ON THE DIFFERENTIAL EQUATION $U_{XXXX} + U_{YYYY} = F$ FOR AN ANISOTROPIC STIFF MATERIAL

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Abstract. We study the differential operator $L = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}$ and investigate positivity preserving properties in the sense that $f \geq 0$ implies that solutions u of $Lu - \lambda u = f$ are nonnegative. Since the operator is of fourth order we have no maximum principle at our disposal. The operator models the deformation of an anisotropic stiff material like a wire fabric, and it has to be complemented by appropriate boundary conditions. Our operator was introduced by Jacob II Bernoulli as the operator that supposedly models the vibrations of an elastic plate. This model was later revised by Kirchhoff, because the operator and its solutions were anisotropic. Modern materials, however, are often anisotropic and therefore the old model of Bernoulli deserves an updated investigation. It turns out that even our simple looking model problem contains some hard analytical challenges.

Key words. orthotropic plate, anisotropic operator, vibrations, spectrum, fourth order elliptic, clamped and hinged boundary condition, positivity of the operator, Green's function, Kirchhoff plate

AMS subject classifications. 35J35, 35J40, 35P10, 74E10, 74G55, 74G40, 74H45, 74K10, 74K20

1. Introduction. Small vertical deformations u of an elastic membrane are usually described by a second order differential equation $-\Delta u = f$ with f denoting the load, whereas the deformation of a plate is commonly modelled by a fourth order equation $\Delta^2 u = f$. Suppose the membrane is replaced by a piece of material or cloth that is woven out of elastic strings. Then the material properties change drastically, and in [4] such a problem was studied for second order differential operators. If a plate is replaced by a stiff woven material (running in cartesian directions) its deformation energy can be described by

$$\int_\Omega \left(u_{xx}^2 + u_{yy}^2\right)\ dx\, dy \tag{1.1}$$

rather than the one for the elastic plate

$$\int_\Omega \left((\Delta u)^2 - (1-\sigma)\left(u_{xx}u_{yy} - u_{xy}^2\right)\right) dx\, dy. \tag{1.2}$$

For the energy that corresponds to the reinforcement or wire fabric that is embedded in for example concrete, a linear combination of (1.1) and (1.2) is appropriate.

In contrast to the plate equation, that is, the Euler equation for (1.2) which contains the operator $\Delta^2 u = u_{xxxx} + 2u_{xxyy} + u_{yyyy}$, the linearized differential equation for a stiff fabric consisting of perpendicular fibers does not contain mixed terms when these fibers run parallel to the x and y-axes. Indeed, if the torsional stiffness can be

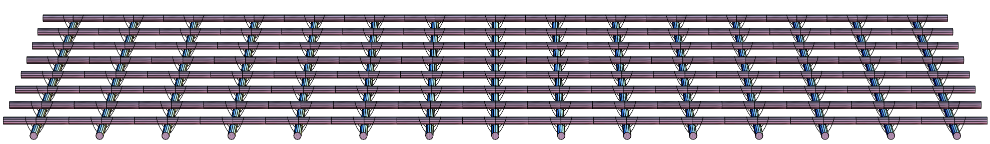


FIG. 1.1. *A rectangular wire fabric with fibers in cartesian directions*

neglected the energy stored in the grid under a vertical load f is supposed to be given by

$$E(u) = \int_\Omega \left\{ \tfrac{1}{2} \left(u_{xx}^2 + u_{yy}^2 \right) - fu \right\} dx\, dy.$$

The corresponding Euler equation is $u_{xxxx} + u_{yyyy} = f$. This equation has to be complemented by suitable boundary conditions, and in the present paper we shall study the problem on a planar domain Ω:

- as a general grid that is hinged at the boundary

$$\begin{cases} u_{xxxx} + u_{yyyy} = f & \text{in } \Omega, \\ u = n_1^2\, u_{xx} + n_2^2\, u_{yy} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

  where $n = n(x,y)$ is the exterior normal at $(x,y) \in \partial\Omega$;
- or as a general grid that is *clamped* at the boundary

$$\begin{cases} u_{xxxx} + u_{yyyy} = f & \text{in } \Omega, \\ u = \frac{\partial}{\partial n} u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

When we checked the literature for this type of equation, we found a remarkable hint in the chapter on the history of plate theory in Szabó's book [37], see p.409. Jacob II Bernoulli, inspired by Chladni's experiments on vibrating plates, had attempted to model their behaviour by our differential equation in [5], but this was later dismissed for isotropic plates and replaced by Kirchhoff's theory [24]. But there is more to it. According to [30] Bernoulli had also studied and absorbed Leonard Euler's idea that an elastic membrane should be modelled as a fabric of one-dimensional orthogonal elastic strings and he tried to carry this idea over to modelling a plate as a fabric of one-dimensional beams. Thus he arrived at

$$\frac{\partial^4 z}{\partial x^4} + \frac{\partial^4 z}{\partial y^4} = \frac{z}{c^4}$$

as *"the fundamental equation of the entire theory"* of plate vibrations. In those days church bells were intended as applications for the theory. Both operators, the isotropic plate operator Δ^2, and the anisotropic

$$L = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} \tag{1.5}$$

retain a certain degree of isotropy. They are special cases of

$$\tilde{L} = \frac{\partial^4}{\partial x^4} + P \frac{\partial^4}{\partial x^2 \partial y^2} + Q \frac{\partial^4}{\partial y^4} \tag{1.6}$$

with $P \geq 0$ and $Q > 0$ denoting material constants. Notice that $\tilde{L}$ is always invariant under reflections across cartesian axes, but not always under rotations. Plates whose deformation is described by such operators are called *orthotropic*, see e.g [29], [31]. By scaling y and not scaling x one can always force Q to be 1. Realistic values for P and Q in the case of plywood material (birch with bakelite glue) can be found in [25, p. 92], [26, p. 269], or in [31]. It is not unrealistic to expect Q to be of order 1-10 and $P \in [0,1)$. Modern (composite) materials like GLARE, see http://www.lr.tudelft.nl/highlights/glare.asp, a composite of layers of fibreglass and aluminium that is also called "plymetal", can be expected to satisfy similar orthotropic equations. Orthotropic plate equations like $\tilde{L}u = f$ have been rigorously derived by homogenization methods as the right macroscopic model for grid structures as the thickness of the structure and the size of its cells goes to zero. To be precise, in [3] p.130, and using our notation the limit equation has coefficients $Q = 1$ and $P = 4/(1+\nu)$, where ν denotes the Poisson ratio of the original (solid and unhomogenized) plate material. For $\nu = -1/3$ one gets $P = 6$ as in (3.6) below. We take the differential equation for granted here and do not address issues of homogenization as in [3].

Section 2 is devoted to proving existence and uniqueness questions, and Section 3 to regularity of solutions to these boundary value problems. Regularity near corners of Ω is delicate, and its discussion will be limited to some special cases. Moreover, we address the subject of representations of solutions by series or by means of a Green function at the end of Section 3.

In Section 4 we study the spectrum of the operator L on rectangular domains and for hinged and clamped boundary conditions. Since the operator is separable, on special domains like rectangles all of its eigenfunctions can be represented in terms of products of one-dimensional eigenfunctions. We learned this from Courant and Hilbert, see [10], Ch.II Par. 1.6, who did it for operators of second order. Therefore the one-dimensional cases will always be treated before the rectangular domains. We present all eigenvalues and eigenfunctions for a number of examples and compare spectra for different (parallel or diagonal) alignments of our anisotropic grid.

Section 5 is dedicated to positivity questions. Suppose the load f on a beam (or grid) is pointing downwards. Does this imply that the deformation u has the same sign everywhere in Ω? The answer is in general negative, unless the geometry of the domain is special or unless the beam (or grid) is embedded in an elastic ambient medium that exerts a restoring force proportional to the deformation. So the modified question is, for which (presumably negative) values of λ one can show that $f \geq 0$ implies positivity of the solution to

$$u_{xxxx} + u_{yyyy} = \lambda u + f \qquad \text{in } \Omega,$$

that satisfies the boundary conditions under consideration. This question turns out to be technically most challenging and its answer is given using different tricks for different alignments or boundary conditions.

For the reader's convenience we finish with a summary in section 6 and an appendix.

2. Existence and uniqueness for hinged and clamped grids. Let $\Omega \subset \mathbb{R}^n$ be a bounded simply connected set. Then the variational problem

$$\text{Minimize:} \quad E(v) = \int_\Omega \left(\tfrac{1}{2} \sum_{i=1}^n v_{x_i x_i}^2 - f\,v \right) dx \quad \text{on } W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega) \tag{2.1}$$

has a unique solution. To see this directly we follow the ideas of [16] and first show that $E(v)$ is coercive on $W^{2,2}(\Omega)$. Obviously $2u_{xx}u_{yy} \leq u_{xx}^2 + u_{yy}^2$, so that

$$E(v) \geq c(n) \int_\Omega (\Delta v)^2 \, dx - \int_\Omega f\, v\, dx.$$

If we denote Δv by g, then a well known a-priori estimate for solutions of Dirichlet problems for second order elliptic differential equations on bounded domains (see e.g.

[14], p.317) implies that

$$||D^2 v||_{L^2(\Omega)} \le ||\Delta v||_{L^2(\Omega)},$$

so that all second derivatives of v are in $L^2(\Omega)$. This and a Poincaré type inequality show the coerciveness of E on $W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$. Now the existence and uniqueness of a solution follow from the direct method in the calculus of variations and from the strict convexity of the functional E. The solution satisfies the Euler-equation $\sum_{i=1}^n u_{x_i x_i x_i x_i} = f$ in Ω. To derive the boundary conditions, we note that a weak solution satisfies

$$\int_\Omega \left(\sum_{i=1}^n u_{x_i x_i} \varphi_{x_i x_i} - f\,\varphi \right) dx = 0, \tag{2.2}$$

and after two integrations by parts we obtain

$$0 = \int_\Omega \left(-\sum_{i=1}^n u_{x_i x_i x_i} \varphi_{x_i} - f\,\varphi \right) dx + \int_{\partial\Omega} \sum_{i=1}^n u_{x_i x_i} \varphi_{x_i} \nu_i \; d\sigma \tag{2.3}$$

$$= 0 + \int_{\partial\Omega} \sum_{i=1}^n u_{x_i x_i} \varphi_{x_i} \nu_i \; d\sigma - \int_{\partial\Omega} \sum_{i=1}^n u_{x_i x_i x_i} \; \varphi \; \nu_i \; d\sigma \tag{2.4}$$

$$= \int_{\partial\Omega} \left(\sum_{i=1}^n u_{x_i x_i} \nu_i^2 \right) \frac{\partial \varphi}{\partial \nu} \; d\sigma \tag{2.5}$$

Notice that the last integral in (2.4) vanishes because φ vanishes on the boundary. Therefore the first boundary integral in (2.4) must vanish too. The vanishing of φ on $\partial\Omega$ implies in particular that the bracket in (2.5) must vanish on $\partial\Omega$. Thus we have formally derived (1.3) in the plane case.

If the grid or stiff fabric is clamped, we consider the variational problem

$$\text{Minimize:} \quad E(v) = \int_\Omega \left(\tfrac{1}{2} \sum_{i=1}^n v_{x_i x_i}^2 - f\,v \right) dx \quad \text{on } W_0^{2,2}(\Omega) \tag{2.6}$$

and observe that the same existence proof works for this problem, too. The solution satisfies

$$\begin{cases} \sum_{i=1}^n u_{x_i x_i x_i x_i} = f & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.7}$$

We have the following existence and uniqueness results..

