particular $R(T - \lambda_0 K)$ is closed, see [12]. The vector φ above is called a K-eigenvector of T. It follows from ii. and iii. that

$$Y = R(T - \lambda_0 K) \oplus \operatorname{span}\{K\varphi\}. \tag{A.1}$$

Lemma A.2 *Let $T, K \in L(X;Y)$ and let $\lambda_0 \in \mathbb{K}$ be a K-simple eigenvalue of T. Then λ_0 is a K'-simple eigenvalue of T', where T', K' denote the adjoint of T, K, acting from Y' to X'.*

Moreover, if $\Psi \in Y'$ is the corresponding K'-eigenfunction, then $N(\Psi) = R(T - \lambda_0 K)$. Hence the eigenvector Ψ can be normalized by $\Psi(K\varphi) = 1$.

Proof. Using (A.1), the closedness of $R(T - \lambda_0 K)$ and Hahn-Banach's Theorem it follows that there exists $\Psi \in Y'$ satisfying $\Psi(K\varphi) = 1$ and $N(\Psi) = R(T - \lambda_0 K)$. Notice that such Ψ is unique. We use Ψ to prove that λ_0 is a K'-simple eigenvalue of T'.
i. Since $N(\Psi) = R(T - \lambda_0 K)$ we find

$$\Big(\left(T' - \lambda_0 K'\right)(\Psi)\Big)(x) = \Psi\Big((T - \lambda_0 K)\,x\Big) = 0 \text{ for all } x \in X.$$

ii. We have for $\Phi \in Y'$ that

$$\Phi \in N\left(T' - \lambda_0 K'\right) \Rightarrow N(\Phi) \supseteq R(T - \lambda_0 K).$$

If $\Phi \neq 0$, then by (A.1) we have $\Phi(K\varphi) \neq 0$ and $\Phi(\cdot) = \Phi(K\varphi)\,\Psi(\cdot)$. It follows that $\dim N(T' - \lambda_0 K') = 1$.

Since $R(T - \lambda_0 K)$ is closed also $R(T' - \lambda_0 K')$ is closed and we have

$$R\left(T' - \lambda_0 K'\right) = \overline{R(T' - \lambda_0 K')} = N(T - \lambda_0 K)^{\perp} = \left\{\Phi \in X'; \Phi(\varphi) = 0\right\}.$$

Hence $\operatorname{codim} R(T' - \lambda_0 K') = 1$.
iii. $(K'\Psi)(\varphi) = \Psi(K\varphi) = 1$ implies $K'\Psi \notin R(T' - \lambda_0 K')$. □

In what follows we shall denote by P_0 the projection in Y on $\operatorname{span}\{K\varphi\}$ corresponding to the decomposition (A.1). We have

$$P_0 y = \Psi(y)\;K\varphi \quad \text{for } y \in Y. \tag{A.2}$$

It turns out that there is a natural projection P_1 in X on $\operatorname{span}\{\varphi\}$ defined by

$$P_1 x = \Psi(Kx)\;\varphi \quad \text{for } x \in X. \tag{A.3}$$

corresponding with the decomposition

$$X = N(\Psi \circ K) \oplus \operatorname{span}\{\varphi\}. \tag{A.4}$$

Indeed we have $P_1\varphi = \Psi(K\varphi)\,\varphi = \varphi$ and $P_1(P_1x) = P_1(\Psi(Kx)\,\varphi) = \Psi(Kx)\,\varphi = P_1x$ for every $x \in X$ and (A.4) follows.

These projections, which are such that $(T - \lambda_0 K)_{|N(P_1)} : N(P_1) \to N(P_0)$ is an isomorphism, will be very convenient in section 4. Indeed, from (A.1-A.4), ii. and the Open Mapping Theorem it follows that the restriction of $T - \lambda_0 K$ to $N(P_1)$ is an isomorphism from $N(P_1)$ onto $R(T - \lambda_0 K)$.

In the next theorem we show that for $\lambda \in \mathbb{K}$ near λ_0 the restriction of $T - \lambda_0 K$ to $N(P_1)$ is also an isomorphism from $N(P_1)$ onto $R(T - \lambda_0 K)$.

