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Among all 2-dimensional convex domains the disk is not optimal for the lifetime of a conditioned Brownian motion.
— *the extended version* —

short version accepted for publication
([10])

Directory

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1. Introduction

Probabilistic background: Let $\Omega \subset \mathbb{R}^2$ be a convex bounded domain and let τ_Ω denote the lifetime of a Brownian motion in Ω, starting at $y \in \overline{\Omega}$, to be stopped at $x \in \overline{\Omega}$ and conditioned to be killed at the boundary $\partial\Omega$. For $n > 1$ the expectation for this lifetime is known to be equal

$$\mathrm{E}_x^y(\tau_\Omega) = \frac{\int_{z\in\Omega} G_\Omega(x,z)\, G_\Omega(z,y)\, dz}{G_\Omega(x,y)}, \tag{1}$$

where $G_\Omega(x,y)$ is the Dirichlet Green function for the laplacian (or, in probabilistic setting, $\frac{1}{2}$laplacian). Indeed, for such a conditioned Brownian motion the corresponding diffusion has transition density

$$p_\Omega^y(t,x,z) = \frac{p_\Omega(t,x,z)\, G_\Omega(z,y)}{G_\Omega(x,y)},$$

where $p_\Omega(t,x,z)$ is the standard kernel of the parabolic Dirichlet problem. We refer to [6], [14], [3] or [2].

The expectation for the lifetime of conditioned Brownian motion plays in important role in the probabilistic approach to Schrödinger operators. Major estimates on $\mathrm{E}_x^y(\tau_\Omega)$ are due to Cranston and McConnell. Indeed, in [4] it is shown that there exists an absolute constant c such that for every two-dimensional domain

$$\mathrm{E}_x^y(\tau_\Omega) \le c\,|\Omega|, \tag{2}$$

where Ω is Lebesgue measure of Ω. In higher dimensions, $n \ge 3$, Cranston [5] proved $\mathrm{E}_x^y(\tau_\Omega) \le c(\Omega)$ and here $c(\Omega)$ does depend on the Lipschitz nature of the boundary. Local estimates of the type $\mathrm{E}_x^y(\tau_\Omega) \le f_\Omega(x,y)$ are established in [12].

A link to elliptic systems: Our interest in this quantity comes from the fact that λ_c, defined by $\lambda_c^{-1} = \sup\{\mathrm{E}_x^y(\tau_\Omega); x,y \in \Omega\}$, appears as a bound for the parameter in noncooperative elliptic systems in order that such a system is positivity preserving (see [12]). For the trivially coupled system

$$\begin{cases} -\Delta u = f - \lambda v & \text{in } \Omega, \\ -\Delta v = f & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

$f > 0$ implies $u > 0$ if and only if $\lambda < \lambda_c$. The positivity preserving property for this system implies that property for more generally coupled systems. We refer to [11]. Note that we use the analyst's $-\Delta$ instead of $-\frac{1}{2}\Delta$. All numbers quoted from the references have been rescaled to $-\Delta$. The reader from probability should add the factor 2.

Relation with domain shape: Let us return to known estimates for this expectation. If one looks for the maximum expected lifetime, heuristically this should be attained in a pair of points $(x,y) \in \Omega^2$ which are, in an appropriate metric, as far apart as possible. If Ω is a rectangle, the pair of points which maximize $\mathrm{E}_x^y(\tau_\Omega)$ is expected to sit in diagonal corners, if it is an ellipse, they should sit on the long axes. In fact, Griffin, McConnell and Verchota have shown in [8, Corollary 2.4] that for planar domains these points are located at the boundary provided one of them is located at the boundary:

$$\sup\left\{\mathrm{E}_x^y(\tau_\Omega); x \in \partial\Omega, y \in \bar{\Omega}\right\} = \sup\left\{\mathrm{E}_x^y(\tau_\Omega); x,y \in \partial\Omega\right\} =: s(\Omega). \tag{3}$$

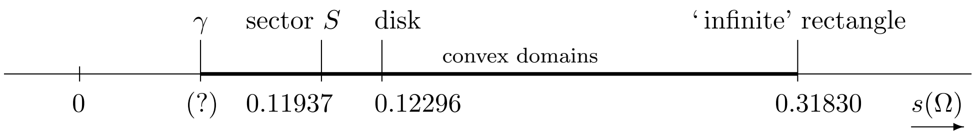
It is believed that a stronger statement holds true:

$$\sup\left\{\mathrm{E}_x^y(\tau_\Omega); x,y \in \bar{\Omega}\right\} = \sup\left\{\mathrm{E}_x^y(\tau_\Omega); x,y \in \partial\Omega\right\}.$$

It was also shown in [8, Theorem 3.1] that $s(\Omega)/|\Omega| < 1/\pi = .318309\ldots$, and that this upper bound is optimal. In other words the best constant c for (2) is $1/\pi$.

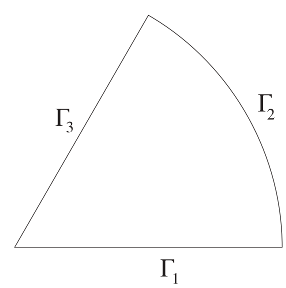
In [13, Theorem 2] Jianming Xu studied estimates from below for $s(\Omega)$. He showed that there exists a universal constant γ for two dimensional convex domains such that $s(\Omega)/|\Omega| \ge \gamma$, and that without a convexity or similar assumption on Ω we cannot expect this result. In fact, Xu showed that amobae-like domains which have many thin necks can have $s(\Omega)/|\Omega|$ as small as we wish. Our paper is sort of a contribution towards determining γ. If a convex domain Ω is "everywhere thin", in the sense that no two points x and y in Ω are too far apart from each other, then one might expect $s(\Omega)/|\Omega$ to be small. This heuristic consideration led Griffin, McConnell and Verchota [8, p. 244] to the open question, whether the optimal constant γ is given by the disk, in other words, if $s(\Omega) \geq s(\Omega^*)$ for any convex plane Ω. Here Ω^* denotes the disk of same area as Ω. It is well known that the disk has many isoperimetric properties: it minimizes for instance the diameter, the perimeter or the first Dirichlet-Laplace eigenvalue of a domain of prescribed area. In [8, Prop. 4.2] it was shown (in our notation with Δ) that $s(\Omega^*)/|\Omega^*| = (2\log 2 - 1)/\pi = 0.12296\ldots$, which implies that $\gamma \leq 0.12296\ldots$.

Main result: We shall show that the disk does not minimize $s(\Omega)$ among all convex planar domains of given area by constructing a counterexample, that is a domain S for which $s(S)/|S| = 3/8\pi = 0.11937\ldots$. Therefore the optimal γ must satisfy $\gamma \leq 0.11937\ldots < 0.12296\ldots$, and so the disk does not minimize $s(\Omega)/|\Omega|$.



Our choice of S was motivated by the consideration that domains with large perimeter but small diameter seem to have small $s(\Omega)/|\Omega|$. A larger perimeter leads to killing more of those Brownian paths that are "wandering around" and tends to decrease the expected lifetime. A good candidate in this respect is an equilateral triangle. Numerical results seemed to confirm, that on an equilateral triangle the maximum of $\mathrm{E}_x^y(\tau_\Omega)$ is attained if x and y sit in corners of the triangle and this maximum value is very close to the one of the disk. The result however was to imprecise and hence unconclusive. A major complication for a direct computation is the fact that we do not have an sufficiently simple explicit Green function for a triangle at our disposal and had to approximate it by a series.

On the other hand, replacing the straight lines of the triangles by convex arcs, even seemed to slightly decrease $s(\Omega)/|\Omega|$. This could hint at a domain like the Reuleaux-triangle [9], a 'triangle' with circular arcs, as a potential minimizer. However, the Green function for a Reuleaux-triangle does not seem to be available in a convenient form either. Therefore we looked for a domain which was close to a triangle or Reuleaux-triangle and for which we could calculate the Green function in an sufficiently simple formula. Such a domain indeed exists, namely a sector. To state our results, we define S to be a sector of the unit disk



$$S := \left\{x \in \mathbb{R}^2; |x| < 1 \text{ and } 0 < \arg x < \tfrac{1}{3}\pi\right\}.$$

The main work of our paper goes into proving

Theorem 1 $\sup\left\{\mathrm{E}_x^y(\tau_S); x \in \partial S, y \in \bar{S}\right\} = \frac{1}{16} = \frac{3}{8\pi}|S|$. *Notice that* $\frac{3}{8\pi} = 0.11937\ldots$.

An immediate consequence is

Corollary 2 *Among all convex two-dimensional domains* Ω *of given area the disk does not minimize* $\sup\left\{\mathrm{E}_x^y(\tau_S); x \in \partial\Omega, y \in \bar{\Omega}\right\}$.

Outline of the proof: In Section 2 we compute the explicit Green function for S. We are not able to explicitly calculate the iterated Green's function, i.e. the enumerator of (1), but due to (3) it is sufficient to analyze the behaviour of (1) for x and y on the boundary ∂S.

Note that both the enumerator and denominator in (1) go to zero on the boundary ∂S. In order to compute it for x and y on the boundary ∂S, we divide both terms in Section 3 in such way, that not only they converge to a nonvanishing function on the boundary. Moreover, we choose the divisor in such a way, that the remaining integrand becomes a rational function of z_1 and z_2. Due to symmetry we have to distinguish just four cases for the locations of x and y, each of which is treated in a separate section, i.e. Sections 4 to 7.

We proceed by rewriting the integral in terms of polar coordinates (θ, r). Since in each case the integrand is rational, it allows us to perform the integration with respect to θ by means of a contour integral in the complex plane. In fact, by grouping appropriate terms together, we are able to come to a closed contour. Distinguishing numerous subcases, we can in each subcase identify the poles of the integrand and apply the residue theorem to evaluate the integral with respect to θ. The resulting expressions contain rational functions and logarithms in r which allows us to perform an explicit integration with respect to r.

As one might guess, the expected lifetime has local maxima when x and y are two distinct corner points. The supremum is indeed attained, when x is at the vertex and y on any place of the circular sector of ∂S. If one restricts attention to these particular x and y only, the computations are much simpler. However, we felt the need to give a proof, that the supremum is attained in those points.

Since all these calculations are very elaborate but hardly contribute to the basic understanding, only a shorter version of this manuscript will be published ([10]).

2. The Green function

The Green function on the unit disk is as follows:

$$G_O(x,y) = \frac{1}{2\pi}\log\left(\frac{[X\ Y]}{|x-y|}\right),$$

where $[X\ Y] = \left|x\,|y| - y\,|y|^{-1}\right| = \sqrt{|x|^2\,|y|^2 - 2x\cdot y + 1}$. The function defined by

$$u(x) = \int_{|y|<1} G_O(x,y)\, f(y)\, dy$$

solves, with $\Omega = O = \left\{x \in \mathbb{R}^2; |x| < 1\right\}$,

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

With this function we may build the Green function for the sector S. Using the notation $\mathcal{R}$ for rotating with $\frac{2}{3}\pi$ around $(0,0)$ and $\mathcal{S}$ for reflecting in $x_2 = 0$ we obtain

$$\begin{aligned} G_S(x,y) &= G_O(x,y) + G_O(\mathcal{R}x,y) + G_O\left(\mathcal{R}^2x,y\right) + \\ &\qquad - G_O(\mathcal{S}x,y) - G_O(\mathcal{R}\mathcal{S}x,y) - G_O\left(\mathcal{R}^2\mathcal{S}x,y\right) \\ &= \frac{1}{2\pi}\log\left(\frac{[X\ Y]\,[\mathcal{R}X\ Y]\,\left[\mathcal{R}^2X\ Y\right]}{|x-y|\,|\mathcal{R}x-y|\,|\mathcal{R}^2x-y|}\,\frac{|\mathcal{S}x-y|\,|\mathcal{R}\mathcal{S}x-y|\,\left|\mathcal{R}^2\mathcal{S}x-y\right|}{[\mathcal{S}X\ Y]\,[\mathcal{R}\mathcal{S}X\ Y]\,[\mathcal{R}^2\mathcal{S}X\ Y]}\right). \end{aligned}$$

Notice that we may rewrite $G_O(x,y)$ as

$$G_O(x,y) = \frac{1}{4\pi}\log\left(1 + \frac{(1-|x|^2)(1-|y|^2)}{|x-y|^2}\right) = \frac{1}{2\pi}\log\left|\frac{1-\bar{\mathbf{x}}\mathbf{y}}{\mathbf{x}-\mathbf{y}}\right|$$

with $\mathbf{x} = x_1 + ix_2 \in \mathbb{C}$ etc.

3. Four boundary combinations and limits of the Green function

We introduce the notation

$$\begin{aligned} \Gamma_1 &= \left\{(a,0)\,; 0 \le a \le 1\right\}, \\ \Gamma_2 &= \left\{(\cos\theta, \sin\theta)\,; 0 \le \theta \le \tfrac{1}{3}\pi\right\}, \\ \Gamma_3 &= \left\{\left(\tau\cos\tfrac{1}{3}\pi, \tau\sin\tfrac{1}{3}\pi\right); 0 \le \tau \le 1\right\}. \end{aligned}$$

Depending on the location of x and y in one of these Γ_i we have nine cases of which we distinguish the following four, since the remaining ones follow by symmetry.

$$1)\ x,y \in \Gamma_1,\quad 2)\ x \in \Gamma_1,\ y \in \Gamma_2,\quad 3)\ x,y \in \Gamma_2,\quad 4)\ x \in \Gamma_1,\ y \in \Gamma_3. \tag{5}$$

In order to evaluate the enumerator of (1), we take the limit inside the integral. Without this procedure the integrand contains logarithms which we are unable to handle. Subsequently we derive expressions for G as one of the arguments approaches the boundary. This will also help us to control the limiting behaviour. Since there are straight and circular parts of the boundary we distinguish two cases. For later reference let us record these.

a) For x near Γ_1 one has x_2 small and one finds by direct calculus that

$$\begin{aligned} G_S(x,y) &= \frac{1}{4\pi}\log\left(1+\frac{4x_2y_2}{|x-y|^2}\right)-\frac{1}{4\pi}\log\left(1+\frac{4x_2y_2}{[XY]^2}\right)+ \\ &+\frac{1}{4\pi}\log\left(1-\frac{4x_2\left(\frac{1}{2}y_2+\frac{1}{2}\sqrt{3}y_1\right)}{|x-\mathcal{R}y|^2}\right)-\frac{1}{4\pi}\log\left(1-\frac{4x_2\left(\frac{1}{2}y_2+\frac{1}{2}\sqrt{3}y_1\right)}{[X\mathcal{R}Y]^2}\right)+ \\ &+\frac{1}{4\pi}\log\left(1-\frac{4x_2\left(\frac{1}{2}y_2-\frac{1}{2}\sqrt{3}y_1\right)}{|x-\mathcal{R}^2y|^2}\right)-\frac{1}{4\pi}\log\left(1-\frac{4x_2\left(\frac{1}{2}y_2-\frac{1}{2}\sqrt{3}y_1\right)}{[X\mathcal{R}^2Y]^2}\right) \end{aligned} \tag{6}$$

with

$$\begin{aligned} \mathcal{R}y &= \left(-\tfrac{1}{2}y_1-\tfrac{1}{2}\sqrt{3}y_2, \tfrac{1}{2}\sqrt{3}y_1-\tfrac{1}{2}y_2\right), \\ \mathcal{R}^2y &= \left(-\tfrac{1}{2}y_1+\tfrac{1}{2}\sqrt{3}y_2, -\tfrac{1}{2}\sqrt{3}y_1-\tfrac{1}{2}y_2\right). \end{aligned}$$

b) For x near Γ_2 one has $1-|x|$ small and it follows that

$$\begin{aligned} G_S(x,y) &= \frac{1}{4\pi}\log\left(1-\frac{(1-|x|^2)(1-|y|^2)}{|x-y|^2}\right)+ \\ &+\frac{1}{4\pi}\log\left(1-\frac{(1-|x|^2)(1-|y|^2)}{|\mathcal{R}x-y|^2}\right)+\frac{1}{4\pi}\log\left(1-\frac{(1-|x|^2)(1-|y|^2)}{|\mathcal{R}^2x-y|^2}\right)+ \\ &-\frac{1}{4\pi}\log\left(1-\frac{(1-|x|^2)(1-|y|^2)}{|\mathcal{S}x-y|^2}\right)-\frac{1}{4\pi}\log\left(1-\frac{(1-|x|^2)(1-|y|^2)}{|\mathcal{R}\mathcal{S}x-y|^2}\right)+ \\ &-\frac{1}{4\pi}\log\left(1-\frac{(1-|x|^2)(1-|y|^2)}{|\mathcal{R}^2\mathcal{S}x-y|^2}\right). \end{aligned} \tag{7}$$

4. The case that $x, y \in \Gamma_1$

In the first step for the computation of (1) we shift the limits inside the integral as follows:

$$\lim_{x_2\downarrow 0,\ y_2\downarrow 0}\frac{\int_{z\in\Omega}G_\Omega(x,z)\,G_\Omega(z,y)\,dz}{G_\Omega(x,y)} = \frac{\int_S \lim\limits_{x_2\downarrow 0}\frac{G_S(x,z)}{x_2}\lim\limits_{y_2\downarrow 0}\frac{G_S(z,y)}{y_2}dz}{\lim\limits_{x_2\downarrow 0,\ y_2\downarrow 0}\frac{G_S(x,y)}{x_2y_2}}. \tag{8}$$

For all three limits in the right hand side we can use representation (6).

4.1. Limit of the Green function

¿From the expression in (6) we find

$$\begin{aligned} \lim_{x_2\downarrow 0}\frac{G_S(x,y)}{x_2} &= \frac{1}{\pi}\left(\frac{y_2}{|x-y|^2}-\frac{y_2}{[XY]^2}-\frac{\frac{1}{2}y_2+\frac{1}{2}\sqrt{3}y_1}{|x-\mathcal{R}y|^2}+\right. \\ &\left.+\frac{\frac{1}{2}y_2+\frac{1}{2}\sqrt{3}y_1}{[X\mathcal{R}Y]^2}-\frac{\frac{1}{2}y_2-\frac{1}{2}\sqrt{3}y_1}{|x-\mathcal{R}^2y|^2}+\frac{\frac{1}{2}y_2-\frac{1}{2}\sqrt{3}y_1}{[X\mathcal{R}^2Y]^2}\right). \end{aligned} \tag{9}$$

Replacing y by $z = (r\cos\theta, r\sin\theta)$ we obtain

$$\begin{aligned}
\lim_{x_2\downarrow 0}\frac{G_S(x,z)}{x_2} &= \frac{1}{\pi}\left(\frac{z_2}{|x|^2-2x_1z_1+|z|^2}-\frac{z_2}{1-2x_1z_1+|x|^2|z|^2}\right)+\\
&+\frac{1}{\pi}\left(\frac{-\frac12\sqrt3 z_1-\frac12 z_2}{|x|^2-2x_1\left(-\frac12 z_1+\frac12\sqrt3 z_2\right)+|z|^2}-\frac{-\frac12\sqrt3 z_1-\frac12 z_2}{1-2x_1\left(-\frac12 z_1+\frac12\sqrt3 z_2\right)+|x|^2|z|^2}\right)+\\
&+\frac{1}{\pi}\left(\frac{\frac12\sqrt3 z_1-\frac12 z_2}{|x|^2-2x_1\left(-\frac12 z_1-\frac12\sqrt3 z_2\right)+|z|^2}-\frac{\frac12\sqrt3 z_1-\frac12 z_2}{1-2x_1\left(-\frac12 z_1-\frac12\sqrt3 z_2\right)+|x|^2|z|^2}\right)\\
&= \frac{1}{\pi}\sum_{k=0}^{2}\left(\frac{r\sin\left(\theta+\frac23 k\pi\right)}{x_1^2-2x_1r\cos\left(\theta+\frac23 k\pi\right)+r^2}-\frac{r\sin\left(\theta+\frac23 k\pi\right)}{1-2x_1r\cos\left(\theta+\frac23 k\pi\right)+x_1^2r^2}\right). \qquad (10)
\end{aligned}$$

Using the symmetry $G_S(x,z) = G_S(z,x)$ and replacing x by y we find a similar formula for

$$\begin{aligned}
\lim_{y_2\downarrow 0}\frac{G_S(z,y)}{y_2} &= \frac{1}{\pi}\left(\frac{z_2}{|y|^2-2y_1z_1+|z|^2}-\frac{z_2}{1-2y_1z_1+|y|^2|z|^2}\right)+\\
&+\frac{1}{\pi}\left(\frac{-\frac12\sqrt3 z_1-\frac12 z_2}{|y|^2-2y_1\left(-\frac12 z_1+\frac12\sqrt3 z_2\right)+|z|^2}-\frac{-\frac12\sqrt3 z_1-\frac12 z_2}{1-2y_1\left(-\frac12 z_1+\frac12\sqrt3 z_2\right)+|y|^2|z|^2}\right)+\\
&+\frac{1}{\pi}\left(\frac{\frac12\sqrt3 z_1-\frac12 z_2}{|y|^2-2y_1\left(-\frac12 z_1-\frac12\sqrt3 z_2\right)+|z|^2}-\frac{\frac12\sqrt3 z_1-\frac12 z_2}{1-2y_1\left(-\frac12 z_1-\frac12\sqrt3 z_2\right)+|y|^2|z|^2}\right)=\\
&= \frac{1}{\pi}\left(\frac{r\sin\theta}{y_1^2-2y_1r\cos\theta+r^2}-\frac{r\sin\theta}{1-2y_1r\cos\theta+y_1^2r^2}\right)+\\
&+\frac{1}{\pi}\left(\frac{r\sin\left(\theta-\frac23\pi\right)}{y_1^2-2y_1r\cos\left(\theta-\frac23\pi\right)+r^2}-\frac{r\sin\left(\theta-\frac23\pi\right)}{1-2y_1r\cos\left(\theta-\frac23\pi\right)+y_1^2r^2}\right)+\\
&+\frac{1}{\pi}\left(\frac{r\sin\left(\theta+\frac23\pi\right)}{y_1^2-2y_1r\cos\left(\theta+\frac23\pi\right)+r^2}-\frac{r\sin\left(\theta+\frac23\pi\right)}{1-2y_1r\cos\left(\theta+\frac23\pi\right)+y_1^2r^2}\right). \qquad (11)
\end{aligned}$$

The last limit in this subsection is the denominator

$$\begin{aligned}
\lim_{y_2\downarrow 0}\lim_{x_2\downarrow 0}\frac{G_S(x,y)}{x_2y_2} &= \lim_{y_2\downarrow 0}\frac{1}{y_2}\left(\frac{1}{\pi}\left(\frac{y_2}{|x|^2-2x_1y_1+|y|^2}-\frac{y_2}{1-2x_1y_1+|x|^2|y|^2}\right)+\right.\\
&+\frac{1}{\pi}\left(\frac{-\frac12\sqrt3 y_1-\frac12 y_2}{|x|^2-2x_1\left(-\frac12 y_1+\frac12\sqrt3 y_2\right)+|y|^2}-\frac{-\frac12\sqrt3 y_1-\frac12 y_2}{1-2x_1\left(-\frac12 y_1+\frac12\sqrt3 y_2\right)+|x|^2|y|^2}\right)+\\
&\left.+\frac{1}{\pi}\left(\frac{\frac12\sqrt3 y_1-\frac12 y_2}{|x|^2-2x_1\left(-\frac12 y_1-\frac12\sqrt3 y_2\right)+|y|^2}-\frac{\frac12\sqrt3 y_1-\frac12 y_2}{1-2x_1\left(-\frac12 y_1-\frac12\sqrt3 y_2\right)+|x|^2|y|^2}\right)\right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\pi}\left(\frac{1}{(x_1-y_1)^2}-\frac{1}{(1-x_1y_1)^2}\right)+\\
&+\frac{1}{\pi}\left(-\frac{1}{x_1^2+x_1y_1+y_1^2}+\frac{1}{1+x_1y_1+x_1^2y_1^2}\right)+\\
&+\lim_{y_2\downarrow 0}\frac{1}{\pi y_2}\left(-\frac{\frac12\sqrt3 y_1}{|x|^2-2x_1\left(-\frac12 y_1+\frac12\sqrt3 y_2\right)+|y|^2}+\frac{\frac12\sqrt3 y_1}{1-2x_1\left(-\frac12 y_1+\frac12\sqrt3 y_2\right)+|x|^2|y|^2}\right)+\\
&+\lim_{y_2\downarrow 0}\frac{1}{\pi y_2}\left(\frac{\frac12\sqrt3 y_1}{|x|^2+2x_1\left(\frac12 y_1+\frac12\sqrt3 y_2\right)+|y|^2}+\frac{-\frac12\sqrt3 y_1}{1+2x_1\left(\frac12 y_1+\frac12\sqrt3 y_2\right)+|x|^2|y|^2}\right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\pi}\left(\frac{1}{(x_1-y_1)^2}-\frac{1}{(1-x_1y_1)^2}\right)+\\
&+\frac{1}{\pi}\left(-\frac{1}{x_1^2+x_1y_1+y_1^2}+\frac{1}{1+x_1y_1+x_1^2y_1^2}\right)+\\
&+\lim_{y_2\downarrow 0}\frac{\frac{1}{2}\sqrt{3}y_1}{\pi y_2}\left(\frac{1}{1-2x_1\left(-\frac{1}{2}y_1+\frac{1}{2}\sqrt{3}y_2\right)+|x|^2|y|^2}-\frac{1}{|x|^2-2x_1\left(-\frac{1}{2}y_1+\frac{1}{2}\sqrt{3}y_2\right)+|y|^2}\right)+\\
&+\lim_{y_2\downarrow 0}\frac{\frac{1}{2}\sqrt{3}y_1}{\pi y_2}\left(\frac{1}{|x|^2+2x_1\left(\frac{1}{2}y_1+\frac{1}{2}\sqrt{3}y_2\right)+|y|^2}-\frac{1}{1+2x_1\left(\frac{1}{2}y_1+\frac{1}{2}\sqrt{3}y_2\right)+|x|^2|y|^2}\right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\pi}\left(\frac{1}{(x_1-y_1)^2}-\frac{1}{(1-x_1y_1)^2}\right)+\\
&+\frac{1}{\pi}\left(-\frac{1}{x_1^2+x_1y_1+y_1^2}+\frac{1}{1+x_1y_1+x_1^2y_1^2}\right)+\\
&+\lim_{y_2\downarrow 0}\frac{\frac{1}{2}\sqrt{3}y_1}{\pi y_2}\left(\frac{1}{|x|^2+2x_1\left(\frac{1}{2}y_1+\frac{1}{2}\sqrt{3}y_2\right)+|y|^2}-\frac{1}{|x|^2-2x_1\left(-\frac{1}{2}y_1+\frac{1}{2}\sqrt{3}y_2\right)+|y|^2}\right)+\\
&+\lim_{y_2\downarrow 0}\frac{\frac{1}{2}\sqrt{3}y_1}{\pi y_2}\left(\frac{1}{1-2x_1\left(-\frac{1}{2}y_1+\frac{1}{2}\sqrt{3}y_2\right)+|x|^2|y|^2}-\frac{1}{1+2x_1\left(\frac{1}{2}y_1+\frac{1}{2}\sqrt{3}y_2\right)+|x|^2|y|^2}\right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\pi}\left(\frac{1}{(x_1-y_1)^2}-\frac{1}{(1-x_1y_1)^2}\right)+\\
&+\frac{1}{\pi}\left(-\frac{1}{x_1^2+x_1y_1+y_1^2}+\frac{1}{1+x_1y_1+x_1^2y_1^2}\right)+\\
&+\lim_{y_2\downarrow 0}\frac{\frac{1}{2}\sqrt{3}y_1}{\pi y_2}\left(\frac{-4x_1\frac{1}{2}\sqrt{3}y_2}{\left(|x|^2+2x_1\left(\frac{1}{2}y_1+\frac{1}{2}\sqrt{3}y_2\right)+|y|^2\right)\left(|x|^2+2x_1\left(\frac{1}{2}y_1-\frac{1}{2}\sqrt{3}y_2\right)+|y|^2\right)}\right)+\\
&+\lim_{y_2\downarrow 0}\frac{\frac{1}{2}\sqrt{3}y_1}{\pi y_2}\left(\frac{4x_1\frac{1}{2}\sqrt{3}y_2}{\left(1+2x_1\left(\frac{1}{2}y_1-\frac{1}{2}\sqrt{3}y_2\right)+|x|^2|y|^2\right)\left(1+2x_1\left(\frac{1}{2}y_1+\frac{1}{2}\sqrt{3}y_2\right)+|x|^2|y|^2\right)}\right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\pi}\left(\frac{1}{(x_1-y_1)^2}-\frac{1}{(1-x_1y_1)^2}\right)+\\
&+\frac{1}{\pi}\left(\frac{1}{1+x_1y_1+x_1^2y_1^2}-\frac{1}{x_1^2+x_1y_1+y_1^2}\right)+\\
&+\frac{1}{\pi}\left(\frac{3x_1y_1}{(1+x_1y_1+x_1^2y_1^2)^2}-\frac{3x_1y_1}{(x_1^2+x_1y_1+y_1^2)^2}\right). \qquad (12)
\end{aligned}$$