THEOREM 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with piecewise smooth boundary and suppose that $f \in L^2(\Omega)$. Then problems (2.1) and (2.6) have a unique minimizer. Moreover, the corresponding boundary value problems, which in the case $n = 2$ are given by (1.3) and (1.4), have a unique weak solution.*

REMARK 2.1. *As usual a weak solution for (1.3) is a function u in $W_0^{2,2}(\Omega)$ that satisfies (2.2) for all $\varphi \in W_0^{2,2}(\Omega)$. A weak solution of (1.4) is a function u in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, satisfying (2.2) for all $\varphi \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$.*

The existence was shown above by variational methods and the uniqueness of the weak solution follows from the strict convexity of the underlying functional E. Notice that the second order boundary condition holds only in the sense of distributions. To see that it holds pointwise in every smooth point of the boundary, we need to know more about its regularity.

3. Regularity. One may use the standard regularity theory for elliptic operators whenever the elliptic system is of an appropriate type and if the boundary is sufficiently smooth. First we will show that our systems are regular elliptic.

3.1. Regular elliptic. The symbol, that is $L = \mathcal{L}(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$, of our fourth order operator can be decomposed as follows:

$$\mathcal{L}(\xi_1, \xi_2) := \xi_1^4 + \xi_2^4 = \left(\xi_1^2 + \sqrt{2}\xi_1\xi_2 + \xi_2^2\right)\left(\xi_1^2 - \sqrt{2}\xi_1\xi_2 + \xi_2^2\right), \tag{3.1}$$

Hence L can be written as the composition of two second order elliptic operators. Notice however that the boundary value problem (1.4) cannot be split into a system of two second order equations with separated boundary conditions. In fact, even for the boundary value problem (1.3) this seems to be out of reach. The boundary operators have the following symbols:

- For (1.3): $\mathcal{B}_1(\xi) = 1$ and $\mathcal{B}_2(\xi) = n_1(x)^2\xi_1^2 + n_2(x)^2\xi_2^2$.
- For (1.4): $\mathcal{B}_1(\xi) = 1$ and $\mathcal{B}_2(\xi) = n_1(x)\xi_1 + n_2(x)\xi_2$.

A necessary condition to have the full classical regularity results is that the corresponding boundary value problem should be regular elliptic in the sense of [27], and this is indeed the case.

Lemma 3.1. *Problems (1.3) and (1.4) are regular elliptic.*

Proof. The differential operator is regular elliptic of order $2k$ if there is $c > 0$ such that $\mathcal{L}(\xi) \geq c\,|\xi|^{2k}$ for $\xi \in \mathbb{R}^2$ which obviously holds true. In order to show that the boundary conditions make it into a regular elliptic one has to consider the factorisation of $\tau \mapsto \mathcal{L}(\xi + \tau\eta)$. One finds that the roots of this polynomial are

$$\tau_k = -\frac{\xi_1 + (-1)^{\frac{2k-1}{4}}\xi_2}{\eta_1 + (-1)^{\frac{2k-1}{4}}\eta_2} \text{ with } k \in \{1, 2, 3, 4\}.$$

We use $(-1)^\alpha = \cos\pi\alpha + \boldsymbol{i}\sin\pi\alpha$. Depending on ξ and η, which should be taken independently, there are two roots, τ_I and τ_{II}, which have positive imaginary part. We find $\mathcal{L}(\xi + \tau\eta) = a^+(\xi, \eta; \tau)a^-(\xi, \eta; \tau)$ with

$$\begin{aligned} a^+(\xi, \eta; \tau) &= \sqrt{\eta_1^4 + \eta_2^4}\,(\tau - \tau_I)(\tau - \tau_{II}), \\ a^-(\xi, \eta; \tau) &= \sqrt{\eta_1^4 + \eta_2^4}\,(\tau - \bar{\tau}_I)(\tau - \bar{\tau}_{II}). \end{aligned}$$

Since the imaginary parts of τ_I and τ_{II} have the same sign the first order term in $a^+(\xi, \eta; \tau)$ has a coefficient with a strictly negative imaginary part, indeed

$$(\tau - \tau_I)(\tau - \tau_{II}) = \tau^2 - (\tau_I + \tau_{II})\tau - \tau_I\tau_{II}.$$

The condition for regularity that has to be verified is that, for ξ a tangential direction and η a normal direction, the polynomials $\tau \mapsto \mathcal{B}_1(\xi + \tau\eta)$ and $\tau \mapsto \mathcal{B}_2(\xi + \tau\eta)$ are independent modulo $\tau \mapsto a^+(\xi, \eta; \tau)$. Therefore we set $\eta = (n_1, n_2)$ and $\xi = (-n_2, n_1)$.

For (1.3) $\mathcal{B}_1(\xi + \tau\eta) = 1$ and

$$\begin{aligned} \mathcal{B}_2(\xi + \tau\eta) &= n_1^2(-n_2 + n_1\tau)^2 + n_2^2(n_1 + n_2\tau)^2 \\ &= 2n_1^2n_2^2 + (n_2^2 - n_1^2)n_1n_2\tau + \left(n_1^4 + n_2^4\right)\tau^2. \end{aligned}$$

This is a polynomial with only real coefficients. Since $a^{+}(\xi,\eta;\tau)$ contains a real second order term and an imaginary first order term so that both polynomials are linearly independent.

For (1.4) $\mathcal{B}_1(\xi+\tau\eta)=1$ and $\mathcal{B}_2(\xi+\tau\eta)=1+\tau$. These are clearly independent modulo any second order polynomial. □

3.2. Regularity for smooth domains. Near the smooth boundary part the standard regularity results e.g. from [27, Ch. 2] may be used, since both the clamped and the hinged problem are regular elliptic. Only the corners need more attention. But to fix the facts let us summarize the regularity results in a Theorem.

THEOREM 3.2. *Let $\Omega\subset\mathbb{R}^2$ be a bounded domain with piecewise smooth boundary and let $\Omega'\subset\Omega$ be a subset such that $\overline{\Omega'}$ contains only the smooth boundary points of $\partial\Omega$. If $f\in W^{k,2}(\Omega)$ and $k\in\{0,1,2,\ldots\}$ then the weak solutions of (1.3) and (1.4) are of class $W^{k+4}(\Omega')$. In particular for $f\in L^2(\Omega)$ the derivatives $u_{x_ix_i}$ are in $W^{3/2,2}(\partial\Omega\cap\partial\Omega')$, so that the boundary condition in (1.3) holds pointwise a.e. on $\partial\Omega$.*

3.3. Regularity near corners. It remains to discuss the regularity near singular points of the boundary, and this will be done for some special but typical situations. First we will give an explanation for a simple case.

3.3.1. The hinged rectangular grid with aligned fibers. Let $R=(0,a)\times(0,b)$. be the rectangle. It will be relatively easy to study the regularity of the *hinged grid* near a corner, say $(0,0)$ when the grid is *aligned with the rectangle* as in Figure 1.

Reflection:. The first approach is through a reflection argument. As an example we will consider the hinged rectangular grid with horizontally and vertically aligned fibers.

Note that the differential operator and boundary conditions all satisfy

$$\mathcal{L}(\pm\tfrac{\partial}{\partial x},\tfrac{\partial}{\partial y})=\mathcal{L}(\tfrac{\partial}{\partial x},\tfrac{\partial}{\partial y})\ \text{ and }\ \mathcal{B}(\pm\tfrac{\partial}{\partial x},\tfrac{\partial}{\partial y})=\mathcal{B}(\tfrac{\partial}{\partial x},\tfrac{\partial}{\partial y}).$$

Instead of considering

$$\left\{\begin{array}{rl} u_{xxxx}+u_{yyyy}=f & \text{in } R,\\ u=u_{xx}=0 & \text{on } \{0,a\}\times[0,b]\,,\\ u=u_{yy}=0 & \text{on } [0,a]\times\{0,b\}\,,\end{array}\right. \tag{3.2}$$

we extend f to $\tilde f$ on $(-a,a)\times(0,b)$ by

$$\tilde f(x,y)=\operatorname{sign}(x)\ f(|x|\,,y)$$

and consider

$$\left\{\begin{array}{rl} \tilde u_{xxxx}+\tilde u_{yyyy}=\tilde f & \text{in } \tilde R=(-a,a)\times(0,b)\,,\\ \tilde u=\tilde u_{xx}=0 & \text{on } \{-a,a\}\times[0,b]\,,\\ \tilde u=\tilde u_{yy}=0 & \text{on } [-a,a]\times\{0,b\}\,.\end{array}\right. \tag{3.3}$$

If $f\in L^p(R)$ then $\tilde f\in L^p(\tilde R)$ and by the result above there is unique solution $\tilde u\in W^{2,p}(\tilde R)$ and $\tilde u\in W^{4,p}(\tilde R\backslash N)$ with N some neighborhood of the four corners of $\tilde R$; $(0,0)$ has become a regular boundary point. Since the solution $\tilde u$ is unique one finds that $\tilde u(x,y)=-\tilde u(-x,y)$ and hence $\tilde u(0,y)=\tilde u_{xx}(0,y)=0$. In other words, $u:=\tilde u_{|R}$ is the solution of (3.2) which is in $W^{4,p}(R\cap B_\varepsilon(0))$. Since we may do so for every corner of R we find that $u\in W^{4,p}(R)$.

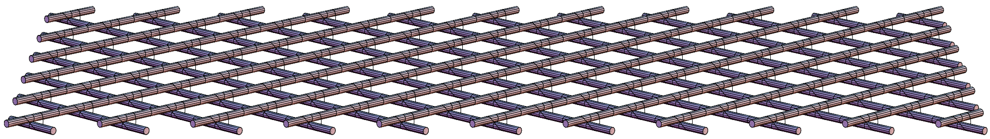


FIG. 3.1. *Rectangular grid with diagonal fabric*

Seperation of eigenfunctions:. A second approach can be used if there is a complete orthonormal system of eigenfunctions of the form $\{\varphi_i(x)\psi_j(y); i,j \in \mathbb{N}\}$. For example for the problem (3.2) the set $\{\Phi_{ij}; i,j \in \mathbb{N}^+\}$ with

$$\Phi_{i,j}(x,y) = \frac{2}{\sqrt{ab}} \sin(i\frac{\pi}{a}x) \sin(j\frac{\pi}{b}y)$$

is a complete orthonormal set of eigenfunctions. Writing $f_{ij} = \langle \Phi_{i,j}, f \rangle$ the solution u is given by

$$u(x,y) = \sum_{i,j=1}^{\infty} \frac{f_{ij}}{\left(i\frac{\pi}{a}\right)^4 + \left(j\frac{\pi}{b}\right)^4} \Phi_{i,j}.$$

Using Parseval a straightforward computation shows that

$$\left\| \left(\frac{\partial}{\partial x}\right)^k \left(\frac{\partial}{\partial y}\right)^\ell u \right\|_{L^2(R)}^2 = \sum_{i,j=1}^{\infty} \frac{\left(i\frac{\pi}{a}\right)^{2k} \left(j\frac{\pi}{b}\right)^{2\ell}}{\left(\left(i\frac{\pi}{a}\right)^4 + \left(j\frac{\pi}{b}\right)^4\right)^2} (f_{ij})^2$$

which is bounded if $f \in L^2(R)$ and $k,l \in \mathbb{N}$ with $k + l \leq 4$. So $u \in W^{4,2}(R)$.