Theorem A.3 *Let $T, K \in L(X;Y)$ and let $\lambda_0 \in \mathbb{K}$ be a K-simple eigenvalue of T. Let $P_0 : Y \to Y$ and $P_1 : X \to X$ be as above. Then*

i. *for every $\lambda \in \mathbb{K}$, $S_\lambda := (T - \lambda_0 K)_{|N(P_1)}$ is a bounded linear operator from $N(P_1)$ into $N(P_0)$.*

ii. *there exists $\delta > 0$ such that for $\lambda \in \mathbb{K}$ with $|\lambda - \lambda_0| < \delta$ we have*
   - *a.* $S_\lambda \in \operatorname{Isom}\Big(N(P_1); N(P_0)\Big)$,
   - *b.* $\lambda \to S_\lambda^{-1} \in L\Big(N(P_0); N(P_1)\Big)$ *is analytic,*
   - *c.* *if $\lambda \neq \lambda_0$ then $T - \lambda K \in \operatorname{Isom}\Big(X, Y\Big)$.*

Remark. We have the following scheme for $|\lambda - \lambda_0| < \delta$.

$$\begin{array}{ccccccc}
 & & X & \overset{T-\lambda K}{\longrightarrow} & Y & & \\
 & & \shortparallel & & \shortparallel & & \\
 & & N(P_1) & \overset{\text{isomorphism}}{\longrightarrow} & N(P_0) & = & R(T-\lambda_0 K) \\
 & & \oplus & & \oplus & & \\
\operatorname{span}\{\varphi\} & = & R(P_1) & \longrightarrow & R(P_0) & = & \operatorname{span}\{K\varphi\}
\end{array}$$

Proof. i. For $x \in N(P_1)$ and $\lambda \neq \lambda_0$ we have $(T - \lambda K)x = (T - \lambda_0 K)x + (\lambda - \lambda_0)Kx$. It suffices to show that $P_0 K x = 0$. And indeed, from (A.2-A.3) we have $P_0 K x = \Psi(Kx)\,K\varphi = K(P_1 x) = 0$.

ii. We have $S_\lambda \in L\Big(N(P_1), N(P_0)\Big)$ for $\lambda \in \mathbb{K}$. Since $S_{\lambda_0} \in \operatorname{Isom}\Big(N(P_1), N(P_0)\Big)$ and since $\lambda \mapsto S_\lambda$ is analytic, the result follows. Note that for $0 < |\lambda - \lambda_0| < \delta$ one finds

$$(T - \lambda K)^{-1} = S_\lambda^{-1} \circ (I - P_0) + \tfrac{1}{\lambda - \lambda_0} P_0. \qquad \square$$

Appendix B - Regular elliptic problems

Let $(\mathcal{A}, \mathcal{B})$ be as in (2.2) and (2.3):

$$\mathcal{A} = \sum_{|\alpha| \le 2m} a_\alpha(x) \left(\frac{\partial}{\partial x}\right)^\alpha \quad \text{and} \quad \mathcal{B}_j = \sum_{|\alpha| \le m_j} b_{j,\alpha}(x) \left(\frac{\partial}{\partial x}\right)^\alpha.$$

Definition B.1 *The elliptic boundary value problem*

$$\begin{cases} \mathcal{A}u &= f & \text{in } \Omega, \\ \mathcal{B}u &= 0 & \text{on } \partial\Omega, \end{cases} \tag{B.1}$$

is called regular *if*

i. $\mathcal{A}$ *is* properly *elliptic;*
ii. $\mathcal{B}$ *is a* normal *system;*
iii. $(\mathcal{A}, \mathcal{B})$ *satisfies the* complementary condition.

See [29, 5.2.1] or [22, Chapter 2, Def. 1.2, 1.4, 1.5]. These three conditions are defined as follows.

- $\mathcal{A}$ is a *properly* elliptic operator if for all $x \in \bar{\Omega}$ and all independent couples $\xi, \eta \in \mathbb{R}^n$ the polynomial
$$\tau \mapsto \sum_{|\alpha|=2m} a_\alpha(x)(\xi + \tau\eta)^\alpha \tag{B.2}$$
has exactly m roots with positive imaginary part. In our case we consider coefficients $a_\alpha(x)$ which are real-valued. Then the condition reduces to $\sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha \neq 0$ for all $\xi \in \mathbb{R}^n \backslash \{0\}$ and $x \in \bar{\Omega}$, the usual ellipticity condition.

