L^n is sharp for the anti-maximum principle

Guido Sweers
Dept. of Pure Mathematics
Delft University of Technology
PObox 5031
2600 GA Delft
The Netherlands
sweers@twi.tudelft.nl

November 6, 1998

Clément and Peletier showed in [3] a result that reads for the Dirichlet Laplacian on bounded smooth domains $\Omega \subset \mathbb{R}^n$ as follows.

- For all $f > 0$ with $f \in L^p(\Omega)$ and $p > n$, there is $\lambda_f > \lambda_1$, where λ_1 is the first eigenvalue, such that one finds for $\lambda \in (\lambda_1, \lambda_f)$ that the solution of

$$\begin{cases} -\Delta u = \lambda u + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

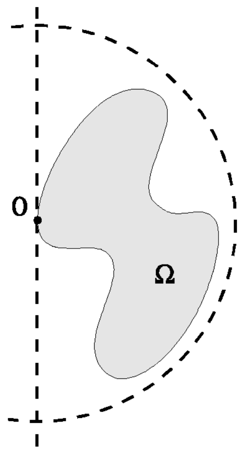
  satisfies $u < 0$.

For $\lambda < \lambda_1$ the maximum principle yields that a solution u, no matter in which space $f > 0$ lies, satisfies $u > 0$. The question remained open if the condition $p > n$ is necessary for the anti-maximum principle. One should notice that the so-called anti-maximum principle is not a uniform result (λ_f depends on f) like the maximum principle is. The fact that some regularity of f is necessary should hence not come as a surprise. We will show that the result above is no longer true for all $f \in L^p(\Omega)$ with $p \leq n$.

Isabeau Birindelli recently extended the anti-maximum principle to general domains ([2]). She uses both $f \in L^p(\Omega)$, with $p > n$, and that the support of f lies outside of the non-smooth boundary. The second condition is necessary on general non-smooth domains. We will consider domains $\Omega \subset \mathbb{R}^n$ with $n \leq 2$ that are bounded and have a C^∞-boundary $\partial\Omega$.

By a moving plane argument one finds that for some boundary point the domain lies on one side of a (hyper)plane through that boundary point. Using some elementary transformations we may hence assume that

$$\begin{aligned}\Omega &\subset B_2(0)\,,\\ \Omega &\subset \{x\in\mathbb{R}^n; x_1>0\}\,,\\ 0 &\in \partial\Omega.\end{aligned}$$



Balls in $\mathbb{R}^n$ are denoted by

$$B_\rho(0)=\{x\in\mathbb{R}^n; |x|<\rho\}\,.$$

Since the boundary is C^∞ there exists $r>0$ and $\psi\in C^\infty(\mathbb{R}^{n-1})$ such that

$$\partial\Omega\cap B_r(0)=\left\{(x_1,x')\in\mathbb{R}\times\mathbb{R}^{n-1}; x_1=\psi(x')\right\}$$

with $\psi(0)=0$, $\nabla\psi(0)=0$ and $\Delta\psi(0)\geq 0$ and we may assume that $\psi(x')\geq 0$ for all $x\in\Omega$.

We will make an extra assumption: there is $c>0$ such that for all $|x'|<r$

$$\begin{aligned}|\nabla\psi(x')| &\leq c\,|x'|\;\Delta\psi(x')\,,\\ \psi(x') &\leq c\,|x'|^2\,\Delta\psi(x')\,.\end{aligned}\tag{2}$$

Both conditions in (2) are satisfied for some small $r>0$ when ψ is analytic.

Since the inverse of the Dirichlet-Laplacian on $L^p(\Omega)$ for smooth bounded domains Ω is compact and strongly positive, standard arguments show the existence of a unique solution u_λ in $W_0^{1,p}(\Omega)\cap W^{2,p}(\Omega)$ of (1) for $f\in L^p(\Omega)$ whenever λ is not one of the countable many (positive) eigenvalues. Here p is any number in $(1,\infty)$. We will also use Hölder type regularity results. Both type of results can be found in [4].