3.3.2. The hinged rectangular grid with diagonal fibers. Now suppose that the grid runs *diagonally* into the horizontal and vertical axis as in Figure 3.3.2, and that $\hat{x} := \frac{1}{2}\sqrt{2}\,(x+y)$ and $\hat{y} = \frac{1}{2}\sqrt{2}\,(y-x)$. Then a straightforward calculation shows that

$$u_{xxxx} + u_{yyyy} = \tfrac{1}{2} u_{\hat{x}\hat{x}\hat{x}\hat{x}} + \tfrac{6}{2} u_{\hat{x}\hat{x}\hat{y}\hat{y}} + \tfrac{1}{2} u_{\hat{y}\hat{y}\hat{y}\hat{y}} = f \ , \tag{3.4}$$

while the boundary condition from (2.4) becomes

$$u_{xx} + u_{yy} = \Delta u = 0 = u_{\hat{x}\hat{x}} + u_{\hat{y}\hat{y}} \tag{3.5}$$

because $(\nu_1)^2 = (\nu_2)^2 = 1/2$ on the sides of the rectangle and because the Laplacian is invariant under rotations. Since also $u = 0$ on the boundary, this implies $u_{\hat{x}\hat{x}} = 0 = u_{\hat{y}\hat{y}}$. Therefore after an obvious change of notation the deformation u of the *hinged diagonal* grid satisfies again a regular elliptic boundary value problem, namely

$$\begin{cases} u_{xxxx} + 6u_{xxyy} + u_{yyyy} = 2f & \text{in } R = (0,a) \times (0,b)\,, \\ u = u_{xx} = 0 & \text{on } \{0,a\} \times [0,b]\,, \\ u = u_{yy} = 0 & \text{on } [0,a] \times \{0,b\}\,. \end{cases} \tag{3.6}$$

Also for this boundary value problem we find that $\mathcal{L}(\pm\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = \mathcal{L}(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ and $\mathcal{B}(\pm\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = \mathcal{B}(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ and hence we may use the odd reflection argument of (3.3) to find $u \in W^{4,2}(R)$ that does satisfy the boundary conditions for $x = 0$.

Incidentally, the transformed elliptic operator has a symbol that can again be factorized as

$$2\hat{\mathcal{L}}(\xi_1,\xi_2) := \xi_1^4 + 6\xi_1^2\xi_2^2 + \xi_2^4 = \left(\xi_1^2 + \left(3-2\sqrt{2}\right)\xi_2^2\right)\left(\xi_1^2 + \left(3+2\sqrt{2}\right)\xi_2^2\right). \tag{3.7}$$

The fact that (3.6) constitutes a regular elliptic boundary value problem does not need to be checked again, since this property is invariant under changes of the coordinate system. Moreover, the boundary conditions fit nicely with this factorization and we find a system of two well-posed second order problems:

$$\begin{cases} u_{xx} + \left(3-2\sqrt{2}\right)u_{yy} = v & \text{in } R, \\ u = 0 & \text{on } \partial R \\ v_{xx} + \left(3+2\sqrt{2}\right)v_{yy} = 2f & \text{in } R, \\ v = 0 & \text{on } \partial R \end{cases} \tag{3.8}$$

Using the result of Kadlec ([22]) for second order operators on convex domains one finds that $f \in L^2(R)$ implies $v \in W^{2,2}(R) \cap W_0^{1,2}(R)$. Since v satisfies the boundary conditions of u (!), we do not only find that $u \in W^{2,2}(R) \cap W_0^{1,2}(R)$ but even that $u \in W^{4,2}(R) \cap W_0^{1,2}(R)$.

We remark that some boundary value problems with different boundary conditions along each side can be treated by a reflection argument. In fact, the same reflection argument works for the aligned rectangular grid, if it is clamped on the horizontal parts of the boundary and hinged on the vertical part. To be specific, for $f \in L^2(\Omega)$ the unique solution u of

$$\begin{cases} u_{xxxx} + u_{yyyy} = f & \text{in } R, \\ u = u_{xx} = 0 & \text{on } \{0,a\} \times [0,b], \\ u = u_y = 0 & \text{on } [0,a] \times \{0,b\}. \end{cases} \tag{3.9}$$

is in $W^{4,2}(R)$.

3.3.3. The clamped rectangular grid with aligned fibers. The regularity of the *clamped* grid near a corner does not follow from such a simple reflection argument, because u_{xx} does in general not vanish on $(0,y)$ with $y \in (0,b)$. However, provided the grid is *aligned* with the rectangle as in Figure 1, we may proceed by 'seperation of eigenfunctions'. To complete this argument we need to borrow some results of subsection 4.2.3 and more specifically Lemma 4.3 and 4.4.

The set $\{\Phi_{ij}\}$ of eigenfunctions in (4.11) is a complete orthonormal system in $L^2(\Omega)$. Then, as above for the hinged rectangular grid, the solution of (1.4) can be represented by

$$u(x,y) = \sum_{i,j=1}^{\infty} \frac{\alpha_{ij}}{\Gamma_{ij}} \Phi_{ij}(x,y)\ ,$$

where α_{ij} are the Fourier coefficients from the representation of f with $\sum_{i,j} \alpha_{ij}^2$ being finite by Parseval's identity. The eigenvalues Γ_{ij} are defined by $\Gamma_{ij} = a^{-4}\lambda_i + b^{-4}\lambda_j$ and we find that

$$\frac{\partial^{n+m}}{\partial x^n \partial y^m} u(x,y) = \sum_{i,j=1}^{\infty} \frac{a^{-n}\lambda_i^{n/4} b^{-m}\lambda_j^{m/4}}{a^{-4}\lambda_i + b^{-4}\lambda_j} \alpha_{ij}\Phi_{ij}(x,y)\ ,$$

which is bounded when $n+m \le 4$. This shows that $u \in W^{4,2}(R)$ even in this case of a *clamped rectangular grid* aligned with R. From Theorem 2.1 we were only allowed to conclude that $u \in W^{2,2}(R)$.

3.3.4. The clamped rectangular grid with diagonal fibers. How to obtain the regularity of u for a *clamped–hinged* or *clamped–clamped* diagonal grid near a corner is a nontrivial problem and will not be discussed here.

One conceivable way to represent a solution would be a Green function $g^{0,\xi}(\cdot) = F(\cdot-\xi)+h(\xi,\cdot)$. It can in principle be obtained by adding a solution h of $Lh(\xi,\cdot)=0$ in R, $h(\xi,\cdot)+F(\cdot-\xi)=0=B_2h(\xi,\cdot)+B_2F(\cdot-\xi)$ on ∂R to a fundamental solution $F(\cdot-\xi)$, i.e. to a distributional solution of $LF(\cdot-\xi)=\delta_0(\cdot)$. Clearly F is not unique, but just for the record let us quote a fundamental solution F (for L as in (3.9)) from [39] or [29]:

$$\begin{aligned} F(x,y) = \quad & -\tfrac{1}{16\pi\sqrt{2}}\Big[(x^2+y^2)\log(x^4+y^4)+2\sqrt{2}xy\log\Big(\tfrac{x^2+y^2+\sqrt{2}xy}{x^2+y^2-\sqrt{2}xy}\Big)+ \\ & \qquad\qquad + 2\sqrt{2}\Big(x^2\arctan\tfrac{x^2}{y^2}+y^2\arctan\tfrac{y^2}{x^2}\Big)\Big] \end{aligned} \tag{3.10}$$

An explicit calculation of the Green function even on a quarter plane seems to be beyond reach.

4. Eigenfunctions and eigenvalues.

4.1. Eigenfunctions for a hinged rectangular grid. The eigenfunctions for the *hinged beam*

$$\begin{cases} \varphi_{xxxx}=\lambda\varphi & \text{in } (0,1), \\ \varphi=\varphi_{xx}=0 & \text{in } \{0,1\}, \end{cases} \tag{4.1}$$

are obviously given by $\phi_i(x)=\sqrt{2}\sin(i\pi x)$ and the eigenvalues are $\lambda_i=i^4\pi^4$.

If a *hinged grid* is rectangular and *aligned with the cartesian coordinates*, then a calculation shows that the eigenfunctions and eigenvalues of

$$\begin{cases} \Phi_{xxxx}+\Phi_{yyyy}=\Lambda\Phi & \text{in } R, \\ \Phi=\Phi_{xx}=0 & \text{on } \{0,a\}\times[0,b], \\ \Phi=\Phi_{yy}=0 & \text{on } [0,a]\times\{0,b\}. \end{cases} \tag{4.2}$$

are given by

$$\Phi_{ij}(x,y)=\frac{2}{\sqrt{ab}}\sin\left(\frac{i\pi x}{a}\right)\sin\left(\frac{j\pi y}{b}\right) \quad \text{and} \quad \Lambda_{ij}=\frac{i^4\pi^4}{a^4}+\frac{j^4\pi^4}{b^4}. \tag{4.3}$$

For $i=j=1$ one finds:

LEMMA 4.1. *The first eigenfunction for (4.2), the hinged rectangular grid with aligned fibers, is of fixed sign.*

Even if the *hinged grid* is *diagonally aligned* we can determine the eigenfunctions and eigenvalues of

$$\begin{cases} \frac{1}{2}\Phi_{xxxx}+3\Phi_{xxyy}+\frac{1}{2}\Phi_{yyyy}=\tilde\Lambda\Phi & \text{in } R, \\ \Phi=\Phi_{xx}=0 & \text{on } \{0,a\}\times[0,b], \\ \Phi=\Phi_{yy}=0 & \text{on } [0,a]\times\{0,b\}. \end{cases} \tag{4.4}$$

by a separation of variables. In fact the eigenfunctions are still given by

$$\Phi_{ij}(x,y)=\frac{2}{\sqrt{ab}}\sin\left(\frac{i\pi x}{a}\right)\sin\left(\frac{j\pi y}{b}\right),$$

but now the eigenvalues are given by

$$2\tilde{\Lambda}_{ij} = \pi^4 \left(\frac{i^4}{a^4} + \frac{6i^2j^2}{a^2b^2} + \frac{j^4}{b^4} \right) . \tag{4.5}$$

We may conclude as before:

LEMMA 4.2. *The first eigenfunction for (4.4), the hinged rectangular grid with diagonal fibers, is fixed sign.*

Notice that

$$\tfrac{1}{2}\Lambda_{ij} \leq \tilde{\Lambda}_{ij} \leq 2\Lambda_{ij}, \tag{4.6}$$

$$\tilde{\Lambda}_{ij} = \frac{\pi^4}{2} \left(\frac{i^4}{a^4} + \frac{6i^2j^2}{a^2b^2} + \frac{j^4}{b^4} \right) \text{ and } \Lambda_{ij} = \pi^4 \left(\frac{i^4}{a^4} + \frac{j^4}{b^4} \right)$$

Notice also that the first eigenfunction is of fixed sign.

4.2. Eigenfunctions for clamped problems.

4.2.1. Eigenfunctions for the clamped beam. The set of all normalized eigenfunctions for

$$\begin{cases} \varphi'''' = \lambda\varphi & \text{in } (0,1)\,, \\ \varphi(0) = \varphi'(0) = 0 = \varphi(1) = \varphi'(1) \end{cases} \tag{4.7}$$

forms a complete orthonormal system in $L^2\,(0,1)\,.$

LEMMA 4.3. *These eigenfunctions and eigenvalues are*

$$\varphi_i\,(x) = \beta_i\, \cosh\nu_i \left(\frac{\cosh\,(\nu_i x) - \cos\,(\nu_i x)}{\cosh\nu_i - \cos\nu_i} - \frac{\sinh\,(\nu_i x) - \sin\,(\nu_i x)}{\sinh\nu_i - \sin\nu_i} \right) \quad \textit{and} \quad \lambda_i = \nu_i^4,$$

with $i = 1,2,\ldots$ *where* ν_i *is the* i^{th} *positive zero of* $\cos\nu - \frac{1}{\cosh\nu} = 0$ *and* β_i *is the normalization factor such that* $\int_0^1 \varphi_i\,(x)^2\,dx = 1$.