4.2. Derivation of a contour integral

We introduce the notation

$$h(\theta)=\frac{r\sin\theta}{x_1^2-2x_1r\cos\theta+r^2}-\frac{r\sin\theta}{1-2x_1r\cos\theta+x_1^2r^2} \qquad (13)$$

$$g(\theta)=\frac{r\sin\theta}{y_1^2-2y_1r\cos\theta+r^2}-\frac{r\sin\theta}{1-2y_1r\cos\theta+y_1^2r^2} \qquad (14)$$

and the integral to be evaluated becomes

$$\begin{aligned}
&\pi^2 \int_{\theta=0}^{\frac{1}{3}\pi} \lim_{x_2\downarrow 0} \frac{G_S(x,z)}{x_2} \lim_{y_2\downarrow 0} \frac{G_S(z,y)}{y_2} d\theta = \\
&= \int_{\theta=0}^{\frac{\pi}{3}} \left(h(\theta) + h\left(\theta+\tfrac{2}{3}\pi\right) + h\left(\theta+\tfrac{4}{3}\pi\right)\right)\left(g(\theta) + g\left(\theta+\tfrac{2}{3}\pi\right) + g\left(\theta+\tfrac{4}{3}\pi\right)\right) d\theta \\
&= \tfrac{1}{2} \int_{\theta=-\frac{\pi}{3}}^{\frac{\pi}{3}} \left(h(\theta) + h\left(\theta+\tfrac{2}{3}\pi\right) + h\left(\theta+\tfrac{4}{3}\pi\right)\right)\left(g(\theta) + g\left(\theta+\tfrac{2}{3}\pi\right) + g\left(\theta+\tfrac{4}{3}\pi\right)\right) d\theta \\
&= \tfrac{1}{6} \int_{\theta=0}^{2\pi} \left(h(\theta) + h\left(\theta+\tfrac{2}{3}\pi\right) + h\left(\theta+\tfrac{4}{3}\pi\right)\right)\left(g(\theta) + g\left(\theta+\tfrac{2}{3}\pi\right) + g\left(\theta+\tfrac{4}{3}\pi\right)\right) d\theta \qquad (15) \\
&= \tfrac{1}{2} \int_{\theta=0}^{2\pi} \left(h(\theta) + h\left(\theta+\tfrac{2}{3}\pi\right) + h\left(\theta+\tfrac{4}{3}\pi\right)\right) g(\theta)\, d\theta. \qquad (16)
\end{aligned}$$

Here we have used that h and g are odd and 2π-periodic. Note that the integrand contains singularities due to h at $\theta \in \left\{0, \frac{2}{3}\pi, \frac{4}{3}\pi\right\}$ when $r = x_1$, and due to g in $\theta = 0$ when $r = y_1$. The singularities are integrable whenever $x_1 \neq y_1$. To apply Cauchy's residue theorem, we introduce complex notation. We call $w = re^{i\theta}$. Then $r\sin\theta = \frac{w-\bar{w}}{2i}$, $r\cos\theta = \frac{w+\bar{w}}{2}$, $id\theta = dw/w$, $\bar{w} = \frac{r^2}{w}$, and after some straightforward computations we arrive at:

$$\begin{aligned}
h(\theta) &= \frac{1}{2i}\left(\frac{w-\bar{w}}{x_1^2 - x_1(w+\bar{w}) + w\bar{w}} - \frac{w-\bar{w}}{1 - x_1(w+\bar{w}) + x_1^2 w\bar{w}}\right) \\
&= \frac{1}{2i}\left(\frac{1}{\bar{w}-x_1} - \frac{1}{w-x_1} - \frac{x_1^{-2}}{\bar{w}-x_1^{-1}} + \frac{x_1^{-2}}{w-x_1^{-1}}\right) \\
&= \frac{1}{2i}\left(\frac{w}{r^2 - x_1 w} - \frac{1}{w-x_1} - \frac{x_1^{-2} w}{r^2 - x_1^{-1} w} + \frac{x_1^{-2}}{w-x_1^{-1}}\right) \\
&= \frac{1}{2i}\left(\frac{r^2}{w-x_1 r^2} + \frac{x_1^{-2}}{w-x_1^{-1}} - \frac{x_1^{-2} r^2}{w-x_1^{-1} r^2} - \frac{1}{w-x_1}\right), \qquad (17)
\end{aligned}$$

$$\begin{aligned}
h\left(\theta+\tfrac{2}{3}\pi\right) = \frac{e^{-\frac{2}{3}\pi i}}{2i} \Bigg(& \frac{r^2}{w - x_1 r^2 e^{-\frac{2}{3}\pi i}} + \frac{x_1^{-2}}{w - x_1^{-1} e^{-\frac{2}{3}\pi i}} + \\
& - \frac{x_1^{-2} r^2}{w - x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}} - \frac{1}{w - x_1 e^{-\frac{2}{3}\pi i}} \Bigg), \qquad (18)
\end{aligned}$$

$$\begin{aligned}
h\left(\theta+\tfrac{4}{3}\pi\right) = \frac{e^{\frac{2}{3}\pi i}}{2i} \Bigg(& \frac{r^2}{w - x_1 r^2 e^{\frac{2}{3}\pi i}} + \frac{x_1^{-2}}{w - x_1^{-1} e^{\frac{2}{3}\pi i}} + \\
& - \frac{x_1^{-2} r^2}{w - x_1^{-1} r^2 e^{\frac{2}{3}\pi i}} - \frac{1}{w - x_1 e^{\frac{2}{3}\pi i}} \Bigg). \qquad (19)
\end{aligned}$$

We find

$$\begin{aligned}
\#_{(16)} &= \frac{i}{8} \oint_{|w|=r} \Big(\Psi_{x_1,0,r}(w) + \Psi_{x_1,1,r}(w) + \Psi_{x_1,2,r}(w)\Big)\ \Psi_{y_1,0,r}(w)\ \frac{dw}{w} \\
&= \frac{-\pi}{4} \sum_{\substack{\text{poles } w_*:\\ |w_*|<r}} \operatorname{Res}\left(\frac{1}{w}\ \Psi_{y_1,0,r}(w) \sum_{m=0}^{2} \Psi_{x_1,m,r}(w)\right)_{w=w_*}, \qquad (20)
\end{aligned}$$

where we define

$$\Psi_{t,m,r}(w) := \frac{r^2 e^{-\frac{2}{3}m\pi i}}{w - t\,r^2 e^{-\frac{2}{3}m\pi i}} + \frac{t^{-2} e^{-\frac{2}{3}m\pi i}}{w - t^{-1} e^{-\frac{2}{3}m\pi i}} + \\ - \frac{t^{-2} r^2 e^{-\frac{2}{3}m\pi i}}{w - t^{-1} r^2 e^{-\frac{2}{3}m\pi i}} - \frac{e^{-\frac{2}{3}m\pi i}}{w - t\,e^{-\frac{2}{3}m\pi i}}\,. \tag{21}$$

For later use we also define

$$J_1(x_1, y_1, r; w) := \frac{1}{w}\,\Psi_{y_1,0,r}(w) \sum_{m=0}^{2} \Psi_{x_1,m,r}(w) \tag{22}$$

Let us remark that the factor $w^{-1}\Psi_{y_1,0,r}(w)$ in (20), (22) has a removable singularity at $w = 0$.

4.3. Computation of the contour integral

Without loss of generality we may assume that $x_1 < y_1$. Then, according to the size of r, the integrand has different sets of poles. In the following table we give a scheme in which we denote how we split the integral (and which range of r corresponds to which poles).

poles due to:		$\Psi_{x_1,0,r}$		$\Psi_{x_1,1,r}$		$\Psi_{x_1,2,r}$		$\Psi_{y_1,0,r}$	
range:		a_1.	a_2.	b_1.	b_2.	c_1.	c_2.	d_1.	d_2.
$r \in (0, x_1)$	I.	$x_1 r^2$	$\frac{r^2}{x_1}$	$x_1 r^2 e^{-\frac{2}{3}\pi i}$	$\frac{r^2 e^{-\frac{2}{3}\pi i}}{x_1}$	$x_1 r^2 e^{\frac{2}{3}\pi i}$	$\frac{r^2 e^{\frac{2}{3}\pi i}}{x_1}$	$y_1 r^2$	$\frac{r^2}{y_1}$
$r \in (x_1, y_1)$	II.	$x_1 r^2$	x_1	$x_1 r^2 e^{-\frac{2}{3}\pi i}$	$x_1 e^{-\frac{2}{3}\pi i}$	$x_1 r^2 e^{\frac{2}{3}\pi i}$	$x_1 e^{\frac{2}{3}\pi i}$	$y_1 r^2$	$\frac{r^2}{y_1}$
$r \in (y_1, 1)$	III.	$x_1 r^2$	x_1	$x_1 r^2 e^{-\frac{2}{3}\pi i}$	$x_1 e^{-\frac{2}{3}\pi i}$	$x_1 r^2 e^{\frac{2}{3}\pi i}$	$x_1 e^{\frac{2}{3}\pi i}$	$y_1 r^2$	y_1

For ease of writing we define for $z \in \mathbb{C}$ the function $\mathrm{Ln}(z) := \ln|z| + i \arg(z)$ with $\arg(z) \in (-\pi, \pi]$. The function $z \mapsto \mathrm{Ln}(z)$ is a primitive of $z \mapsto z^{-1}$ on $\mathbb{C}\setminus(-\infty, 0]$.

Next we calculate the residues, one by one, as listed in the table.

I, II and III, a_1: pole at $w = x_1 r^2$.

$$\begin{aligned} &\mathrm{Res}\Big(J_1(x_1, y_1, r; w)\Big)_{w=x_1r^2} = \\ &= \frac{1}{x_1}\left(\frac{r^2}{x_1 r^2 - y_1 r^2} + \frac{y_1^{-2}}{x_1 r^2 - y_1^{-1}} - \frac{y_1^{-2} r^2}{x_1 r^2 - y_1^{-1} r^2} - \frac{1}{x_1 r^2 - y_1}\right) \\ &= \frac{1}{x_1 y_1}\left(\frac{1}{1 - x_1 y_1} - \frac{1}{1 - \frac{x_1}{y_1}}\right) + \frac{1}{x_1^2 y_1^2}\frac{1}{r^2 - \frac{1}{x_1 y_1}} - \frac{1}{x_1^2}\frac{1}{r^2 - \frac{y_1}{x_1}} \end{aligned} \tag{23}$$

I, a_2: pole at $w = x_1^{-1} r^2$.

$$\begin{aligned} &\mathrm{Res}\Big(J_1(x_1, y_1, r; w)\Big)_{w=x_1^{-1}r^2} = \\ &= \frac{-x_1^{-2} r^2}{x_1^{-1} r^2}\left(\frac{r^2}{x_1^{-1} r^2 - y_1 r^2} + \frac{y_1^{-2}}{x_1^{-1} r^2 - y_1^{-1}} - \frac{y_1^{-2} r^2}{x_1^{-1} r^2 - y_1^{-1} r^2} - \frac{1}{x_1^{-1} r^2 - y_1}\right) \\ &= \frac{1}{y_1^2}\frac{1}{1 - \frac{x_1}{y_1}} - \frac{1}{1 - x_1 y_1} + \frac{1}{r^2 - x_1 y_1} - \frac{1}{y_1^2}\frac{1}{r^2 - \frac{x_1}{y_1}} \end{aligned} \tag{24}$$

II and III, a_2**:** pole at $w = x_1$.

$$
\begin{aligned}
&\operatorname{Res}\left(J_1\left(x_1, y_1, r; w\right)\right)_{w=x_1} = \\
= & -\frac{1}{x_1}\left(\frac{r^2}{x_1 - y_1 r^2} + \frac{y_1^{-2}}{x_1 - y_1^{-1}} - \frac{y_1^{-2} r^2}{x_1 - y_1^{-1} r^2} - \frac{1}{x_1 - y_1}\right) \\
= & -\frac{1}{x_1 y_1}\left(\frac{1}{1 - \frac{x_1}{y_1}} - \frac{1}{1 - x_1 y_1}\right) - \frac{1}{r^2 - x_1 y_1} + \frac{1}{y_1^2}\frac{1}{r^2 - \frac{x_1}{y_1}}
\end{aligned} \tag{25}
$$

I, II and III, b_1**:** pole at $w = x_1 r^2 e^{-\frac{2}{3}\pi i}$.

$$
\begin{aligned}
&\operatorname{Res}\left(J_1\left(x_1, y_1, r; w\right)\right)_{w=x_1 r^2 e^{-\frac{2}{3}\pi i}} = \\
= & \frac{1}{x_1}\left(\frac{r^2}{x_1 r^2 e^{-\frac{2}{3}\pi i} - y_1 r^2} + \frac{y_1^{-2}}{x_1 r^2 e^{-\frac{2}{3}\pi i} - y_1^{-1}} - \frac{y_1^{-2} r^2}{x_1 r^2 e^{-\frac{2}{3}\pi i} - y_1^{-1} r^2} - \frac{1}{x_1 r^2 e^{-\frac{2}{3}\pi i} - y_1}\right) \\
= & \frac{e^{\frac{2}{3}\pi i}}{x_1 y_1}\left(\frac{1}{e^{\frac{2}{3}\pi i} - x_1 y_1} - \frac{1}{e^{\frac{2}{3}\pi i} - \frac{x_1}{y_1}}\right) + \frac{e^{\frac{2}{3}\pi i}}{x_1^2 y_1^2}\frac{1}{r^2 - \frac{1}{x_1 y_1} e^{\frac{2}{3}\pi i}} - \frac{e^{\frac{2}{3}\pi i}}{x_1^2}\frac{1}{r^2 - \frac{y_1}{x_1} e^{\frac{2}{3}\pi i}}
\end{aligned} \tag{26}
$$

I, b_2**:** pole at $w = x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}$.

$$
\begin{aligned}
&\operatorname{Res}\left(J_1\left(x_1, y_1, r; w\right)\right)_{w=x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}} = \\
= & -\frac{x_1^{-2} r^2 e^{-\frac{2}{3}\pi i}}{x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}}\left(\frac{r^2}{x_1^{-1} r^2 e^{-\frac{2}{3}\pi i} - y_1 r^2} + \frac{y_1^{-2}}{x_1^{-1} r^2 e^{-\frac{2}{3}\pi i} - y_1^{-1}} - \frac{y_1^{-2} r^2}{x_1^{-1} r^2 e^{-\frac{2}{3}\pi i} - y_1^{-1} r^2} - \frac{1}{x_1^{-1} r^2 e^{-\frac{2}{3}\pi i} - y_1}\right) \\
= & \frac{e^{\frac{2}{3}\pi i}}{y_1^2}\frac{1}{1 - \frac{x_1}{y_1} e^{\frac{2}{3}\pi i}} - e^{\frac{2}{3}\pi i}\frac{1}{1 - x_1 y_1 e^{\frac{2}{3}\pi i}} + e^{\frac{2}{3}\pi i}\frac{1}{r^2 - x_1 y_1 e^{\frac{2}{3}\pi i}} - \frac{e^{\frac{2}{3}\pi i}}{y_1^2}\frac{1}{r^2 - \frac{x_1}{y_1} e^{\frac{2}{3}\pi i}}
\end{aligned} \tag{27}
$$

II and III, b_2**:** pole at $w = x_1 e^{-\frac{2}{3}\pi i}$.

$$
\begin{aligned}
&\operatorname{Res}\left(J_1\left(x_1, y_1, r; w\right)\right)_{w=x_1 e^{-\frac{2}{3}\pi i}} = \\
= & \frac{-e^{-\frac{2}{3}\pi i}}{x_1 e^{-\frac{2}{3}\pi i}}\left(\frac{r^2}{x_1 e^{-\frac{2}{3}\pi i} - y_1 r^2} + \frac{y_1^{-2}}{x_1 e^{-\frac{2}{3}\pi i} - y_1^{-1}} - \frac{y_1^{-2} r^2}{x_1 e^{-\frac{2}{3}\pi i} - y_1^{-1} r^2} - \frac{1}{x_1 e^{-\frac{2}{3}\pi i} - y_1}\right) \\
= & \frac{1}{x_1 y_1}\left(\frac{1}{1 - x_1 y_1 e^{-\frac{2}{3}\pi i}} - \frac{1}{1 - \frac{x_1}{y_1} e^{-\frac{2}{3}\pi i}}\right) - e^{-\frac{2}{3}\pi i}\frac{1}{r^2 - x_1 y_1 e^{-\frac{2}{3}\pi i}} + \frac{e^{-\frac{2}{3}\pi i}}{y_1^2}\frac{1}{r^2 - \frac{x_1}{y_1} e^{-\frac{2}{3}\pi i}}
\end{aligned} \tag{28}
$$

I, II and III, c_1: pole at $w = x_1 r^2 e^{\frac{2}{3}\pi i}$.

$$\begin{aligned}
&\operatorname{Res}\left(J_1(x_1, y_1, r; w)\right)_{w=x_1 r^2 e^{\frac{2}{3}\pi i}} = \\
&= \frac{1}{x_1}\left(\frac{r^2}{x_1 r^2 e^{\frac{2}{3}\pi i} - y_1 r^2} + \frac{y_1^{-2}}{x_1 r^2 e^{\frac{2}{3}\pi i} - y_1^{-1}} - \frac{y_1^{-2} r^2}{x_1 r^2 e^{\frac{2}{3}\pi i} - y_1^{-1} r^2} - \frac{1}{x_1 r^2 e^{\frac{2}{3}\pi i} - y_1}\right) \\
&= \frac{e^{-\frac{2}{3}\pi i}}{x_1 y_1}\left(\frac{1}{e^{-\frac{2}{3}\pi i} - x_1 y_1} - \frac{1}{e^{-\frac{2}{3}\pi i} - \frac{x_1}{y_1}}\right) + \frac{e^{-\frac{2}{3}\pi i}}{x_1^2 y_1^2}\frac{1}{r^2 - \frac{1}{x_1 y_1} e^{-\frac{2}{3}\pi i}} - \frac{e^{-\frac{2}{3}\pi i}}{x_1^2}\frac{1}{r^2 - \frac{y_1}{x_1} e^{-\frac{2}{3}\pi i}} \qquad (29)
\end{aligned}$$

I, c_2: pole at $w = x_1^{-1} r^2 e^{\frac{2}{3}\pi i}$.

$$\begin{aligned}
&\operatorname{Res}\left(J_1(x_1, y_1, r; w)\right)_{w=x_1^{-1} r^2 e^{\frac{2}{3}\pi i}} = \\
&= -\frac{1}{x_1}\left(\frac{r^2}{x_1^{-1} r^2 e^{\frac{2}{3}\pi i} - y_1 r^2} + \frac{y_1^{-2}}{x_1^{-1} r^2 e^{\frac{2}{3}\pi i} - y_1^{-1}} - \frac{y_1^{-2} r^2}{x_1^{-1} r^2 e^{\frac{2}{3}\pi i} - y_1^{-1} r^2} - \frac{1}{x_1^{-1} r^2 e^{\frac{2}{3}\pi i} - y_1}\right) \\
&= \frac{e^{-\frac{2}{3}\pi i}}{y_1^2}\frac{1}{1 - \frac{x_1}{y_1} e^{-\frac{2}{3}\pi i}} - e^{-\frac{2}{3}\pi i}\frac{1}{1 - x_1 y_1 e^{-\frac{2}{3}\pi i}} + e^{-\frac{2}{3}\pi i}\frac{1}{r^2 - x_1 y_1 e^{-\frac{2}{3}\pi i}} - \frac{e^{-\frac{2}{3}\pi i}}{y_1^2}\frac{1}{r^2 - \frac{x_1}{y_1} e^{-\frac{2}{3}\pi i}} \qquad (30)
\end{aligned}$$

II and III, c_2: pole at $w = x_1 e^{\frac{2}{3}\pi i}$.

$$\begin{aligned}
&\operatorname{Res}\left(J_1(x_1, y_1, r; w)\right)_{w=x_1 e^{\frac{2}{3}\pi i}} = \\
&= -\frac{1}{x_1}\left(\frac{r^2}{x_1 e^{\frac{2}{3}\pi i} - y_1 r^2} + \frac{y_1^{-2}}{x_1 e^{\frac{2}{3}\pi i} - y_1^{-1}} - \frac{y_1^{-2} r^2}{x_1 e^{\frac{2}{3}\pi i} - y_1^{-1} r^2} - \frac{1}{x_1 e^{\frac{2}{3}\pi i} - y_1}\right) \\
&= \frac{1}{x_1 y_1}\left(\frac{1}{\frac{x_1}{y_1} e^{\frac{2}{3}\pi i} - 1} - \frac{1}{x_1 y_1 e^{\frac{2}{3}\pi i} - 1}\right) - e^{\frac{2}{3}\pi i}\frac{1}{r^2 - x_1 y_1 e^{\frac{2}{3}\pi i}} + \frac{e^{\frac{2}{3}\pi i}}{y_1^2}\frac{1}{r^2 - \frac{x_1}{y_1} e^{\frac{2}{3}\pi i}} \qquad (31)
\end{aligned}$$

I, II and III, d_1: pole at $w = y_1 r^2$.

$$
\begin{aligned}
&\operatorname{Res}\Big(J_1\left(x_1, y_1, r; w\right)\Big)_{w=y_1 r^2} = \\
= \;& \frac{1}{y_1}\left(\frac{1}{y_1 - x_1} + \frac{x_1^{-2}}{y_1 r^2 - x_1^{-1}} - \frac{x_1^{-2}}{y_1 - x_1^{-1}} - \frac{1}{y_1 r^2 - x_1} + \right. \\
&+ \frac{e^{-\frac{2}{3}\pi i}}{y_1 - x_1 e^{-\frac{2}{3}\pi i}} + \frac{x_1^{-2} e^{-\frac{2}{3}\pi i}}{y_1 r^2 - x_1^{-1} e^{-\frac{2}{3}\pi i}} - \frac{x_1^{-2} e^{-\frac{2}{3}\pi i}}{y_1 - x_1^{-1} e^{-\frac{2}{3}\pi i}} - \frac{e^{-\frac{2}{3}\pi i}}{y_1 r^2 - x_1 e^{-\frac{2}{3}\pi i}} \\
&\left. + \frac{e^{\frac{2}{3}\pi i}}{y_1 - x_1 e^{\frac{2}{3}\pi i}} + \frac{x_1^{-2} e^{\frac{2}{3}\pi i}}{y_1 r^2 - x_1^{-1} e^{\frac{2}{3}\pi i}} - \frac{x_1^{-2} e^{\frac{2}{3}\pi i}}{y_1 - x_1^{-1} e^{\frac{2}{3}\pi i}} - \frac{e^{\frac{2}{3}\pi i}}{y_1 r^2 - x_1 e^{\frac{2}{3}\pi i}}\right) \\
= \;& \sum_{k=0}^{2}\left(\frac{1}{y_1^2} \frac{e^{\frac{2}{3}k\pi i}}{1 - \frac{x_1}{y_1} e^{\frac{2}{3}k\pi i}} + \frac{1}{x_1 y_1} \frac{e^{\frac{2}{3}k\pi i}}{e^{\frac{2}{3}k\pi i} - x_1 y_1}\right) + \\
&\qquad + \sum_{k=0}^{2}\left(\frac{1}{x_1^2 y_1^2} \frac{e^{\frac{2}{3}k\pi i}}{r^2 - \frac{1}{x_1 y_1} e^{\frac{2}{3}k\pi i}} - \frac{1}{y_1^2} \frac{e^{\frac{2}{3}k\pi i}}{r^2 - \frac{x_1}{y_1} e^{\frac{2}{3}k\pi i}}\right)
\end{aligned} \tag{32}
$$

I and II, d_2: pole at $w = y_1^{-1} r^2$.

$$
\begin{aligned}
&\operatorname{Res}\Big(J_1\left(x_1, y_1, r; w\right)\Big)_{w=y_1^{-1} r^2} = \\
= \;& -\frac{1}{y_1}\left(\frac{1}{y_1^{-1} - x_1} + \frac{x_1^{-2}}{y_1^{-1} r^2 - x_1^{-1}} - \frac{x_1^{-2}}{y_1^{-1} - x_1^{-1}} - \frac{1}{y_1^{-1} r^2 - x_1} + \right. \\
&+ \frac{e^{-\frac{2}{3}\pi i}}{y_1^{-1} - x_1 e^{-\frac{2}{3}\pi i}} + \frac{x_1^{-2} e^{-\frac{2}{3}\pi i}}{y_1^{-1} r^2 - x_1^{-1} e^{-\frac{2}{3}\pi i}} - \frac{x_1^{-2} e^{-\frac{2}{3}\pi i}}{y_1^{-1} - x_1^{-1} e^{-\frac{2}{3}\pi i}} - \frac{e^{-\frac{2}{3}\pi i}}{y_1^{-1} r^2 - x_1 e^{-\frac{2}{3}\pi i}} \\
&\left. + \frac{e^{\frac{2}{3}\pi i}}{y_1^{-1} - x_1 e^{\frac{2}{3}\pi i}} + \frac{x_1^{-2} e^{\frac{2}{3}\pi i}}{y_1^{-1} r^2 - x_1^{-1} e^{\frac{2}{3}\pi i}} - \frac{x_1^{-2} e^{\frac{2}{3}\pi i}}{y_1^{-1} - x_1^{-1} e^{\frac{2}{3}\pi i}} - \frac{e^{\frac{2}{3}\pi i}}{y_1^{-1} r^2 - x_1 e^{\frac{2}{3}\pi i}}\right) \\
= \;& -\sum_{k=0}^{2}\left(\frac{1}{x_1 y_1} \frac{1}{1 - \frac{x_1}{y_1} e^{\frac{2}{3}k\pi i}} + \frac{e^{\frac{2}{3}k\pi i}}{1 - x_1 y_1 e^{\frac{2}{3}k\pi i}}\right) + \\
&\qquad + \sum_{k=0}^{2}\left(\frac{e^{\frac{2}{3}k\pi i}}{r^2 - x_1 y_1 e^{\frac{2}{3}k\pi i}} - \frac{1}{x_1^2} \frac{e^{\frac{2}{3}k\pi i}}{r^2 - \frac{y_1}{x_1} e^{\frac{2}{3}k\pi i}}\right)
\end{aligned} \tag{33}
$$

III, d_2: pole at $w = y_1$.

$$\begin{aligned}
&\operatorname{Res}\Big(J_1(x_1, y_1, r; w)\Big)_{w=y_1} = \\
= \; & -\frac{1}{y_1}\left(\frac{r^2}{y_1 - x_1 r^2} + \frac{x_1^{-2}}{y_1 - x_1^{-1}} - \frac{x_1^{-2} r^2}{y_1 - x_1^{-1} r^2} - \frac{1}{y_1 - x_1} + \right. \\
& + \frac{r^2 e^{-\frac{2}{3}\pi i}}{y_1 - x_1 r^2 e^{-\frac{2}{3}\pi i}} + \frac{x_1^{-2} e^{-\frac{2}{3}\pi i}}{y_1 - x_1^{-1} e^{-\frac{2}{3}\pi i}} - \frac{x_1^{-2} r^2 e^{-\frac{2}{3}\pi i}}{y_1 - x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}} - \frac{e^{-\frac{2}{3}\pi i}}{y_1 - x_1 e^{-\frac{2}{3}\pi i}} \\
& \left. + \frac{r^2 e^{\frac{2}{3}\pi i}}{y_1 - x_1 r^2 e^{\frac{2}{3}\pi i}} + \frac{x_1^{-2} e^{\frac{2}{3}\pi i}}{y_1 - x_1^{-1} e^{\frac{2}{3}\pi i}} - \frac{x_1^{-2} r^2 e^{\frac{2}{3}\pi i}}{y_1 - x_1^{-1} r^2 e^{\frac{2}{3}\pi i}} - \frac{e^{\frac{2}{3}\pi i}}{y_1 - x_1 e^{\frac{2}{3}\pi i}} \right) \\
= \; & \sum_{k=0}^{2}\left(\frac{1}{x_1 y_1}\frac{1}{1 - x_1 y_1 e^{\frac{2}{3}k\pi i}} + \frac{1}{y_1^2}\frac{e^{\frac{2}{3}k\pi i}}{1 - \frac{x_1}{y_1} e^{\frac{2}{3}k\pi i}}\right) + \\
& \qquad + \sum_{k=0}^{2}\left(-\frac{e^{\frac{2}{3}k\pi i}}{r^2 - x_1 y_1 e^{\frac{2}{3}k\pi i}} + \frac{1}{x_1^2}\frac{e^{\frac{2}{3}k\pi i}}{r^2 - \frac{y_1}{x_1} e^{\frac{2}{3}k\pi i}}\right)
\end{aligned} \tag{34}$$

4.4. Integration in the radial direction

It remains to combine the appropriate residues and to integrate them with respect to r. In doing so, we integrate r in three steps; from 0 to x_1, from x_1 to y_1 and from y_1 to 1. In each step we list the appropriate residues and we rewrite them as partial fractions with each denominator linear in r^2.