Proposition *Let $n\geq 2$. There exists $f\in L^n(\Omega)$ with $f>0$ such that, for all $\lambda>\lambda_1$ and λ not an eigenvalue, the solution u_λ of (1) changes sign.*

Proof. We will proceed in several steps.

i. First we will assume that $\psi(x')\equiv 0$ on $B_r(0)$. In this case we will use for the right hand side

$$f(x)=\frac{x_1}{|x|^2(2-\log|x|)}\left(n+\frac{1}{2-\log|x|}\right).$$

Since

$$\tfrac{1}{2}\sqrt{|x|}\leq\frac{1}{2-\log|x|}\leq 1 \text{ for } x\in\Omega\tag{3}$$

we find that

$$f(x) \geq x_1 |x|^{-3/2} > 0 \text{ in } \Omega. \tag{4}$$

Let us define the function v on Ω by

$$v(x) = x_1 \log(2 - \log|x|). \tag{5}$$

Then $v \in C^\gamma(\overline{\Omega}) \cap C^\infty(\overline{\Omega} \setminus \{0\})$ for all $\gamma \in (0,1)$ and $v > 0$ holds on Ω. Moreover

$$-\Delta v = -2\nabla x_1 \cdot \nabla \log(2 - \log|x|) - x_1 \Delta \log(2 - \log|x|) =$$

$$= -2\nabla x_1 \cdot \frac{-x|x|^{-2}}{2 - \log|x|} - x_1 \left(\frac{-(n-2)|x|^{-2}}{2 - \log|x|} - \frac{|x|^{-2}}{(2 - \log|x|)^2} \right) =$$

$$= \frac{x_1}{|x|^2 (2 - \log|x|)} \left(n + \frac{1}{2 - \log|x|} \right) = f(x). \tag{6}$$

Note that $f \in C^\infty(\overline{\Omega} \setminus \{0\})$ and since

$$\int_\Omega |f(x)|^n \, dx \leq \int_{r=0}^{2} \int_{\varphi=-\pi/2}^{\pi/2} \left(\frac{r\cos(\varphi)(n+1)}{r^2(2 - \log r)} \right)^n r^{n-1} d\varphi dr \leq$$

$$\leq \pi(n+1)^n \int_{r=0}^{2} \frac{r^{-1}}{(2 - \log r)^n} dr = \pi(n+1)^n \int_{t=2-\log 2}^{\infty} t^{-n} \, dt < \infty,$$

we find that $f \in L^n(\Omega)$.

We take $\lambda > \lambda_1$ and let u_λ denote the solution of (1). Set $w_\lambda = u_\lambda - \chi v$ where χ is a nonnegative C^∞-function on $\mathbb{R}^n$ such that $\chi(x) = 1$ for $|x| \leq \frac{1}{2}r$ and $\chi(x) = 0$ for $|x| \geq r$. Then w_λ satisfies

$$\begin{cases} -\Delta w = \lambda w + (\lambda\chi + \Delta\chi) v + 2\nabla\chi \cdot \nabla v + (1-\chi) f & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{7}$$

Note that $(\lambda\chi + \Delta\chi) v + 2\nabla\chi \cdot \nabla v + (1-\chi) f \in C^\gamma(\overline{\Omega})$ which implies for λ not an eigenvalue that $w_\lambda \in C^{2+\gamma}(\overline{\Omega}) \cap C_0(\overline{\Omega})$ and hence that $|w_\lambda(x)| \leq c_\lambda \, d(x, \partial\Omega)$ for some constant c_λ depending on λ. Here $d(x, \partial\Omega)$ is the distance function to $\partial\Omega$. Since

$$\lim_{x_1 \downarrow 0} \frac{|w_\lambda(x_1, 0)|}{v(x_1, 0)} \leq \lim_{x_1 \downarrow 0} \frac{c_\lambda}{\log(2 - \log x_1)} = 0,$$

we find that $u_\lambda(x_1, 0) > \frac{1}{2} v(x_1, 0) > 0$ for $x_1 > 0$ sufficiently small. Hence u_λ is somewhere positive.

ii. The case that ψ is not identical zero. First we use the following transformation

$$\begin{aligned} y_1 &= x_1 - \psi\left(x'\right), \\ y_i &= x_i \quad \text{for } i \geq 2, \end{aligned}$$

and we consider the function $\tilde{v}\left(x\right) := v\left(y\left(x\right)\right)$ where v is the function in (5). One finds that

$$\begin{aligned} -\Delta \tilde{v}\left(x\right) &= -\Delta\left(v\left(y\left(x\right)\right)\right) = \\ &= -\left(\Delta v\right)\left(y\left(x\right)\right) + \left(\Delta\psi - \left|\nabla\psi\right|^2 \frac{\partial}{\partial y_1} + 2\nabla\psi\cdot\nabla_y\right)\frac{\partial v}{\partial y_1}\left(y\left(x\right)\right). \end{aligned}$$