Note that the first eigenfunction is of fixed sign.

The statement of this Lemma is shown by a lengthy but straightforward calculation.

LEMMA 4.4. *The sequences* ν_i *and* β_i *as above have the following asymptotics*

- $\lim_{i\to\infty} i\pi - \nu_i = \frac{1}{2}\pi$ *and hence* $\lambda_i \approx (i-1/2)^4\,\pi^4$;
- $\lim_{i\to\infty} \beta_i = 1$.

Proof. For obvious reasons two subsequent zeroes ν_i and ν_{i+1} of $\cos\nu - \frac{1}{\cosh\nu} = 0$ are in the interval $((i-\frac{1}{2})\pi, (i+\frac{1}{2})\pi)$ and close to its boundaries. Since $\frac{1}{\cosh\nu_i} \leq 2e^{-\nu_i}$ and $|\sin x| > \frac{1}{2}$ in a sufficiently small neighborhood of $(i-\frac{1}{2})\pi$ we have

$$|\nu_i - (i-\tfrac{1}{2})\pi| < 4\; e^{\pi/2}\; e^{-i\pi}.$$

This proves the first statement, and the following table illustrates it:

λ_i :	500.56390	3803.5370	14617.630	39943.799	89135.406	173881.31	308208.45
$\left(i-\frac{1}{2}\right)^4\pi^4$:	493.13352	3805.0426	14617.451	39943.815	89135.406	173881.31	308208.45

TABLE 4.1
Comparison of the of the eigenvalues λ_i *and the approximation in Lemma 4.4.*

Let us now turn to the second statement. With the help of mathematica one sees that

$$\beta_i = \frac{-Z_i(\cosh\nu_i)^2}{4\,\nu_i\ (\cos\nu_i - \cosh\nu_i)^2\ (\sin\nu_i - \sinh\nu_i)^2}$$

with

$$\begin{aligned} Z_i = \quad & 2\,\nu_i\ \cos 2\,\nu_i + 4\ \cosh\nu_i\ \sin\nu_i - \sin 2\,\nu_i - \cosh 2\,\nu_i\ (2\,\nu_i + \sin 2\,\nu_i) \\ & -4\ \cos\nu_i\ \sinh\nu_i + 8\,\nu_i\ \sin\nu_i\ \sinh\nu_i + \sinh 2\,\nu_i + \cos 2\,\nu_i\ \sinh 2\,\nu_i \end{aligned}$$

Now the second statement follows by a straightforward computation. □

4.2.2. Comparing eigenvalues of clamped plates and grids. In [33] Philippin, following ideas of Hersch [20], obtained estimates for the eigenvalues of a clamped plate through the ones for clamped rectangular and diagonal grid. Let us state a special result in this direction that compares the first eigenvalues of a clamped grid

$$\begin{cases} \Phi_{xxxx} + \Phi_{yyyy} = \Gamma\Phi & \text{in } \Omega, \\ \Phi = |\nabla\Phi| = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.8}$$

with those of the clamped plate:

$$\begin{cases} \Delta^2\Phi = \Upsilon\Phi & \text{in } \Omega, \\ \Phi = |\nabla\Phi| = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.9}$$

LEMMA 4.5. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a $C^{0,1}$-boundary. Let Γ_1 and Υ_1 be the first eigenvalues of (4.8), respectively (4.9). Then it holds that $\frac{1}{2}\Upsilon_1 \le \Gamma_1 \le \Upsilon_1$.*

Proof. The result follows from the definition of the eigenvalue by Rayleigh's quotient and some energy estimates. For the first inequality one uses

$$\tfrac{1}{4}\int_\Omega (\Delta u)^2 dx\,dy = \tfrac{1}{4}\int_\Omega \left(u_{xx}^2 + 2u_{xx}u_{yy} + u_{yy}^2\right) dx\,dy \le \tfrac{1}{2}\int_\Omega \left(u_{xx}^2 + u_{yy}^2\right) dx\,dy.$$

For the second one proceeds via an integration by part that shows, due to the clamped boundary conditions,

$$\begin{aligned} \int_\Omega u_{xx}u_{yy}\ dx\,dy &= \int_{\partial\Omega} (u_x u_{yy} n_1 - u_x u_{xy} n_2)\, d\sigma + \int_\Omega u_{xy}^2\ dx\,dy \\ &= \int_\Omega u_{xy}^2\ dx\,dy \ge 0, \end{aligned}$$

and hence

$$\tfrac{1}{2}\int_\Omega \left(u_{xx}^2 + u_{yy}^2\right)\ dx\,dy \le \tfrac{1}{2}\int_\Omega \left(u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2\right)\ dx\,dy = \tfrac{1}{2}\int_\Omega (\Delta u)^2\, dx\,dy.$$

This completes the proof. □

4.2.3. Eigenfunctions for the clamped rectangular grid. A complete orthonormal system of eigenfunctions and eigenvalues for the grid *aligned with the cartesian coordinates*

$$\begin{cases} \Phi_{xxxx} + \Phi_{yyyy} = \Gamma\Phi & \text{in } R, \\ \Phi = |\nabla\Phi| = 0 & \text{on } \partial R. \end{cases} \tag{4.10}$$

with $R = (0, a) \times (0, b)$ is given in terms of the one-dimensional eigenfunctions and eigenvalues φ_j and λ_j from Lemma 4.3 by

$$\Phi_{ij}(x, y) = \frac{1}{\sqrt{ab}} \varphi_i \left(\frac{x}{a}\right) \varphi_j \left(\frac{y}{b}\right) \quad \text{and} \quad \Gamma_{ij} = a^{-4}\lambda_i + b^{-4}\lambda_j. \tag{4.11}$$

LEMMA 4.6. *The first eigenfunction for (4.10), the clamped rectangular grid with aligned fibers, is of fixed sign.*

This is in marked contrast to the biharmonic operator, whose first eigenfunction under Dirichlet conditions on a rectangle is known to change sign infinitely often (see [8]), and positivity of the ground state for our anisotropic operator cannot be expected for a general domain.

Γ_{ij} :	1001.13	4304.10	15118.2	40444.4	89636.0
		7607.07	18421.2	43747.3	92938.9
			29235.3	54561.	103753.
				79887.	129079.
					178271.

TABLE 4.2
Numerical values for the eigenvalues Λ_{ij} with $i, j \leq 5$ of a clamped square grid of length 1 that is aligned with cartesian coordinates are (without repeating the multiple ones like $\Lambda_{1,2} = \Lambda_{2,1}$)

Υ_{ij} :	1294.93	5386.63
		11710.3

TABLE 4.3
Numerical eigenvalues for a clamped square plate of length 1. We used the values found in [15, p. 79] and scaled these.

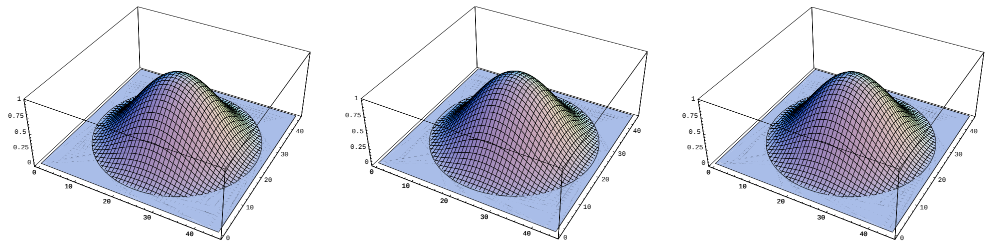
An explicit determination of all eigenfunctions and eigenvalues for the *diagonally aligned clamped grid*, however,

$$\begin{cases} \frac{1}{2}\Phi_{xxxx} + 3\Phi_{xxyy} + \frac{1}{2}\Phi_{yyyy} = \tilde{\Gamma}\Phi & \text{in } R, \\ \Phi = \Phi_x = 0 & \text{on } \{0, a\} \times [0, b], \\ \Phi = \Phi_y = 0 & \text{on } [0, a] \times \{0, b\}. \end{cases} \tag{4.12}$$

seems to be a nontrivial problem. From Lemma 4.5 we can find an estimate, namely $\Gamma_1 \leq 2\tilde{\Gamma}_1$, by using $\Gamma_1 \leq \Upsilon_1$ and $\frac{1}{2}\Upsilon_1 \leq \tilde{\Gamma}_1$, and similarly $\tilde{\Gamma}_1 \leq 2\Gamma_1$. This is consistent with inequality (4.6) for hinged grids. Note that the estimate $\frac{1}{2}\tilde{\Gamma}_1 \leq \Gamma_1 \leq 2\tilde{\Gamma}_1$ even holds on general domains.

REMARK 4.1. *We do not know if the first eigenfunction for (4.12), the clamped rectangular grid with diagonal fibers, is of fixed sign. Some evidence against a fixed sign follows from Coffman's result in [8].*

4.2.4. Eigenfunctions for the clamped circular grid. For a clamped circular plate there are radially symmetric eigenfunctions and these can be expressed in terms of the (modified) Bessel functions J_0 and I_0. Since Boggio [2] gave an explicit formula for the Dirichlet biharmonic on a circular disk Jentzsch' Theorem implies that the first eigenfunction is positive (of fixed sign) and unique and hence radially symmetric. Although a numerical approximation shows that the first eigenfunction



$L_1 = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}$ $\qquad$ $L_2 = \frac{1}{2}\frac{\partial^4}{\partial x^4} + 3\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{1}{2}\frac{\partial^4}{\partial y^4}$ $\qquad$ $L_3 = \Delta^2$

FIG. 4.1. *Numerical approximations of the first 'clamped' eigenfunctions on a disk for L_i, $i = 1, 2, 3$. One sees hardly any difference. We remark that the eigenfunctions for the first and the second operator differ 'analytically' just by a 45° rotation. The finite difference scheme however is different since in each case the discrete version of the corresponding operator L_i has been used.*

of the clamped grid looks similar to the one for the clamped plate this eigenfunction is not radially symmetric.