- $\mathcal{B} = \{\mathcal{B}_1, \ldots, \mathcal{B}_m\}$ is a *normal* system of boundary conditions if: denoting by m_j the order of $\mathcal{B}_j$, the $\mathcal{B}_j$ can be ordered such that
$$0 \le m_1 < m_2 < \ldots < m_j < 2m;$$
and, with n_x the outwards normal direction at $x \in \partial\Omega$,
$$\sum_{|\alpha|=m_j} b_{j,\alpha}(x)\, n_x^\alpha \neq 0 \text{ for all } x \in \partial\Omega \text{ and } j = 1, \ldots, m.$$

Before stating third condition we need to fix some notations. Let $\xi, \eta \in \mathbb{R}^n$ and $\tau \in \mathbb{R}$. If (B.2) holds we may write

$$\sum_{|\alpha|=2m} a_\alpha(x)(\xi + \tau\eta)^\alpha = a^+(x; \xi, \eta, \tau)\, a^-(x; \xi, \eta, \tau)$$

in such a way that for each $x \in \Omega$, $\xi, \eta \in \mathbb{R}^n$ the m zeroes τ_j^+ of $\tau \to a^+(x; \xi, \eta, \tau)$ satisfy $\operatorname{Re}\tau_j^+ > 0$ and the m zeroes τ_j^- of $\tau \to a^-(x; \xi, \eta, \tau)$ satisfy $\operatorname{Re}\tau_j^- < 0$. Define the polynomials $\{p_j\}_{j=1}^m$ by

$$p_j(x; \xi_x, \nu_x, \tau) = \sum_{|\alpha|=m_j} b_{j,\alpha}(x)(\xi_x + \tau\nu_x)^\alpha.$$

- $\mathcal{B}$ satisfies *the complementary condition* with respect to $\mathcal{A}$ if: for each $x \in \partial\Omega$, each tangential direction ξ_x and normal direction ν_x the polynomials $\{p_j(x; \xi_x, \nu_x, \cdot)\}_{j=1}^m$ are linearly independent modulo $a^+(x; \xi_x, \nu_x, \cdot)$.

Notice that we may replace $\mathcal{A}u$ by $(\mathcal{A} - \lambda)u$ in (B.1) without affecting the regularity.

Appendix C - The adjoint problem

Definition C.1 *Let C be a densely defined operator C in $L^p(\Omega)$ with $1 < p < \infty$. Let $\frac{1}{p}+\frac{1}{q}=1$. The adjoint C^* is defined by*

$$\begin{aligned} D(C^*) &= \left\{ v \in L^q(\Omega);\ \exists M > 0 \text{ s.t. } \left|\textstyle\int_\Omega Cu\, v\, dx\right| \le M \|u\|_{L^p}\ \ \forall u \in D(C) \right\}, \\ C^* v &= w, \text{ where } w \text{ is the only element of } L^q(\Omega) \text{ satisfying} \\ &\quad \textstyle\int_\Omega Cu\, v\, dx = \int_\Omega u\, w\, dx \quad \text{for all } u \in D(C). \end{aligned}$$

One has

$$R(C)^\perp = N(C^*) \quad \text{and} \quad R(C^*)^\perp = N(C)$$

where $R(C)^\perp = \left\{ v \in L^q(\Omega); \int_\Omega f\, v\, dx \quad \text{for all } f \in R(C) \right\}$. The operator C^* is closed in $L^q(\Omega)$ and moreover, if the operator C is closed then we have, since $L^p(\Omega)$ is reflexive, that $(C^*)^* = C$. See [20, Theorem 5.29].

Proof of Theorem 8, parts i.-v. Let $(\mathcal{A},\mathcal{B})$ be regular elliptic, let $p,q \in (1,\infty)$ with $\frac{1}{p}+\frac{1}{q}=1$ and let the corresponding realization A in $L^p(\Omega)$ be defined as in section 3.4. In the same fashion we define for the adjoint system $(\mathcal{A}^*,\mathcal{B}^*)$ the unbounded operator $A_\star$ in $L^q(\Omega)$ by $D(A_\star) = H^{2m,q}_{\mathcal{B}^*}(\Omega)$ and $A_\star u = \mathcal{A}^* u$ for $u \in D(A_\star)$. Observe that in view of (2.7) $H^{2m,q}_{\mathcal{B}^*}(\Omega)$ does not depend on the choice of the complementing boundary operators $\mathcal{B}_c$, but only on the pair $(\mathcal{A},\mathcal{B})$. From Green's formula (2.6) we also have

$$\int_\Omega Au\, v\, dx = \int u\, A_\star v\, dx \text{ for all } u \in D(A) \text{ and } v \in D(A_\star). \tag{C.1}$$

In particular we may choose $(\mathcal{B}^*)^* = \mathcal{B}$ implying $(A_\star)_\star = A$.