- For $r \in (0, x_1)$ this procedure leads to the integrand

$$I_1(x_1, y_1; r) = \#_{(23)} + \#_{(24)} + \#_{(26)} + \#_{(27)} + \#_{(29)} + \#_{(30)} + \#_{(32)} + \#_{(33)} =$$

$$\begin{aligned}
= \; & \frac{1}{x_1 y_1}\left(\frac{1}{1 - x_1 y_1} - \frac{1}{1 - \frac{x_1}{y_1}}\right) + \frac{1}{x_1^2 y_1^2}\frac{1}{r^2 - \frac{1}{x_1 y_1}} - \frac{1}{x_1^2}\frac{1}{r^2 - \frac{y_1}{x_1}} + \\
& + \frac{1}{y_1^2}\frac{1}{1 - \frac{x_1}{y_1}} - \frac{1}{1 - x_1 y_1} + \frac{1}{r^2 - x_1 y_1} - \frac{1}{y_1^2}\frac{1}{r^2 - \frac{x_1}{y_1}} + \\
& + \frac{e^{\frac{2}{3}\pi i}}{x_1 y_1}\left(\frac{1}{e^{\frac{2}{3}\pi i} - x_1 y_1} - \frac{1}{e^{\frac{2}{3}\pi i} - \frac{x_1}{y_1}}\right) + \frac{e^{\frac{2}{3}\pi i}}{x_1^2 y_1^2}\frac{1}{r^2 - \frac{1}{x_1 y_1} e^{\frac{2}{3}\pi i}} - \frac{e^{\frac{2}{3}\pi i}}{x_1^2}\frac{1}{r^2 - \frac{y_1}{x_1} e^{\frac{2}{3}\pi i}} + \\
& + \frac{e^{\frac{2}{3}\pi i}}{y_1^2}\frac{1}{1 - \frac{x_1}{y_1} e^{\frac{2}{3}\pi i}} - e^{\frac{2}{3}\pi i}\frac{1}{1 - x_1 y_1 e^{\frac{2}{3}\pi i}} + e^{\frac{2}{3}\pi i}\frac{1}{r^2 - x_1 y_1 e^{\frac{2}{3}\pi i}} - \frac{e^{\frac{2}{3}\pi i}}{y_1^2}\frac{1}{r^2 - \frac{x_1}{y_1} e^{\frac{2}{3}\pi i}} + \\
& + \frac{e^{-\frac{2}{3}\pi i}}{x_1 y_1}\left(\frac{1}{e^{-\frac{2}{3}\pi i} - x_1 y_1} - \frac{1}{e^{-\frac{2}{3}\pi i} - \frac{x_1}{y_1}}\right) + \frac{e^{-\frac{2}{3}\pi i}}{x_1^2 y_1^2}\frac{1}{r^2 - \frac{1}{x_1 y_1} e^{-\frac{2}{3}\pi i}} - \frac{e^{-\frac{2}{3}\pi i}}{x_1^2}\frac{1}{r^2 - \frac{y_1}{x_1} e^{-\frac{2}{3}\pi i}} + \\
& + \frac{e^{-\frac{2}{3}\pi i}}{y_1^2}\frac{1}{1 - \frac{x_1}{y_1} e^{-\frac{2}{3}\pi i}} - e^{-\frac{2}{3}\pi i}\frac{1}{1 - x_1 y_1 e^{-\frac{2}{3}\pi i}} + e^{-\frac{2}{3}\pi i}\frac{1}{r^2 - x_1 y_1 e^{-\frac{2}{3}\pi i}} - \frac{e^{-\frac{2}{3}\pi i}}{y_1^2}\frac{1}{r^2 - \frac{x_1}{y_1} e^{-\frac{2}{3}\pi i}} +
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{y_1^2}\left(\frac{1}{1-\frac{x_1}{y_1}}+\frac{e^{-\frac{2}{3}\pi i}}{1-\frac{x_1}{y_1}e^{-\frac{2}{3}\pi i}}+\frac{e^{\frac{2}{3}\pi i}}{1-\frac{x_1}{y_1}e^{\frac{2}{3}\pi i}}\right)+\frac{1}{x_1y_1}\left(\frac{1}{1-x_1y_1}+\frac{e^{-\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i}-x_1y_1}+\frac{e^{\frac{2}{3}\pi i}}{e^{\frac{2}{3}\pi i}-x_1y_1}\right)+\\
&+\frac{1}{x_1^2y_1^2}\left(\frac{1}{r^2-\frac{1}{x_1y_1}}+\frac{e^{-\frac{2}{3}\pi i}}{r^2-\frac{1}{x_1y_1}e^{-\frac{2}{3}\pi i}}+\frac{e^{\frac{2}{3}\pi i}}{r^2-\frac{1}{x_1y_1}e^{\frac{2}{3}\pi i}}\right)+\\
&-\frac{1}{y_1^2}\left(\frac{1}{r^2-\frac{x_1}{y_1}}+\frac{e^{-\frac{2}{3}\pi i}}{r^2-\frac{x_1}{y_1}e^{-\frac{2}{3}\pi i}}+\frac{e^{\frac{2}{3}\pi i}}{r^2-\frac{x_1}{y_1}e^{\frac{2}{3}\pi i}}\right)+\\
&-\frac{1}{x_1y_1}\left(\frac{1}{1-\frac{x_1}{y_1}}+\frac{1}{1-\frac{x_1}{y_1}e^{\frac{2}{3}\pi i}}+\frac{1}{1-\frac{x_1}{y_1}e^{-\frac{2}{3}\pi i}}\right)-\frac{1}{1-x_1y_1}-\frac{e^{-\frac{2}{3}\pi i}}{1-x_1y_1e^{-\frac{2}{3}\pi i}}-\frac{e^{\frac{2}{3}\pi i}}{1-x_1y_1e^{\frac{2}{3}\pi i}}+\\
&+\frac{1}{r^2-x_1y_1}+\frac{e^{\frac{2}{3}\pi i}}{r^2-x_1y_1e^{\frac{2}{3}\pi i}}+\frac{e^{-\frac{2}{3}\pi i}}{r^2-x_1y_1e^{-\frac{2}{3}\pi i}}-\frac{1}{x_1^2}\left(\frac{1}{r^2-\frac{y_1}{x_1}}+\frac{e^{\frac{2}{3}\pi i}}{r^2-\frac{y_1}{x_1}e^{\frac{2}{3}\pi i}}+\frac{e^{-\frac{2}{3}\pi i}}{r^2-\frac{y_1}{x_1}e^{-\frac{2}{3}\pi i}}\right)\\
&\quad=\frac{2}{x_1y_1}\frac{1}{1-x_1y_1}-\frac{2}{1-x_1y_1}-\frac{2}{x_1y_1}\frac{1}{1-\frac{x_1}{y_1}}+\frac{2}{y_1^2}\frac{1}{1-\frac{x_1}{y_1}}+\\
&\quad+\frac{2}{r^2-x_1y_1}-\frac{2}{x_1^2}\frac{1}{r^2-\frac{y_1}{x_1}}-\frac{2}{y_1^2}\frac{1}{r^2-\frac{x_1}{y_1}}+\frac{2}{x_1^2y_1^2}\frac{1}{r^2-\frac{1}{x_1y_1}}+\\
&\quad+\frac{2}{x_1y_1}\frac{e^{\frac{2}{3}\pi i}}{e^{\frac{2}{3}\pi i}-x_1y_1}-\frac{2}{x_1y_1}\frac{e^{\frac{2}{3}\pi i}}{e^{\frac{2}{3}\pi i}-\frac{x_1}{y_1}}+\frac{2}{y_1^2}\frac{e^{\frac{2}{3}\pi i}}{1-\frac{x_1}{y_1}e^{\frac{2}{3}\pi i}}-\frac{2e^{\frac{2}{3}\pi i}}{1-x_1y_1e^{\frac{2}{3}\pi i}}+\\
&\quad+\frac{2e^{\frac{2}{3}\pi i}}{r^2-x_1y_1e^{\frac{2}{3}\pi i}}-\frac{2}{x_1^2}\frac{e^{\frac{2}{3}\pi i}}{r^2-\frac{y_1}{x_1}e^{\frac{2}{3}\pi i}}-\frac{2}{y_1^2}\frac{e^{\frac{2}{3}\pi i}}{r^2-\frac{x_1}{y_1}e^{\frac{2}{3}\pi i}}+\frac{2}{x_1^2y_1^2}\frac{e^{\frac{2}{3}\pi i}}{r^2-\frac{1}{x_1y_1}e^{\frac{2}{3}\pi i}}+\\
&\quad+\frac{2}{x_1y_1}\frac{e^{-\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i}-x_1y_1}-\frac{2}{x_1y_1}\frac{e^{-\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i}-\frac{x_1}{y_1}}-\frac{2e^{-\frac{2}{3}\pi i}}{1-x_1y_1e^{-\frac{2}{3}\pi i}}+\frac{2}{y_1^2}\frac{e^{-\frac{2}{3}\pi i}}{1-\frac{x_1}{y_1}e^{-\frac{2}{3}\pi i}}+\\
&\quad+\frac{2e^{-\frac{2}{3}\pi i}}{r^2-x_1y_1e^{-\frac{2}{3}\pi i}}-\frac{2}{x_1^2}\frac{e^{-\frac{2}{3}\pi i}}{r^2-\frac{y_1}{x_1}e^{-\frac{2}{3}\pi i}}-\frac{2}{y_1^2}\frac{e^{-\frac{2}{3}\pi i}}{r^2-\frac{x_1}{y_1}e^{-\frac{2}{3}\pi i}}+\frac{2}{x_1^2y_1^2}\frac{e^{-\frac{2}{3}\pi i}}{r^2-\frac{1}{x_1y_1}e^{-\frac{2}{3}\pi i}}.\\
&=\sum_{k=0}^{2}\left(\frac{2e^{\frac{2}{3}k\pi i}}{r^2-x_1y_1e^{\frac{2}{3}k\pi i}}+\frac{2}{x_1^2y_1^2}\frac{e^{\frac{2}{3}k\pi i}}{r^2-\frac{1}{x_1y_1}e^{\frac{2}{3}k\pi i}}-\frac{2}{x_1^2}\frac{e^{\frac{2}{3}k\pi i}}{r^2-\frac{y_1}{x_1}e^{\frac{2}{3}k\pi i}}+\frac{2}{y_1^2}\frac{e^{\frac{2}{3}k\pi i}}{r^2-\frac{x_1}{y_1}e^{\frac{2}{3}k\pi i}}\right),
\end{aligned}
$$

so that, after a tedious calculation

$$\int_{r=0}^{x_1} I_1\left(x_1, y_1; r\right)\, r\, dr \;=$$

$$
\begin{aligned}
= \; & \left[\ln\left(x_1y_1 - r^2\right) - \frac{1}{x_1^2}\ln\left(\frac{y_1}{x_1} - r^2\right) - \frac{1}{y_1^2}\ln\left(\frac{x_1}{y_1} - r^2\right) + \frac{1}{x_1^2y_1^2}\ln\left(\frac{1}{x_1y_1} - r^2\right)\right]_{r=0}^{x_1} + \\
& + \left[e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(x_1y_1 - r^2e^{-\frac{2}{3}\pi i}\right) - \frac{e^{\frac{2}{3}\pi i}}{x_1^2}\mathrm{Ln}\left(\frac{y_1}{x_1} - r^2e^{-\frac{2}{3}\pi i}\right)\right]_{r=0}^{x_1} + \\
& + \left[-\frac{e^{\frac{2}{3}\pi i}}{y_1^2}\mathrm{Ln}\left(\frac{x_1}{y_1} - r^2e^{-\frac{2}{3}\pi i}\right) + \frac{e^{\frac{2}{3}\pi i}}{x_1^2y_1^2}\mathrm{Ln}\left(\frac{1}{x_1y_1} - r^2e^{-\frac{2}{3}\pi i}\right)\right]_{r=0}^{x_1} + \\
& + \left[e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(x_1y_1 - r^2e^{\frac{2}{3}\pi i}\right) - \frac{e^{-\frac{2}{3}\pi i}}{x_1^2}\mathrm{Ln}\left(\frac{y_1}{x_1} - r^2e^{\frac{2}{3}\pi i}\right)\right]_{r=0}^{x_1} + \\
& + \left[-\frac{e^{-\frac{2}{3}\pi i}}{y_1^2}\mathrm{Ln}\left(\frac{x_1}{y_1} - r^2e^{\frac{2}{3}\pi i}\right) + \frac{e^{-\frac{2}{3}\pi i}}{x_1^2y_1^2}\mathrm{Ln}\left(\frac{1}{x_1y_1} - r^2e^{\frac{2}{3}\pi i}\right)\right]_{r=0}^{x_1}
\end{aligned}
$$

$$
\begin{aligned}
= \; & \ln\left(1 - \frac{x_1}{y_1}\right) - \frac{1}{x_1^2}\ln\left(1 - \frac{x_1^3}{y_1}\right) - \frac{1}{y_1^2}\ln\left(1 - x_1y_1\right) + \frac{1}{x_1^2y_1^2}\ln\left(1 - x_1^3y_1\right) + \\
& + e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(1 - \frac{x_1e^{-\frac{2}{3}\pi i}}{y_1}\right) - \frac{e^{\frac{2}{3}\pi i}}{x_1^2}\mathrm{Ln}\left(1 - \frac{x_1^3e^{-\frac{2}{3}\pi i}}{y_1}\right) + \\
& - \frac{e^{\frac{2}{3}\pi i}}{y_1^2}\mathrm{Ln}\left(1 - x_1y_1e^{-\frac{2}{3}\pi i}\right) + \frac{e^{\frac{2}{3}\pi i}}{x_1^2y_1^2}\mathrm{Ln}\left(1 - x_1^3y_1e^{-\frac{2}{3}\pi i}\right) + \\
& + e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(1 - \frac{x_1e^{\frac{2}{3}\pi i}}{y_1}\right) - \frac{e^{-\frac{2}{3}\pi i}}{x_1^2}\mathrm{Ln}\left(1 - \frac{x_1^3e^{\frac{2}{3}\pi i}}{y_1}\right) + \\
& - \frac{e^{-\frac{2}{3}\pi i}}{y_1^2}\mathrm{Ln}\left(1 - x_1y_1e^{\frac{2}{3}\pi i}\right) + \frac{e^{-\frac{2}{3}\pi i}}{x_1^2y_1^2}\mathrm{Ln}\left(1 - x_1^3y_1e^{\frac{2}{3}\pi i}\right)
\end{aligned}
$$

$$
\begin{aligned}
= \; & \ln\left(1 - \frac{x_1}{y_1}\right) - \frac{1}{x_1^2}\ln\left(1 - \frac{x_1^3}{y_1}\right) - \frac{1}{y_1^2}\ln\left(1 - x_1y_1\right) + \frac{1}{x_1^2y_1^2}\ln\left(1 - x_1^3y_1\right) + \\
& + e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(1 + \frac{x_1}{2y_1} + i\frac{x_1\sqrt{3}}{2y_1}\right) - \frac{e^{\frac{2}{3}\pi i}}{x_1^2}\mathrm{Ln}\left(1 + \frac{x_1^3}{2y_1} + i\frac{x_1^3\sqrt{3}}{2y_1}\right) + \\
& - \frac{e^{\frac{2}{3}\pi i}}{y_1^2}\mathrm{Ln}\left(1 + \frac{x_1y_1}{2} + i\frac{x_1y_1\sqrt{3}}{2}\right) + \frac{e^{\frac{2}{3}\pi i}}{x_1^2y_1^2}\mathrm{Ln}\left(1 + \frac{x_1^3y_1}{2} + i\frac{x_1^3y_1\sqrt{3}}{2}\right) + \\
& + e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(1 + \frac{x_1}{2y_1} - i\frac{x_1\sqrt{3}}{2y_1}\right) - \frac{e^{-\frac{2}{3}\pi i}}{x_1^2}\mathrm{Ln}\left(1 + \frac{x_1^3}{2y_1} - i\frac{x_1^3\sqrt{3}}{2y_1}\right) + \\
& - \frac{e^{-\frac{2}{3}\pi i}}{y_1^2}\mathrm{Ln}\left(1 + \frac{x_1y_1}{2} - i\frac{x_1y_1\sqrt{3}}{2}\right) + \frac{e^{-\frac{2}{3}\pi i}}{x_1^2y_1^2}\mathrm{Ln}\left(1 + \frac{x_1^3y_1}{2} - i\frac{x_1^3y_1\sqrt{3}}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
= \;& \ln\left(1-\frac{x_1}{y_1}\right)-\frac{1}{x_1^2}\ln\left(1-\frac{x_1^3}{y_1}\right)-\frac{1}{y_1^2}\ln\left(1-x_1y_1\right)+\frac{1}{x_1^2y_1^2}\ln\left(1-x_1^3y_1\right)+\\
&+e^{\frac{2}{3}\pi i}\ln\sqrt{\left(1+\frac{x_1}{2y_1}\right)^2+\left(\frac{x_1\sqrt{3}}{2y_1}\right)^2}+ie^{\frac{2}{3}\pi i}\arctan\left(\frac{\frac{x_1\sqrt{3}}{2y_1}}{1+\frac{x_1}{2y_1}}\right)+\\
&-\frac{e^{\frac{2}{3}\pi i}}{x_1^2}\ln\sqrt{\left(1+\frac{x_1^3}{2y_1}\right)^2+\left(\frac{x_1^3\sqrt{3}}{2y_1}\right)^2}-i\frac{e^{\frac{2}{3}\pi i}}{x_1^2}\arctan\left(\frac{\frac{x_1^3\sqrt{3}}{2y_1}}{1+\frac{x_1^3}{2y_1}}\right)+\\
&-\frac{e^{\frac{2}{3}\pi i}}{y_1^2}\ln\sqrt{\left(1+\frac{x_1y_1}{2}\right)^2+\left(\frac{x_1y_1\sqrt{3}}{2}\right)^2}-i\frac{e^{\frac{2}{3}\pi i}}{y_1^2}\arctan\left(\frac{\frac{x_1y_1\sqrt{3}}{2}}{1+\frac{x_1y_1}{2}}\right)+\\
&+\frac{e^{\frac{2}{3}\pi i}}{x_1^2y_1^2}\ln\sqrt{\left(1+\frac{x_1^3y_1}{2}\right)^2+\left(\frac{x_1^3y_1\sqrt{3}}{2}\right)^2}+i\frac{e^{\frac{2}{3}\pi i}}{x_1^2y_1^2}\arctan\left(\frac{\frac{x_1^3y_1\sqrt{3}}{2}}{1+\frac{x_1^3y_1}{2}}\right)+\\
&+e^{-\frac{2}{3}\pi i}\ln\sqrt{\left(1+\frac{x_1}{2y_1}\right)^2+\left(-\frac{x_1\sqrt{3}}{2y_1}\right)^2}+ie^{-\frac{2}{3}\pi i}\arctan\left(\frac{-\frac{x_1\sqrt{3}}{2y_1}}{1+\frac{x_1}{2y_1}}\right)+\\
&-\frac{e^{-\frac{2}{3}\pi i}}{x_1^2}\ln\sqrt{\left(1+\frac{x_1^3}{2y_1}\right)^2+\left(-\frac{x_1^3\sqrt{3}}{2y_1}\right)^2}-i\frac{e^{-\frac{2}{3}\pi i}}{x_1^2}\arctan\left(\frac{-\frac{x_1^3\sqrt{3}}{2y_1}}{1+\frac{x_1^3}{2y_1}}\right)+\\
&-\frac{e^{-\frac{2}{3}\pi i}}{y_1^2}\ln\sqrt{\left(1+\frac{x_1y_1}{2}\right)^2+\left(-\frac{x_1y_1\sqrt{3}}{2}\right)^2}-i\frac{e^{-\frac{2}{3}\pi i}}{y_1^2}\arctan\left(\frac{-\frac{x_1y_1\sqrt{3}}{2}}{1+\frac{x_1y_1}{2}}\right)+\\
&+\frac{e^{-\frac{2}{3}\pi i}}{x_1^2y_1^2}\ln\sqrt{\left(1+\frac{x_1^3y_1}{2}\right)^2+\left(-\frac{x_1^3y_1\sqrt{3}}{2}\right)^2}+i\frac{e^{-\frac{2}{3}\pi i}}{x_1^2y_1^2}\arctan\left(\frac{-\frac{x_1^3y_1\sqrt{3}}{2}}{1+\frac{x_1^3y_1}{2}}\right)
\end{aligned}
$$

$$
\begin{aligned}
= \;& \ln\left(1-\frac{x_1}{y_1}\right)-\frac{1}{x_1^2}\ln\left(1-\frac{x_1^3}{y_1}\right)-\frac{1}{y_1^2}\ln\left(1-x_1y_1\right)+\frac{1}{x_1^2y_1^2}\ln\left(1-x_1^3y_1\right)+\\
&-\ln\sqrt{\frac{y_1^2+x_1y_1+x_1^2}{y_1^2}}-\sqrt{3}\arctan\left(\frac{x_1\sqrt{3}}{2y_1+x_1}\right)+\\
&+\frac{1}{x_1^2}\ln\sqrt{\frac{y_1^2+y_1x_1^3+x_1^6}{y_1^2}}+\frac{\sqrt{3}}{x_1^2}\arctan\left(\frac{x_1^3\sqrt{3}}{2y_1+x_1^3}\right)+\\
&+\frac{1}{y_1^2}\ln\sqrt{1+x_1y_1+x_1^2y_1^2}+\frac{\sqrt{3}}{y_1^2}\arctan\left(\frac{x_1y_1\sqrt{3}}{2+x_1y_1}\right)+\\
&-\frac{1}{x_1^2y_1^2}\ln\sqrt{1+y_1x_1^3+x_1^6y_1^2}-\frac{\sqrt{3}}{x_1^2y_1^2}\arctan\left(\frac{x_1^3y_1\sqrt{3}}{2+y_1x_1^3}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \ln\left(1-\frac{x_1}{y_1}\right) - \ln\sqrt{\frac{y_1^2+x_1y_1+x_1^2}{y_1^2}} + \\
&\quad + \frac{1}{x_1^2}\left(\ln\sqrt{\frac{y_1^2+y_1x_1^3+x_1^6}{y_1^2}} - \ln\left(1-\frac{x_1^3}{y_1}\right)\right) + \\
&\quad + \frac{1}{y_1^2}\left(\ln\sqrt{1+x_1y_1+x_1^2y_1^2} - \ln\left(1-x_1y_1\right)\right) + \\
&\quad + \frac{1}{x_1^2y_1^2}\left(\ln\left(1-x_1^3y_1\right) - \ln\sqrt{1+y_1x_1^3+x_1^6y_1^2}\right) + \\
&\quad - \sqrt{3}\arctan\left(\frac{x_1\sqrt{3}}{2y_1+x_1}\right) + \frac{\sqrt{3}}{x_1^2}\arctan\left(\frac{x_1^3\sqrt{3}}{2y_1+x_1^3}\right) + \\
&\quad + \frac{\sqrt{3}}{y_1^2}\arctan\left(\frac{x_1y_1\sqrt{3}}{2+x_1y_1}\right) - \frac{\sqrt{3}}{x_1^2y_1^2}\arctan\left(\frac{x_1^3y_1\sqrt{3}}{2+y_1x_1^3}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\ln\left(\frac{x_1^2-2x_1y_1+y_1^2}{x_1^2+x_1y_1+y_1^2}\right) - \frac{1}{2y_1^2}\ln\left(\frac{1-2x_1y_1+x_1^2y_1^2}{1+x_1y_1+x_1^2y_1^2}\right) + \\
&\quad - \frac{1}{2x_1^2}\ln\left(\frac{x_1^6-2x_1^3y_1+y_1^2}{x_1^6+x_1^3y_1+y_1^2}\right) + \frac{1}{2x_1^2y_1^2}\ln\left(\frac{1-2x_1^3y_1+y_1^2x_1^6}{1+x_1^3y_1+x_1^6y_1^2}\right) + \\
&\quad - \sqrt{3}\arctan\left(\frac{x_1\sqrt{3}}{2y_1+x_1}\right) + \frac{\sqrt{3}}{x_1^2}\arctan\left(\frac{x_1^3\sqrt{3}}{2y_1+x_1^3}\right) + \\
&\quad + \frac{\sqrt{3}}{y_1^2}\arctan\left(\frac{x_1y_1\sqrt{3}}{2+x_1y_1}\right) - \frac{\sqrt{3}}{x_1^2y_1^2}\arctan\left(\frac{x_1^3y_1\sqrt{3}}{2+y_1x_1^3}\right).
\end{aligned}
\tag{35}
$$

- For $r \in (x_1, y_1)$ the integrand becomes

$$
I_2(x_1, y_1; r) = \#_{(23)} + \#_{(25)} + \#_{(26)} + \#_{(28)} + \#_{(29)} + \#_{(31)} + \#_{(32)} + \#_{(33)}
$$

$$
\begin{aligned}
&= \frac{1}{x_1y_1}\left(\frac{1}{1-x_1y_1} - \frac{1}{1-\frac{x_1}{y_1}}\right) + \frac{1}{x_1^2y_1^2}\frac{1}{r^2-\frac{1}{x_1y_1}} - \frac{1}{x_1^2}\frac{1}{r^2-\frac{y_1}{x_1}} + \\
&\quad - \frac{1}{x_1y_1}\left(\frac{1}{1-\frac{x_1}{y_1}} - \frac{1}{1-x_1y_1}\right) - \frac{1}{r^2-x_1y_1} + \frac{1}{y_1^2}\frac{1}{r^2-\frac{x_1}{y_1}} + \\
&\quad + \frac{e^{\frac{2}{3}\pi i}}{x_1y_1}\left(\frac{1}{e^{\frac{2}{3}\pi i}-x_1y_1} - \frac{1}{e^{\frac{2}{3}\pi i}-\frac{x_1}{y_1}}\right) + \frac{e^{\frac{2}{3}\pi i}}{x_1^2y_1^2}\frac{1}{r^2-\frac{1}{x_1y_1}e^{\frac{2}{3}\pi i}} - \frac{e^{\frac{2}{3}\pi i}}{x_1^2}\frac{1}{r^2-\frac{y_1}{x_1}e^{\frac{2}{3}\pi i}} + \\
&\quad + \frac{1}{x_1y_1}\left(\frac{1}{1-x_1y_1e^{-\frac{2}{3}\pi i}} - \frac{1}{1-\frac{x_1}{y_1}e^{-\frac{2}{3}\pi i}}\right) - \frac{e^{-\frac{2}{3}\pi i}}{r^2-x_1y_1e^{-\frac{2}{3}\pi i}} + \frac{e^{-\frac{2}{3}\pi i}}{y_1^2}\frac{1}{r^2-\frac{x_1}{y_1}e^{-\frac{2}{3}\pi i}} + \\
&\quad + \frac{e^{-\frac{2}{3}\pi i}}{x_1y_1}\left(\frac{1}{e^{-\frac{2}{3}\pi i}-x_1y_1} - \frac{1}{e^{-\frac{2}{3}\pi i}-\frac{x_1}{y_1}}\right) + \frac{e^{-\frac{2}{3}\pi i}}{x_1^2y_1^2}\frac{1}{r^2-\frac{1}{x_1y_1}e^{-\frac{2}{3}\pi i}} - \frac{e^{-\frac{2}{3}\pi i}}{x_1^2}\frac{1}{r^2-\frac{y_1}{x_1}e^{-\frac{2}{3}\pi i}} +
\end{aligned}
$$

$$+\frac{1}{x_1y_1}\left(\frac{1}{\frac{x_1}{y_1}e^{\frac{2}{3}\pi i}-1}-\frac{1}{x_1y_1e^{\frac{2}{3}\pi i}-1}\right)-\frac{e^{\frac{2}{3}\pi i}}{r^2-x_1y_1e^{\frac{2}{3}\pi i}}+\frac{e^{\frac{2}{3}\pi i}}{y_1^2}\frac{1}{r^2-\frac{x_1}{y_1}e^{\frac{2}{3}\pi i}}+$$
$$+\frac{1}{y_1^2}\left(\frac{1}{1-\frac{x_1}{y_1}}+\frac{e^{-\frac{2}{3}\pi i}}{1-\frac{x_1}{y_1}e^{-\frac{2}{3}\pi i}}+\frac{e^{\frac{2}{3}\pi i}}{1-\frac{x_1}{y_1}e^{\frac{2}{3}\pi i}}\right)+\frac{1}{x_1y_1}\left(\frac{1}{1-x_1y_1}+\frac{e^{-\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i}-x_1y_1}+\frac{e^{\frac{2}{3}\pi i}}{e^{\frac{2}{3}\pi i}-x_1y_1}\right)+$$
$$+\frac{1}{x_1^2y_1^2}\left(\frac{1}{r^2-\frac{1}{x_1y_1}}+\frac{e^{-\frac{2}{3}\pi i}}{r^2-\frac{1}{x_1y_1}e^{-\frac{2}{3}\pi i}}+\frac{e^{\frac{2}{3}\pi i}}{r^2-\frac{1}{x_1y_1}e^{\frac{2}{3}\pi i}}\right)-\frac{1}{y_1^2}\left(\frac{1}{r^2-\frac{x_1}{y_1}}+\frac{e^{-\frac{2}{3}\pi i}}{r^2-\frac{x_1}{y_1}e^{-\frac{2}{3}\pi i}}+\frac{e^{\frac{2}{3}\pi i}}{r^2-\frac{x_1}{y_1}e^{\frac{2}{3}\pi i}}\right)+$$
$$-\frac{1}{x_1y_1}\left(\frac{1}{1-\frac{x_1}{y_1}}+\frac{1}{1-\frac{x_1}{y_1}e^{\frac{2}{3}\pi i}}+\frac{1}{1-\frac{x_1}{y_1}e^{-\frac{2}{3}\pi i}}\right)-\frac{1}{1-x_1y_1}-\frac{e^{-\frac{2}{3}\pi i}}{1-x_1y_1e^{-\frac{2}{3}\pi i}}-\frac{e^{\frac{2}{3}\pi i}}{1-x_1y_1e^{\frac{2}{3}\pi i}}+$$
$$+\frac{1}{r^2-x_1y_1}+\frac{e^{\frac{2}{3}\pi i}}{r^2-x_1y_1e^{\frac{2}{3}\pi i}}+\frac{e^{-\frac{2}{3}\pi i}}{r^2-x_1y_1e^{-\frac{2}{3}\pi i}}-\frac{1}{x_1^2}\left(\frac{1}{r^2-\frac{y_1}{x_1}}+\frac{e^{\frac{2}{3}\pi i}}{r^2-\frac{y_1}{x_1}e^{\frac{2}{3}\pi i}}+\frac{e^{-\frac{2}{3}\pi i}}{r^2-\frac{y_1}{x_1}e^{-\frac{2}{3}\pi i}}\right)$$