LEMMA 4.7. *Let D denote the unit disk. There is no radial eigenfunction for*

$$\begin{cases} \Phi_{xxxx} + \Phi_{yyyy} = \Gamma\Phi & \text{in } D, \\ \Phi = |\nabla\Phi| = 0 & \text{on } \partial D. \end{cases} \tag{4.13}$$

REMARK 4.2. *Of course, since the differential equation $u_{xxxx} + u_{yyyy} = \lambda u$ is not rotation invariant, this result should not come as a surprise. A nasty consequence however, is that the first eigenfunction does not seem to have an 'easy' explicit representation. We do not even have analytical proof that this eigenfunction has a fixed sign or that it is unique.*

REMARK 4.3. *The first eigenvalue $\lambda_{cp,1} \approx 104.363$ for the cicular clamped plate one may find in [1]. The first one for the clamped grid is approximately 75% of this value.*

Proof. [Proof of Lemma 4.7] Suppose that Φ is a radial eigenfunction. Then we can rotate this eigenfunction by $\pi/4$ and it is still an eigenfunction. However, the rotated Φ satisfies now, see (3.6),

$$\begin{cases} \frac{1}{2}\Phi_{xxxx} + 3\Phi_{xxyy} + \frac{1}{2}\Phi_{yyyy} = \Gamma\Phi & \text{in } D, \\ \Phi = |\nabla\Phi| = 0 & \text{on } \partial D. \end{cases} \tag{4.14}$$

Consequently we can add (4.13) to (4.14) to arrive at $\frac{3}{2}\Phi_{xxxx} + 3\Phi_{xxyy} + \frac{3}{2}\Phi_{yyyy} = 2\Gamma\Phi$ or

$$\begin{cases} \Delta^2\ \Phi = \frac{4}{3}\Gamma\Phi & \text{in } D, \\ \Phi = |\nabla\Phi| = 0 & \text{on } \partial D. \end{cases} \tag{4.15}$$

But then Φ must be a radial eigenfunction of the plate equation, an unlikely coincidence. To show that this cannot be the case suppose that $\Phi(r)$ is such a radial function. Then

$$\Phi_{xx} = \Phi'' \tfrac{x^2}{r^2} + \Phi' \tfrac{y^2}{r^3} \text{ and } \Phi_{xx} = \Phi'' \tfrac{y^2}{r^2} + \Phi' \tfrac{x^2}{r^3}$$

and

$$\Phi_{xxxx} + \Phi_{yyyy} =$$
$$\Phi'''' \frac{r^4 - 2x^2y^2}{r^4} + \Phi''' \frac{12x^2y^2}{r^5} + \Phi'' \frac{3r^4 - 30x^2y^2}{r^6} + \Phi' \frac{-3r^4 + 30x^2y^2}{r^7} = \Gamma\Phi\ ,$$

or equivalently

$$2x^2y^2\left(-\Phi'''' + 6r^{-1}\Phi''' - 15r^{-2}\Phi'' + 15r^{-3}\Phi'\right) = r^4\Gamma\Phi - r^4\Phi'''' - 3r^2\Phi'' + 3r\Phi'.$$

But this implies that either x^2y^2 is a function of r, or both sides are identical 0. So we have to show that this second case cannot occur. Suppose both sides are identical zero. The general solution of

$$-\Phi'''' + 6r^{-1}\Phi''' - 15r^{-2}\Phi'' + 15r^{-3}\Phi' = 0$$

is a linear combination of r^{ν_i} with four distinct numbers $\nu_i \in \{0,\ 2.32219,\ 1.83891 \pm 1.75438i\}$. There is no way that such a combination will make the right hand side identically zero, a contradiction. □

5. Positivity questions. From the Krein-Rutman theorem one knows that for a regular elliptic problem strong positivity of the solution operator implies that the first eigenfunction has multiplicity one and moreover is of fixed sign. If the solution operator has an integral kernel one may even use a much earlier result of Jentzsch [21]. Let us be more precise and consider:

$$\begin{cases} Lu = \lambda u + f & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

If the solution operator $(L-\lambda)_B^{-1} : X \mapsto X$ for (5.1) in the Banach lattice X is compact, positive and irreducible for some λ_0, then there exists an eigenvalue $\lambda_1 \in (\lambda_0, \infty)$ with a positive eigenfunction. For a precise statement see [6]. Moreover, for all $\lambda \in [\lambda_0, \lambda_1)$ and $f \in X$ one finds that there is a solution $u_\lambda \in X$ and

$$f > 0 \text{ implies } u_\lambda > 0.$$

5.1. Known results for plates. Let us recall some of the known positivity preserving results for plates.

For the *hinged plate*

$$\begin{cases} \Delta^2 u = \lambda u + f & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.2}$$

this question was studied in [23] on a general bounded domain Ω. The problem is positivity preserving if $\lambda \in [-\lambda_c(\Omega), \lambda_1(\Omega)^2)$. Here λ_c is a critical number which is bounded above by $\lambda_1(\Omega)\lambda_2(\Omega)$ and $\lambda_i(\Omega)$ are the eigenvalues of the Laplacian operator under Dirichlet conditions. If Ω is a rectangle R with sides a and $b < a$ one calculates easily $\lambda_1(R) = \pi^2(a^2+b^2)$ and $\lambda_2(R) = \pi^2(a^2+4b^2)$, so that (5.2) with $\Omega = R$ is positivity preserving for

$$-\pi^4(a^2+b^2)(a^2+4b^2) \le -\lambda_c(R) \le \lambda < \pi^4(a^2+b^2)^2. \tag{5.3}$$

The *clamped plate*

$$\begin{cases} \Delta^2 u = \lambda u + f & \text{in } \Omega, \\ u = |\nabla u| = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.4}$$

is a more delicate problem. In general (5.4) is not positivity preserving for $\lambda = 0$, see [13] or [36]. The boundary value problem in (5.4) is positivity preserving only in the case of some special domains Ω:

- If Ω is a ball or a disk Boggio's explicit formula for the solution operator with $\lambda = 0$ implies positivity.
- For small perturbations of the disk positivity has been shown in [19].
- For Ω some limaçons positivity can be found in [11].
- For a combination of the above results with Möbius transformations see [12].

5.2. Positivity under hinged boundary conditions.

5.2.1. Hinged beam. For the *hinged beam*

$$\begin{cases} u_{xxxx} = \lambda u + f & \text{in } (0,1), \\ u = u_{xx} = 0 & \text{in } \{0,1\}, \end{cases} \tag{5.5}$$

the boundary value problem (5.5) is positivity preserving, provided (see [23])

$$-950.884 \approx \lambda_c \leq \lambda < \pi^4 \approx 97.409. \tag{5.6}$$

Here the lower bound λ_c equals $4(\kappa_0)^4$ where κ_0 is the first positive zero of $\tan(x) + \tanh(x)$.

The Green function of the hinged beam. For the sake of completeness we list some facts about the Green function of the hinged beam problem (5.5). Set $\nu = \sqrt[4]{\lambda}$ and $\mu = \sqrt[4]{-\frac{1}{4}\lambda}$

$$\phi(\lambda; x) = \begin{cases} \frac{\sinh \nu x - \sin \nu x}{\nu^3} & \text{if } \lambda > 0 \\ \frac{1}{3} x^3 & \text{if } \lambda = 0 \\ \frac{\cosh(\mu x) \sin \mu x - \cos(\mu x) \sinh \mu x}{2\mu^3} & \text{if } \lambda < 0 \end{cases}$$

$$\psi(\lambda; x) = \begin{cases} \frac{\sinh \nu x + \sin \nu x}{2\nu} & \text{if } \lambda > 0 \\ x & \text{if } \lambda = 0 \\ \frac{\cosh(\mu x) \sin \mu x + \cos(\mu x) \sinh \mu x}{2\mu} & \text{if } \lambda < 0 \end{cases}$$

With $g_\lambda(x,y)$ as follows

$$g_\lambda(x,y) = \begin{cases} \begin{array}{l} \alpha_\lambda\ \phi(\lambda;x)\,\phi(\lambda;1-y) + \beta_\lambda\ \psi(\lambda;x)\,\phi(\lambda;1-y) + \\ +\ \beta_\lambda\ \phi(\lambda;x)\,\phi'(\lambda;1-y) + \gamma_\lambda\ \psi(\lambda;x)\,\psi(\lambda;1-y) \end{array} & \text{if } 0 \leq x \leq y \leq 1, \\ \begin{array}{l} \alpha_\lambda\ \phi(\lambda;y)\,\phi(\lambda;1-x) + \beta_\lambda\ \psi(\lambda;y)\,\phi(\lambda;1-x) + \\ +\ \beta_\lambda\ \phi(\lambda;y)\,\psi(\lambda;1-x) + \gamma_\lambda\ \psi(\lambda;y)\,\psi(\lambda;1-x) \end{array} & \text{if } 0 \leq y < x \leq 1. \end{cases} \tag{5.7}$$

with appropriate constants to accommodate the boundary values in 1 and the continuity requirements of g_λ. Some tedious calculations lead to the coefficients in the following table.

For $\lambda \geq 0$ formula (5.7) can be simplified to

$$\begin{array}{ll} g_\lambda(x,y) = \frac{\sin(x\,\nu)\,\sin(\nu(1-y))}{2\,\nu^3 \sin \nu} - \frac{\sinh(x\,\nu)\,\sinh(\nu(1-y))}{2\,\nu^3 \sinh \nu} & \text{if } 0 \leq x \leq y \leq 1, \\ g_0(x,y) = \frac{1}{6} x\,(1-y) - \frac{1}{6} x\,(1-y)^3 - \frac{1}{6} x^3\,(1-y) & \text{if } 0 \leq x \leq y \leq 1. \end{array} \tag{5.8}$$

5.2.2. Hinged rectangular grid with aligned fibers. In this section it will be convenient to use (x_1, x_2) instead of (x, y). An investigation of positivity preserving properties for the *hinged rectangular grid* that is *aligned with the cartesian axes* seems to be difficult. The eigenfunctions are

$$\Phi_{ij}(x_1, x_2) = \frac{1}{\sqrt{ab}} \varphi_i\left(a^{-1} x_1\right) \varphi_j\left(b^{-1} x_2\right)$$

	$0<\lambda\neq\lambda_i$ set $\nu=\sqrt[4]{\lambda}$	$\lambda=0$	$\lambda<0$ set $\mu=\sqrt[4]{-\frac{1}{4}\lambda}$
α_λ	$\frac{\nu^3}{8}\left(\frac{1}{\sin\nu}-\frac{1}{\sinh\nu}\right)\ \frac{\nu}{2\sin\nu}$	0	$\mu^3\frac{\cos\mu\sinh\mu-\cosh\mu\sin\mu}{\cosh 2\mu-\cos 2\mu}$
β_λ	$-\frac{\nu}{4}\left(\frac{1}{\sin\nu}+\frac{1}{\sinh\nu}\right)$	$-\frac{1}{2}$	$-\mu\frac{\sinh\mu\cos\mu+\cosh\mu\sin\mu}{\cosh 2\mu-\cos 2\mu}$
γ_λ	$-\frac{1}{2\nu}\ \left(\frac{1}{\sin\nu}-\frac{1}{\sinh\nu}\right)$	$\frac{1}{6}$	$\frac{\cosh\mu\sin\mu-\cos\mu\sinh\mu}{\mu(\cosh 2\mu-\cos 2\mu)}$

TABLE 5.1
These values were obtained using Mathematica.

with $\varphi_i(t)=\sqrt{2}\sin(i\pi t)$. Recall that the first eigenfunction is of fixed sign and has multiplicity one. Using these eigenfunctions and the Green function g_λ from (5.7) or (5.8) the solution of

$$\begin{cases} \left(\frac{\partial^4}{\partial x_1^4}+\frac{\partial^4}{\partial x_2^4}\right)u=\lambda u+f & \text{in } R, \\ u=\Delta u=0 & \text{on } \partial R, \end{cases} \tag{5.9}$$

can be written as

$$u(x)=\sum_{i,j=1}^{\infty}\frac{1}{a^{-4}\lambda_i+b^{-4}\lambda_j-\lambda}\langle\Phi_{ij},f\rangle\,\Phi_{ij}(x)=$$
$$=\frac{1}{\sqrt{ab}}\sum_{j=1}^{\infty}\ \left\langle\varphi_j\left(\frac{\circ}{b}\right),\int_{s=0}^{a}g_{b^{-4}\lambda_j-\lambda}(x_1,s)\,f(s,\circ)\,ds\right\rangle_{(0,b)}\varphi_j\left(\frac{x_2}{b}\right).$$

An inspection of the series representation above suggests that for nonnegative and nontrivial f, for $\lambda<\Lambda_{11}$ and λ close to Λ_{11} the coefficient in front of Φ_{11} becomes very large and positive. This suggests that the first term in the series decides about the sign of u. But estimating the remainder of the series in terms of $\Phi_{1,1}$ turns out to be a hard technical problem.

In order to verify that problem (5.9) is positivity preserving at least for λ in some interval $[\Lambda_{11}-\gamma,\Lambda_{11})$ it suffices to show that the solution of (5.9) with $f=\delta_y$ is positive for every $y\in R$, where δ_y is the delta function at y.