Part i. See [29, Theorem 5.3.4] and the references therein.

Part ii. See [29, Theorem 5.4.1].

Part iii.-iv. The closedness of A is a consequence of estimate (3.10) for $r = 0$. The closedness of $R(A)$, together with $\dim N(A) < \infty$, follows from (3.10) and the compactness of the imbedding of $H^{2m,p}$ into L^p (see [22, Chapter 2, Lemma 5.1]). From ii. we obtain that if $u \in N(A)$ then $u \in \bigcap_{r=0}^\infty H^{2m+r,p} = C^\infty(\bar{\Omega})$. Hence $N(A)$ is independent of p.

Part v. This statement is equivalent with $A^* = A_\star$. From (C.1) we have $A_\star \subseteq A^*$. It remains to prove that $D(A^*) \subseteq D(A_\star)$. We shall proceed as in [22, Chapter 2, Theorem 8.4].

First we will show that

$$R(A) \supseteq N(A_\star)^\perp, \tag{C.2}$$

where $N(A_\star)^\perp = \{u \in L^p; \langle u,v\rangle = 0 \text{ for all } v \in N(A_\star)\}$. Set $M = N(A_\star)^\perp \cap C^\infty(\bar{\Omega})$. From Theorem 7 we have $M \subset R(A)$ and hence also that $\overline{M} \subseteq R(A)$ where $\overline{M}$ is the closure of M in L^p. We claim $N(A_\star)^\perp \subseteq \overline{M}$. Observe that $N(A_\star) \subset C^\infty(\bar{\Omega}) \subset L^p$ with $\dim N(A_\star) < \infty$ and hence $L^p = N(A_\star) \oplus N(A_\star)^\perp$. Let P denote the associated projection on $N(A_\star)$. Since $\dim N(A_\star) < \infty$ and $N(A_\star)^\perp$ is closed in L^p, the projection P is continuous. Given $f \in N(A_\star)^\perp$, that is $Pf = 0$, we can find a sequence $\{g_n\}_{n=0}^\infty \subset C^\infty(\bar{\Omega})$ such that $g_n \to f$ in L^p. Setting $f_n = (I-P)g_n$, we have $f_n \in N(A_\star)^\perp \cap C^\infty(\bar{\Omega}) = M$ and

$$\lim_{n\to\infty} f_n = (I-P) \lim_{n\to\infty} g_n = (I-P) f = f.$$

Hence $N(A_\star)^\perp \subseteq \overline{M} \subseteq R(A)$.

Next we will show that (C.2) implies

$$N(A^*) \subseteq N(A_\star), \tag{C.3}$$
$$R(A^*) \subseteq R(A_\star). \tag{C.4}$$

Since (C.2) holds we have $N(A^*) = R(A)^\perp \subseteq N(A_\star)^{\perp\perp} = N(A_\star)$. The inclusion (C.2) for $A_\star$ reads $R(A_\star) \supseteq N((A_\star)_\star)^\perp$ and implies $R(A^*) \subseteq R(A^*)^{\perp\perp} = N(A)^\perp = N((A_\star)_\star)^\perp \subseteq R(A_\star)$.

Finally we show that from (C.3) and (C.4) it follows that $D(A^*) \subseteq D(A_\star)$. Indeed, if (C.3) and (C.4) hold, let $v \in D(A^*)$ and set $f = A^* v$. By assumption there exists $\tilde{v} \in D(A_\star)$ such that $A_\star \tilde{v} = f$. Since $0 = A^* v - A_\star \tilde{v} = A^*(v - \tilde{v})$ we find $v - \tilde{v} \in N(A^*) \subseteq N(A_\star) \subseteq D(A_\star)$ and consequently that $v = \tilde{v} + v - \tilde{v} \in D(A_\star)$. □