$$= \frac{1}{x_1y_1}\left(\frac{1}{1-x_1y_1}-\frac{1}{1-\frac{x_1}{y_1}}\right)+\frac{1}{x_1^2y_1^2}\frac{1}{r^2-\frac{1}{x_1y_1}}-\frac{1}{x_1^2}\frac{1}{r^2-\frac{y_1}{x_1}}+$$
$$-\frac{1}{x_1y_1}\left(\frac{1}{1-\frac{x_1}{y_1}}-\frac{1}{1-x_1y_1}\right)+$$
$$+\frac{e^{\frac{2}{3}\pi i}}{x_1y_1}\left(\frac{1}{e^{\frac{2}{3}\pi i}-x_1y_1}-\frac{1}{e^{\frac{2}{3}\pi i}-\frac{x_1}{y_1}}\right)+\frac{e^{\frac{2}{3}\pi i}}{x_1^2y_1^2}\frac{1}{r^2-\frac{1}{x_1y_1}e^{\frac{2}{3}\pi i}}-\frac{e^{\frac{2}{3}\pi i}}{x_1^2}\frac{1}{r^2-\frac{y_1}{x_1}e^{\frac{2}{3}\pi i}}+$$
$$+\frac{1}{x_1y_1}\left(\frac{1}{1-x_1y_1e^{-\frac{2}{3}\pi i}}-\frac{1}{1-\frac{x_1}{y_1}e^{-\frac{2}{3}\pi i}}\right)+$$
$$+\frac{e^{-\frac{2}{3}\pi i}}{x_1y_1}\left(\frac{1}{e^{-\frac{2}{3}\pi i}-x_1y_1}-\frac{1}{e^{-\frac{2}{3}\pi i}-\frac{x_1}{y_1}}\right)+\frac{e^{-\frac{2}{3}\pi i}}{x_1^2y_1^2}\frac{1}{r^2-\frac{1}{x_1y_1}e^{-\frac{2}{3}\pi i}}-\frac{e^{-\frac{2}{3}\pi i}}{x_1^2}\frac{1}{r^2-\frac{y_1}{x_1}e^{-\frac{2}{3}\pi i}}+$$
$$+\frac{1}{x_1y_1}\left(\frac{1}{\frac{x_1}{y_1}e^{\frac{2}{3}\pi i}-1}-\frac{1}{x_1y_1e^{\frac{2}{3}\pi i}-1}\right)+$$
$$+\frac{1}{y_1^2}\left(\frac{1}{1-\frac{x_1}{y_1}}+\frac{e^{-\frac{2}{3}\pi i}}{1-\frac{x_1}{y_1}e^{-\frac{2}{3}\pi i}}+\frac{e^{\frac{2}{3}\pi i}}{1-\frac{x_1}{y_1}e^{\frac{2}{3}\pi i}}\right)+\frac{1}{x_1y_1}\left(\frac{1}{1-x_1y_1}+\frac{e^{-\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i}-x_1y_1}+\frac{e^{\frac{2}{3}\pi i}}{e^{\frac{2}{3}\pi i}-x_1y_1}\right)+$$
$$+\frac{1}{x_1^2y_1^2}\left(\frac{1}{r^2-\frac{1}{x_1y_1}}+\frac{e^{-\frac{2}{3}\pi i}}{r^2-\frac{1}{x_1y_1}e^{-\frac{2}{3}\pi i}}+\frac{e^{\frac{2}{3}\pi i}}{r^2-\frac{1}{x_1y_1}e^{\frac{2}{3}\pi i}}\right)+$$
$$-\frac{1}{x_1y_1}\left(\frac{1}{1-\frac{x_1}{y_1}}+\frac{1}{1-\frac{x_1}{y_1}e^{\frac{2}{3}\pi i}}+\frac{1}{1-\frac{x_1}{y_1}e^{-\frac{2}{3}\pi i}}\right)-\frac{1}{1-x_1y_1}-\frac{e^{-\frac{2}{3}\pi i}}{1-x_1y_1e^{-\frac{2}{3}\pi i}}-\frac{e^{\frac{2}{3}\pi i}}{1-x_1y_1e^{\frac{2}{3}\pi i}}+$$
$$-\frac{1}{x_1^2}\left(\frac{1}{r^2-\frac{y_1}{x_1}}+\frac{e^{\frac{2}{3}\pi i}}{r^2-\frac{y_1}{x_1}e^{\frac{2}{3}\pi i}}+\frac{e^{-\frac{2}{3}\pi i}}{r^2-\frac{y_1}{x_1}e^{-\frac{2}{3}\pi i}}\right)$$

$$\begin{aligned}
= \quad & \frac{3}{x_1 y_1}\left(\frac{1}{1-x_1y_1}-\frac{1}{1-\frac{x_1}{y_1}}\right)+\frac{1}{y_1^2}\frac{1}{1-\frac{x_1}{y_1}}-\frac{1}{1-x_1y_1}+ \\
&+\frac{3}{x_1y_1}\left(\frac{e^{\frac{2}{3}\pi i}}{e^{\frac{2}{3}\pi i}-x_1y_1}-\frac{e^{\frac{2}{3}\pi i}}{e^{\frac{2}{3}\pi i}-\frac{x_1}{y_1}}+\frac{e^{-\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i}-x_1y_1}-\frac{e^{-\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i}-\frac{x_1}{y_1}}\right)+ \\
&+\frac{1}{y_1^2}\frac{e^{-\frac{2}{3}\pi i}}{1-\frac{x_1}{y_1}e^{-\frac{2}{3}\pi i}}+\frac{1}{y_1^2}\frac{e^{\frac{2}{3}\pi i}}{1-\frac{x_1}{y_1}e^{\frac{2}{3}\pi i}}-\frac{e^{-\frac{2}{3}\pi i}}{1-x_1y_1e^{-\frac{2}{3}\pi i}}-\frac{e^{\frac{2}{3}\pi i}}{1-x_1y_1e^{\frac{2}{3}\pi i}}+ \\
&+\frac{2}{x_1^2}\left(\frac{1}{y_1^2}\frac{1}{r^2-\frac{1}{x_1y_1}}-\frac{1}{r^2-\frac{y_1}{x_1}}\right)+ \\
&+\frac{2}{x_1^2}\left(\frac{1}{y_1^2}\frac{e^{\frac{2}{3}\pi i}}{r^2-\frac{1}{x_1y_1}e^{\frac{2}{3}\pi i}}-\frac{e^{\frac{2}{3}\pi i}}{r^2-\frac{y_1}{x_1}e^{\frac{2}{3}\pi i}}\right)+ \\
&+\frac{2}{x_1^2}\left(\frac{1}{y_1^2}\frac{e^{-\frac{2}{3}\pi i}}{r^2-\frac{1}{x_1y_1}e^{-\frac{2}{3}\pi i}}-\frac{e^{-\frac{2}{3}\pi i}}{r^2-\frac{y_1}{x_1}e^{-\frac{2}{3}\pi i}}\right).
\end{aligned}$$

$$\begin{aligned}
= \quad & \frac{3}{x_1 y_1}\left(\frac{1}{1-x_1y_1}-\frac{1}{1-\frac{x_1}{y_1}}\right)+\frac{1}{y_1^2}\frac{1}{1-\frac{x_1}{y_1}}-\frac{1}{1-x_1y_1}+ \\
&+\frac{3}{x_1y_1}\left(\frac{e^{\frac{2}{3}\pi i}}{e^{\frac{2}{3}\pi i}-x_1y_1}-\frac{e^{\frac{2}{3}\pi i}}{e^{\frac{2}{3}\pi i}-\frac{x_1}{y_1}}+\frac{e^{-\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i}-x_1y_1}-\frac{e^{-\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i}-\frac{x_1}{y_1}}\right)+ \\
&+\frac{1}{y_1^2}\frac{e^{-\frac{2}{3}\pi i}}{1-\frac{x_1}{y_1}e^{-\frac{2}{3}\pi i}}+\frac{1}{y_1^2}\frac{e^{\frac{2}{3}\pi i}}{1-\frac{x_1}{y_1}e^{\frac{2}{3}\pi i}}-\frac{e^{-\frac{2}{3}\pi i}}{1-x_1y_1e^{-\frac{2}{3}\pi i}}-\frac{e^{\frac{2}{3}\pi i}}{1-x_1y_1e^{\frac{2}{3}\pi i}}+ \\
&+\frac{2}{x_1^2}\left(\frac{1}{y_1^2}\frac{1}{r^2-\frac{1}{x_1y_1}}-\frac{1}{r^2-\frac{y_1}{x_1}}\right)+ \\
&+\frac{2}{x_1^2}\left(\frac{1}{y_1^2}\frac{e^{\frac{2}{3}\pi i}}{r^2-\frac{1}{x_1y_1}e^{\frac{2}{3}\pi i}}-\frac{e^{\frac{2}{3}\pi i}}{r^2-\frac{y_1}{x_1}e^{\frac{2}{3}\pi i}}\right)+ \\
&+\frac{2}{x_1^2}\left(\frac{1}{y_1^2}\frac{e^{-\frac{2}{3}\pi i}}{r^2-\frac{1}{x_1y_1}e^{-\frac{2}{3}\pi i}}-\frac{e^{-\frac{2}{3}\pi i}}{r^2-\frac{y_1}{x_1}e^{-\frac{2}{3}\pi i}}\right)
\end{aligned}$$

$$\begin{aligned}
= \; & \frac{3}{x_1y_1}\sum_{k=0}^{2}\left(\frac{e^{\frac{2}{3}k\pi i}}{e^{\frac{2}{3}k\pi i}-x_1y_1}-\frac{e^{\frac{2}{3}k\pi i}}{e^{\frac{2}{3}k\pi i}-\frac{x_1}{y_1}}\right)+ \\
&+\sum_{k=0}^{2}\left(\frac{1}{y_1^2}\frac{e^{\frac{2}{3}k\pi i}}{1-\frac{x_1}{y_1}e^{\frac{2}{3}k\pi i}}-\frac{e^{\frac{2}{3}k\pi i}}{1-x_1y_1e^{\frac{2}{3}k\pi i}}\right)\;+ \\
&+\frac{2}{x_1^2}\sum_{k=0}^{2}\left(\frac{1}{y_1^2}\frac{e^{\frac{2}{3}k\pi i}}{r^2-\frac{1}{x_1y_1}e^{\frac{2}{3}k\pi i}}-\frac{e^{\frac{2}{3}k\pi i}}{r^2-\frac{y_1}{x_1}e^{\frac{2}{3}k\pi i}}\right),
\end{aligned}$$

and, after another lengthy calculation

$$\int_{r=x_1}^{y_1} I_2\left(x_1,y_1;r\right)\,r\,dr=$$

$$
\begin{aligned}
= \quad & \left[\left(\frac{3}{x_1y_1}\left(\frac{1}{1-x_1y_1}-\frac{1}{1-\frac{x_1}{y_1}}\right)+\frac{1}{y_1^2}\frac{1}{1-\frac{x_1}{y_1}}-\frac{1}{1-x_1y_1}\right)\frac{1}{2}r^2\right]_{r=x_1}^{y_1} + \\
& + \left[\frac{3}{x_1y_1}\left(\frac{e^{\frac{2}{3}\pi i}}{e^{\frac{2}{3}\pi i}-x_1y_1}-\frac{e^{\frac{2}{3}\pi i}}{e^{\frac{2}{3}\pi i}-\frac{x_1}{y_1}}+\frac{e^{-\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i}-x_1y_1}-\frac{e^{-\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i}-\frac{x_1}{y_1}}\right)\frac{1}{2}r^2\right]_{r=x_1}^{y_1} + \\
& + \left[\left(\frac{1}{y_1^2}\frac{e^{-\frac{2}{3}\pi i}}{1-\frac{x_1}{y_1}e^{-\frac{2}{3}\pi i}}+\frac{1}{y_1^2}\frac{e^{\frac{2}{3}\pi i}}{1-\frac{x_1}{y_1}e^{\frac{2}{3}\pi i}}-\frac{e^{-\frac{2}{3}\pi i}}{1-x_1y_1e^{-\frac{2}{3}\pi i}}-\frac{e^{\frac{2}{3}\pi i}}{1-x_1y_1e^{\frac{2}{3}\pi i}}\right)\frac{1}{2}r^2\right]_{r=x_1}^{y_1} + \\
& + \left[\frac{1}{x_1^2}\left(\frac{1}{y_1^2}\ln\left(\frac{1}{x_1y_1}-r^2\right)-\ln\left(\frac{y_1}{x_1}-r^2\right)\right)\right]_{r=x_1}^{y_1} + \\
& + \left[\frac{1}{x_1^2}\left(\frac{e^{\frac{2}{3}\pi i}}{y_1^2}\mathrm{Ln}\left(\frac{1}{x_1y_1}e^{\frac{2}{3}\pi i}-r^2\right)-e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{y_1}{x_1}e^{\frac{2}{3}\pi i}-r^2\right)\right)\right]_{r=x_1}^{y_1} + \\
& + \left[\frac{1}{x_1^2}\left(\frac{e^{-\frac{2}{3}\pi i}}{y_1^2}\mathrm{Ln}\left(\frac{1}{x_1y_1}e^{-\frac{2}{3}\pi i}-r^2\right)-e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{y_1}{x_1}e^{-\frac{2}{3}\pi i}-r^2\right)\right)\right]_{r=x_1}^{y_1}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \left(\frac{3}{x_1y_1}\left(\frac{1}{1-x_1y_1}-\frac{1}{1-\frac{x_1}{y_1}}\right)+\frac{1}{y_1^2}\frac{1}{1-\frac{x_1}{y_1}}-\frac{1}{1-x_1y_1}\right)\frac{1}{2}\left(y_1^2-x_1^2\right) + \\
& + \frac{3}{x_1y_1}\left(\frac{e^{\frac{2}{3}\pi i}}{e^{\frac{2}{3}\pi i}-x_1y_1}-\frac{e^{\frac{2}{3}\pi i}}{e^{\frac{2}{3}\pi i}-\frac{x_1}{y_1}}+\frac{e^{-\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i}-x_1y_1}-\frac{e^{-\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i}-\frac{x_1}{y_1}}\right)\frac{1}{2}\left(y_1^2-x_1^2\right) + \\
& + \left(\frac{1}{y_1^2}\frac{e^{-\frac{2}{3}\pi i}}{1-\frac{x_1}{y_1}e^{-\frac{2}{3}\pi i}}+\frac{1}{y_1^2}\frac{e^{\frac{2}{3}\pi i}}{1-\frac{x_1}{y_1}e^{\frac{2}{3}\pi i}}-\frac{e^{-\frac{2}{3}\pi i}}{1-x_1y_1e^{-\frac{2}{3}\pi i}}-\frac{e^{\frac{2}{3}\pi i}}{1-x_1y_1e^{\frac{2}{3}\pi i}}\right)\frac{1}{2}\left(y_1^2-x_1^2\right) + \\
& + \frac{1}{x_1^2}\left(\frac{1}{y_1^2}\ln\left(\frac{\frac{1}{x_1y_1}-y_1^2}{\frac{1}{x_1y_1}-x_1^2}\right)-\ln\left(\frac{\frac{y_1}{x_1}-y_1^2}{\frac{y_1}{x_1}-x_1^2}\right)\right) + \\
& + \frac{1}{x_1^2}\left(\frac{e^{\frac{2}{3}\pi i}}{y_1^2}\mathrm{Ln}\left(\frac{\frac{1}{x_1y_1}e^{\frac{2}{3}\pi i}-y_1^2}{\frac{1}{x_1y_1}e^{\frac{2}{3}\pi i}-x_1^2}\right)-e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{\frac{y_1}{x_1}e^{\frac{2}{3}\pi i}-y_1^2}{\frac{y_1}{x_1}e^{\frac{2}{3}\pi i}-x_1^2}\right)\right) + \\
& + \frac{1}{x_1^2}\left(\frac{e^{-\frac{2}{3}\pi i}}{y_1^2}\mathrm{Ln}\left(\frac{\frac{1}{x_1y_1}e^{-\frac{2}{3}\pi i}-y_1^2}{\frac{1}{x_1y_1}e^{-\frac{2}{3}\pi i}-x_1^2}\right)-e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{\frac{y_1}{x_1}e^{-\frac{2}{3}\pi i}-y_1^2}{\frac{y_1}{x_1}e^{-\frac{2}{3}\pi i}-x_1^2}\right)\right)
\end{aligned}
$$

$$\begin{aligned}
= \quad & -\frac{(x_1+y_1)\left(1-y_1^2\right)}{y_1\left(1-x_1y_1\right)}+ \\
& +\frac{3}{x_1y_1}\left(\frac{e^{\frac{2}{3}\pi i}}{e^{\frac{2}{3}\pi i}-x_1y_1}+\frac{e^{-\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i}-x_1y_1}\right)\frac{1}{2}\left(y_1^2-x_1^2\right)+ \\
& -\frac{3}{x_1y_1}\left(\frac{e^{\frac{2}{3}\pi i}}{e^{\frac{2}{3}\pi i}-\frac{x_1}{y_1}}+\frac{e^{-\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i}-\frac{x_1}{y_1}}\right)\frac{1}{2}\left(y_1^2-x_1^2\right)+ \\
& +\frac{1}{y_1^2}\left(\frac{e^{-\frac{2}{3}\pi i}}{1-\frac{x_1}{y_1}e^{-\frac{2}{3}\pi i}}+\frac{e^{\frac{2}{3}\pi i}}{1-\frac{x_1}{y_1}e^{\frac{2}{3}\pi i}}\right)\frac{1}{2}\left(y_1^2-x_1^2\right)+ \\
& -\left(\frac{e^{-\frac{2}{3}\pi i}}{1-x_1y_1e^{-\frac{2}{3}\pi i}}+\frac{e^{\frac{2}{3}\pi i}}{1-x_1y_1e^{\frac{2}{3}\pi i}}\right)\frac{1}{2}\left(y_1^2-x_1^2\right)+ \\
& +\frac{1}{x_1^2}\left(\frac{1}{y_1^2}\ln\left(\frac{\frac{1}{x_1y_1}-y_1^2}{\frac{1}{x_1y_1}-x_1^2}\right)-\ln\left(\frac{\frac{y_1}{x_1}-y_1^2}{\frac{y_1}{x_1}-x_1^2}\right)\right)+ \\
& +\frac{1}{x_1^2}\left(\frac{e^{\frac{2}{3}\pi i}}{y_1^2}\operatorname{Ln}\left(\frac{\frac{1}{x_1y_1}e^{\frac{2}{3}\pi i}-y_1^2}{\frac{1}{x_1y_1}e^{\frac{2}{3}\pi i}-x_1^2}\right)-e^{\frac{2}{3}\pi i}\operatorname{Ln}\left(\frac{\frac{y_1}{x_1}e^{\frac{2}{3}\pi i}-y_1^2}{\frac{y_1}{x_1}e^{\frac{2}{3}\pi i}-x_1^2}\right)\right)+ \\
& +\frac{1}{x_1^2}\left(\frac{e^{-\frac{2}{3}\pi i}}{y_1^2}\operatorname{Ln}\left(\frac{\frac{1}{x_1y_1}e^{-\frac{2}{3}\pi i}-y_1^2}{\frac{1}{x_1y_1}e^{-\frac{2}{3}\pi i}-x_1^2}\right)-e^{-\frac{2}{3}\pi i}\operatorname{Ln}\left(\frac{\frac{y_1}{x_1}e^{-\frac{2}{3}\pi i}-y_1^2}{\frac{y_1}{x_1}e^{-\frac{2}{3}\pi i}-x_1^2}\right)\right)
\end{aligned}$$

$$\begin{aligned}
= \quad & -\frac{(x_1+y_1)\left(1-y_1^2\right)}{y_1\left(1-x_1y_1\right)}+ \\
& +\frac{3}{x_1y_1}\left(\frac{2+x_1y_1}{1+x_1y_1+x_1^2y_1^2}\right)\frac{1}{2}\left(y_1^2-x_1^2\right)+ \\
& -\frac{3}{x_1y_1}\left(\frac{2y_1^2+x_1y_1}{y_1^2+x_1y_1+x_1^2}\right)\frac{1}{2}\left(y_1^2-x_1^2\right)+ \\
& -\frac{1}{y_1^2}\left(\frac{y_1^2+2x_1y_1}{y_1^2+x_1y_1+x_1^2}\right)\frac{1}{2}\left(y_1^2-x_1^2\right)+ \\
& +\left(\frac{1+2x_1y_1}{1+x_1y_1+x_1^2y_1^2}\right)\frac{1}{2}\left(y_1^2-x_1^2\right)+ \\
& +\frac{1}{x_1^2y_1^2}\ln\left(\frac{1-x_1y_1^3}{1-x_1^3y_1}\right)-\frac{1}{x_1^2}\ln\left(\frac{1-x_1y_1}{1-x_1^2\frac{x_1}{y_1}}\right)+ \\
& +\frac{e^{\frac{2}{3}\pi i}}{x_1^2y_1^2}\operatorname{Ln}\left(\frac{1-x_1y_1^3e^{-\frac{2}{3}\pi i}}{1-x_1^3y_1e^{-\frac{2}{3}\pi i}}\right)-\frac{e^{\frac{2}{3}\pi i}}{x_1^2}\operatorname{Ln}\left(\frac{1-x_1y_1e^{-\frac{2}{3}\pi i}}{1-x_1^2\frac{x_1}{y_1}e^{-\frac{2}{3}\pi i}}\right)+ \\
& +\frac{e^{-\frac{2}{3}\pi i}}{x_1^2y_1^2}\operatorname{Ln}\left(\frac{1-x_1y_1^3e^{\frac{2}{3}\pi i}}{1-x_1^3y_1e^{\frac{2}{3}\pi i}}\right)-\frac{e^{-\frac{2}{3}\pi i}}{x_1^2}\operatorname{Ln}\left(\frac{1-x_1y_1e^{\frac{2}{3}\pi i}}{1-x_1^2\frac{x_1}{y_1}e^{\frac{2}{3}\pi i}}\right)
\end{aligned}$$

$$
\begin{aligned}
= \; & - \frac{(x_1+y_1)\left(1-y_1^2\right)}{y_1\left(1-x_1y_1\right)} + \\
& + \frac{y_1^2-x_1^2}{x_1y_1}\frac{3+2x_1y_1+x_1^2y_1^2}{1+x_1y_1+x_1^2y_1^2} + \\
& - \frac{y_1^2-x_1^2}{x_1y_1}\frac{3y_1^2+2x_1y_1+x_1^2}{y_1^2+x_1y_1+x_1^2} + \\
& + \frac{1}{x_1^2y_1^2}\ln\left(\frac{1-x_1y_1^3}{1-x_1^3y_1}\right) - \frac{1}{x_1^2}\ln\left(\frac{1-x_1y_1}{1-x_1^2\frac{x_1}{y_1}}\right) + \\
& + \frac{e^{\frac{2}{3}\pi i}}{x_1^2y_1^2}\mathrm{Ln}\left(2+y_1^3x_1+iy_1^3x_1\sqrt{3}\right) - \frac{e^{\frac{2}{3}\pi i}}{x_1^2y_1^2}\mathrm{Ln}\left(2+x_1^3y_1+ix_1^3y_1\sqrt{3}\right) + \\
& - \frac{e^{\frac{2}{3}\pi i}}{x_1^2}\mathrm{Ln}\left(y_1\left(2+x_1y_1+ix_1y_1\sqrt{3}\right)\right) + \frac{e^{\frac{2}{3}\pi i}}{x_1^2}\mathrm{Ln}\left(2y_1+x_1^3+ix_1^3\sqrt{3}\right) + \\
& + \frac{e^{-\frac{2}{3}\pi i}}{x_1^2y_1^2}\mathrm{Ln}\left(2+y_1^3x_1-iy_1^3x_1\sqrt{3}\right) - \frac{e^{-\frac{2}{3}\pi i}}{x_1^2y_1^2}\mathrm{Ln}\left(2+x_1^3y_1-ix_1^3y_1\sqrt{3}\right) + \\
& - \frac{e^{-\frac{2}{3}\pi i}}{x_1^2}\mathrm{Ln}\left(y_1\left(2+x_1y_1-ix_1y_1\sqrt{3}\right)\right) + \frac{e^{-\frac{2}{3}\pi i}}{x_1^2}\mathrm{Ln}\left(2y_1+x_1^3-ix_1^3\sqrt{3}\right)
\end{aligned}
$$

$$
\begin{aligned}
= \; & - \frac{(x_1+y_1)\left(1-y_1^2\right)}{y_1\left(1-x_1y_1\right)} + \\
& + \frac{y_1^2-x_1^2}{x_1y_1}\frac{3+2x_1y_1+x_1^2y_1^2}{1+x_1y_1+x_1^2y_1^2} + \\
& - \frac{y_1^2-x_1^2}{x_1y_1}\frac{3y_1^2+2x_1y_1+x_1^2}{y_1^2+x_1y_1+x_1^2} + \\
& + \frac{1}{x_1^2y_1^2}\ln\left(\frac{1-x_1y_1^3}{1-x_1^3y_1}\right) - \frac{1}{x_1^2}\ln\left(\frac{1-x_1y_1}{1-x_1^2\frac{x_1}{y_1}}\right) + \\
& + \frac{e^{\frac{2}{3}\pi i}+e^{-\frac{2}{3}\pi i}}{x_1^2y_1^2}\ln\sqrt{\left(2+y_1^3x_1\right)^2+\left(y_1^3x_1\sqrt{3}\right)^2} + i\frac{e^{\frac{2}{3}\pi i}-e^{-\frac{2}{3}\pi i}}{x_1^2y_1^2}\arctan\left(\frac{y_1^3x_1\sqrt{3}}{2+y_1^3x_1}\right) + \\
& + \frac{e^{\frac{2}{3}\pi i}+e^{-\frac{2}{3}\pi i}}{x_1^2}\ln\sqrt{\left(2y_1+x_1^3\right)^2+\left(x_1^3\sqrt{3}\right)^2} + i\frac{e^{\frac{2}{3}\pi i}-e^{-\frac{2}{3}\pi i}}{x_1^2}\arctan\left(\frac{x_1^3\sqrt{3}}{2y_1+x_1^3}\right) + \\
& - \frac{e^{\frac{2}{3}\pi i}+e^{-\frac{2}{3}\pi i}}{x_1^2y_1^2}\ln\sqrt{\left(2+x_1^3y_1\right)^2+\left(x_1^3y_1\sqrt{3}\right)^2} - i\frac{e^{\frac{2}{3}\pi i}-e^{-\frac{2}{3}\pi i}}{x_1^2y_1^2}\arctan\left(\frac{x_1^3y_1\sqrt{3}}{2+x_1^3y_1}\right) + \\
& - \frac{e^{\frac{2}{3}\pi i}+e^{-\frac{2}{3}\pi i}}{x_1^2}\ln\sqrt{y_1^2\left(2+x_1y_1\right)^2+y_1^2\left(x_1y_1\sqrt{3}\right)^2} - i\frac{e^{\frac{2}{3}\pi i}-e^{-\frac{2}{3}\pi i}}{x_1^2}\arctan\left(\frac{x_1y_1\sqrt{3}}{2+x_1y_1}\right)
\end{aligned}
$$

$$\begin{aligned}
= \quad & -\frac{(x_1+y_1)\left(1-y_1^2\right)}{y_1\left(1-x_1y_1\right)} + \\
& + \frac{y_1^2-x_1^2}{x_1y_1}\frac{3+2x_1y_1+x_1^2y_1^2}{1+x_1y_1+x_1^2y_1^2} + \\
& - \frac{y_1^2-x_1^2}{x_1y_1}\frac{3y_1^2+2x_1y_1+x_1^2}{y_1^2+x_1y_1+x_1^2} + \\
& + \frac{1}{x_1^2y_1^2}\ln\left(\frac{1-x_1y_1^3}{1-x_1^3y_1}\right) - \frac{1}{x_1^2}\ln\left(\frac{1-x_1y_1}{1-x_1^2\frac{x_1}{y_1}}\right) + \\
& - \frac{1}{x_1^2y_1^2}\ln\left(2\sqrt{(1+y_1^3x_1+y_1^6x_1^2)}\right) - \frac{\sqrt{3}}{x_1^2y_1^2}\arctan\left(\frac{y_1^3x_1\sqrt{3}}{2+y_1^3x_1}\right) + \\
& - \frac{1}{x_1^2}\ln\left(2\sqrt{(y_1^2+x_1^3y_1+x_1^6)}\right) - \frac{\sqrt{3}}{x_1^2}\arctan\left(\frac{x_1^3\sqrt{3}}{2y_1+x_1^3}\right) + \\
& + \frac{1}{x_1^2y_1^2}\ln\left(2\sqrt{(1+x_1^3y_1+x_1^6y_1^2)}\right) + \frac{\sqrt{3}}{x_1^2y_1^2}\arctan\left(\frac{x_1^3y_1\sqrt{3}}{2+x_1^3y_1}\right) + \\
& + \frac{1}{x_1^2}\ln\left(2y_1\sqrt{(1+x_1y_1+x_1^2y_1^2)}\right) + \frac{\sqrt{3}}{x_1^2}\arctan\left(\frac{x_1y_1\sqrt{3}}{2+x_1y_1}\right)
\end{aligned}$$

$$\begin{aligned}
= \quad & -\frac{(x_1+y_1)\left(1-y_1^2\right)}{y_1\left(1-x_1y_1\right)} + \\
& + \frac{y_1^2-x_1^2}{x_1y_1}\frac{3+2x_1y_1+x_1^2y_1^2}{1+x_1y_1+x_1^2y_1^2} + \\
& - \frac{y_1^2-x_1^2}{x_1y_1}\frac{3y_1^2+2x_1y_1+x_1^2}{y_1^2+x_1y_1+x_1^2} + \\
& + \frac{1}{x_1^2y_1^2}\ln\left(\frac{1-x_1y_1^3}{1-x_1^3y_1}\right) - \frac{1}{x_1^2}\ln\left(\frac{1-x_1y_1}{1-x_1^2\frac{x_1}{y_1}}\right) + \\
& - \frac{1}{2x_1^2y_1^2}\ln\left(1+y_1^3x_1+y_1^6x_1^2\right) - \frac{\sqrt{3}}{x_1^2y_1^2}\arctan\left(\frac{y_1^3x_1\sqrt{3}}{2+y_1^3x_1}\right) + \\
& - \frac{1}{2x_1^2}\ln\left(y_1^2+x_1^3y_1+x_1^6\right) - \frac{\sqrt{3}}{x_1^2}\arctan\left(\frac{x_1^3\sqrt{3}}{2y_1+x_1^3}\right) + \\
& + \frac{1}{2x_1^2y_1^2}\ln\left(1+x_1^3y_1+x_1^6y_1^2\right) + \frac{\sqrt{3}}{x_1^2y_1^2}\arctan\left(\frac{x_1^3y_1\sqrt{3}}{2+x_1^3y_1}\right) + \\
& + \frac{1}{2x_1^2}\ln\left(y_1^2\left(1+x_1y_1+x_1^2y_1^2\right)\right) + \frac{\sqrt{3}}{x_1^2}\arctan\left(\frac{x_1y_1\sqrt{3}}{2+x_1y_1}\right)
\end{aligned}$$

$$
\begin{aligned}
= \quad & -\frac{(x_1+y_1)\left(1-y_1^2\right)}{y_1\left(1-x_1y_1\right)} \; + \\
& + \frac{y_1^2-x_1^2}{x_1y_1}\left(\frac{3+2x_1y_1+x_1^2y_1^2}{1+x_1y_1+x_1^2y_1^2}-\frac{3y_1^2+2x_1y_1+x_1^2}{y_1^2+x_1y_1+x_1^2}\right) \; + \\
& + \frac{1}{x_1^2y_1^2}\left(\ln\left(\frac{1-x_1y_1^3}{1-x_1^3y_1}\right)-\tfrac{1}{2}\ln\left(\frac{1+x_1y_1^3+y_1^6x_1^2}{1+x_1^3y_1+x_1^6y_1^2}\right)\right) \; + \\
& + \frac{1}{x_1^2}\left(\tfrac{1}{2}\ln\left(\frac{1+x_1y_1+x_1^2y_1^2}{y_1^2+x_1^3y_1+x_1^6}\right)-\ln\left(\frac{1-x_1y_1}{y_1-x_1^3}\right)\right) \; + \\
& + \frac{\sqrt{3}}{x_1^2y_1^2}\left(\arctan\left(\frac{x_1^3y_1\sqrt{3}}{2+x_1^3y_1}\right)-\arctan\left(\frac{y_1^3x_1\sqrt{3}}{2+y_1^3x_1}\right)\right) \; + \\
& + \frac{\sqrt{3}}{x_1^2}\left(\arctan\left(\frac{x_1y_1\sqrt{3}}{2+x_1y_1}\right)-\arctan\left(\frac{x_1^3\sqrt{3}}{2y_1+x_1^3}\right)\right) \; .
\end{aligned}
\tag{36}
$$