Since the first eigenfunction is strictly positive in the interior we may prove the following result, in which we use some notation for a domain Ω:

- the ε-interior: $A_\varepsilon=\{x\in A;d(x,\partial\Omega)>\varepsilon\}$
- the ε-neighborhood: $A+B^\varepsilon=\{x\in\Omega;d(x,\partial A)<\varepsilon\}$

LEMMA 5.1. *Let u^λ be the solution of (5.9). For every $\varepsilon>0$ there is a positive $\gamma>0$ such that for $\lambda\in[\Lambda_{11}-\gamma,\Lambda_{11})$ and $f\geq 0$ the following two statements hold (here C denotes the set of corner points):*

- *if* support $f\in R_\varepsilon$, *then $u^\lambda(x)\geq 0$ for all $x\in R\setminus(C+B^\varepsilon)$,*
- *if* support $f\in R\setminus(C+B^\varepsilon)$, *then $u^\lambda(x)\geq 0$ for all $x\in R_\varepsilon$.*

Proof. It is sufficient to show such a result for $f=\delta_y$, the delta function, with $y\in R_\varepsilon$. Formally we have

$$\delta_y(\cdot)=\sum_{i,j=1}^{\infty}\Phi_{ij}(y)\,\Phi_{ij}(\cdot)$$

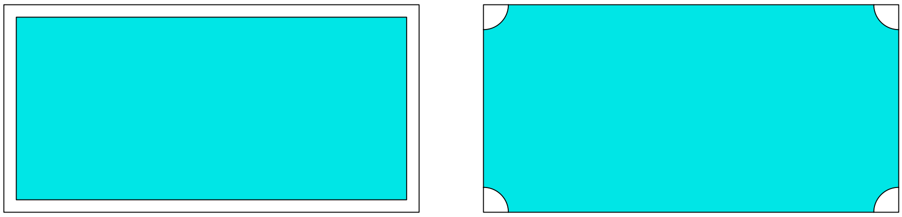


FIG. 5.1. *The sets R_ε and $R\setminus(C+B^\varepsilon)$ from Lemma 5.1*

but since $\delta_y\notin L^2(R)$ this series does not converge. The distributional solution $u^{y,\lambda}$ of (5.9) with $f=\delta_y$, that is

$$u^{y,\lambda}(\cdot)=\sum_{i,j=1}^{\infty}\frac{\Phi_{ij}(y)}{\Lambda_{ij}-\lambda}\Phi_{ij}(\cdot)\,, \tag{5.10}$$

lies in $L^2(R)$ since its coefficients are in ℓ_2 :

$$\sum_{i,j=1}^{\infty}\left(\frac{\Phi_{ij}(y)}{\Lambda_{ij}-\lambda}\right)^2\le\sum_{i,j=1}^{\infty}\left(\frac{1}{\pi^4\left(\frac{i^4}{a^4}+\frac{j^4}{b^4}\right)-\lambda}\right)^2<\infty.$$

We even find for $\alpha+\beta<3$ that

$$\left\{i^\alpha j^\beta\frac{\Phi_{ij}(y)}{\Lambda_{ij}-\lambda}\right\}\in\ell_2.$$

and hence $u^{y,\lambda}\in W^{3-t,2}(R)$ for all $t>0$.

We split $u^{y,\lambda}=u_1^{y,\lambda}+u_2^{y,\lambda}$ where

$$u_1^{y,\lambda}(\cdot)=\frac{\Phi_{11}(y)}{\Lambda_{11}-\lambda}\Phi_{11}(\cdot)\,.$$

By our assumption we have

$$|\Phi_{ij}(y)|\le\frac{c}{\varepsilon}\Phi_{11}(y)\ \text{ and }\ |\Phi_{ij}(x)|\le c\max(i,j)\,\Phi_{11}(x).$$

This implies that we find

$$\left|u_2^{y,\lambda}(x)\right|\le\sum_{\substack{i,j=1\\(i,j)\ne(1,1)}}^{\infty}\left|\frac{\Phi_{ij}(y)}{\Lambda_{ij}-\lambda}\Phi_{ij}(x)\right|\le$$

$$\le\frac{c^3}{\varepsilon^2}\Phi_{11}(x)\,\Phi_{11}(y)\sum_{\substack{i,j=1\\(i,j)\ne(1,1)}}^{\infty}\frac{\max(i,j)}{\pi^4\left(\frac{i^4}{a^4}+\frac{j^4}{b^4}\right)-\lambda}.$$

Since Λ_{12} and Λ_{21} are greater than Λ_{11}, a straightforward computation shows that the last sum is bounded uniformly with respect to $\lambda<\Lambda_{11}$ by a constant $\gamma=C(a,b)\,\varepsilon^{-2}$. For $\lambda\in[\Lambda_{11}-\gamma,\Lambda_{11})$ we find

$$\left|u_2^{y,\lambda}(x)\right|\le\gamma\Phi_{11}(x)\,\Phi_{11}(y)\le u_1^{y,\lambda}(x)$$

and hence that $u^{y,\lambda}(x) > 0$. □

With Proposition A.1 we may conclude that the follwing holds.

LEMMA 5.2. *For every $\varepsilon > 0$ there is a $\gamma > 0$ such that if $\lambda \in [\Lambda_{11} - \gamma, \Lambda_{11})$ and $f \geq 0$, then the solution of (5.9) satisfies $u^\lambda(x) \geq 0$ for all $x \in R_\varepsilon \cup (R \backslash (\mathrm{supp} f + B_\varepsilon))$.*

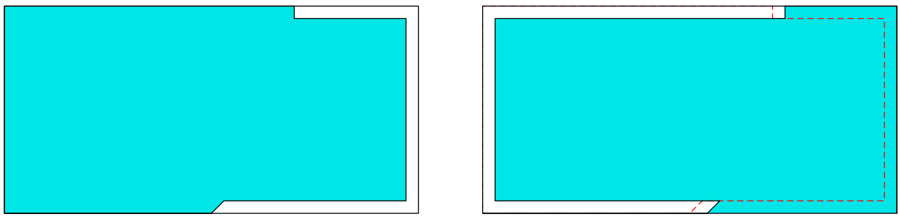


FIG. 5.2. *The sets* $\mathrm{supp} f$ *and* $R_\varepsilon \cup (R \setminus (\mathrm{supp} f + B_\varepsilon))$ *from Lemma 5.2*

Proof. If $\mathrm{supp} f \in R_\varepsilon$ then the previous lemma yields that $u^\lambda(x) \geq 0$ except near the corners C. By the Proposition A.3 and using duality we find that $\left\| u_2^\lambda \right\|_{W^{28,2}(C + B_{\varepsilon/2})} \leq c(\varepsilon) \left\| f \right\|_{W^{-4,2}(\Omega)}$. Let us denote by $d_h(x)$ and $d_v(x)$ the distance of $x \in R$ to the horizontal and vertical part of its boundary and by $\langle v, f \rangle$ the $L^2(R)$ product, when applicable. Then one continues with the imbedding $W_0^{4,2}(\Omega)$ in $C^2(\bar{\Omega}) \cap C_0^1(\bar{\Omega})$, through

$$\begin{aligned} \|f\|_{W^{-4,2}(\Omega)} &= \sup \left\{ \langle v, f \rangle ; v \in W_0^{4,2}(\Omega) \text{ with } \|v\|_{W^{4,2}(\Omega)} \leq 1 \right\} \leq \\ &\leq c \sup \left\{ \langle v, f \rangle ; v \in C^2(\bar{\Omega}) \cap C_0^1(\bar{\Omega}) \text{ with } \|v\|_{C^2(\bar{\Omega})} \leq 1 \right\} \leq \\ &\leq c \sup \left\{ \langle v, f \rangle ; |v(x)| \leq d_h(x) d_v(x) | \right\} \\ &\leq c' \langle \Phi_{11}, f \rangle . \end{aligned}$$

The last inequality is due to the fact that Φ_{11} can be bounded above and below by multiples of $d_h(x)d_v(x)$.

Similarly, again with an imbedding, we find for the function $u_2^\lambda \in C^2\bar{\Omega}) \cap C_0^1(\bar{\Omega})$ and for $x \in C + B_{\varepsilon/2}$ that

$$\begin{aligned} u_2^\lambda(x) &\leq c_1 \left\| u_2^\lambda \right\|_{C^2(C+B_{\varepsilon/2})} \Phi_{11}(x) \leq c_2 \left\| u_2^\lambda \right\|_{W^{28,2}(C+B_{\varepsilon/2})} \Phi_{11}(x) \leq \\ &\leq c(\varepsilon) \|f\|_{W^{-3,2}(\Omega)} \Phi_{11}(x) \leq c'(\varepsilon) \langle \Phi_{11}, f \rangle \Phi_{11}(x) \end{aligned}$$

Since $u_1^\lambda(x) = (\Lambda_{11} - \lambda)^{-1} \langle \Phi_{11}, f \rangle \Phi_{11}(x)$ we find $u^\lambda(x) > 0$ near the corners for $\Lambda_{11} - \lambda$ chosen sufficiently small. A similar proof does it for the remaining claim. □

Let us summarize our results in terms of positivity for the Green function $u^{y,\lambda}$ from (5.10) belonging to the *hinged rectangular grid*.

COROLLARY 5.3. *For every $\varepsilon > 0$ there is a $\gamma(\varepsilon) > 0$ such that $u^{y,\lambda}(x) \geq 0$ for all $x \in R$, all $y \in R_{2\varepsilon}$ and all $\lambda \in [\Lambda_{11} - \gamma(\varepsilon), \Lambda_{11})$.*

Proof. We approximate $\delta_y(\cdot)$ in $\mathcal{D}'(\Omega)$ by a sequence of smooth f_n with support in $B_\varepsilon(y)$ and note that the corresponding solutions $u_n^{y,\lambda}(x)$ of (5.9) are nonnegative for all $x \in R$ and all $y \in R_{2\varepsilon}$ due to Lemma 5.2. Then we send $n \to \infty$. Since f_n converges in $W^{-1,2}(R)$, the sequence u_n converges pointwise. □

Notice that when sending ε to zero, it is conceivable (although it seems unlikely) that $\gamma(\varepsilon) \to 0$. In that case, as $\varepsilon_n \to 0$, there exists sequences $\lambda_n < \Lambda_{11}$ with

$\lambda_n \to \Lambda_{11}$, $y_n \in R \setminus R_{2\varepsilon_n}$ with $y_n \to y_0 \in \partial R$ and $x_n \to x_0 \in R$ such that

$$z_n := u^{y_n,\lambda_n}(x_n) < 0 \quad \text{for all } n \in \mathbb{N} .$$

At present we are unable to derive a contradiction from this.

We wil end this section by a another nonuniform positivity result near Λ_{11} by using the fact that the projection on the first eigenfunction will dominate near Λ_{11}. We proceed as for the non-uniform version of the anti-maximum principle in [7] to obtain the following nonuniform result.