Lemma C.2 *Let λ is a geometrically simple eigenvalue of A and of A^* with corresponding eigenfunctions φ and φ^*. Then the following are equivalent:*

i. λ is an algebraically simple (in the sense of [20]) eigenvalue of A and hence of A^*;

ii. the pair of eigenfunctions satisfies

$$\int_\Omega \varphi(x)\, \varphi^*(x)\, dx \neq 0. \tag{C.5}$$

Proof. i ⇒ ii. Indeed, suppose this integral would be zero. Since every $g \in L^q$ can be written as $g = g_1 + c\varphi^*$ with $g_1 = (A^* - \lambda)\, v$ for some $v \in D(A^*)$ we would find

$$\int_\Omega \varphi(x)\, g(x)\, dx = \int_\Omega \varphi(x)\, (\mathcal{A}^* - \lambda)\, v(x)\, dx + c \int_\Omega \varphi(x)\, \varphi^*(x)\, dx =$$

$$= \int_\Omega (\mathcal{A} - \lambda)\, \varphi(x)\ v(x)\, dx + c \int_\Omega \varphi(x)\, \varphi^*(x)\, dx = 0.$$

Hence $\varphi \equiv 0$, a contradiction.

ii ⇒ i. Again we proceed by contradiction. Suppose there is $\psi \in D(A)$ such that $(\mathcal{A} - \lambda)\, \psi = \varphi$. Then

$$\int_\Omega \varphi(x)\, \varphi^*(x)\, dx = \int_\Omega (\mathcal{A} - \lambda)\, \psi(x)\, \varphi^*(x)\, dx = \int_\Omega \psi(x)\, (\mathcal{A}^* - \lambda)\, \varphi^*(x)\, dx = 0.$$

□

Appendix D - An imbedding

Lemma D.1 *Let $n \geq 1$, $p \in (1, \infty)$ and suppose that $s - \frac{n}{p} > m_1 \in \mathbb{N}$. Then there exists $c > 0$ such that for all $w \in H^{s,p}(\Omega) \cap H_0^{m_1,p}(\Omega)$ it follows that*

$$|w(x)| \leq c\, \|w\|_{H^{s,p}(\Omega)}\, d(x, \partial\Omega)^{m_1} \ \text{ for all } x \in \Omega. \tag{D.1}$$

Proof. If $w \in H^{s,p}(\Omega)$ and $s - \frac{n}{p} > m_1$ then $w \in C^{m_1}(\bar{\Omega})$ by Sobolev's imbedding. Moreover, there exists c_1 such that for all $u \in H^{s,p}(\Omega)$

$$\|w\|_{C^{m_1}(\bar{\Omega})} \leq c_1\, \|w\|_{H^{s,p}(\Omega)}. \tag{D.2}$$

For $m_1 = 0$ we are done. If $m_1 \geq 1$ then, since $\left|\left(\frac{\partial}{\partial x}\right)^\alpha w(x)\right| \leq \|w\|_{C^{m_1}(\bar{\Omega})}$ for all $x \in \Omega$ and $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq m_1$, it follows that

$$\left|\left(\frac{\partial}{\partial \nu_x}\right)^{|\alpha|} w(x)\right| \leq \|w\|_{C^{m_1}(\bar{\Omega})} \text{ for all } x \in \Omega,$$

where we take ν_x the direction along a shortest line connecting x with the boundary $\partial\Omega$. Using that $\left(\frac{\partial}{\partial \nu_x}\right)^{|\alpha|} w = 0$ on $\partial\Omega$ for $|\alpha| \leq m_1 - 1$ and integrating along that line yields

$$|w(x)| \leq \frac{1}{m_1!} \|w\|_{C^{m_1}(\bar{\Omega})} \, d(x, \partial\Omega)^{m_1}. \tag{D.3}$$

Combining (D.2) and (D.3) implies (D.1). □

Appendix E - A version of Hopf's boundary point Lemma

For the sake of completeness we give a proof of a version of Hopf's boundary point Lemma that we need. See e.g. [14, Lemma 3.4 and (3.11)] for the standard version.