- For $r \in (y_1, 1)$ the integrand turns out to be

$$
I_3\left(x_1,y_1;r\right) = \#_{(23)}+\#_{(25)}+\#_{(26)}+\#_{(28)}+\#_{(29)}+\#_{(31)}+\#_{(32)}+\#_{(34)} =
$$

$$
\begin{aligned}
= \; & \frac{1}{x_1y_1}\left(\frac{1}{1-x_1y_1}-\frac{1}{1-\frac{x_1}{y_1}}\right)+\frac{1}{x_1^2y_1^2}\frac{1}{r^2-\frac{1}{x_1y_1}}-\frac{1}{x_1^2}\frac{1}{r^2-\frac{y_1}{x_1}}+ \\
- \; & \frac{1}{x_1y_1}\left(\frac{1}{1-\frac{x_1}{y_1}}-\frac{1}{1-x_1y_1}\right)-\frac{1}{r^2-x_1y_1}+\frac{1}{y_1^2}\frac{1}{r^2-\frac{x_1}{y_1}}+ \\
+ \; & \frac{e^{\frac{2}{3}\pi i}}{x_1y_1}\left(\frac{1}{e^{\frac{2}{3}\pi i}-x_1y_1}-\frac{1}{e^{\frac{2}{3}\pi i}-\frac{x_1}{y_1}}\right)+\frac{e^{\frac{2}{3}\pi i}}{x_1^2y_1^2}\frac{1}{r^2-\frac{1}{x_1y_1}e^{\frac{2}{3}\pi i}}-\frac{e^{\frac{2}{3}\pi i}}{x_1^2}\frac{1}{r^2-\frac{y_1}{x_1}e^{\frac{2}{3}\pi i}}+ \\
+ \; & \frac{1}{x_1y_1}\left(\frac{1}{1-x_1y_1e^{-\frac{2}{3}\pi i}}-\frac{1}{1-\frac{x_1}{y_1}e^{-\frac{2}{3}\pi i}}\right)-\frac{e^{-\frac{2}{3}\pi i}}{r^2-x_1y_1e^{-\frac{2}{3}\pi i}}+\frac{e^{-\frac{2}{3}\pi i}}{y_1^2}\frac{1}{r^2-\frac{x_1}{y_1}e^{-\frac{2}{3}\pi i}}+ \\
+ \; & \frac{e^{-\frac{2}{3}\pi i}}{x_1y_1}\left(\frac{1}{e^{-\frac{2}{3}\pi i}-x_1y_1}-\frac{1}{e^{-\frac{2}{3}\pi i}-\frac{x_1}{y_1}}\right)+\frac{e^{-\frac{2}{3}\pi i}}{x_1^2y_1^2}\frac{1}{r^2-\frac{1}{x_1y_1}e^{-\frac{2}{3}\pi i}}-\frac{e^{-\frac{2}{3}\pi i}}{x_1^2}\frac{1}{r^2-\frac{y_1}{x_1}e^{-\frac{2}{3}\pi i}}+ \\
+ \; & \frac{1}{x_1y_1}\left(\frac{1}{\frac{x_1}{y_1}e^{\frac{2}{3}\pi i}-1}-\frac{1}{x_1y_1e^{\frac{2}{3}\pi i}-1}\right)-e^{\frac{2}{3}\pi i}\frac{1}{r^2-x_1y_1e^{\frac{2}{3}\pi i}}+\frac{e^{\frac{2}{3}\pi i}}{y_1^2}\frac{1}{r^2-\frac{x_1}{y_1}e^{\frac{2}{3}\pi i}}+ \\
+ \; & \frac{1}{y_1^2}\left(\frac{1}{1-\frac{x_1}{y_1}}+\frac{e^{-\frac{2}{3}\pi i}}{1-\frac{x_1}{y_1}e^{-\frac{2}{3}\pi i}}+\frac{e^{\frac{2}{3}\pi i}}{1-\frac{x_1}{y_1}e^{\frac{2}{3}\pi i}}\right)+\frac{1}{x_1y_1}\left(\frac{1}{1-x_1y_1}+\frac{e^{-\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i}-x_1y_1}+\frac{e^{\frac{2}{3}\pi i}}{e^{\frac{2}{3}\pi i}-x_1y_1}\right)+ \\
+ \; & \frac{1}{x_1^2y_1^2}\left(\frac{1}{r^2-\frac{1}{x_1y_1}}+\frac{e^{-\frac{2}{3}\pi i}}{r^2-\frac{1}{x_1y_1}e^{-\frac{2}{3}\pi i}}+\frac{e^{\frac{2}{3}\pi i}}{r^2-\frac{1}{x_1y_1}e^{\frac{2}{3}\pi i}}\right)-\frac{1}{y_1^2}\left(\frac{1}{r^2-\frac{x_1}{y_1}}+\frac{e^{-\frac{2}{3}\pi i}}{r^2-\frac{x_1}{y_1}e^{-\frac{2}{3}\pi i}}+\frac{e^{\frac{2}{3}\pi i}}{r^2-\frac{x_1}{y_1}e^{\frac{2}{3}\pi i}}\right)+ \\
+ \; & \frac{1}{x_1y_1}\left(\frac{1}{1-x_1y_1}+\frac{1}{1-x_1y_1e^{-\frac{2}{3}\pi i}}+\frac{1}{1-x_1y_1e^{\frac{2}{3}\pi i}}\right)+\frac{1}{y_1^2}\left(\frac{1}{1-\frac{x_1}{y_1}}+\frac{e^{-\frac{2}{3}\pi i}}{1-\frac{x_1}{y_1}e^{-\frac{2}{3}\pi i}}+\frac{e^{\frac{2}{3}\pi i}}{1-\frac{x_1}{y_1}e^{\frac{2}{3}\pi i}}\right)+ \\
- \; & \frac{1}{r^2-x_1y_1}-\frac{e^{-\frac{2}{3}\pi i}}{r^2-x_1y_1e^{-\frac{2}{3}\pi i}}-\frac{e^{\frac{2}{3}\pi i}}{r^2-x_1y_1e^{\frac{2}{3}\pi i}}+\frac{1}{x_1^2}\left(\frac{1}{r^2-\frac{y_1}{x_1}}+\frac{e^{-\frac{2}{3}\pi i}}{r^2-\frac{y_1}{x_1}e^{-\frac{2}{3}\pi i}}+\frac{e^{\frac{2}{3}\pi i}}{r^2-\frac{y_1}{x_1}e^{\frac{2}{3}\pi i}}\right)
\end{aligned}
$$

$$= \sum_{k=0}^{2}\left(\frac{2}{x_1y_1}\left(\frac{2e^{\frac{2}{3}k\pi i}}{e^{\frac{2}{3}k\pi i}-x_1y_1}-\frac{e^{\frac{2}{3}k\pi i}}{e^{\frac{2}{3}k\pi i}-\frac{x_1}{y_1}}\right)+\frac{2}{y_1^2}\left(\frac{e^{\frac{2}{3}k\pi i}}{1-\frac{x_1}{y_1}e^{\frac{2}{3}k\pi i}}\right)\right)+$$
$$+\sum_{k=0}^{2}\left(\frac{2}{x_1^2y_1^2}\left(\frac{e^{\frac{2}{3}k\pi i}}{r^2-\frac{1}{x_1y_1}e^{\frac{2}{3}k\pi i}}\right)-\frac{2e^{\frac{2}{3}k\pi i}}{r^2-x_1y_1e^{\frac{2}{3}k\pi i}}\right),$$

while

$$\int_{r=y_1}^{1} I_3\left(x_1,y_1;r\right)\,r\,dr =$$

$$= \left[\left(\frac{2}{x_1y_1}\left(\frac{1}{1-x_1y_1}-\frac{1}{1-\frac{x_1}{y_1}}\right)+\frac{2}{x_1y_1}\frac{1}{1-x_1y_1}+\frac{2}{y_1^2}\frac{1}{1-\frac{x_1}{y_1}}\right)\frac{1}{2}r^2\right]_{r=y_1}^{1}+$$
$$+\left[\frac{1}{x_1^2y_1^2}\ln\left(\frac{1}{x_1y_1}-r^2\right)-\ln\left(r^2-x_1y_1\right)\right]_{r=y_1}^{1}+$$
$$+\left[\frac{4}{x_1y_1}\left(\frac{e^{\frac{2}{3}\pi i}}{e^{\frac{2}{3}\pi i}-x_1y_1}+\frac{e^{-\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i}-x_1y_1}\right)\frac{1}{2}r^2\right]_{r=y_1}^{1}+$$
$$+\left[\left(-\frac{2}{x_1y_1}\left(\frac{e^{-\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i}-\frac{x_1}{y_1}}+\frac{e^{\frac{2}{3}\pi i}}{e^{\frac{2}{3}\pi i}-\frac{x_1}{y_1}}\right)+\frac{2}{y_1^2}\left(\frac{e^{-\frac{2}{3}\pi i}}{1-\frac{x_1}{y_1}e^{-\frac{2}{3}\pi i}}+\frac{e^{\frac{2}{3}\pi i}}{1-\frac{x_1}{y_1}e^{\frac{2}{3}\pi i}}\right)\right)\frac{1}{2}r^2\right]_{r=y_1}^{1}+$$
$$+\left[\frac{1}{x_1^2y_1^2}\left(e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{1}{x_1y_1}e^{-\frac{2}{3}\pi i}-r^2\right)+e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{1}{x_1y_1}e^{\frac{2}{3}\pi i}-r^2\right)\right)\right]_{r=y_1}^{1}+$$
$$+\left[-e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(r^2-x_1y_1e^{-\frac{2}{3}\pi i}\right)-e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(r^2-x_1y_1e^{\frac{2}{3}\pi i}\right)\right]_{r=y_1}^{1}$$

$$= \left(\frac{2}{x_1y_1}\left(\frac{1}{1-x_1y_1}-\frac{1}{1-\frac{x_1}{y_1}}\right)+\frac{2}{x_1y_1}\frac{1}{1-x_1y_1}+\frac{2}{y_1^2}\frac{1}{1-\frac{x_1}{y_1}}\right)\frac{1}{2}\left(1-y_1^2\right)+$$
$$+\frac{1}{x_1^2y_1^2}\ln\left(\frac{\frac{1}{x_1y_1}-1}{\frac{1}{x_1y_1}-y_1^2}\right)-\ln\left(\frac{x_1y_1-1}{x_1y_1-y_1^2}\right)+$$
$$+\frac{4}{x_1y_1}\left(\frac{e^{\frac{2}{3}\pi i}}{e^{\frac{2}{3}\pi i}-x_1y_1}+\frac{e^{-\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i}-x_1y_1}\right)\frac{1}{2}\left(1-y_1^2\right)+$$
$$+\left(-\frac{2}{x_1y_1}\left(\frac{e^{-\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i}-\frac{x_1}{y_1}}+\frac{e^{\frac{2}{3}\pi i}}{e^{\frac{2}{3}\pi i}-\frac{x_1}{y_1}}\right)+\frac{2}{y_1^2}\left(\frac{e^{-\frac{2}{3}\pi i}}{1-\frac{x_1}{y_1}e^{-\frac{2}{3}\pi i}}+\frac{e^{\frac{2}{3}\pi i}}{1-\frac{x_1}{y_1}e^{\frac{2}{3}\pi i}}\right)\right)\frac{1}{2}\left(1-y_1^2\right)+$$
$$+\frac{1}{x_1^2y_1^2}\left(e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{\frac{1}{x_1y_1}e^{-\frac{2}{3}\pi i}-1}{\frac{1}{x_1y_1}e^{-\frac{2}{3}\pi i}-y_1^2}\right)+e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{\frac{1}{x_1y_1}e^{\frac{2}{3}\pi i}-1}{\frac{1}{x_1y_1}e^{\frac{2}{3}\pi i}-y_1^2}\right)\right)+$$
$$-e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{1-x_1y_1e^{-\frac{2}{3}\pi i}}{y_1^2-x_1y_1e^{-\frac{2}{3}\pi i}}\right)-e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{1-x_1y_1e^{\frac{2}{3}\pi i}}{y_1^2-x_1y_1e^{\frac{2}{3}\pi i}}\right)$$

$$
\begin{aligned}
= \; & \frac{1}{x_1y_1}\frac{x_1y_1+1}{(1-x_1y_1)}\left(1-y_1^2\right)+ \\
& + \frac{1}{x_1^2y_1^2}\ln\left(\frac{1-x_1y_1}{1-x_1y_1^3}\right)-\ln\left(\frac{x_1y_1-1}{x_1y_1-y_1^2}\right)+ \\
& + \left(\frac{2}{x_1y_1}\frac{2+x_1y_1}{1+x_1y_1+x_1^2y_1^2}-\frac{1}{x_1}\frac{2y_1+x_1}{y_1^2+x_1y_1+x_1^2}-\frac{1}{y_1}\frac{y_1+2x_1}{y_1^2+x_1y_1+x_1^2}\right)\left(1-y_1^2\right)+ \\
& + \frac{e^{-\frac{2}{3}\pi i}}{x_1^2y_1^2}\mathrm{Ln}\left(\frac{1+i\sqrt{3}+2x_1y_1}{1+i\sqrt{3}+2y_1^3x_1}\right)+\frac{e^{\frac{2}{3}\pi i}}{x_1^2y_1^2}\mathrm{Ln}\left(\frac{1-i\sqrt{3}+2x_1y_1}{1-i\sqrt{3}+2y_1^3x_1}\right)+ \\
& - e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{2+x_1y_1+ix_1y_1\sqrt{3}}{y_1\left(2y_1+x_1+ix_1\sqrt{3}\right)}\right)-e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{-2-x_1y_1+ix_1y_1\sqrt{3}}{y_1\left(-2y_1-x_1+ix_1\sqrt{3}\right)}\right)
\end{aligned}
$$

$$
\begin{aligned}
= \; & \frac{1}{x_1y_1}\frac{x_1y_1+1}{(1-x_1y_1)}\left(1-y_1^2\right)+ \\
& + \frac{1}{x_1^2y_1^2}\ln\left(\frac{1-x_1y_1}{1-x_1y_1^3}\right)-\ln\left(\frac{x_1y_1-1}{x_1y_1-y_1^2}\right)+ \\
& + \left(\frac{2}{x_1y_1}\frac{2+x_1y_1}{1+x_1y_1+x_1^2y_1^2}-\frac{1}{x_1}\frac{2y_1+x_1}{y_1^2+x_1y_1+x_1^2}-\frac{1}{y_1}\frac{y_1+2x_1}{y_1^2+x_1y_1+x_1^2}\right)\left(1-y_1^2\right)+ \\
& + \frac{e^{-\frac{2}{3}\pi i}}{x_1^2y_1^2}\mathrm{Ln}\left(1+i\sqrt{3}+2x_1y_1\right)+\frac{e^{\frac{2}{3}\pi i}}{x_1^2y_1^2}\mathrm{Ln}\left(1-i\sqrt{3}+2x_1y_1\right)+ \\
& - \frac{e^{-\frac{2}{3}\pi i}}{x_1^2y_1^2}\mathrm{Ln}\left(1+i\sqrt{3}+2y_1^3x_1\right)-\frac{e^{\frac{2}{3}\pi i}}{x_1^2y_1^2}\mathrm{Ln}\left(1-i\sqrt{3}+2y_1^3x_1\right)+ \\
& + e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(y_1\left(2y_1+x_1+ix_1\sqrt{3}\right)\right)+e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(y_1\left(2y_1+x_1-ix_1\sqrt{3}\right)\right) \\
& - e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(2+x_1y_1+ix_1y_1\sqrt{3}\right)-e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(2+x_1y_1-ix_1y_1\sqrt{3}\right)
\end{aligned}
$$

$$
\begin{aligned}
= \; & \frac{1}{x_1y_1}\frac{x_1y_1+1}{(1-x_1y_1)}\left(1-y_1^2\right)+ \\
& + \frac{1}{x_1^2y_1^2}\ln\left(\frac{1-x_1y_1}{1-x_1y_1^3}\right)-\ln\left(\frac{x_1y_1-1}{x_1y_1-y_1^2}\right)+ \\
& + \left(\frac{2}{x_1y_1}\frac{2+x_1y_1}{1+x_1y_1+x_1^2y_1^2}-\frac{1}{x_1}\frac{2y_1+x_1}{y_1^2+x_1y_1+x_1^2}-\frac{1}{y_1}\frac{y_1+2x_1}{y_1^2+x_1y_1+x_1^2}\right)\left(1-y_1^2\right)+ \\
& + \frac{e^{-\frac{2}{3}\pi i}+e^{\frac{2}{3}\pi i}}{x_1^2y_1^2}\ln\sqrt{\left(1+2x_1y_1\right)^2+3}+i\frac{e^{-\frac{2}{3}\pi i}-e^{\frac{2}{3}\pi i}}{x_1^2y_1^2}\arctan\left(\frac{\sqrt{3}}{1+2x_1y_1}\right)+ \\
& - \frac{e^{-\frac{2}{3}\pi i}+e^{\frac{2}{3}\pi i}}{x_1^2y_1^2}\ln\sqrt{\left(1+2y_1^3x_1\right)^2+3}-i\frac{e^{-\frac{2}{3}\pi i}-e^{\frac{2}{3}\pi i}}{x_1^2y_1^2}\arctan\left(\frac{\sqrt{3}}{1+2y_1^3x_1}\right)+ \\
& + \left(e^{-\frac{2}{3}\pi i}+e^{\frac{2}{3}\pi i}\right)\ln\sqrt{y_1^2\left(2y_1+x_1\right)^2+3x_1^2y_1^2}+i\left(e^{-\frac{2}{3}\pi i}-e^{\frac{2}{3}\pi i}\right)\arctan\left(\frac{\sqrt{3}x_1}{2y_1+x_1}\right) \\
& - \left(e^{-\frac{2}{3}\pi i}+e^{\frac{2}{3}\pi i}\right)\ln\sqrt{\left(2+x_1y_1\right)^2+3x_1^2y_1^2}-i\left(e^{-\frac{2}{3}\pi i}-e^{\frac{2}{3}\pi i}\right)\arctan\left(\frac{x_1y_1\sqrt{3}}{2+x_1y_1}\right)
\end{aligned}
$$

$$
\begin{aligned}
= \;& \frac{1}{x_1 y_1}\frac{x_1 y_1 + 1}{(1 - x_1 y_1)}\left(1 - y_1^2\right) + \\
&+ \frac{1}{x_1^2 y_1^2}\ln\left(\frac{1 - x_1 y_1}{1 - x_1 y_1^3}\right) - \ln\left(\frac{x_1 y_1 - 1}{x_1 y_1 - y_1^2}\right) + \\
&+ \left(\frac{2}{x_1 y_1}\frac{2 + x_1 y_1}{1 + x_1 y_1 + x_1^2 y_1^2} - \frac{1}{x_1}\frac{2y_1 + x_1}{y_1^2 + x_1 y_1 + x_1^2} - \frac{1}{y_1}\frac{y_1 + 2x_1}{y_1^2 + x_1 y_1 + x_1^2}\right)\left(1 - y_1^2\right) + \\
&- \frac{1}{x_1^2 y_1^2}\ln\sqrt{\left(1 + 2x_1 y_1\right)^2 + 3} + \frac{\sqrt{3}}{x_1^2 y_1^2}\arctan\left(\frac{\sqrt{3}}{1 + 2x_1 y_1}\right) + \\
&+ \frac{1}{x_1^2 y_1^2}\ln\sqrt{\left(1 + 2y_1^3 x_1\right)^2 + 3} - \frac{\sqrt{3}}{x_1^2 y_1^2}\arctan\left(\frac{\sqrt{3}}{1 + 2y_1^3 x_1}\right) + \\
&- \ln\sqrt{y_1^2\left(2y_1 + x_1\right)^2 + 3x_1^2 y_1^2} + \sqrt{3}\arctan\left(\frac{\sqrt{3}x_1}{2y_1 + x_1}\right) \\
&+ \ln\sqrt{\left(2 + x_1 y_1\right)^2 + 3x_1^2 y_1^2} - \sqrt{3}\arctan\left(\frac{x_1 y_1\sqrt{3}}{2 + x_1 y_1}\right)
\end{aligned}
$$

$$
\begin{aligned}
= \;& \frac{1}{x_1 y_1}\frac{x_1 y_1 + 1}{(1 - x_1 y_1)}\left(1 - y_1^2\right) + \\
&+ \frac{1}{x_1^2 y_1^2}\ln\left(\frac{1 - x_1 y_1}{1 - x_1 y_1^3}\right) - \ln\left(\frac{x_1 y_1 - 1}{x_1 y_1 - y_1^2}\right) + \\
&+ \left(\frac{2}{x_1 y_1}\frac{2 + x_1 y_1}{1 + x_1 y_1 + x_1^2 y_1^2} - \frac{1}{x_1}\frac{2y_1 + x_1}{y_1^2 + x_1 y_1 + x_1^2} - \frac{1}{y_1}\frac{y_1 + 2x_1}{y_1^2 + x_1 y_1 + x_1^2}\right)\left(1 - y_1^2\right) + \\
&- \frac{1}{x_1^2 y_1^2}\ln\left(2\sqrt{(1 + x_1 y_1 + x_1^2 y_1^2)}\right) + \frac{\sqrt{3}}{x_1^2 y_1^2}\arctan\left(\frac{\sqrt{3}}{1 + 2x_1 y_1}\right) + \\
&+ \frac{1}{x_1^2 y_1^2}\ln\left(2\sqrt{(1 + y_1^3 x_1 + y_1^6 x_1^2)}\right) - \frac{\sqrt{3}}{x_1^2 y_1^2}\arctan\left(\frac{\sqrt{3}}{1 + 2y_1^3 x_1}\right) + \\
&- \ln\left(2y_1\sqrt{(y_1^2 + x_1 y_1 + x_1^2)}\right) + \sqrt{3}\arctan\left(\frac{\sqrt{3}x_1}{2y_1 + x_1}\right) \\
&+ \ln\left(2\sqrt{(1 + x_1 y_1 + x_1^2 y_1^2)}\right) - \sqrt{3}\arctan\left(\frac{x_1 y_1\sqrt{3}}{2 + x_1 y_1}\right)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{1}{x_1 y_1} \frac{1 + x_1 y_1}{1 - x_1 y_1} \left(1 - y_1^2\right) + \\
& + \frac{1}{x_1^2 y_1^2} \ln\left(\frac{1 - x_1 y_1}{1 - x_1 y_1^3}\right) - \ln\left(\frac{x_1 y_1 - 1}{x_1 y_1 - y_1^2}\right) + \\
& + \frac{2}{x_1 y_1} \frac{1 - x_1^2 y_1^2}{1 + x_1 y_1 + x_1^2 y_1^2} \left(1 - y_1^2\right) + \\
& - \frac{1}{x_1^2 y_1^2} \ln\left(\sqrt{(1 + x_1 y_1 + x_1^2 y_1^2)}\right) + \frac{\sqrt{3}}{x_1^2 y_1^2} \arctan\left(\frac{\sqrt{3}}{1 + 2 x_1 y_1}\right) + \\
& + \frac{1}{x_1^2 y_1^2} \ln\left(\sqrt{(1 + y_1^3 x_1 + y_1^6 x_1^2)}\right) - \frac{\sqrt{3}}{x_1^2 y_1^2} \arctan\left(\frac{\sqrt{3}}{1 + 2 y_1^3 x_1}\right) + \\
& - \ln\left(y_1 \sqrt{(y_1^2 + x_1 y_1 + x_1^2)}\right) + \sqrt{3} \arctan\left(\frac{\sqrt{3} x_1}{2 y_1 + x_1}\right) \\
& + \ln\left(\sqrt{(1 + x_1 y_1 + x_1^2 y_1^2)}\right) - \sqrt{3} \arctan\left(\frac{x_1 y_1 \sqrt{3}}{2 + x_1 y_1}\right) \\
= \quad & \frac{1 - y_1^2}{x_1 y_1} \left(\frac{1 + x_1 y_1}{1 - x_1 y_1} + 2 \frac{1 - x_1^2 y_1^2}{1 + x_1 y_1 + x_1^2 y_1^2}\right) + \\
& - \ln\left(\frac{1 - x_1 y_1}{y_1 - x_1}\right) + \tfrac{1}{2} \ln\left(\frac{1 + x_1 y_1 + x_1^2 y_1^2}{x_1^2 + x_1 y_1 + y_1^2}\right) + \\
& + \frac{1}{x_1^2 y_1^2} \left(\ln\left(\frac{1 - x_1 y_1}{1 - x_1 y_1^3}\right) - \tfrac{1}{2} \ln\left(\frac{1 + x_1 y_1 + x_1^2 y_1^2}{1 + x_1 y_1^3 + x_1^2 y_1^6}\right)\right) + \\
& + \frac{\sqrt{3}}{x_1^2 y_1^2} \left(\arctan\left(\frac{\sqrt{3}}{1 + 2 x_1 y_1}\right) - \arctan\left(\frac{\sqrt{3}}{1 + 2 y_1^3 x_1}\right)\right) + \\
& + \sqrt{3} \left(\arctan\left(\frac{\sqrt{3} x_1}{2 y_1 + x_1}\right) - \arctan\left(\frac{x_1 y_1 \sqrt{3}}{2 + x_1 y_1}\right)\right) . \qquad (37)
\end{aligned}
$$

4.5. Conclusion of case 1)

Recall that we want to evaluate (8). And when adding the three integrals we find, separated according to a main expression:

- algebraic terms

$$
\begin{aligned}
& - \frac{(x_1 + y_1)\left(1 - y_1^2\right)}{y_1 (1 - x_1 y_1)} + \frac{y_1^2 - x_1^2}{x_1 y_1} \frac{3 + 2 x_1 y_1 + x_1^2 y_1^2}{1 + x_1 y_1 + x_1^2 y_1^2} - \frac{y_1^2 - x_1^2}{x_1 y_1} \frac{3 y_1^2 + 2 x_1 y_1 + x_1^2}{y_1^2 + x_1 y_1 + x_1^2} + \\
& + \frac{1}{x_1 y_1} \frac{1 + x_1 y_1}{1 - x_1 y_1} \left(1 - y_1^2\right) + \frac{2}{x_1 y_1} \frac{1 - x_1^2 y_1^2}{1 + x_1 y_1 + x_1^2 y_1^2} \left(1 - y_1^2\right) \\
= \quad & \frac{3\left(1 - x_1^2\right)\left(1 - y_1^2\right)}{x_1 y_1 \left(1 + x_1 y_1 + x_1^2 y_1^2\right)} \left(\frac{1}{1 - x_1 y_1} + \frac{x_1^2 y_1^2}{y_1^2 + x_1 y_1 + x_1^2}\right)
\end{aligned}
$$

- pure logarithmic terms

$$
\frac{1}{2} \ln\left(\frac{x_1^2 - 2 x_1 y_1 + y_1^2}{x_1^2 + x_1 y_1 + y_1^2}\right) + \left(- \ln\left(\frac{1 - x_1 y_1}{y_1 - x_1}\right) + \frac{1}{2} \ln\left(\frac{1 + x_1 y_1 + x_1^2 y_1^2}{x_1^2 + x_1 y_1 + y_1^2}\right)\right)
$$

$$= \frac{1}{2}\ln\left(\frac{x_1^2-2x_1y_1+y_1^2}{x_1^2+x_1y_1+y_1^2}\left(\frac{y_1-x_1}{1-x_1y_1}\right)^2\frac{1+x_1y_1+x_1^2y_1^2}{x_1^2+x_1y_1+y_1^2}\right)$$
$$= \frac{1}{2}\ln\left(\frac{(1-x_1y_1)^2}{1+x_1y_1+x_1^2y_1^2}\right)-2\ln\left(\frac{1-x_1y_1}{y_1-x_1}\right)+\ln\left(\frac{1+x_1y_1+x_1^2y_1^2}{x_1^2+x_1y_1+y_1^2}\right)$$

- logarithmic terms with x_1^{-2}

$$-\frac{1}{2x_1^2}\ln\left(\frac{x_1^6-2x_1^3y_1+y_1^2}{x_1^6+x_1^3y_1+y_1^2}\right)+\frac{1}{2x_1^2}\ln\left(\frac{1+x_1y_1+x_1^2y_1^2}{y_1^2+x_1^3y_1+x_1^6}\right)-\frac{1}{x_1^2}\ln\left(\frac{1-x_1y_1}{y_1-x_1^3}\right)$$
$$= \frac{1}{2x_1^2}\ln\left(\frac{1+x_1y_1+x_1^2y_1^2}{y_1^2+x_1^3y_1+x_1^6}\,\frac{x_1^6+x_1^3y_1+y_1^2}{x_1^6-2x_1^3y_1+y_1^2}\left(\frac{y_1-x_1^3}{1-x_1y_1}\right)^2\right)$$
$$= -\frac{1}{2x_1^2}\ln\left(\frac{(1-x_1y_1)^2}{1+x_1y_1+x_1^2y_1^2}\right)$$