PROPOSITION 5.4. *For all $f \in L^2(R)$ with $f \geq 0$ there exists $\lambda_f < \Lambda_{11}$ such that for $\lambda \in [\lambda_f, \Lambda_{11})$ the solution u_λ of (5.9) satisfies $u_\lambda \geq 0$.*

Proof. We will adjust the arguments in [7] for the present situation. Let $\mathcal{L}$: $W^{4,2}(R) \cap W_0^{2,2}(R) \to L^2(R)$ be the operator that corresponds to (5.9). Fix P_0 to be the projection on the first eigenfunction, that is, $P_0 f = \langle \Phi_{11}, f \rangle_R \ \Phi_{11}$ and set $\tilde{\Lambda} \in (\Lambda_{11}, \min(\Lambda_{12}, \Lambda_{21}))$. Then using our regularity result for (5.9) as in [7], we find that there exists a constant C such that for all $\lambda \in \left[0, \tilde{\Lambda}\right]$ the following holds

$$\left\| (\mathcal{L} - \lambda)^{-1} (I - P_0) f \right\|_{W^{4,2}(R)} \leq C \|f\|_{L^2(R)} .$$

Since the domain R satisfies a uniform interior cone condition we find by [18, Theorem 7.26] that $W^{4,2}(R)$ is imbedded in $C^{2,\alpha}(\bar{\Omega})$ for any $\alpha \in (0,1)$. Since

$$(\mathcal{L} - \lambda)^{-1} (I - P_0) f \in W_0^{2,2}(\Omega)$$

we find that $u \in C_0(\bar{\Omega})$ and hence that

$$\left\| \frac{(\mathcal{L} - \lambda)^{-1} (I - P_0) f}{\Phi_{11}} \right\|_\infty \leq C' \left\| (\mathcal{L} - \lambda)^{-1} (I - P_0) f \right\|_{W^{4,2}(R)} .$$

The solution u_λ of (5.9) can be written as

$$\begin{aligned} u_\lambda(x) &= \frac{\langle \Phi_{11}, f \rangle_R}{\Lambda_{11} - \lambda} \Phi_{11}(x) + \left((\mathcal{L} - \lambda)^{-1} (I - P_0) f \right)(x) \\ &\geq \left(\frac{\langle \Phi_{11}, f \rangle_R}{\Lambda_{11} - \lambda} - C'' \|f\|_{L^2(R)} \right) \Phi_{11}(x) \end{aligned}$$

which is positive for $0 \leq \Lambda_{11} - \lambda$ sufficiently small. □

5.2.3. Hinged rectangular grid with diagonal fibers. The positivity question is much simpler to decide if the grid runs diagonally. For the *diagonally hinged grid on the rectangle R* as in (3.4)–(3.7),

$$\begin{cases} \frac{1}{2}u_{xxxx} + 3u_{xxyy} + \frac{1}{2}u_{yyyy} = \lambda u + f & \text{in } R, \\ u = u_{xx} = 0 & \text{on } \{0,a\} \times [0,b] , \\ u = u_{yy} = 0 & \text{on } [0,a] \times \{0,b\} . \end{cases} \tag{5.11}$$

one may decouple the fourth order equation (3.6) (or (5.11) with $\lambda = 0$) into a system of two second order equations by using (3.7).

Since the boundary conditions decouple nicely with the two second order operators, one may use the substitution $v := -u_{xx} - (3 + 2\sqrt{2})u_{yy}$ and two iterations of the standard maximum principle for second order differential operators to find that (5.11) is positivity preserving for $\lambda = 0$.

Going back to the fourth order problem one has a strongly positive and compact solution operator that maps $f \in C(\overline{\Omega})$ to $u \in C(\overline{\Omega})$. From Krein-Rutman's Theorem one finds that there exists a first eigenvalue and this eigenvalue corresponds to an eigenfunction of fixed sign. But then one can show the following as in [35].

PROPOSITION 5.5. *For* $\lambda \in \left[0, \pi^4 \left(\frac{1}{2}a^{-4} + 3a^{-2}b^{-2} + \frac{1}{2}b^{-4}\right)\right)$ *the problem (5.11) is positivity preserving.*

The upper bound for λ is the first eigenvalue Γ_{11} given in (4.5).

5.2.4. Numerical comparison for hinged rectangles. For the hinged rectangular plate and grids we obtained the following numerical result by a finite difference method.

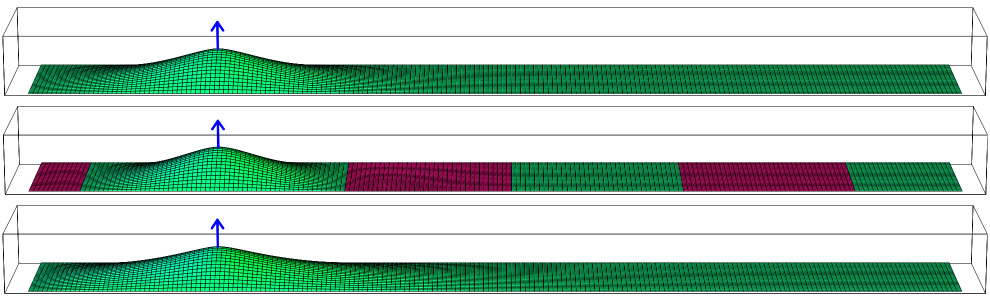


FIG. 5.3. *A hinged plate, a hinged grid with rectangular fibers and a hinged grid with diagonal fibers. The arrow denotes the location of the pointed force and the red (dark) part represents the part of the grid with a negative deviation.*

5.3. Positivity under clamped boundary conditions.

5.3.1. Clamped beam. What can be said about positivity preservation for the *clamped beam* (5.12)?

$$\begin{cases} u_{xxxx} = \lambda u + f & \text{in } (0,1), \\ u = u_x = 0 & \text{in } \{0,1\}. \end{cases} \tag{5.12}$$

This requires more efforts. If λ is not an eigenvalue there exists a Green function g_λ for the clamped beam problem (5.12) such that the solution can be represented as

$$u(x) = \int_0^1 g_\lambda(x,y) f(y) \, dy.$$

Let us define

$$\phi(\lambda; x) = \begin{cases} \nu^{-3} \left(\sinh(\nu x) - \sin(\nu x)\right) & \text{if } \lambda > 0, \\ \frac{1}{3}x^3 & \text{if } \lambda = 0, \\ \frac{1}{2}\mu^{-3} \left(\cosh(\mu x)\sin(\mu x) - \sinh(\mu x)\cos(\mu x)\right) & \text{if } \lambda < 0. \end{cases} \tag{5.13}$$

where $\nu = \sqrt[4]{\lambda}$ and $\mu = \sqrt[4]{-\frac{1}{4}\lambda}$. The functions $\phi(\lambda; \cdot)$ and $\frac{\partial}{\partial x}\phi(\lambda; \cdot)$ are two linearly independent solutions of the differential equation and the boundary conditions of

(5.12) in the left end point 0. By the definition of the Green function it follows that

$$g_\lambda(x,y)=\begin{cases}\begin{array}{l}\alpha_\lambda\ \phi(\lambda;x)\,\phi(\lambda;1-y)+\beta_\lambda\ \phi'(\lambda;x)\,\phi(\lambda;1-y)+\\ +\ \beta_\lambda\ \phi(\lambda;x)\,\phi'(\lambda;1-y)+\gamma_\lambda\ \phi'(\lambda;x)\,\phi'(\lambda;1-y)\end{array} & \text{if } 0\le x\le y\le 1,\\[2ex] \begin{array}{l}\alpha_\lambda\ \phi(\lambda;y)\,\phi(\lambda;1-x)+\beta_\lambda\ \phi'(\lambda;y)\,\phi(\lambda;1-x)+\\ +\ \beta_\lambda\ \phi(\lambda;y)\,\phi'(\lambda;1-x)+\gamma_\lambda\ \phi'(\lambda;y)\,\phi'(\lambda;1-x)\end{array} & \text{if } 0\le y< x\le 1.\end{cases}\tag{5.14}$$

with appropriate constants to accommodate the boundary values in 1 and the continuity requirements of g_λ. Some tedious calculations lead to the coefficients in the following table.

	$0<\lambda\neq\lambda_i$ set $\nu=\sqrt[4]{\lambda}$	$\lambda=0$	$\lambda<0$ set $\mu=\sqrt[4]{-\frac{1}{4}\lambda}$
α_λ	$\frac{\nu^3(\sinh\nu+\sin\nu)}{4-4\cosh\nu\cos\nu}$	3	$\frac{2\mu^3(\cos\mu\sinh\mu+\cosh\mu\sin\mu)}{\cosh 2\mu+\cos 2\mu-2}$
β_λ	$\frac{\nu^2(\cos\nu-\cosh\nu)}{4-4\cosh\nu\cos\nu}$	$-\frac{3}{2}$	$\frac{-2\mu^2\sinh\mu\sin\mu}{\cosh 2\mu+\cos 2\mu-2}$
γ_λ	$\frac{\nu(\sinh\nu-\sin\nu)}{4-4\cosh\nu\cos\nu}$	$\frac{1}{2}$	$\frac{\mu(\cosh\mu\sin\mu-\cos\mu\sinh\mu)}{\cosh 2\mu+\cos 2\mu-2}$

TABLE 5.2
These values were obtained using Mathematica.

For $\lambda=0$ formula (5.14) can be simplified to

$$g_0(x,y)=\begin{cases}\frac{1}{2}x^2(1-y)^2\left(y-x+\frac{2}{3}x(1-y)\right) & \text{if } 0\le x\le y\le 1,\\ \frac{1}{2}y^2(1-x)^2\left(x-y+\frac{2}{3}y(1-x)\right) & \text{if } 0\le y< x\le 1.\end{cases}$$

Problem (5.12) is positivity preserving if and only if the Green function is positive and for g_0 this is now easily seen to be the case. Instead of directly computing for which λ the Green function g_λ is in fact positive one may proceed through the results of Schröder in [34]. The Green function changes sign for some λ if and only if this λ is an eigenvalue of either (4.7) or of

$$\begin{cases}\varphi''''=\lambda\varphi \text{ in } (0,1),\\ \varphi(0)=\varphi'(0)=\varphi'''(0)=0=\varphi(1).\end{cases}\tag{5.15}$$

The 'first' solution of (5.15) is $g_\lambda(x,1)$ with $\lambda_c=-4\nu_0^4$ where ν_0 is the first positive zero of $\tanh\nu=\tan\nu$.

LEMMA 5.6. *Problem (5.12) is positivity preserving if and only if $\lambda\in[\lambda_c,\lambda_1)$ where*

- *λ_1 is the first eigenvalue of (4.7), that is, the fourth power of the first positive solution of*

$$\cos\lambda=\frac{1}{\cosh\lambda},$$

- *λ_c is the 'first' eigenvalue of (5.15), that is, the first negative solution of*

$$\tan\sqrt[4]{-\tfrac{1}{4}\lambda}=\tanh\sqrt[4]{-\tfrac{1}{4}\lambda}\tag{5.16}$$

The numerical approximations are $\lambda_1 \approx 4.7300$ and $\lambda_c \approx -950.884$. Notice that this is the same λ_c as in (5.6) for Problem (5.5).

Proof. The arguments are similar to the ones in [23] and reflect the ideas from [34].

Direct inspection shows that g_0 is strictly positive. To study the case of positive λ, notice that (5.12 can be rewritten as $(I - \lambda L^{-1})u = f$, where $Lu = u_{xxxx}$, so that by a Neumann series argument $u = \sum_{k=1}^{\infty} (\lambda L^{-1})^k f$ converges and is positive for all $\lambda \in [0, \lambda_1)$. For $\lambda = \lambda_1$ no solution exists when $f = \varphi_1$. For $\lambda > \lambda_1$ and $f = \varphi_1$ the solution is $u = (\lambda_1 - \lambda)^{-1} \varphi_1$ and this is negative.