We will need to estimate some derivatives of v for $|y| \to 0$:

$$\begin{aligned} \tfrac{\partial}{\partial y_1} v\left(y\right) &= \log\left(2-\log|y|\right) - \frac{y_1^2}{|y|^2}\frac{1}{2-\log|y|} = \log\left(2-\log|y|\right) + o\left(1\right); \\ \left(\tfrac{\partial}{\partial y_1}\right)^2 v\left(y\right) &= \frac{-3y_1 + 2\frac{y_1^3}{|y|^2}\left(1-\frac{1}{2-\log|y|}\right)}{|y|^2\left(2-\log|y|\right)} = o\left(|y|^{-1}\right); \\ \tfrac{\partial}{\partial y_j}\tfrac{\partial}{\partial y_1} v\left(y\right) &= \frac{-y_j + 2\frac{y_1^2 y_j}{|y|^2}\left(1-\frac{1}{2-\log|y|}\right)}{|y|^2\left(2-\log|y|\right)} = o\left(|y|^{-1}\right) \quad \text{for } j \neq 1; \end{aligned}$$

With the assumptions in (2) it follows that for $|x| \to 0$ we have

$$-\Delta\tilde{v}\left(x\right) = f\left(y\left(x\right)\right) + \Delta\psi\left(x\right)\Big(\log\left(2-\log|y\left(x\right)|\right) + o\left(1\right)\Big)$$

which is hence positive for small $|x|$. Now we set $f^*\left(x\right) = \max\left(-\Delta\tilde{v}\left(x\right), 0\right)$ and consider (1) with f replaced by f^*. Denoting v^* the solution of

$$\begin{cases} -\Delta v^* = f^* & \text{in } \Omega, \\ \quad v^* = \tilde{v} & \text{on } \partial\Omega, \end{cases}$$

one finds by the maximum principle that $v^* \geq \tilde{v}$. The remaining arguments are as in the case $\psi \equiv 0$.

iii. Although it is not the main point of the counterexample we still have to show that $u_\lambda \geq 0$ doesn't hold everywhere in Ω. We will proceed by contradiction using the Barta inequality. Barta ([1]) states that for any $w \in C^2(\overline{\Omega})$ with $w > 0$ the following holds

$$\lambda_1 \geq \inf_{x\in\Omega} \frac{-\Delta w\left(x\right)}{w\left(x\right)}. \tag{8}$$

This inequality can be generalized to more general elliptic operators as well as to $w \in C^2(\Omega) \cap C_0(\overline{\Omega})$ with $w > 0$ in Ω (see [5], [6]). Suppose that $u_\lambda \geq 0$. Then by the strong maximum principle it follows from

$$-\Delta u_\lambda = \lambda u_\lambda + f \geq 0$$

and not identical zero, that $u_\lambda > 0$. By using (8) with u_λ one finds that

$$\lambda_1 \geq \inf_{x\in\Omega} \frac{-\Delta u_\lambda (x)}{u_\lambda (x)} = \inf_{x\in\Omega} \frac{(\lambda u_\lambda + f)(x)}{u_\lambda (x)} \geq \lambda$$

contradicting $\lambda > \lambda_1$. Hence u_λ changes sign for $\lambda > \lambda_1$. □

Remark: The proposition shows that for $p \leq n$ there is no anti-maximum principle. If one only wants to see that n is critical one may use for $p < n$ the functions $f = x_1 |x|^{\alpha-2}$ and $v = \frac{1}{-\alpha(\alpha+n)} x_1 |x|^{\alpha}$ with $\alpha \in (-1, 0)$ satisfying $\alpha > 1 - n/p$.

References

[1] J. Barta, Sur la vibration d'une membrane, *C.R. Acad. Sci. Paris* **204** (1937), 472-473.

[2] I. Birindelli, Hopf's lemma and anti-maximum principle in general domains, *J.Differ. Equations* **119** (1995), 450-472.

[3] Ph. Clément and L.A. Peletier, An anti-maximum principle for second order elliptic operators. *J. Differ. Equations* **34** (1979), 218-229.

[4] D. Gilbarg and N.S. Trudinger, "Elliptic partial differential equations of second order, 2^{nd} edition", Springer, Berlin Heidelberg New York, 1983.

[5] M.H. Protter and H.F. Weinberger, On the spectrum of general second order operators, *Bulletin A.M.S.* **72** (1966), 251-255.

[6] G. Sweers, Strong positivity in $C(\bar{\Omega})$ for elliptic systems, *Math. Z.* **209** (1992), 251-271.