- logarithmic terms with y_1^{-2}

$$-\frac{1}{2y_1^2}\ln\left(\frac{(1-x_1y_1)^2}{1+x_1y_1+x_1^2y_1^2}\right)$$

- logarithmic terms with $x_1^{-2}y_1^{-2}$

$$\frac{1}{2x_1^2y_1^2}\ln\left(\frac{1-2x_1^3y_1+y_1^2x_1^6}{1+x_1^3y_1+x_1^6y_1^2}\right)+\frac{1}{x_1^2y_1^2}\ln\left(\frac{1-x_1y_1^3}{1-x_1^3y_1}\right)-\frac{1}{2x_1^2y_1^2}\ln\left(\frac{1+x_1y_1^3+y_1^6x_1^2}{1+x_1^3y_1+x_1^6y_1^2}\right)+$$
$$+\frac{1}{x_1^2y_1^2}\ln\left(\frac{1-x_1y_1}{1-x_1y_1^3}\right)-\frac{1}{2x_1^2y_1^2}\ln\left(\frac{1+x_1y_1+x_1^2y_1^2}{1+x_1y_1^3+x_1^2y_1^6}\right)$$
$$=\frac{1}{2x_1^2y_1^2}\ln\left(\frac{1-2x_1^3y_1+y_1^2x_1^6}{1+x_1^3y_1+x_1^6y_1^2}\left(\frac{1-x_1y_1^3}{1-x_1^3y_1}\right)^2\frac{1+x_1^3y_1+x_1^6y_1^2}{1+x_1y_1^3+y_1^6x_1^2}\left(\frac{1-x_1y_1}{1-x_1y_1^3}\right)^2\frac{1+x_1y_1^3+x_1^2y_1^6}{1+x_1y_1+x_1^2y_1^2}\right)$$
$$=\frac{1}{2x_1^2y_1^2}\ln\left(\frac{(1-x_1y_1)^2}{1+x_1y_1+x_1^2y_1^2}\right)$$

- terms containing arctan:

$$\sqrt{3}\arctan\left(\frac{\sqrt{3}x_1}{2y_1+x_1}\right)-\sqrt{3}\arctan\left(\frac{x_1\sqrt{3}}{2y_1+x_1}\right)+$$
$$+\frac{\sqrt{3}}{x_1^2}\arctan\left(\frac{x_1^3\sqrt{3}}{2y_1+x_1^3}\right)-\frac{\sqrt{3}}{x_1^2}\arctan\left(\frac{x_1^3\sqrt{3}}{2y_1+x_1^3}\right)+$$
$$+\frac{\sqrt{3}}{x_1^2}\arctan\left(\frac{x_1y_1\sqrt{3}}{2+x_1y_1}\right)+\frac{\sqrt{3}}{y_1^2}\arctan\left(\frac{x_1y_1\sqrt{3}}{2+x_1y_1}\right)+$$
$$+\frac{\sqrt{3}}{x_1^2y_1^2}\arctan\left(\frac{x_1^3y_1\sqrt{3}}{2+x_1^3y_1}\right)-\frac{\sqrt{3}}{x_1^2y_1^2}\arctan\left(\frac{x_1^3y_1\sqrt{3}}{2+y_1x_1^3}\right)+$$
$$-\frac{\sqrt{3}}{x_1^2y_1^2}\arctan\left(\frac{y_1^3x_1\sqrt{3}}{2+y_1^3x_1}\right)-\sqrt{3}\arctan\left(\frac{x_1y_1\sqrt{3}}{2+x_1y_1}\right)+$$
$$+\frac{\sqrt{3}}{x_1^2y_1^2}\arctan\left(\frac{\sqrt{3}}{1+2x_1y_1}\right)-\frac{\sqrt{3}}{x_1^2y_1^2}\arctan\left(\frac{\sqrt{3}}{1+2y_1^3x_1}\right)$$

$$= \quad \sqrt{3}\left(\frac{1}{x_1^2}+\frac{1}{y_1^2}-1\right)\arctan\left(\frac{x_1y_1\sqrt{3}}{2+x_1y_1}\right)+$$
$$+\ \frac{\sqrt{3}}{x_1^2y_1^2}\left(\arctan\left(\frac{\sqrt{3}}{1+2x_1y_1}\right)-\arctan\left(\frac{\sqrt{3}}{1+2y_1^3x_1}\right)-\arctan\left(\frac{y_1^3x_1\sqrt{3}}{2+y_1^3x_1}\right)\right)$$
$$= -\sqrt{3}\left(1-\frac{1}{x_1^2}-\frac{1}{y_1^2}+\frac{1}{x_1^2y_1^2}\right)\arctan\left(\frac{x_1y_1\sqrt{3}}{2+x_1y_1}\right)$$

Combining the expressions in (35), (36) and (37), listed as above, the enumerator of (8) becomes

$$\int_{|z|<1}\lim_{x_2\downarrow 0}\frac{G(x,z)}{x_2}\lim_{y_2\downarrow 0}\frac{G(z,y)}{y_2}dz \ =$$
$$= \left(\frac{-\frac{1}{4}\pi}{\pi^2}\right)\Bigg(\frac{3\left(1-x_1^2\right)\left(1-y_1^2\right)}{x_1y_1\left(1+x_1y_1+x_1^2y_1^2\right)}\left(\frac{1}{1-x_1y_1}+\frac{x_1^2y_1^2}{y_1^2+x_1y_1+x_1^2}\right)+$$
$$-\,2\ \ln\left(\frac{1-x_1y_1}{y_1-x_1}\right)+\ln\left(\frac{1+x_1y_1+x_1^2y_1^2}{x_1^2+x_1y_1+y_1^2}\right)+$$
$$+\,\frac{1}{2}\left(1-\frac{1}{x_1^2}-\frac{1}{y_1^2}+\frac{1}{x_1^2y_1^2}\right)\ln\left(\frac{\left(1-x_1y_1\right)^2}{1+x_1y_1+x_1^2y_1^2}\right)+$$
$$-\sqrt{3}\left(1-\frac{1}{x_1^2}-\frac{1}{y_1^2}+\frac{1}{x_1^2y_1^2}\right)\arctan\left(\frac{x_1y_1\sqrt{3}}{2+x_1y_1}\right)\Bigg)\ , \qquad (38)$$

while the denominator is given by (12). If we define

$$T_{11}\left(x_1,y_1\right):=\frac{\int_{|z|<1}\lim_{x_2\downarrow 0}\frac{G(x,z)}{x_2}\lim_{y_2\downarrow 0}\frac{G(z,y)}{y_2}dz}{\lim_{x_2\downarrow 0}\lim_{y_2\downarrow 0}\frac{G(x,y)}{x_2y_2}}=$$

$$= \left(\frac{-\frac{1}{4}\pi}{\pi^2}\right)\left(\frac{\frac{3\left(1-x_1^2\right)\left(1-y_1^2\right)}{x_1y_1\left(1+x_1y_1+x_1^2y_1^2\right)}\left(\frac{1}{1-x_1y_1}+\frac{x_1^2y_1^2}{y_1^2+x_1y_1+x_1^2}\right)-2\ln\left(\frac{1-x_1y_1}{y_1-x_1}\right)+\ln\left(\frac{1+x_1y_1+x_1^2y_1^2}{x_1^2+x_1y_1+y_1^2}\right)}{\frac{1}{\pi}\left(\frac{1}{(x_1-y_1)^2}-\frac{1}{(1-x_1y_1)^2}+\frac{1}{1+x_1y_1+x_1^2y_1^2}-\frac{1}{x_1^2+x_1y_1+y_1^2}+\frac{3x_1y_1}{\left(1+x_1y_1+x_1^2y_1^2\right)^2}-\frac{3x_1y_1}{\left(x_1^2+x_1y_1+y_1^2\right)^2}\right)}+\right.$$
$$\left.+\frac{\frac{1}{2}\left(1-\frac{1}{x_1^2}-\frac{1}{y_1^2}+\frac{1}{x_1^2y_1^2}\right)\ln\left(\frac{(1-x_1y_1)^2}{1+x_1y_1+x_1^2y_1^2}\right)-\sqrt{3}\left(1-\frac{1}{x_1^2}-\frac{1}{y_1^2}+\frac{1}{x_1^2y_1^2}\right)\arctan\left(\frac{x_1y_1\sqrt{3}}{2+x_1y_1}\right)}{\frac{1}{\pi}\left(\frac{1}{(x_1-y_1)^2}-\frac{1}{(1-x_1y_1)^2}+\frac{1}{1+x_1y_1+x_1^2y_1^2}-\frac{1}{x_1^2+x_1y_1+y_1^2}+\frac{3x_1y_1}{\left(1+x_1y_1+x_1^2y_1^2\right)^2}-\frac{3x_1y_1}{\left(x_1^2+x_1y_1+y_1^2\right)^2}\right)}\right)$$

then this function represents $\mathrm{E}_x^y\left(\tau_S\right)$ for $x, y \in \Gamma_1$ and $x_1 < y_1$. By symmetry we obtain this function for $x_1 > y_1$ as $T_{11}(x_1, y_1) = T_{11}(y_1, x_1)$. A close inspection reveals that

$$\sup_{x_1,y_1\in(0,1)} T_{11}(x_1,y_1)=\lim_{x_1\downarrow 0,y_1\uparrow 1} T_{11}(x_1,y_1)=\frac{1}{16}\ . \qquad (39)$$

See Figure 1 on p. 32.

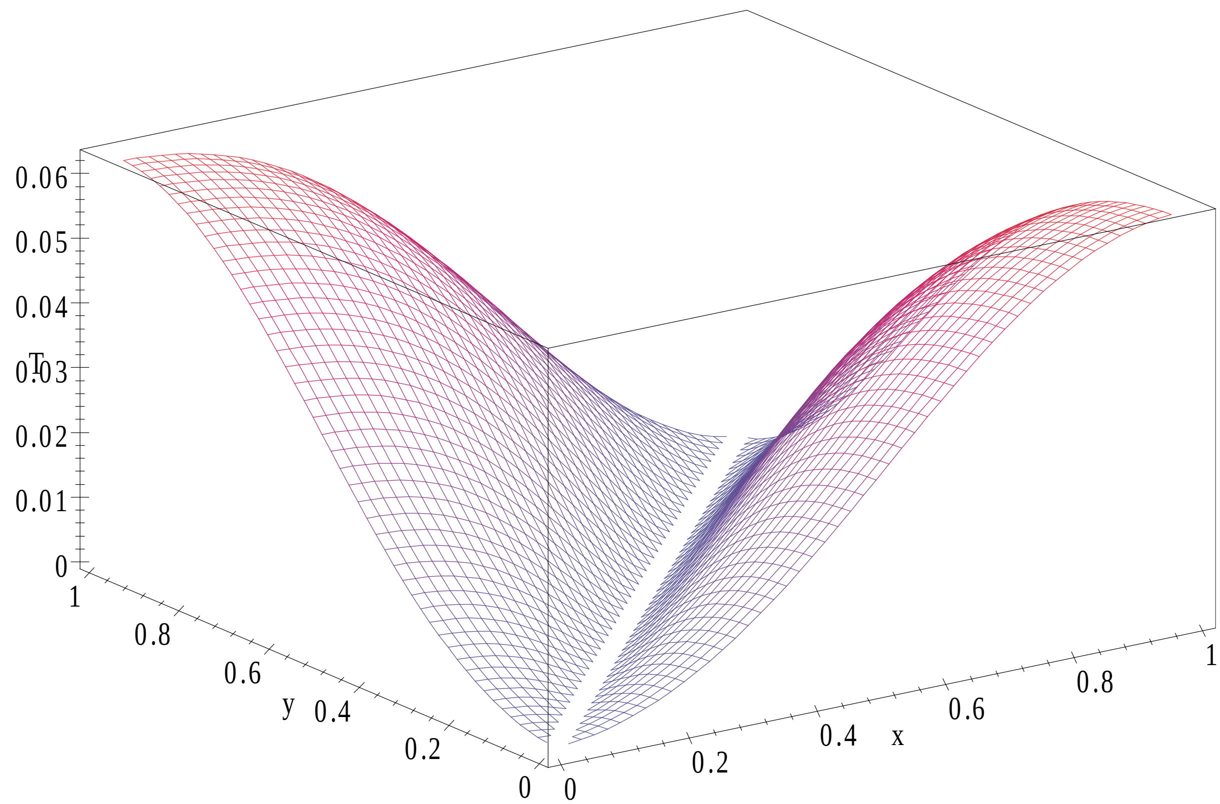


Figure 1: x and y both on Γ_1: $T_{11}(x_1, y_1)$.

5. The case that $x \in \Gamma_1,\ y \in \Gamma_2$

In this section we let $y \to (\cos\psi, \sin\psi)$ and $x \to (x_1, 0)$ and we will consider

$$T_{12}(x_1,\psi) := \frac{\displaystyle\int_{z\in S} \lim_{x_2\downarrow 0}\frac{G_S(x,z)}{x_2}\lim_{|y|\uparrow 1}\frac{G_S(z,y)}{1-|y|^2}dz}{\displaystyle\lim_{|y|\uparrow 1\ \&\ x_2\downarrow 0}\frac{G_S(x,y)}{x_2(1-|y|^2)}}\ .$$

Again the limits are computed from the expression in (6).

5.1. Limit of the Green function

Next to the expression in (9), with y replaced by z,

$$\begin{aligned}
&\lim_{x_2\downarrow 0}\frac{G_S(x,z)}{x_2} = \\
= \quad & \frac{1}{\pi}\left(\frac{r\sin\theta}{x_1^2-2x_1r\cos\theta+r^2}-\frac{r\sin\theta}{1-2x_1r\cos\theta+x_1^2r^2}\right)+ \\
&+\frac{1}{\pi}\left(\frac{r\sin\left(\theta-\frac{2}{3}\pi\right)}{x_1^2-2x_1r\cos\left(\theta-\frac{2}{3}\pi\right)+r^2}-\frac{r\sin\left(\theta-\frac{2}{3}\pi\right)}{1-2x_1r\cos\left(\theta-\frac{2}{3}\pi\right)+x_1^2r^2}\right)+ \\
&+\frac{1}{\pi}\left(\frac{r\sin\left(\theta+\frac{2}{3}\pi\right)}{x_1^2-2x_1r\cos\left(\theta+\frac{2}{3}\pi\right)+r^2}-\frac{r\sin\left(\theta+\frac{2}{3}\pi\right)}{1-2x_1r\cos\left(\theta+\frac{2}{3}\pi\right)+x_1^2r^2}\right),
\end{aligned}$$

we need

$$\lim_{y\to(\cos\psi,\sin\psi)} \frac{G_S(z,y)}{1-|y|^2} =$$

$$= \frac{1-|z|^2}{4\pi}\left(-\frac{1}{|z-y|^2}-\frac{1}{|\mathcal{R}z-y|^2}-\frac{1}{|\mathcal{R}^2z-y|^2}+\frac{1}{|\mathcal{S}z-y|^2}+\frac{1}{|\mathcal{R}\mathcal{S}z-y|^2}+\frac{1}{|\mathcal{R}^2\mathcal{S}z-y|^2}\right)_{y=(\cos\psi,\sin\psi)}$$

$$= \frac{1-r^2}{4\pi}\left(\frac{1}{r^2-2r\cos(\theta+\psi)+1}-\frac{1}{r^2-2r\cos(\theta-\psi)+1}\right)$$

$$+\frac{1-r^2}{4\pi}\left(\frac{1}{r^2-2r\cos\left(\theta+\psi+\frac{2}{3}\pi\right)+1}-\frac{1}{r^2-2r\cos\left(\theta-\psi+\frac{2}{3}\pi\right)+1}\right)+$$

$$+\frac{1-r^2}{4\pi}\left(+\frac{1}{r^2-2r\cos\left(\theta+\psi-\frac{2}{3}\pi\right)+1}-\frac{1}{r^2-2r\cos\left(\theta-\psi-\frac{2}{3}\pi\right)+1}\right)=$$

$$= \frac{1-r^2}{4\pi}\sum_{k=0}^{2}\sum_{\sigma=\pm1}\frac{\sigma}{r^2-2r\cos\left(\theta+\sigma\psi+\frac{2}{3}k\pi\right)+1}\ , \tag{40}$$

and the double limit of the denominator

$$\lim_{|y|\uparrow1}\lim_{x_2\downarrow0}\frac{G_S(x,y)}{\left(1-|y|^2\right)x_2}=$$

$$= \lim_{|y|\uparrow1}\left(\frac{1}{\pi(1-r^2)}\left(\frac{r\sin\psi}{x_1^2-2x_1r\cos\psi+r^2}-\frac{r\sin\psi}{1-2x_1r\cos\psi+x_1^2r^2}\right)+\right.$$

$$+\frac{1}{\pi}\left(\frac{r\sin\left(\psi+\frac{2}{3}\pi\right)}{x_1^2-2x_1r\cos\left(\psi+\frac{2}{3}\pi\right)+r^2}-\frac{r\sin\left(\psi+\frac{2}{3}\pi\right)}{1-2x_1\cos\left(\psi+\frac{2}{3}\pi\right)+x_1^2r^2}\right)+$$

$$\left.+\frac{1}{\pi}\left(\frac{r\sin\left(\psi-\frac{2}{3}\pi\right)}{x_1^2-2x_1r\cos\left(\psi-\frac{2}{3}\pi\right)+r^2}-\frac{r\sin\left(\psi-\frac{2}{3}\pi\right)}{1-2x_1r\cos\left(\psi-\frac{2}{3}\pi\right)+x_1^2r^2}\right)\right)$$

$$= \frac{\left(1-x_1^2\right)\sin\psi}{\pi\left(x_1^2-2x_1\cos\psi+1\right)^2}+\frac{\left(1-x_1^2\right)\sin\left(\psi+\frac{2}{3}\pi\right)}{\pi\left(x_1^2-2x_1\cos\left(\psi+\frac{2}{3}\pi\right)+1\right)^2}+\frac{\left(1-x_1^2\right)\sin\left(\psi-\frac{2}{3}\pi\right)}{\pi\left(x_1^2-2x_1\cos\left(\psi-\frac{2}{3}\pi\right)+1\right)^2}. \tag{41}$$

5.2. Derivation of a contour integral

In addition to the h as in (17,18,19) we define

$$f(\theta) = \frac{1-r^2}{4}\left(\frac{1}{r^2-2r\cos(\theta+\psi)+1}-\frac{1}{r^2-2r\cos(\theta-\psi)+1}\right), \tag{42}$$

$$\begin{aligned} f_\psi(\theta) &= \frac{1-r^2}{4}\left(\frac{1}{r^2-2r\cos(\theta+\psi)+1}\right) \\ &= \frac{-e^{-i\psi}}{4}\left(\frac{1}{w-e^{-i\psi}}-\frac{r^2}{w-r^2e^{-i\psi}}\right). \end{aligned} \tag{43}$$

Note that $h(-\theta)=-h(\theta)$ and also $f(\theta)=f(-\theta)$. Hence $\theta\mapsto\sum_{k=0}^{2}h\left(\theta+k\frac{2}{3}\pi\right)$ and also $\theta\mapsto\sum_{k=0}^{2}f\left(\theta+k\frac{2}{3}\pi\right)$ are odd and also periodic with period $\frac{2}{3}\pi$. Using h and f (respectively f_ψ we may rewrite the angular integral in the enumerator of (5) as follows

$$\begin{aligned}
&\pi^2 \int_{\theta=0}^{\frac{1}{3}\pi} \lim_{x\to(x_1,0)} \frac{G_S(x,z)}{x_2} \lim_{y\to(\cos\psi,\sin\psi)} \frac{G_S(z,y)}{1-|y|^2}\, d\theta \;= \\
&= \int_{\theta=0}^{\frac{1}{3}\pi} \sum_{k=0}^{2} h\left(\theta + k\frac{2}{3}\pi\right) \sum_{m=0}^{2} f\left(\theta + m\tfrac{2}{3}\pi\right) d\theta \;= \qquad (44)\\
&= \frac{1}{6}\int_{\theta=0}^{2\pi} \sum_{m=0}^{2} h\left(\theta + m\frac{2}{3}\pi\right) \sum_{m=0}^{2} f\left(\theta + m\tfrac{2}{3}\pi\right) d\theta \;= \\
&= \frac{1}{2}\int_{\theta=0}^{2\pi} \sum_{k=0}^{2} h\left(\theta + k\frac{2}{3}\pi\right) f(\theta)\, d\theta \;= \\
&= \int_{\theta=0}^{2\pi} \sum_{k=0}^{2} h\left(\theta + k\tfrac{2}{3}\pi\right) f_\psi(\theta)\, d\theta\ . \qquad (45)
\end{aligned}$$

It leaves us to compute:

$$\begin{aligned}
&\pi^2 \int_{z\in S} \lim_{\substack{x\to(x_1,0)\\ x\in S}} \frac{G_S(x,z)}{x_2} \lim_{\substack{y\to(\cos\psi,\sin\psi)\\ y\in S}} \frac{G_S(z,y)}{1-|y|^2} dz = \\
&= \int_{r=0}^{1}\int_{\theta=0}^{2\pi} \left(h(\theta) + h\left(\theta+\tfrac{2}{3}\pi\right) + h\left(\theta + \frac{4}{3}\pi\right)\right) f_\psi(\theta)\ d\theta\, r\, dr \\
&= \int_{r=0}^{1}\oint_{|w|=r} \left(\frac{1}{2i}\left(\frac{r^2}{w-x_1r^2} + \frac{x_1^{-2}}{w-x_1^{-1}} - \frac{x_1^{-2}r^2}{w-x_1^{-1}r^2} - \frac{1}{w-x_1}\right) + \right.\\
&\quad + \frac{e^{-\frac{2}{3}\pi i}}{2i}\left(\frac{r^2}{w-x_1r^2e^{-\frac{2}{3}\pi i}} + \frac{x_1^{-2}}{w-x_1^{-1}e^{-\frac{2}{3}\pi i}} - \frac{x_1^{-2}r^2}{w-x_1^{-1}r^2e^{-\frac{2}{3}\pi i}} - \frac{1}{w-x_1e^{-\frac{2}{3}\pi i}}\right) + \\
&\quad \left. + \frac{e^{\frac{2}{3}\pi i}}{2i}\left(\frac{r^2}{w-x_1r^2e^{\frac{2}{3}\pi i}} + \frac{x_1^{-2}}{w-x_1^{-1}e^{\frac{2}{3}\pi i}} - \frac{x_1^{-2}r^2}{w-x_1^{-1}r^2e^{\frac{2}{3}\pi i}} - \frac{1}{w-x_1e^{\frac{2}{3}\pi i}}\right)\right) \times \\
&\quad \left(\frac{-e^{-i\psi}}{4}\left(\frac{1}{w-e^{-i\psi}} - \frac{r^2}{w-r^2e^{-i\psi}}\right)\right) \frac{dw}{iw}\, r\, dr. \\
&= \frac{1}{8}\int_{r=0}^{1}\oint_{|w|=r} e^{-i\psi} \left(\left(\frac{r^2}{w-x_1r^2} + \frac{x_1^{-2}}{w-x_1^{-1}} - \frac{x_1^{-2}r^2}{w-x_1^{-1}r^2} - \frac{1}{w-x_1}\right) + \right.\\
&\quad + e^{-\frac{2}{3}\pi i}\left(\frac{r^2}{w-x_1r^2e^{-\frac{2}{3}\pi i}} + \frac{x_1^{-2}}{w-x_1^{-1}e^{-\frac{2}{3}\pi i}} - \frac{x_1^{-2}r^2}{w-x_1^{-1}r^2e^{-\frac{2}{3}\pi i}} - \frac{1}{w-x_1e^{-\frac{2}{3}\pi i}}\right) + \\
&\quad \left. + e^{\frac{2}{3}\pi i}\left(\frac{r^2}{w-x_1r^2e^{\frac{2}{3}\pi i}} + \frac{x_1^{-2}}{w-x_1^{-1}e^{\frac{2}{3}\pi i}} - \frac{x_1^{-2}r^2}{w-x_1^{-1}r^2e^{\frac{2}{3}\pi i}} - \frac{1}{w-x_1e^{\frac{2}{3}\pi i}}\right)\right) \times \\
&\quad \left(\frac{1}{w-e^{-i\psi}} - \frac{r^2}{w-r^2e^{-i\psi}}\right) \frac{dw}{w}\, r\, dr.
\end{aligned}$$

Using the expression from (21) and a newly defined

$$\Phi_{\psi,r}(w) := \frac{1}{w-e^{-i\psi}} - \frac{r^2}{w-r^2e^{-i\psi}}\ , \qquad (46)$$

we obtain

$$\#_{(45)} = \frac{e^{-i\psi}}{8} \oint_{|w|=r} \Big(\Psi_{x_1,0,r}(w) + \Psi_{x_1,1,r}(w) + \Psi_{x_1,2,r}(w)\Big)\, \Phi_{\psi,r}(w)\, \frac{dw}{w}\,. \tag{47}$$

In this section we use

$$J_2(x_1,\psi,r;w) := \frac{1}{w}\ \Phi_{\psi,0,r}(w) \sum_{m=0}^{2} \Psi_{x_1,m,r}(w) \tag{48}$$

5.3. Computation of the contour integral

Two cases have to be distinguished, namely $r < x_1$ and $r > x_1$. Again $w = 0$ does not contribute and the scheme for the residues is as follows:

poles due to:		$\Psi_{x_1,0,r}$		$\Psi_{x_1,1,r}$		$\Psi_{x_1,2,r}$		$\Phi_{\psi,r}$
range:		$a_1.$	$a_2.$	$b_1.$	$b_2.$	$c_1.$	$c_2.$	$d.$
$r \in (0,x_1)$	I.	x_1r^2	$\frac{r^2}{x_1}$	$x_1r^2e^{-\frac{2}{3}\pi i}$	$\frac{r^2}{x_1}e^{-\frac{2}{3}\pi i}$	$x_1r^2e^{\frac{2}{3}\pi i}$	$\frac{r^2}{x_1}e^{\frac{2}{3}\pi i}$	$r^2e^{-i\psi}$
$r \in (x_1,1)$	II.	x_1r^2	x_1	$x_1r^2e^{-\frac{2}{3}\pi i}$	$x_1e^{-\frac{2}{3}\pi i}$	$x_1r^2e^{\frac{2}{3}\pi i}$	$x_1e^{\frac{2}{3}\pi i}$	$r^2e^{-i\psi}$

We proceed as before.