For $\lambda < 0$ one finds from (5.14)-(5.13) and the coefficients in Table 5.1 that $\lambda \mapsto g_\lambda(x, y)$ is continuous for $\lambda \leq 0$ in almost every sense. Let $\lambda_c < 0$ be the first number after which positivity fails. Suppose that for a fixed $y \in (0, 1)$ the value of $g_{\lambda_c}(x, y)$ is nonnegative but equals 0 for some $x_y \in (0, 1)$. And suppose w.l.o.g. that $x_y \leq y$. Then $g_{\lambda_c}(x_y, y) = \frac{\partial}{\partial x} g_{\lambda_c}(x_y, y) = \frac{\partial}{\partial x} g_{\lambda_c}(0, y) = \frac{\partial}{\partial x} g_{\lambda_c}(0, y) = 0$ and we have found an eigenfunction scaled to $[0, x_y]$, a contradiction. Hence $x_y = 0$. Using the symmetry $g_\lambda(x, y) = g_\lambda(y, x)$ we may assume that y is at the boundary, say $y = 1$. We may repeat the argument above for $\tilde{g}_{\lambda_c}$ defined by $\tilde{g}_\lambda(x) = \lim_{y \uparrow 1} (1 - y)^{-2} g_\lambda(x, y)$ which is a nontrivial function. Again if $\tilde{g}(x_1) = 0$ for some $x_1 \in (0, 1)$ we find an eigenfunction by scaling on $[0, x_1]$. Since $\tilde{g}'(1) < 0 = \tilde{g}(1)$ it remains that $x_1 = 0$. One finds that $\tilde{g}$ is an eigenfunction of (5.15). The first eigenfunction of that eigenvalue problem is

$$\psi_1(x) = \cosh(\mu x) \sin(\mu x) - \sinh(\mu x) \cos(\mu x)$$

with μ the first positive root of $\cosh \mu \sin \mu = \sinh \mu \cos \mu$ and $\lambda_c = 4\mu^2$. This can be rephrased to (5.16). For $\lambda < \lambda_c$ one finds that $\tilde{g}_\lambda$ is sign changing implying that for y near 1 the function $x \mapsto g_\lambda(x, y)$ is sign changing. □

5.3.2. Clamped rectangular grid with aligned fibers. In this section we investigate the problem

$$\begin{cases} u_{xxxx} + u_{yyyy} = \lambda u + f & \text{in } R, \\ u = |\nabla u| = 0 & \text{on } \partial R. \end{cases} \tag{5.17}$$

Numerical calculations suggest that for $\lambda = 0$ a point load $f = \delta_y(\cdot)$ can lead to a sign changing solution, see Figure 5.4 in which the sign of u is colour coded. This behaviour is also known and recorded in [9] for isotropic rectangular plates, whose deformation solves $\Delta^2 u = f$ instead.

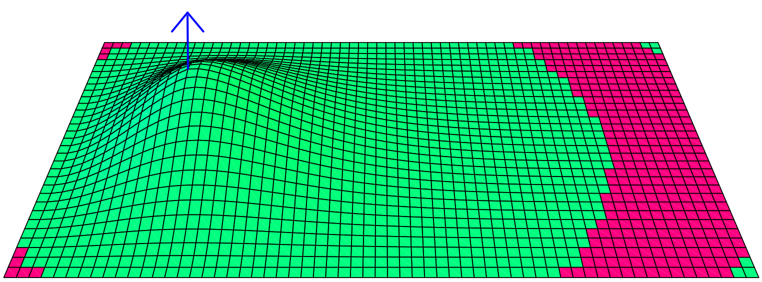


FIG. 5.4. *Numerical simulation of a clamped rectangularly aligned grid; (5.17) with $\mu = 0$ and a point source f.*

However, since the first eigenfunction is positive, by using the eigenfunction expansion one finds the following solution formula for (5.17):

$$u(x,y) = \sum_{i,j=1}^{\infty} \frac{1}{\Lambda_{ij} - \lambda} \langle \Phi_{ij}, f \rangle_R \; \Phi_{ij}(x,y).$$

As for the hinged plate one might hope that for λ near Λ_{11} the projection on the first eigenfunction will dominate the sign. But to find such a result we would need a C^4 estimate near corner points which, unfortunately, we do not have at our disposal.

5.3.3. Clamped diagonal grid. Since we do not know if the first eigenfunction is of fixed sign for this grid we can only give some numerical evidence. With the same point source and domain as in rectangularly aligned grid from Figure 5.4 the area where the solutions changes sign seems to be much smaller for the diagonally aligned grid.

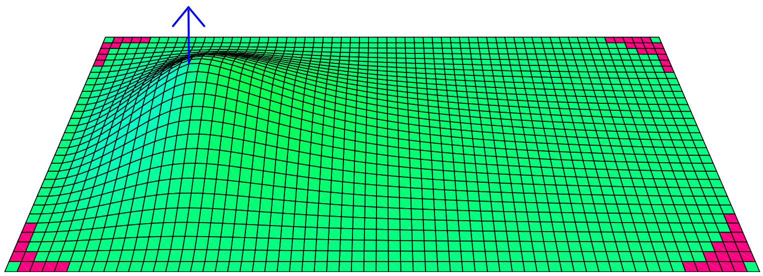


FIG. 5.5. *Numerical simulation of a clamped diagonally aligned grid.*

5.3.4. Numerical comparison for clamped rectangles. Duffin's famous counterexample in [13] for the conjecture of Boggio-Hadamard (the clamped plate problem on convex domains is positivity preserving) uses a long thin rectangle. Here we present numerical results for long clamped rectangular plate and grids. Rather surprisingly the numerical result for long thin rectangle with a diagonal fabric hardly shows any sign change.

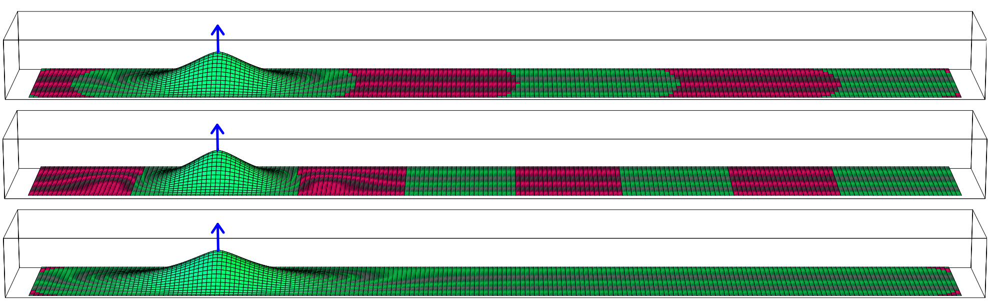


FIG. 5.6. *A clamped plate, a clamped grid with rectangular fibers, and a clamped grid with diagonal fibers. The arrow denotes the location of the pointed force and the red (dark) part represents the part of the grid with a negative deviation.*

The numerical illustrations have been obtained using a finite difference method.

6. Summary for rectangular grids. We set out to study positivity for rectangular grids with aligned and with diagonal fibers. An overview of the results we obtained for those problems can be found in Table 6.1. For the sake of comparison we include the known results for the rectangular plate.

		positive eigenfunction	positivity preserving
hinged	plate	$\Phi_1 > 0$	for $\lambda \in [0, \Lambda_1)$
	grid aligned with sides	$\Phi_1 > 0$	conditionally near Λ_1
	grid with diagonal fibers	$\Phi_1 > 0$	for $\lambda \in [0, \Lambda_1)$
clamped	plate	Φ_1 changes sign	no
	grid aligned with sides	$\Phi_1 > 0$	conditionally near Λ_1
	grid with diagonal fibers	?	?

TABLE 6.1
Overview for rectangular plates and grids

Numerics for the clamped plate with diagonal fabric suggest that the first question mark in the table above should be answered affirmatively; the second question mark might have a positive answer for λ near Λ_1. Of course 'near Λ_1' always means in a left neighbourhood of Λ_1.

Appendix A. Nonlocal smoothness.

The standard regularity statement for $2m-$th order elliptic problems is usually a statement of the form $f \in W^{k,p}(\Omega)$ implies $u \in W^{k+2m,p}(\Omega)$ or $f \in C^{k,\gamma}(\bar{\Omega})$ implies $u \in C^{k+2m,\gamma}(\bar{\Omega})$. Such a maximal regularity result is optimal. However, for a function $f \in L^p(\Omega)$ which has its support in $\Omega' \subset \Omega$ one may show that the corresponding solution is smooth outside of Ω'. Although this result is well-known we are not aware of any reference. So allow us to formulate a corresponding statement.

Consider a regular elliptic problem with L of order $2m$ and Ω a domain in $\mathbb{R}^n$:

$$\begin{cases} Lu = f & \text{in } \Omega, \\ B_i u = 0 & \text{on } \partial\Omega \text{ for } i = 0, \ldots, m. \end{cases} \tag{A.1}$$

PROPOSITION A.1. *Let Ω_1, Ω_2 be two disjoint subdomains of Ω having a positive distance r, that is, $r := \inf\{|x-y|\,; x \in \Omega_1, y \in \Omega_2\} > 0$. Suppose that there exists $c > 0$ such that for all $k \in \{0, \ldots, \kappa\}$ and all $f \in W^{k,2}(\Omega)$ there is a solution $u \in W^{2m+k,2}(\Omega)$ of (A.1) with*

$$\|u\|_{W^{2m+k,2}(\Omega)} \leq c\,\|f\|_{W^{k,2}(\Omega)}\,, \tag{A.2}$$

then there exists $C(c, \kappa, r)$ such that for all $f \in L_2(\Omega)$ with support$f$ $\subset \Omega_1$ *the following holds true:*

$$\|u\|_{W^{2m+\kappa,2}(\Omega_2)} \leq C(c, \kappa, r)\,\|f\|_{L_2(\Omega_1)}\,. \tag{A.3}$$

Proof. We will prove this by induction. For $k = 0$ the estimate (A.3) follows from (A.2) and the fact that supp $f \subset \Omega_1$. Next we do the induction from k to $k+1$ and suppose that $\|u\|_{W^{2m+k,2}(\Omega_2)} \leq C(c,k,r)\|f\|_{L_2(\Omega_1)}$ for some $k \geq 0$. One may construct a cut-off functions χ such that for some $c_1 \in \mathbb{R}^+$

1. $\chi \in C^\infty(\bar{\Omega})$ with $\chi_{|\Omega_1} = 0$ and $\chi_{|\Omega_2} = 1$;
2. $\bar{\Omega}_2 \Subset \text{support}\chi \Subset \bar{\Omega}\backslash\Omega_1$;
3. $\|\chi\|_{C^i(\bar{\Omega})} \leq c_1 r^{-i}$ for $i \in \{0,\ldots,k\}$.

Note that $L(\chi u) = \chi Lu + l.o.t. = 0 + l.o.t.$ and that χu satisfies the boundary conditions from (A.1). Since the right hand side $l.o.t.$ lies in $W^{k+1,2}(\Omega)$ we find $\chi u \in W^{2m+k+1,2}(\Omega)$. Moreover

$$\|u\|_{W^{2m+k+1,2}(\Omega_2)} \leq \|\chi u\|_{W^{2m+k+1,2}(\Omega)} \leq c_1\|l.o.t.\|_{W^{k+1,2}(\Omega)} =$$
$$= c_1\|l.o.t.\|_{W^{k+1,2}(\text{support}\chi)} \leq c(r)\ \|u\|_{W^{2m+k,2}(\tilde{\Omega}_2)} \leq C'(c,k,r/2)\|f\|_{L_2(\Omega_1)}.$$

Here $\tilde{\Omega}_2$ can be chosen so that $d(\Omega_1, \tilde{\Omega_2}) < r/(2k)$. □

Acknowledgment:
This research was financially supported through the RiP program of the Mathematisches Forschungsinstitut Oberwolfach, where most of this paper was completed during a stay in June 2004. Special thanks go to W. Jäger for bringing [17] to our attention and also to P. Seidel for her help in locating some of the older literature.

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