Lemma E.1 *Let* $\partial\Omega \in C^2$ *and suppose that* $u \in C^1(\bar{\Omega})$ *is superharmonic in* Ω. *Assume* $\inf_{x\in\Omega} u = 0$. *Then either*

i. $u \equiv 0$, *or*

ii. $u > 0$ *in* Ω *and for every* $x_0 \in \partial\Omega$ *with* $u(x_0) = 0$ *one has*

$$\frac{\partial}{\partial n} u(x_0) < 0.$$

Proof. A superharmonic function satisfies the strong maximum principle (see [14]), that is, either $u \equiv 0$ or $u(x) > \min_{y\in\partial\Omega} u(y) = 0$ for all $x \in \Omega$. Let $x_0 \in \partial\Omega$ be such that $u(x_0) = 0$. Since $\partial\Omega \in C^2$ there exists a ball $B \subset \Omega$ with $\partial B \cap \partial\Omega = \{x_0\}$. Set h to be harmonic in B and to satisfy $h = u$ on ∂B. Then $h \in C^\infty(B) \cap C(\bar{B})$ and the standard Hopf's boundary lemma yields that $c > 0$ exists such that $h(x) \geq c\,|x - x_0|$ for $x \in [x_0, x_B]$, where $[x_0, x_B]$ is the segment from x_0 to the center x_B of B. Then $u - h$ is superharmonic, hence $u \geq h$ in B implying $\frac{\partial}{\partial n} u(x_0) < -c$. □

Appendix F - On the Krein-Rutman Theorem

We state some extension of the Krein-Rutman Theorem which can be found by combining results in [21]. First we recall the notion of u_0-bounded for operators in L^p, $p \in (1, \infty)$.

Definition F.1 *An operator* $\mathcal{T} : L^p \to L^p$ *for which there exists* $0 < u_0 \in L^p$ *such that for all* $f \in L^p$ *with* $0 \neq f > 0$ *there are* $c_f, d_f > 0$ *such that* $c_f\, u_0(x) \leq \mathcal{T} f(x) \leq d_f\, u_0(x)$ *in* Ω, *is called* u_0*-bounded.*

Clearly such $\mathcal{T}$ is positive. The theorem that will be convenient for our purposes is the following.

Theorem F.2 *Suppose that $\mathcal{T} : L^p \to L^p$ is compact and u_0-bounded for some $0 < u_0 \in L^p$. Then the following holds.*

i. *The spectral radius $\nu(\mathcal{T}) > 0$ is a geometrically simple eigenvalue of $\mathcal{T}$.*

ii. *There exists a corresponding eigenfunction φ which satisfies for some $c, d > 0$:*

$$c\, u_0(x) \leq \varphi(x) \leq d\, u_0(x). \tag{F.1}$$

iii. *Every positive eigenfunction of $\mathcal{T}$ is a multiple of φ.*

Proof. Since $\mathcal{T}$ is u_0-bounded there is $c_0 > 0$ such that $c_0 u_0 \leq \mathcal{T} u_0$. By [21, Lemma 9.1] one finds $\nu(\mathcal{T}) \geq c_0$.

The cone of positive functions in L^p is *reproducing:* every $f \in L^p$ can be written as $f^+ - f^-$ where f^+, f^- are nonnegative functions in L^p. Hence [21, Theorem 9.2] states that $\nu(\mathcal{T})$ is an eigenvalue with corresponding eigenfunction φ being positive. Since condition b) of [21, Theorem 11.1] is satisfied $\nu(\mathcal{T})$ is geometrically simple and φ is the unique positive eigenfunction up to normalization. The u_0-boundedness of $\mathcal{T}$ implies (F.1). □

References

1. R.A. Adams, "Sobolev spaces", Academic Press, New York-London, 1975.

2. S. Agmon, On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems. *Comm. Pure Appl. Math.* **15** (1962), 119-147.

3. H. Amann, Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems, *in* "Function Spaces, Differential Operators and Nonlinear Analysis", Teubner-Texte zur Mathematik 133, 9-126, Stuttgart-Leipzig 1993.

4. H. Amann, "Linear and quasilinear parabolic problems", Vol. 1. Abstract linear theory, Birkhäuser, Basel, 1995; Vol. 2. to appear.

5. M.S. Baouendi and C. Goulaouic, Commutation de l'intersection et des foncteurs d'interpolation. *Compt. Rend. Acad. Sci. Paris* **266** (1967), 313-315.

6. I. Birindelli, Hopf's lemma and anti-maximum principle in general domains. *J. Differ. Equations* **119** (1995), no. 2, 450–472.