I and II, a_1: pole at $w = x_1r^2$.

$$\begin{aligned}
&\operatorname{Res}\Big(J_2(x_1,\psi,r;w)\Big)_{w=x_1r^2} = \\
&= \frac{r^2}{w}\left(\frac{1}{w-e^{-i\psi}} - \frac{r^2}{w-r^2e^{-i\psi}}\right)_{w=x_1r^2} \\
&= \frac{1}{x_1}\left(\frac{1}{x_1r^2-e^{-i\psi}} - \frac{r^2}{x_1r^2-r^2e^{-i\psi}}\right) \\
&= \frac{1}{x_1^2}\left(\frac{1}{r^2-x_1^{-1}e^{-i\psi}} - \frac{1}{1-x_1^{-1}e^{-i\psi}}\right).
\end{aligned} \tag{49}$$

I, a_2: pole at $w = x_1^{-1}r^2$.

$$\begin{aligned}
&\operatorname{Res}\Big(J_2(x_1,\psi,r;w)\Big)_{w=x_1^{-1}r^2} \\
&= \frac{-x_1^{-2}r^2}{w}\left(\frac{1}{w-e^{-i\psi}} - \frac{r^2}{w-r^2e^{-i\psi}}\right)_{w=x_1^{-1}r^2} \\
&= \frac{-x_1^{-2}r^2}{x_1^{-1}r^2}\left(\frac{1}{x_1^{-1}r^2-e^{-i\psi}} - \frac{r^2}{x_1^{-1}r^2-r^2e^{-i\psi}}\right) \\
&= \frac{1}{1-x_1e^{-i\psi}} - \frac{1}{r^2-x_1e^{-i\psi}}.
\end{aligned} \tag{50}$$

II, a_2: pole at $w = x_1$.

$$\begin{aligned}
&\operatorname{Res}\Big(J_2(x_1,\psi,r;w)\Big)_{w=x_1} = \\
&= -\frac{1}{w}\left(\frac{1}{w-e^{-i\psi}} - \frac{r^2}{w-r^2e^{-i\psi}}\right)_{w=x_1} \\
&= -\frac{1}{x_1}\left(\frac{1}{x_1-e^{-i\psi}} - \frac{r^2}{x_1-r^2e^{-i\psi}}\right) \\
&= e^{2i\psi}\left(\frac{1}{1-x_1e^{i\psi}} - \frac{1}{r^2-x_1e^{i\psi}}\right)
\end{aligned} \tag{51}$$

I and II, b_1: pole at $w = x_1 r^2 e^{-\frac{2}{3}\pi i}$.

$$\operatorname{Res}\left(J_2\left(x_1,\psi,r;w\right)\right)_{w=x_1 r^2 e^{-\frac{2}{3}\pi i}} =$$

$$\begin{aligned}
&= \frac{r^2 e^{-\frac{2}{3}\pi i}}{w}\left(\frac{1}{w-e^{-i\psi}} - \frac{r^2}{w-r^2e^{-i\psi}}\right)_{w=x_1 r^2 e^{-\frac{2}{3}\pi i}} \\
&= \frac{1}{x_1}\left(\frac{1}{x_1 r^2 e^{-\frac{2}{3}\pi i} - e^{-i\psi}} - \frac{r^2}{x_1 r^2 e^{-\frac{2}{3}\pi i} - r^2 e^{-i\psi}}\right) \\
&= \frac{e^{\frac{2}{3}\pi i}}{x_1^2}\left(\frac{1}{r^2 - x_1^{-1} e^{\frac{2}{3}\pi i - i\psi}} - \frac{1}{1 - x_1^{-1} e^{\frac{2}{3}\pi i - i\psi}}\right). \qquad (52)
\end{aligned}$$

I, b_2: pole at $w = x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}$.

$$\operatorname{Res}\left(J_2\left(x_1,\psi,r;w\right)\right)_{w=x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}} =$$

$$\begin{aligned}
&= -\frac{x_1^{-2} r^2 e^{-\frac{2}{3}\pi i}}{w}\left(\frac{1}{w-e^{-i\psi}} - \frac{r^2}{w-r^2e^{-i\psi}}\right)_{w=x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}} \\
&= -\frac{x_1^{-2} r^2 e^{-\frac{2}{3}\pi i}}{x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}}\left(\frac{1}{x_1^{-1} r^2 e^{-\frac{2}{3}\pi i} - e^{-i\psi}} - \frac{r^2}{x_1^{-1} r^2 e^{-\frac{2}{3}\pi i} - r^2 e^{-i\psi}}\right) \\
&= e^{\frac{2}{3}\pi i}\left(\frac{1}{1 - x_1 e^{\frac{2}{3}\pi i - i\psi}} - \frac{1}{r^2 - x_1 e^{\frac{2}{3}\pi i - i\psi}}\right). \qquad (53)
\end{aligned}$$

II, b_2: pole at $w = x_1 e^{-\frac{2}{3}\pi i}$.

$$\operatorname{Res}\left(J_2\left(x_1,\psi,r;w\right)\right)_{w=x_1 e^{-\frac{2}{3}\pi i}} =$$

$$\begin{aligned}
&= \frac{-e^{-\frac{2}{3}\pi i}}{w}\left(\frac{1}{w-e^{-i\psi}} - \frac{r^2}{w-r^2e^{-i\psi}}\right)_{w=x_1 e^{-\frac{2}{3}\pi i}} \\
&= \frac{-e^{-\frac{2}{3}\pi i}}{x_1 e^{-\frac{2}{3}\pi i}}\left(\frac{1}{x_1 e^{-\frac{2}{3}\pi i} - e^{-i\psi}} - \frac{r^2}{x_1 e^{-\frac{2}{3}\pi i} - r^2 e^{-i\psi}}\right) \\
&= \frac{-1}{x_1}\left(\frac{1}{x_1 e^{-\frac{2}{3}\pi i} - e^{-i\psi}} - e^{i\psi}\frac{r^2 e^{-i\psi}}{x_1 e^{-\frac{2}{3}\pi i} - r^2 e^{-i\psi}}\right) \\
&= \frac{-1}{x_1}\left(\frac{1}{x_1 e^{-\frac{2}{3}\pi i} - e^{-i\psi}} - e^{i\psi}\left(-1 + \frac{x_1 e^{-\frac{2}{3}\pi i}}{x_1 e^{-\frac{2}{3}\pi i} - r^2 e^{-i\psi}}\right)\right) \\
&= \frac{-1}{x_1}\left(\frac{x_1 e^{i\psi} e^{-\frac{2}{3}\pi i}}{x_1 e^{-\frac{2}{3}\pi i} - e^{-i\psi}} - e^{i\psi}\frac{x_1 e^{-\frac{2}{3}\pi i}}{x_1 e^{-\frac{2}{3}\pi i} - r^2 e^{-i\psi}}\right) \\
&= e^{-\frac{2}{3}\pi i + 2i\psi}\left(\frac{1}{1 - x_1 e^{-\frac{2}{3}\pi i + i\psi}} - \frac{1}{r^2 - x_1 e^{-\frac{2}{3}\pi i + i\psi}}\right) \qquad (54)
\end{aligned}$$

I and II, c_1**:** pole at $w = x_1 r^2 e^{\frac{2}{3}\pi i}$.

$$\operatorname{Res}\left(J_2\left(x_1,\psi,r;w\right)\right)_{w=x_1 r^2 e^{\frac{2}{3}\pi i}} =$$

$$\begin{aligned}
&= \frac{r^2 e^{\frac{2}{3}\pi i}}{w}\left(\frac{1}{w-e^{-i\psi}} - \frac{r^2}{w-r^2e^{-i\psi}}\right)_{w=x_1 r^2 e^{\frac{2}{3}\pi i}} \\
&= \frac{1}{x_1}\left(\frac{1}{x_1 r^2 e^{\frac{2}{3}\pi i} - e^{-i\psi}} - \frac{r^2}{x_1 r^2 e^{\frac{2}{3}\pi i} - r^2 e^{-i\psi}}\right) \\
&= \frac{e^{-\frac{2}{3}\pi i}}{x_1^2}\left(\frac{1}{r^2 - x_1^{-1}e^{-\frac{2}{3}\pi i - i\psi}} - \frac{1}{1 - x_1^{-1}e^{-\frac{2}{3}\pi i - i\psi}}\right)
\end{aligned} \tag{55}$$

I, c_2**:** pole at $w = x_1^{-1} r^2 e^{\frac{2}{3}\pi i}$.

$$\operatorname{Res}\left(J_2\left(x_1,\psi,r;w\right)\right)_{w=x_1^{-1} r^2 e^{\frac{2}{3}\pi i}} =$$

$$\begin{aligned}
&= -\frac{x_1^{-2} r^2 e^{\frac{2}{3}\pi i}}{w}\left(\frac{1}{w-e^{-i\psi}} - \frac{r^2}{w-r^2e^{-i\psi}}\right)_{w=x_1^{-1} r^2 e^{\frac{2}{3}\pi i}} \\
&= -\frac{1}{x_1}\left(\frac{1}{x_1^{-1} r^2 e^{\frac{2}{3}\pi i} - e^{-i\psi}} - \frac{r^2}{x_1^{-1} r^2 e^{\frac{2}{3}\pi i} - r^2 e^{-i\psi}}\right) \\
&= e^{-\frac{2}{3}\pi i}\left(\frac{1}{1 - x_1 e^{-\frac{2}{3}\pi i - i\psi}} - \frac{1}{r^2 - x_1 e^{-\frac{2}{3}\pi i - i\psi}}\right).
\end{aligned} \tag{56}$$

II, c_2**:** pole at $w = x_1 e^{\frac{2}{3}\pi i}$.

$$\operatorname{Res}\left(J_2\left(x_1,\psi,r;w\right)\right)_{w=x_1 e^{\frac{2}{3}\pi i}} =$$

$$\begin{aligned}
&= -\frac{e^{\frac{2}{3}\pi i}}{w}\left(\frac{1}{w-e^{-i\psi}} - \frac{r^2}{w-r^2e^{-i\psi}}\right)_{w=x_1 e^{\frac{2}{3}\pi i}} \\
&= -\frac{1}{x_1}\left(\frac{1}{x_1 e^{\frac{2}{3}\pi i} - e^{-i\psi}} - \frac{r^2}{x_1 e^{\frac{2}{3}\pi i} - r^2 e^{-i\psi}}\right) \\
&= -\frac{1}{x_1}\left(\frac{1}{x_1 e^{\frac{2}{3}\pi i} - e^{-i\psi}} + e^{i\psi}\left(1 - \frac{x_1 e^{\frac{2}{3}\pi i}}{x_1 e^{\frac{2}{3}\pi i} - r^2 e^{-i\psi}}\right)\right) \\
&= -\frac{1}{x_1}\left(\frac{x_1 e^{\frac{2}{3}\pi i} e^{i\psi}}{x_1 e^{\frac{2}{3}\pi i} - e^{-i\psi}} - \frac{x_1 e^{\frac{2}{3}\pi i} e^{i\psi}}{x_1 e^{\frac{2}{3}\pi i} - r^2 e^{-i\psi}}\right) \\
&= e^{\frac{2}{3}\pi i} e^{2i\psi}\left(\frac{1}{1 - x_1 e^{\frac{2}{3}\pi i} e^{i\psi}} - \frac{1}{r^2 - x_1 e^{\frac{2}{3}\pi i} e^{i\psi}}\right).
\end{aligned} \tag{57}$$

I and II, d**:** pole at $w = r^2 e^{-i\psi}$.

$$\operatorname{Res}\left(J_2\left(x_1,y_1,r;w\right)\right)_{w=r^2 e^{-i\psi}} =$$

$$
\begin{aligned}
&= \frac{-r^2}{w}\Bigg(\left(\frac{r^2}{w-x_1r^2}+\frac{x_1^{-2}}{w-x_1^{-1}}-\frac{x_1^{-2}r^2}{w-x_1^{-1}r^2}-\frac{1}{w-x_1}\right)+\\
&\quad + e^{-\frac{2}{3}\pi i}\left(\frac{r^2}{w-x_1r^2e^{-\frac{2}{3}\pi i}}+\frac{x_1^{-2}}{w-x_1^{-1}e^{-\frac{2}{3}\pi i}}-\frac{x_1^{-2}r^2}{w-x_1^{-1}r^2e^{-\frac{2}{3}\pi i}}-\frac{1}{w-x_1e^{-\frac{2}{3}\pi i}}\right)+\\
&\quad + e^{\frac{2}{3}\pi i}\left(\frac{r^2}{w-x_1r^2e^{\frac{2}{3}\pi i}}+\frac{x_1^{-2}}{w-x_1^{-1}e^{\frac{2}{3}\pi i}}-\frac{x_1^{-2}r^2}{w-x_1^{-1}r^2e^{\frac{2}{3}\pi i}}-\frac{1}{w-x_1e^{\frac{2}{3}\pi i}}\right)\Bigg)_{w=r^2e^{-i\psi}}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{-r^2}{r^2e^{-i\psi}}\Bigg(\left(\frac{r^2}{r^2e^{-i\psi}-x_1r^2}+\frac{x_1^{-2}}{r^2e^{-i\psi}-x_1^{-1}}-\frac{x_1^{-2}r^2}{r^2e^{-i\psi}-x_1^{-1}r^2}-\frac{1}{r^2e^{-i\psi}-x_1}\right)+\\
&\quad + e^{-\frac{2}{3}\pi i}\left(\frac{r^2}{r^2e^{-i\psi}-x_1r^2e^{-\frac{2}{3}\pi i}}+\frac{x_1^{-2}}{r^2e^{-i\psi}-x_1^{-1}e^{-\frac{2}{3}\pi i}}-\frac{x_1^{-2}r^2}{r^2e^{-i\psi}-x_1^{-1}r^2e^{-\frac{2}{3}\pi i}}-\frac{1}{r^2e^{-i\psi}-x_1e^{-\frac{2}{3}\pi i}}\right)+\\
&\quad + e^{\frac{2}{3}\pi i}\left(\frac{r^2}{r^2e^{-i\psi}-x_1r^2e^{\frac{2}{3}\pi i}}+\frac{x_1^{-2}}{r^2e^{-i\psi}-x_1^{-1}e^{\frac{2}{3}\pi i}}-\frac{x_1^{-2}r^2}{r^2e^{-i\psi}-x_1^{-1}r^2e^{\frac{2}{3}\pi i}}-\frac{1}{r^2e^{-i\psi}-x_1e^{\frac{2}{3}\pi i}}\right)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&= -e^{2i\psi}\Bigg(\left(\frac{1}{1-x_1e^{i\psi}}+\frac{x_1^{-2}}{r^2-x_1^{-1}e^{i\psi}}-\frac{x_1^{-2}}{1-x_1^{-1}e^{i\psi}}-\frac{1}{r^2-x_1e^{i\psi}}\right)+\\
&\quad + e^{-\frac{2}{3}\pi i}\left(\frac{1}{1-x_1e^{-\frac{2}{3}\pi i}e^{i\psi}}+\frac{x_1^{-2}}{r^2-x_1^{-1}e^{-\frac{2}{3}\pi i}e^{i\psi}}-\frac{x_1^{-2}}{1-x_1^{-1}e^{-\frac{2}{3}\pi i}e^{i\psi}}-\frac{1}{r^2-x_1e^{-\frac{2}{3}\pi i}e^{i\psi}}\right)+\\
&\quad + e^{\frac{2}{3}\pi i}\left(\frac{1}{1-x_1e^{\frac{2}{3}\pi i}e^{i\psi}}+\frac{x_1^{-2}}{r^2-x_1^{-1}e^{\frac{2}{3}\pi i}e^{i\psi}}-\frac{x_1^{-2}}{1-x_1^{-1}e^{\frac{2}{3}\pi i}e^{i\psi}}-\frac{1}{r^2-x_1e^{\frac{2}{3}\pi i}e^{i\psi}}\right)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&= -e^{2i\psi}\left(\frac{1}{1-x_1e^{i\psi}}+\frac{e^{-\frac{2}{3}\pi i}}{1-x_1e^{-\frac{2}{3}\pi i}e^{i\psi}}+\frac{e^{\frac{2}{3}\pi i}}{1-x_1e^{\frac{2}{3}\pi i}e^{i\psi}}\right)+\\
&\quad + \frac{e^{2i\psi}}{x_1^2}\left(\frac{1}{1-x_1^{-1}e^{i\psi}}+\frac{e^{-\frac{2}{3}\pi i}}{1-x_1^{-1}e^{-\frac{2}{3}\pi i}e^{i\psi}}+\frac{e^{\frac{2}{3}\pi i}}{1-x_1^{-1}e^{\frac{2}{3}\pi i}e^{i\psi}}\right)+\\
&\quad + e^{2i\psi}\left(\frac{1}{r^2-x_1e^{i\psi}}+\frac{e^{-\frac{2}{3}\pi i}}{r^2-x_1e^{-\frac{2}{3}\pi i}e^{i\psi}}+\frac{e^{\frac{2}{3}\pi i}}{r^2-x_1e^{\frac{2}{3}\pi i}e^{i\psi}}\right)+\\
&\quad - \frac{e^{2i\psi}}{x_1^2}\left(\frac{1}{r^2-x_1^{-1}e^{i\psi}}+\frac{e^{-\frac{2}{3}\pi i}}{r^2-x_1^{-1}e^{-\frac{2}{3}\pi i}e^{i\psi}}+\frac{e^{\frac{2}{3}\pi i}}{r^2-x_1^{-1}e^{\frac{2}{3}\pi i}e^{i\psi}}\right).
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=0}^{2}\left(-\frac{e^{\frac{2}{3}k\pi i+2i\psi}}{1-x_1e^{\frac{2}{3}k\pi i}e^{i\psi}}+\frac{1}{x_1^2}\frac{e^{\frac{2}{3}k\pi i+2i\psi}}{1-x_1^{-1}e^{\frac{2}{3}k\pi i}e^{i\psi}}\right)+\\
&\quad + \sum_{k=0}^{2}\left(\frac{e^{\frac{2}{3}k\pi i+2i\psi}}{r^2-x_1e^{\frac{2}{3}k\pi i}e^{i\psi}}-\frac{1}{x_1^2}\frac{e^{\frac{2}{3}k\pi i+2i\psi}}{r^2-x_1^{-1}e^{\frac{2}{3}k\pi i}e^{i\psi}}\right)
\end{aligned}
\tag{58}
$$

5.4. Integration in the radial direction

- For $r \in (0, x_1)$ we have to compute

$$\int_{r=0}^{x_1} e^{-i\psi} I_4\left(x_1, \psi, r\right)\, r\, dr,$$

where

$$\frac{e^{-i\psi} I_4\left(x_1, \psi, r\right)}{2\pi i} = e^{-i\psi}\left(\#_{(49)} + \#_{(50)} + \#_{(52)} + \#_{(53)} + \#_{(55)} + \#_{(56)} + \#_{(58)}\right) =$$

$$\begin{aligned}
= \quad & e^{-i\psi}\left(\frac{1}{x_1^2}\left(\frac{1}{r^2 - x_1^{-1}e^{-i\psi}} - \frac{1}{1 - x_1^{-1}e^{-i\psi}}\right) + \right.\\
& + \frac{1}{1 - x_1 e^{-i\psi}} - \frac{1}{r^2 - x_1 e^{-i\psi}} + \\
& + \frac{e^{\frac{2}{3}\pi i}}{x_1^2}\left(\frac{1}{r^2 - x_1^{-1}e^{\frac{2}{3}\pi i - i\psi}} - \frac{1}{1 - x_1^{-1}e^{\frac{2}{3}\pi i - i\psi}}\right) + \\
& + e^{\frac{2}{3}\pi i}\left(\frac{1}{1 - x_1 e^{\frac{2}{3}\pi i - i\psi}} - \frac{1}{r^2 - x_1 e^{\frac{2}{3}\pi i - i\psi}}\right) + \\
& + \frac{e^{-\frac{2}{3}\pi i}}{x_1^2}\left(\frac{1}{r^2 - x_1^{-1}e^{-\frac{2}{3}\pi i - i\psi}} - \frac{1}{1 - x_1^{-1}e^{-\frac{2}{3}\pi i - i\psi}}\right) + \\
& + e^{-\frac{2}{3}\pi i}\left(\frac{1}{1 - x_1 e^{-\frac{2}{3}\pi i - i\psi}} - \frac{1}{r^2 - x_1 e^{-\frac{2}{3}\pi i - i\psi}}\right) + \\
& - e^{2i\psi}\left(\frac{1}{1 - x_1 e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i}}{1 - x_1 e^{-\frac{2}{3}\pi i}e^{i\psi}} + \frac{e^{\frac{2}{3}\pi i}}{1 - x_1 e^{\frac{2}{3}\pi i}e^{i\psi}}\right) + \\
& + \frac{e^{2i\psi}}{x_1^2}\left(\frac{1}{1 - x_1^{-1}e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i}}{1 - x_1^{-1}e^{-\frac{2}{3}\pi i}e^{i\psi}} + \frac{e^{\frac{2}{3}\pi i}}{1 - x_1^{-1}e^{\frac{2}{3}\pi i}e^{i\psi}}\right) + \\
& + e^{2i\psi}\left(\frac{1}{r^2 - x_1 e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i}}{r^2 - x_1 e^{-\frac{2}{3}\pi i}e^{i\psi}} + \frac{e^{\frac{2}{3}\pi i}}{r^2 - x_1 e^{\frac{2}{3}\pi i}e^{i\psi}}\right) + \\
& \left. - \frac{e^{2i\psi}}{x_1^2}\left(\frac{1}{r^2 - x_1^{-1}e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i}}{r^2 - x_1^{-1}e^{-\frac{2}{3}\pi i}e^{i\psi}} + \frac{e^{\frac{2}{3}\pi i}}{r^2 - x_1^{-1}e^{\frac{2}{3}\pi i}e^{i\psi}}\right)\right) \\
= \quad & e^{-i\psi}\left(\frac{1}{1 - x_1 e^{-i\psi}} + \frac{e^{\frac{2}{3}\pi i}}{1 - x_1 e^{\frac{2}{3}\pi i}e^{-i\psi}} + \frac{e^{-\frac{2}{3}\pi i}}{1 - x_1 e^{-\frac{2}{3}\pi i}e^{-i\psi}}\right) + \\
& - e^{i\psi}\left(\frac{1}{1 - x_1 e^{i\psi}} + \frac{e^{\frac{2}{3}\pi i}}{1 - x_1 e^{\frac{2}{3}\pi i}e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i}}{1 - x_1 e^{-\frac{2}{3}\pi i}e^{i\psi}}\right) + \\
& - \frac{e^{-i\psi}}{x_1^2}\left(\frac{1}{1 - x_1^{-1}e^{-i\psi}} + \frac{e^{\frac{2}{3}\pi i}}{1 - x_1^{-1}e^{\frac{2}{3}\pi i - i\psi}} + \frac{e^{-\frac{2}{3}\pi i}}{1 - x_1^{-1}e^{-\frac{2}{3}\pi i - i\psi}}\right) + \\
& + \frac{e^{i\psi}}{x_1^2}\left(\frac{1}{1 - x_1^{-1}e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i}}{1 - x_1^{-1}e^{-\frac{2}{3}\pi i}e^{i\psi}} + \frac{e^{\frac{2}{3}\pi i}}{1 - x_1^{-1}e^{\frac{2}{3}\pi i}e^{i\psi}}\right) +
\end{aligned}$$

$$
\begin{aligned}
&- e^{-i\psi}\left(\frac{1}{r^2 - x_1 e^{-i\psi}} + \frac{e^{-\frac{2}{3}\pi i}}{r^2 - x_1 e^{-\frac{2}{3}\pi i}e^{-i\psi}} + \frac{e^{\frac{2}{3}\pi i}}{r^2 - x_1 e^{\frac{2}{3}\pi i}e^{-i\psi}}\right) + \\
&+ e^{i\psi}\left(\frac{1}{r^2 - x_1 e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i}}{r^2 - x_1 e^{-\frac{2}{3}\pi i}e^{i\psi}} + \frac{e^{\frac{2}{3}\pi i}}{r^2 - x_1 e^{\frac{2}{3}\pi i}e^{i\psi}}\right) + \\
&+ \frac{e^{-i\psi}}{x_1^2}\left(\frac{1}{r^2 - x_1^{-1} e^{-i\psi}} + \frac{e^{\frac{2}{3}\pi i}}{r^2 - x_1^{-1} e^{\frac{2}{3}\pi i - i\psi}} + \frac{e^{-\frac{2}{3}\pi i}}{r^2 - x_1^{-1} e^{-\frac{2}{3}\pi i - i\psi}}\right) + \\
&- \frac{e^{i\psi}}{x_1^2}\left(\frac{1}{r^2 - x_1^{-1} e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i}}{r^2 - x_1^{-1} e^{-\frac{2}{3}\pi i}e^{i\psi}} + \frac{e^{\frac{2}{3}\pi i}}{r^2 - x_1^{-1} e^{\frac{2}{3}\pi i}e^{i\psi}}\right) \\
= \ &\left(\frac{e^{-i\psi}}{1 - x_1 e^{-i\psi}} + \frac{e^{\frac{2}{3}\pi i}e^{-i\psi}}{1 - x_1 e^{\frac{2}{3}\pi i}e^{-i\psi}} + \frac{e^{-\frac{2}{3}\pi i}e^{-i\psi}}{1 - x_1 e^{-\frac{2}{3}\pi i}e^{-i\psi}}\right) + \\
&- \left(\frac{e^{i\psi}}{1 - x_1 e^{i\psi}} + \frac{e^{\frac{2}{3}\pi i}e^{i\psi}}{1 - x_1 e^{\frac{2}{3}\pi i}e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i}e^{i\psi}}{1 - x_1 e^{-\frac{2}{3}\pi i}e^{i\psi}}\right) + \\
&- \frac{1}{x_1^2}\left(\frac{e^{-i\psi}}{1 - x_1^{-1} e^{-i\psi}} + \frac{e^{\frac{2}{3}\pi i}e^{-i\psi}}{1 - x_1^{-1} e^{\frac{2}{3}\pi i - i\psi}} + \frac{e^{-\frac{2}{3}\pi i}e^{-i\psi}}{1 - x_1^{-1} e^{-\frac{2}{3}\pi i - i\psi}}\right) + \\
&+ \frac{1}{x_1^2}\left(\frac{e^{i\psi}}{1 - x_1^{-1} e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i}e^{i\psi}}{1 - x_1^{-1} e^{-\frac{2}{3}\pi i}e^{i\psi}} + \frac{e^{\frac{2}{3}\pi i}e^{i\psi}}{1 - x_1^{-1} e^{\frac{2}{3}\pi i}e^{i\psi}}\right) + \\
&- \left(\frac{e^{-i\psi}}{r^2 - x_1 e^{-i\psi}} + \frac{e^{-\frac{2}{3}\pi i}e^{-i\psi}}{r^2 - x_1 e^{-\frac{2}{3}\pi i}e^{-i\psi}} + \frac{e^{\frac{2}{3}\pi i}e^{-i\psi}}{r^2 - x_1 e^{\frac{2}{3}\pi i}e^{-i\psi}}\right) + \\
&+ \left(\frac{e^{i\psi}}{r^2 - x_1 e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i}e^{i\psi}}{r^2 - x_1 e^{-\frac{2}{3}\pi i}e^{i\psi}} + \frac{e^{\frac{2}{3}\pi i}e^{i\psi}}{r^2 - x_1 e^{\frac{2}{3}\pi i}e^{i\psi}}\right) + \\
&+ \frac{1}{x_1^2}\left(\frac{e^{-i\psi}}{r^2 - x_1^{-1} e^{-i\psi}} + \frac{e^{\frac{2}{3}\pi i}e^{-i\psi}}{r^2 - x_1^{-1} e^{\frac{2}{3}\pi i - i\psi}} + \frac{e^{-\frac{2}{3}\pi i}e^{-i\psi}}{r^2 - x_1^{-1} e^{-\frac{2}{3}\pi i - i\psi}}\right) + \\
&- \frac{1}{x_1^2}\left(\frac{1e^{i\psi}}{r^2 - x_1^{-1} e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i}e^{i\psi}}{r^2 - x_1^{-1} e^{-\frac{2}{3}\pi i}e^{i\psi}} + \frac{e^{\frac{2}{3}\pi i}e^{i\psi}}{r^2 - x_1^{-1} e^{\frac{2}{3}\pi i}e^{i\psi}}\right)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{e^{-i\psi}}{1-x_1e^{-i\psi}} - \frac{e^{i\psi}}{1-x_1e^{i\psi}} + \frac{e^{\frac{2}{3}\pi i}e^{-i\psi}}{1-x_1e^{\frac{2}{3}\pi i}e^{-i\psi}} - \frac{e^{\frac{2}{3}\pi i}e^{i\psi}}{1-x_1e^{\frac{2}{3}\pi i}e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i}e^{-i\psi}}{1-x_1e^{-\frac{2}{3}\pi i}e^{-i\psi}} - \frac{e^{-\frac{2}{3}\pi i}e^{i\psi}}{1-x_1e^{-\frac{2}{3}\pi i}e^{i\psi}} + \\
& + \frac{1}{x_1^2}\left(\frac{e^{i\psi}}{1-x_1^{-1}e^{i\psi}} - \frac{e^{-i\psi}}{1-x_1^{-1}e^{-i\psi}} + \frac{e^{-\frac{2}{3}\pi i}e^{i\psi}}{1-x_1^{-1}e^{-\frac{2}{3}\pi i}e^{i\psi}} + \right. \\
& \qquad \left. - \frac{e^{-\frac{2}{3}\pi i}e^{-i\psi}}{1-x_1^{-1}e^{-\frac{2}{3}\pi i-i\psi}} + \frac{e^{\frac{2}{3}\pi i}e^{i\psi}}{1-x_1^{-1}e^{\frac{2}{3}\pi i}e^{i\psi}} - \frac{e^{\frac{2}{3}\pi i}e^{-i\psi}}{1-x_1^{-1}e^{\frac{2}{3}\pi i-i\psi}} \right) + \\
& + \left(\frac{e^{i\psi}}{r^2-x_1e^{i\psi}} - \frac{e^{-i\psi}}{r^2-x_1e^{-i\psi}} + \frac{e^{-\frac{2}{3}\pi i}e^{i\psi}}{r^2-x_1e^{-\frac{2}{3}\pi i}e^{i\psi}} + \right. \\
& \qquad \left. - \frac{e^{-\frac{2}{3}\pi i}e^{-i\psi}}{r^2-x_1e^{-\frac{2}{3}\pi i}e^{-i\psi}} + \frac{e^{\frac{2}{3}\pi i}e^{i\psi}}{r^2-x_1e^{\frac{2}{3}\pi i}e^{i\psi}} - \frac{e^{\frac{2}{3}\pi i}e^{-i\psi}}{r^2-x_1e^{\frac{2}{3}\pi i}e^{-i\psi}} \right) + \\
& + \frac{1}{x_1^2}\left(\frac{e^{-i\psi}}{r^2-x_1^{-1}e^{-i\psi}} - \frac{e^{i\psi}}{r^2-x_1^{-1}e^{i\psi}} + \frac{e^{\frac{2}{3}\pi i}e^{-i\psi}}{r^2-x_1^{-1}e^{\frac{2}{3}\pi i-i\psi}} + \right. \\
& \qquad \left. - \frac{e^{\frac{2}{3}\pi i}e^{i\psi}}{r^2-x_1^{-1}e^{\frac{2}{3}\pi i}e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i}e^{-i\psi}}{r^2-x_1^{-1}e^{-\frac{2}{3}\pi i-i\psi}} - \frac{e^{-\frac{2}{3}\pi i}e^{i\psi}}{r^2-x_1^{-1}e^{-\frac{2}{3}\pi i}e^{i\psi}} \right)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{-e^{2i\psi}+1}{\left(-e^{i\psi}+x_1\right)\left(-1+x_1e^{i\psi}\right)} + \frac{e^{\frac{2}{3}\pi i}e^{-i\psi}}{1-x_1e^{\frac{2}{3}\pi i}e^{-i\psi}} - \frac{e^{-\frac{2}{3}\pi i}e^{i\psi}}{1-x_1e^{-\frac{2}{3}\pi i}e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i}e^{-i\psi}}{1-x_1e^{-\frac{2}{3}\pi i}e^{-i\psi}} - \frac{e^{\frac{2}{3}\pi i}e^{i\psi}}{1-x_1e^{\frac{2}{3}\pi i}e^{i\psi}} + \\
& + \frac{1}{x_1^2}\left(\frac{e^{i\psi}}{1-x_1^{-1}e^{i\psi}} - \frac{e^{-i\psi}}{1-x_1^{-1}e^{-i\psi}} + \frac{e^{-\frac{2}{3}\pi i}e^{i\psi}}{1-x_1^{-1}e^{-\frac{2}{3}\pi i}e^{i\psi}} + \right. \\
& \qquad \left. - \frac{e^{\frac{2}{3}\pi i}e^{-i\psi}}{1-x_1^{-1}e^{\frac{2}{3}\pi i-i\psi}} + \frac{e^{\frac{2}{3}\pi i}e^{i\psi}}{1-x_1^{-1}e^{\frac{2}{3}\pi i}e^{i\psi}} - \frac{e^{-\frac{2}{3}\pi i}e^{-i\psi}}{1-x_1^{-1}e^{-\frac{2}{3}\pi i-i\psi}} \right) + \\
& + \left(\frac{e^{i\psi}}{r^2-x_1e^{i\psi}} - \frac{e^{-i\psi}}{r^2-x_1e^{-i\psi}} + \frac{e^{-\frac{2}{3}\pi i}e^{i\psi}}{r^2-x_1e^{-\frac{2}{3}\pi i}e^{i\psi}} + \right. \\
& \qquad \left. - \frac{e^{\frac{2}{3}\pi i}e^{-i\psi}}{r^2-x_1e^{\frac{2}{3}\pi i}e^{-i\psi}} + \frac{e^{\frac{2}{3}\pi i}e^{i\psi}}{r^2-x_1e^{\frac{2}{3}\pi i}e^{i\psi}} - \frac{e^{-\frac{2}{3}\pi i}e^{-i\psi}}{r^2-x_1e^{-\frac{2}{3}\pi i}e^{-i\psi}} \right) + \\
& + \frac{1}{x_1^2}\left(\frac{e^{-i\psi}}{r^2-x_1^{-1}e^{-i\psi}} - \frac{e^{i\psi}}{r^2-x_1^{-1}e^{i\psi}} + \frac{e^{\frac{2}{3}\pi i}e^{-i\psi}}{r^2-x_1^{-1}e^{\frac{2}{3}\pi i-i\psi}} + \right. \\
& \qquad \left. - \frac{e^{-\frac{2}{3}\pi i}e^{i\psi}}{r^2-x_1^{-1}e^{-\frac{2}{3}\pi i}e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i}e^{-i\psi}}{r^2-x_1^{-1}e^{-\frac{2}{3}\pi i-i\psi}} - \frac{e^{\frac{2}{3}\pi i}e^{i\psi}}{r^2-x_1^{-1}e^{\frac{2}{3}\pi i}e^{i\psi}} \right)
\end{aligned}
$$

$$= \sum_{k=0}^{2} \sum_{\sigma=\pm 1} \sigma \left(\frac{e^{(\frac{2}{3}k\pi - \sigma\psi)i}}{1 - x_1 e^{(\frac{2}{3}k\pi - \sigma\psi)i}} - \frac{1}{x_1^2} \frac{e^{(\frac{2}{3}k\pi - \sigma\psi)i}}{1 - x_1^{-1} e^{(\frac{2}{3}k\pi - \sigma\psi)i}} \right) +$$
$$- \sum_{k=0}^{2} \sum_{\sigma=\pm 1} \sigma \left(\frac{e^{(\frac{2}{3}k\pi - \sigma\psi)i}}{r^2 - x_1 e^{(\frac{2}{3}k\pi - \sigma\psi)i}} - \frac{1}{x_1^2} \frac{e^{(\frac{2}{3}k\pi - \sigma\psi)i}}{r^2 - x_1^{-1} e^{(\frac{2}{3}k\pi - \sigma\psi)i}} \right) .$$