7. T. Boggio (1905), Sulle funzioni di Green d'ordine m, *Rend. Circ. Mat. Palermo* **20**, 97-135.

8. Ph. Clément and L.A. Peletier, An anti-maximum principle for second order elliptic operators. *J. Differ. Equations* **34** (1979), 218-229.

9. C. V. Coffman, R. J. Duffin and D.H. Shaffer (1979), The fundamental mode of vibration of a clamped annular plate is not of one sign, *in* C. V. Coffman and G. J. Fix (eds.), "Constructive approaches to mathematical models" (Proc. Conf. in honor of R.J. Duffin, Pittsburgh, Pa., 1978), pp. 267–277, Academic Press, New York etc.

10. C. V. Coffman, R. J. Duffin (1980), On the structure of biharmonic functions satisfying the clamped plate conditions on a right angle, *Adv. Appl. Math.* **1**, 373-389.

11. C. V. Coffman (1982), On the structure of solutions to $\Delta^2 u = \lambda u$ which satisfy the clamped plate condition on a right angle, *SIAM J. Math. Anal.* **13**, 746-757.

12. M.G. Crandall and P.H. Rabinowitz, Bifurcation, perturbation of simple eigenvalues and linearized stability, *Arch. Rational Mech. Anal.* 52 (1973), 161–180.

13. J. Fleckinger, J.-P. Gossez, P. Takáč and F. de Thélin, Nonexistence of solutions and an anti-maximum principle for cooperative systems with the p-Laplacien, *Math. Nachrichten* ***194*** (1998), 49-78.

14. D. Gilbarg and N.S. Trudinger, "Elliptic partial differential equations of second order", 2^{nd} edition, Springer, Berlin, 1983.

15. P. Grisvard, Espaces intermédiaires entre espaces de Sobolev avec poids, *Ann. Scuola Norm. Sup. Pisa* **17** (1963), 255-296.

16. H.-Ch. Grunau and G. Sweers, Positivity for perturbations of polyharmonic operators with Dirichlet boundary conditions in two dimensions, *Math. Nachr.* **179** (1996), 89-102.

17. H.-Ch. Grunau and G. Sweers, Positivity for equations involving polyharmonic elliptic operators with Dirichlet boundary conditions, *Math. Ann.* **307** (1997), 589-626.

18. H.-Ch. Grunau and G. Sweers, The maximum principle and positive principal eigenfunctions for polyharmonic equations, *in* "Reaction Diffusion systems", Marcel Dekker Inc., New York 1997, 163-182.

19. H.-Ch. Grunau and G. Sweers, Positivity properties of elliptic Dirichlet problems of higher order, *Nonlinear Analysis, T.M.A.* **30** (1997), 5251-5258 (Proc. 2nd World Congress of Nonlinear Analysts).

20. T. Kato, "Perturbation theory for linear operators", Springer-Verlag, Berlin, 1966 (corrected printing of the second edition 1980).

21. M.A. Krasnosel'skij, Je.A. Lifshits and A.V. Sobolev, "Positive Linear Systems", Heldermann Verlag, Berlin, 1989.

22. J.L. Lions and E. Magenes, "Non-homogeneous Boundary Value Problems and Applications I", Springer, Berlin, 1972.

23. J.L. Lions and E. Magenes, Problèmes aux limites nonhomogènes VI, *J. d' Anal. Math.* **11** (1963), 165-188.

24. M. Schechter, On L^p estimates and regularity, *Ricerche Mat.* **13** (1964), 192–206

25. R. Seeley, Fractional powers of boundary problems, *Actes, Congrès. Intern. Math.* **93** (1971), 299-309.

26. R. Seeley, Interpolation in L^P with boundary conditions, *Studia Math.* **44** (1972), 47-60.

27. G. Sweers, L^N is sharp for the antimaximum principle, *J. Differential Equations* **134** (1997), 148-153.

28. P. Takáč, An abstract form of maximum and anti-maximum principles of Hopf's type, *J. Math. Anal. Appl.* **201** (1996), no. 2, 339–364.

29. H. Triebel, "Interpolation Theory, Function Spaces, Differential Operators", North-Holland, Amsterdam, 1978.

30. H. Triebel, "Theory of Function Spaces", Birkhauser, Basel, 1983.

31. K. Yosida, "Functional Analysis", Springer-Verlag, Berlin, 1965 (sixth edition 1980).