We obtain

$$\frac{1}{2\pi i} \int_{r=0}^{x_1} e^{-i\psi} I_4 (x_1, \psi, r)\ r\, dr =$$

$$= \frac{1}{2} \left(\frac{e^{-i\psi}}{1 - x_1 e^{-i\psi}} - \frac{e^{i\psi}}{1 - x_1 e^{i\psi}} + \frac{e^{\frac{2}{3}\pi i} e^{-i\psi}}{1 - x_1 e^{\frac{2}{3}\pi i} e^{-i\psi}} + \right.$$
$$\left. - \frac{e^{-\frac{2}{3}\pi i} e^{i\psi}}{1 - x_1 e^{-\frac{2}{3}\pi i} e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i} e^{-i\psi}}{1 - x_1 e^{-\frac{2}{3}\pi i} e^{-i\psi}} - \frac{e^{\frac{2}{3}\pi i} e^{i\psi}}{1 - x_1 e^{\frac{2}{3}\pi i} e^{i\psi}} \right) x_1^2 +$$
$$+ \frac{1}{2} \left(\frac{e^{i\psi}}{1 - x_1^{-1} e^{i\psi}} - \frac{e^{-i\psi}}{1 - x_1^{-1} e^{-i\psi}} + \frac{e^{-\frac{2}{3}\pi i} e^{i\psi}}{1 - x_1^{-1} e^{-\frac{2}{3}\pi i} e^{i\psi}} + \right.$$
$$\left. - \frac{e^{\frac{2}{3}\pi i} e^{-i\psi}}{1 - x_1^{-1} e^{\frac{2}{3}\pi i - i\psi}} + \frac{e^{\frac{2}{3}\pi i} e^{i\psi}}{1 - x_1^{-1} e^{\frac{2}{3}\pi i} e^{i\psi}} - \frac{e^{-\frac{2}{3}\pi i} e^{-i\psi}}{1 - x_1^{-1} e^{-\frac{2}{3}\pi i - i\psi}} \right) +$$
$$+ \frac{1}{2} \left[e^{i\psi} \mathrm{Ln} \left(x_1 e^{i\psi} - r^2 \right) - e^{-i\psi} \mathrm{Ln} \left(x_1 e^{-i\psi} - r^2 \right) \right]_0^{x_1} +$$
$$+ \frac{1}{2} \left[e^{-\frac{2}{3}\pi i} e^{i\psi} \mathrm{Ln} \left(x_1 e^{-\frac{2}{3}\pi i} e^{i\psi} - r^2 \right) - e^{\frac{2}{3}\pi i} e^{-i\psi} \mathrm{Ln} \left(x_1 e^{\frac{2}{3}\pi i} e^{-i\psi} - r^2 \right) \right]_0^{x_1} +$$
$$+ \frac{1}{2} \left[e^{\frac{2}{3}\pi i} e^{i\psi} \mathrm{Ln} \left(x_1 e^{\frac{2}{3}\pi i} e^{i\psi} - r^2 \right) - e^{-\frac{2}{3}\pi i} e^{-i\psi} \mathrm{Ln} \left(x_1 e^{-\frac{2}{3}\pi i} e^{-i\psi} - r^2 \right) \right]_0^{x_1} +$$
$$- \frac{1}{2x_1^2} \left[e^{i\psi} \mathrm{Ln} \left(x_1^{-1} e^{i\psi} - r^2 \right) - e^{-i\psi} \mathrm{Ln} \left(x_1^{-1} e^{-i\psi} - r^2 \right) \right]_0^{x_1} +$$
$$- \frac{1}{2} \left[e^{-\frac{2}{3}\pi i} e^{i\psi} \mathrm{Ln} \left(x_1^{-1} e^{-\frac{2}{3}\pi i} e^{i\psi} - r^2 \right) - e^{\frac{2}{3}\pi i} e^{-i\psi} \mathrm{Ln} \left(x_1^{-1} e^{\frac{2}{3}\pi i} e^{-i\psi} - r^2 \right) \right]_0^{x_1} +$$
$$- \frac{1}{2} \left[e^{\frac{2}{3}\pi i} e^{i\psi} \mathrm{Ln} \left(x_1^{-1} e^{\frac{2}{3}\pi i} e^{i\psi} - r^2 \right) - e^{-\frac{2}{3}\pi i} e^{-i\psi} \mathrm{Ln} \left(x_1^{-1} e^{-\frac{2}{3}\pi i} e^{-i\psi} - r^2 \right) \right]_0^{x_1}$$

$$
\begin{aligned}
= \quad & \frac{1}{2}\left(\frac{e^{-i\psi}}{1-x_1e^{-i\psi}}-\frac{e^{i\psi}}{1-x_1e^{i\psi}}+\frac{e^{\frac{2}{3}\pi i}e^{-i\psi}}{1-x_1e^{\frac{2}{3}\pi i}e^{-i\psi}}+\right.\\
& \qquad\left.-\ \frac{e^{\frac{2}{3}\pi i}e^{i\psi}}{1-x_1e^{\frac{2}{3}\pi i}e^{i\psi}}+\frac{e^{-\frac{2}{3}\pi i}e^{-i\psi}}{1-x_1e^{-\frac{2}{3}\pi i}e^{-i\psi}}-\frac{e^{-\frac{2}{3}\pi i}e^{i\psi}}{1-x_1e^{-\frac{2}{3}\pi i}e^{i\psi}}\right)x_1^2+\\
& +\frac{1}{2}\left(\frac{e^{i\psi}}{1-x_1^{-1}e^{i\psi}}-\frac{e^{-i\psi}}{1-x_1^{-1}e^{-i\psi}}+\frac{e^{-\frac{2}{3}\pi i}e^{i\psi}}{1-x_1^{-1}e^{-\frac{2}{3}\pi i}e^{i\psi}}+\right.\\
& \qquad\left.-\ \frac{e^{-\frac{2}{3}\pi i}e^{-i\psi}}{1-x_1^{-1}e^{-\frac{2}{3}\pi i-i\psi}}+\frac{e^{\frac{2}{3}\pi i}e^{i\psi}}{1-x_1^{-1}e^{\frac{2}{3}\pi i}e^{i\psi}}-\frac{e^{\frac{2}{3}\pi i}e^{-i\psi}}{1-x_1^{-1}e^{\frac{2}{3}\pi i-i\psi}}\right)+\\
& +\frac{1}{2}\left(e^{i\psi}\mathrm{Ln}\left(\frac{x_1e^{i\psi}-x_1^2}{x_1e^{i\psi}}\right)-e^{-i\psi}\mathrm{Ln}\left(\frac{x_1e^{-i\psi}-x_1^2}{x_1e^{-i\psi}}\right)\right)+\\
& +\frac{1}{2}\left(e^{-\frac{2}{3}\pi i}e^{i\psi}\mathrm{Ln}\left(\frac{x_1e^{-\frac{2}{3}\pi i}e^{i\psi}-x_1^2}{x_1e^{-\frac{2}{3}\pi i}e^{i\psi}}\right)-e^{\frac{2}{3}\pi i}e^{-i\psi}\mathrm{Ln}\left(\frac{x_1e^{\frac{2}{3}\pi i}e^{-i\psi}-x_1^2}{x_1e^{\frac{2}{3}\pi i}e^{-i\psi}}\right)\right)+\\
& +\frac{1}{2}\left(e^{\frac{2}{3}\pi i}e^{i\psi}\mathrm{Ln}\left(\frac{x_1e^{\frac{2}{3}\pi i}e^{i\psi}-x_1^2}{x_1e^{\frac{2}{3}\pi i}e^{i\psi}}\right)-e^{-\frac{2}{3}\pi i}e^{-i\psi}\mathrm{Ln}\left(\frac{x_1e^{-\frac{2}{3}\pi i}e^{-i\psi}-x_1^2}{x_1e^{-\frac{2}{3}\pi i}e^{-i\psi}}\right)\right)+\\
& -\frac{1}{2x_1^2}\left(e^{i\psi}\mathrm{Ln}\left(\frac{x_1^{-1}e^{i\psi}-x_1^2}{x_1^{-1}e^{i\psi}}\right)-e^{-i\psi}\mathrm{Ln}\left(\frac{x_1^{-1}e^{-i\psi}-x_1^2}{x_1^{-1}e^{-i\psi}}\right)\right)+\\
& -\frac{1}{2}\left(e^{-\frac{2}{3}\pi i}e^{i\psi}\mathrm{Ln}\left(\frac{x_1^{-1}e^{-\frac{2}{3}\pi i}e^{i\psi}-x_1^2}{x_1^{-1}e^{-\frac{2}{3}\pi i}e^{i\psi}}\right)-e^{\frac{2}{3}\pi i}e^{-i\psi}\mathrm{Ln}\left(\frac{x_1^{-1}e^{\frac{2}{3}\pi i}e^{-i\psi}-x_1^2}{x_1^{-1}e^{\frac{2}{3}\pi i}e^{-i\psi}}\right)\right)+\\
& -\frac{1}{2}\left(e^{\frac{2}{3}\pi i}e^{i\psi}\mathrm{Ln}\left(\frac{x_1^{-1}e^{\frac{2}{3}\pi i}e^{i\psi}-x_1^2}{x_1^{-1}e^{\frac{2}{3}\pi i}e^{i\psi}}\right)-e^{-\frac{2}{3}\pi i}e^{-i\psi}\mathrm{Ln}\left(\frac{x_1^{-1}e^{-\frac{2}{3}\pi i}e^{-i\psi}-x_1^2}{x_1^{-1}e^{-\frac{2}{3}\pi i}e^{-i\psi}}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & -ix_1^2\left(\frac{\sin\psi}{1-2x_1\cos\psi+x_1^2}+\frac{\sin\left(\psi-\frac{2}{3}\pi\right)}{1-2x_1\cos\left(\psi-\frac{2}{3}\pi\right)+x_1^2}+\frac{\sin\left(\psi+\frac{2}{3}\pi\right)}{1-2x_1\cos\left(\psi+\frac{2}{3}\pi\right)+x_1^2}\right)+\\
& +i\left(\frac{x_1^2\sin\psi}{1-2x_1\cos\psi+x_1^2}+\frac{x_1^2\sin\left(\psi-\frac{2}{3}\pi\right)}{1-2x_1\cos\left(\psi-\frac{2}{3}\pi\right)+x_1^2}+\frac{x_1^2\sin\left(\psi+\frac{2}{3}\pi\right)}{1-2x_1\cos\left(\psi+\frac{2}{3}\pi\right)+x_1^2}\right)+\\
& +\frac{1}{2}\left(e^{i\psi}\mathrm{Ln}\left(1-x_1e^{-i\psi}\right)-e^{-i\psi}\mathrm{Ln}\left(1-x_1e^{i\psi}\right)\right)+\\
& +\frac{1}{2}\left(e^{-\frac{2}{3}\pi i+i\psi}\mathrm{Ln}\left(1-x_1e^{\frac{2}{3}\pi i-i\psi}\right)-e^{\frac{2}{3}\pi i-i\psi}\mathrm{Ln}\left(1-x_1e^{-\frac{2}{3}\pi i+i\psi}\right)\right)+\\
& +\frac{1}{2}\left(e^{\frac{2}{3}\pi i+i\psi}\mathrm{Ln}\left(1-x_1e^{-\frac{2}{3}\pi i-i\psi}\right)-e^{-\frac{2}{3}\pi i-i\psi}\mathrm{Ln}\left(1-x_1e^{\frac{2}{3}\pi i+i\psi}\right)\right)+\\
& -\frac{1}{2x_1^2}\left(e^{i\psi}\mathrm{Ln}\left(1-x_1^3e^{-i\psi}\right)-e^{-i\psi}\mathrm{Ln}\left(1-x_1^3e^{i\psi}\right)\right)+\\
& -\frac{1}{2}\left(e^{-\frac{2}{3}\pi i+i\psi}\mathrm{Ln}\left(1-x_1^3e^{\frac{2}{3}\pi i-i\psi}\right)-e^{\frac{2}{3}\pi i-i\psi}\mathrm{Ln}\left(1-x_1^3e^{-\frac{2}{3}\pi i+i\psi}\right)\right)+\\
& -\frac{1}{2}\left(e^{\frac{2}{3}\pi i+i\psi}\mathrm{Ln}\left(1-x_1^3e^{-\frac{2}{3}\pi i-i\psi}\right)-e^{-\frac{2}{3}\pi i-i\psi}\mathrm{Ln}\left(1-x_1^3e^{\frac{2}{3}\pi i+i\psi}\right)\right)
\end{aligned}
$$

$$\begin{aligned}
= \quad & i\sin\psi \ln\sqrt{1-2x_1\cos\psi + x_1^2} + i\cos\psi \arctan\left(\frac{x_1\sin\psi}{1-x_1\cos\psi}\right) + \\
& + i\sin\left(\psi-\tfrac{2}{3}\pi\right)\ln\sqrt{1-2x_1\cos\left(\psi-\tfrac{2}{3}\pi\right)+x_1^2} + i\cos\left(\psi-\tfrac{2}{3}\pi\right)\arctan\left(\frac{x_1\sin\left(\psi-\frac{2}{3}\pi\right)}{1-x_1\cos\left(\psi-\frac{2}{3}\pi\right)}\right) + \\
& + i\sin\left(\psi+\tfrac{2}{3}\pi\right)\ln\sqrt{1-2x_1\cos\left(\psi+\tfrac{2}{3}\pi\right)+x_1^2} + i\cos\left(\psi+\tfrac{2}{3}\pi\right)\arctan\left(\frac{x_1\sin\left(\psi+\frac{2}{3}\pi\right)}{1-x_1\cos\left(\psi+\frac{2}{3}\pi\right)}\right) + \\
& - \frac{i\sin\psi}{x_1^2}\ln\sqrt{1-2x_1^3\cos\psi+x_1^6} - i\frac{\cos\psi}{x_1^2}\arctan\left(\frac{x_1^3\sin\psi}{1-x_1^3\cos\psi}\right) + \\
& - \frac{i\sin\left(\psi-\frac{2}{3}\pi\right)}{x_1^2}\ln\sqrt{1-2x_1^3\cos\left(\psi-\tfrac{2}{3}\pi\right)+x_1^6} - i\frac{\cos\left(\psi-\frac{2}{3}\pi\right)}{x_1^2}\arctan\left(\frac{x_1^3\sin\left(\psi-\frac{2}{3}\pi\right)}{1-x_1^3\cos\left(\psi-\frac{2}{3}\pi\right)}\right) + \\
& - i\frac{\sin\left(\psi+\frac{2}{3}\pi\right)}{x_1^2}\ln\sqrt{1-2x_1^3\cos\left(\psi+\tfrac{2}{3}\pi\right)+x_1^6} - i\frac{\cos\left(\psi+\frac{2}{3}\pi\right)}{x_1^2}\arctan\left(\frac{x_1^3\sin\left(\psi+\frac{2}{3}\pi\right)}{1-x_1^3\cos\left(\psi+\frac{2}{3}\pi\right)}\right)
\end{aligned}$$

$$\begin{aligned}
= \ & i\sum_{k=0}^{2}\sin\left(\psi+\tfrac{2}{3}k\pi\right)\ln\sqrt{1-2x_1\cos\left(\psi+\tfrac{2}{3}k\pi\right)+x_1^2} + \\
& + i\sum_{k=0}^{2}\cos\left(\psi+\tfrac{2}{3}k\pi\right)\arctan\left(\frac{x_1\sin\left(\psi+\frac{2}{3}k\pi\right)}{1-x_1\cos\left(\psi+\frac{2}{3}k\pi\right)}\right) + \\
& - i\sum_{k=0}^{2}\frac{\sin\left(\psi+\frac{2}{3}k\pi\right)}{x_1^2}\ln\sqrt{1-2x_1^3\cos\left(\psi+\tfrac{2}{3}k\pi\right)+x_1^6} + \\
& - i\sum_{k=0}^{2}\frac{\cos\left(\psi+\frac{2}{3}k\pi\right)}{x_1^2}\arctan\left(\frac{x_1^3\sin\left(\psi+\frac{2}{3}k\pi\right)}{1-x_1^3\cos\left(\psi+\frac{2}{3}k\pi\right)}\right)
\end{aligned} \tag{59}$$

• For $r \in (x_1, 1)$ the integral becomes

$$\int_{r=x_1}^{1} I_5\left(x_1,\psi,r\right)\, r\, dr,$$

where

$$\frac{I_5\left(x_1,\psi,r\right)}{2\pi i} = e^{-i\psi}\left(\#_{(49)} + \#_{(51)} + \#_{(52)} + \#_{(54)} + \#_{(55)} + \#_{(57)} + \#_{(58)}\right)$$

$$
\begin{aligned}
= \quad & \frac{e^{-i\psi}}{x_1^2}\left(\frac{1}{r^2 - x_1^{-1}e^{-i\psi}} - \frac{1}{1 - x_1^{-1}e^{-i\psi}}\right) + \\
& + e^{i\psi}\left(\frac{1}{1 - x_1 e^{i\psi}} - \frac{1}{r^2 - x_1 e^{i\psi}}\right) + \\
& + \frac{e^{\frac{2}{3}\pi i - i\psi}}{x_1^2}\left(\frac{1}{r^2 - x_1^{-1}e^{\frac{2}{3}\pi i - i\psi}} - \frac{1}{1 - x_1^{-1}e^{\frac{2}{3}\pi i - i\psi}}\right) + \\
& + e^{-\frac{2}{3}\pi i + i\psi}\left(\frac{1}{1 - x_1 e^{-\frac{2}{3}\pi i + i\psi}} - \frac{1}{r^2 - x_1 e^{-\frac{2}{3}\pi i + i\psi}}\right) + \\
& + \frac{e^{-\frac{2}{3}\pi i - i\psi}}{x_1^2}\left(\frac{1}{r^2 - x_1^{-1}e^{-\frac{2}{3}\pi i - i\psi}} - \frac{1}{1 - x_1^{-1}e^{-\frac{2}{3}\pi i - i\psi}}\right) + \\
& + e^{\frac{2}{3}\pi i + i\psi}\left(\frac{1}{1 - x_1 e^{\frac{2}{3}\pi i + i\psi}} - \frac{1}{r^2 - x_1 e^{\frac{2}{3}\pi i + i\psi}}\right) + \\
& - e^{i\psi}\left(\frac{1}{1 - x_1 e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i}}{1 - x_1 e^{-\frac{2}{3}\pi i + i\psi}} + \frac{e^{\frac{2}{3}\pi i}}{1 - x_1 e^{\frac{2}{3}\pi i + i\psi}}\right) + \\
& + \frac{e^{i\psi}}{x_1^2}\left(\frac{1}{1 - x_1^{-1}e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i}}{1 - x_1^{-1}e^{-\frac{2}{3}\pi i + i\psi}} + \frac{e^{\frac{2}{3}\pi i}}{1 - x_1^{-1}e^{\frac{2}{3}\pi i + i\psi}}\right) + \\
& + e^{i\psi}\left(\frac{1}{r^2 - x_1 e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i}}{r^2 - x_1 e^{-\frac{2}{3}\pi i + i\psi}} + \frac{e^{\frac{2}{3}\pi i}}{r^2 - x_1 e^{\frac{2}{3}\pi i + i\psi}}\right) + \\
& - \frac{e^{i\psi}}{x_1^2}\left(\frac{1}{r^2 - x_1^{-1}e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i}}{r^2 - x_1^{-1}e^{-\frac{2}{3}\pi i + i\psi}} + \frac{e^{\frac{2}{3}\pi i}}{r^2 - x_1^{-1}e^{\frac{2}{3}\pi i + i\psi}}\right)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & + \frac{1}{x_1^2}\left(\frac{e^{i\psi}}{1 - x_1^{-1}e^{i\psi}} - \frac{e^{-i\psi}}{1 - x_1^{-1}e^{-i\psi}} + \frac{e^{-\frac{2}{3}\pi i + i\psi}}{1 - x_1^{-1}e^{-\frac{2}{3}\pi i + i\psi}} + \right. \\
& \left. - \frac{e^{\frac{2}{3}\pi i - i\psi}}{1 - x_1^{-1}e^{\frac{2}{3}\pi i - i\psi}} + \frac{e^{\frac{2}{3}\pi i + i\psi}}{1 - x_1^{-1}e^{\frac{2}{3}\pi i + i\psi}} - \frac{e^{-\frac{2}{3}\pi i - i\psi}}{1 - x_1^{-1}e^{-\frac{2}{3}\pi i - i\psi}}\right) + \\
& + \frac{1}{x_1^2}\left(\frac{e^{-i\psi}}{r^2 - x_1^{-1}e^{-i\psi}} - \frac{e^{i\psi}}{r^2 - x_1^{-1}e^{i\psi}} + \frac{e^{\frac{2}{3}\pi i - i\psi}}{r^2 - x_1^{-1}e^{\frac{2}{3}\pi i - i\psi}} - + \right. \\
& \left. - \frac{e^{-\frac{2}{3}\pi i + i\psi}}{r^2 - x_1^{-1}e^{-\frac{2}{3}\pi i + i\psi}} + \frac{e^{-\frac{2}{3}\pi i - i\psi}}{r^2 - x_1^{-1}e^{-\frac{2}{3}\pi i - i\psi}} - \frac{e^{\frac{2}{3}\pi i + i\psi}}{r^2 - x_1^{-1}e^{\frac{2}{3}\pi i + i\psi}}\right).
\end{aligned}
$$

$$
= \sum_{k=0}^{2} \sum_{\sigma = \pm 1} \frac{\sigma}{x_1^2}\left(\frac{e^{\left(\frac{2}{3}k\pi - \sigma\psi\right)i}}{r^2 - x_1^{-1}e^{\left(\frac{2}{3}k\pi - \sigma\psi\right)i}} - \frac{e^{\left(\frac{2}{3}k\pi - \sigma\psi\right)i}}{1 - x_1^{-1}e^{\left(\frac{2}{3}k\pi - \sigma\psi\right)i}}\right) . \qquad (60)
$$

We obtain

$$
\frac{1}{2\pi i}\int_{r=x_1}^{1} e^{-i\psi} I_5\left(x_1, \psi, r\right)\, r\, dr =
$$

$$
\begin{aligned}
= \quad & \frac{1}{2x_1^2}\left(\frac{e^{i\psi}}{1-x_1^{-1}e^{i\psi}} - \frac{e^{-i\psi}}{1-x_1^{-1}e^{-i\psi}} + \frac{e^{-\frac{2}{3}\pi i+i\psi}}{1-x_1^{-1}e^{-\frac{2}{3}\pi i+i\psi}} + \right.\\
& \left. - \frac{e^{\frac{2}{3}\pi i-i\psi}}{1-x_1^{-1}e^{\frac{2}{3}\pi i-i\psi}} + \frac{e^{\frac{2}{3}\pi i+i\psi}}{1-x_1^{-1}e^{\frac{2}{3}\pi i+i\psi}} - \frac{e^{-\frac{2}{3}\pi i-i\psi}}{1-x_1^{-1}e^{-\frac{2}{3}\pi i-i\psi}}\right)\left(1-x_1^2\right) + \\
& + \frac{1}{2x_1^2}\left[e^{-i\psi}\mathrm{Ln}\left(x_1^{-1}e^{-i\psi}-r^2\right) - e^{i\psi}\mathrm{Ln}\left(x_1^{-1}e^{i\psi}-r^2\right) + \ + \right.\\
& \quad + e^{\frac{2}{3}\pi i-i\psi}\mathrm{Ln}\left(x_1^{-1}e^{\frac{2}{3}\pi i-i\psi}-r^2\right) - e^{-\frac{2}{3}\pi i+i\psi}\mathrm{Ln}\left(x_1^{-1}e^{-\frac{2}{3}\pi i+i\psi}-r^2\right) + \\
& \quad \left. + e^{-\frac{2}{3}\pi i-i\psi}\mathrm{Ln}\left(x_1^{-1}e^{-\frac{2}{3}\pi i-i\psi}-r^2\right) - e^{\frac{2}{3}\pi i+i\psi}\mathrm{Ln}\left(x_1^{-1}e^{\frac{2}{3}\pi i+i\psi}-r^2\right)\right]_{r=x_1}^{1}.
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{1}{2x_1^2}\left(\frac{e^{i\psi}}{1-x_1^{-1}e^{i\psi}} - \frac{e^{-i\psi}}{1-x_1^{-1}e^{-i\psi}} + \frac{e^{-\frac{2}{3}\pi i+i\psi}}{1-x_1^{-1}e^{-\frac{2}{3}\pi i+i\psi}} + \right.\\
& \left. - \frac{e^{\frac{2}{3}\pi i-i\psi}}{1-x_1^{-1}e^{\frac{2}{3}\pi i-i\psi}} + \frac{e^{\frac{2}{3}\pi i+i\psi}}{1-x_1^{-1}e^{\frac{2}{3}\pi i+i\psi}} - \frac{e^{-\frac{2}{3}\pi i-i\psi}}{1-x_1^{-1}e^{-\frac{2}{3}\pi i-i\psi}}\right)\left(1-x_1^2\right) + \\
& + \frac{1}{2x_1^2}\left(e^{-i\psi}\mathrm{Ln}\left(\frac{x_1^{-1}e^{-i\psi}-1}{x_1^{-1}e^{-i\psi}-x_1^2}\right) - e^{i\psi}\mathrm{Ln}\left(\frac{x_1^{-1}e^{i\psi}-1}{x_1^{-1}e^{i\psi}-x_1^2}\right) + \right.\\
& \quad + e^{\frac{2}{3}\pi i-i\psi}\mathrm{Ln}\left(\frac{x_1^{-1}e^{\frac{2}{3}\pi i-i\psi}-1}{x_1^{-1}e^{\frac{2}{3}\pi i-i\psi}-x_1^2}\right) - e^{-\frac{2}{3}\pi i+i\psi}\mathrm{Ln}\left(\frac{x_1^{-1}e^{-\frac{2}{3}\pi i+i\psi}-1}{x_1^{-1}e^{-\frac{2}{3}\pi i+i\psi}-x_1^2}\right) + \\
& \quad \left. + e^{-\frac{2}{3}\pi i-i\psi}\mathrm{Ln}\left(\frac{x_1^{-1}e^{-\frac{2}{3}\pi i-i\psi}-1}{x_1^{-1}e^{-\frac{2}{3}\pi i-i\psi}-x_1^2}\right) - e^{\frac{2}{3}\pi i+i\psi}\mathrm{Ln}\left(\frac{x_1^{-1}e^{\frac{2}{3}\pi i+i\psi}-1}{x_1^{-1}e^{\frac{2}{3}\pi i+i\psi}-x_1^2}\right)\right).
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{1}{2x_1^2}\left(\frac{2i\sin\psi}{1-2x_1^{-1}\cos\psi+x_1^{-2}} + \frac{2i\sin\left(\psi-\frac{2}{3}\pi\right)}{1-2x_1^{-1}\cos\left(\psi-\frac{2}{3}\pi\right)+x_1^{-2}} + \frac{2i\sin\left(\psi+\frac{2}{3}\pi\right)}{1-2x_1^{-1}\cos\left(\psi+\frac{2}{3}\pi\right)+x_1^{-2}}\right)\left(1-x_1^2\right) + \\
& + \frac{1}{2x_1^2}\left(e^{-i\psi}\mathrm{Ln}\left(\frac{1-x_1e^{i\psi}}{1-x_1^3e^{i\psi}}\right) - e^{i\psi}\mathrm{Ln}\left(\frac{1-x_1e^{-i\psi}}{1-x_1^3e^{-i\psi}}\right) + e^{\frac{2}{3}\pi i-i\psi}\mathrm{Ln}\left(\frac{1-x_1e^{-\frac{2}{3}\pi i+i\psi}}{1-x_1^3e^{-\frac{2}{3}\pi i+i\psi}}\right) + \right.\\
& \left. - e^{-\frac{2}{3}\pi i+i\psi}\mathrm{Ln}\left(\frac{1-x_1e^{\frac{2}{3}\pi i-i\psi}}{1-x_1^3e_1^{\frac{2}{3}\pi i-i\psi}}\right) + e^{-\frac{2}{3}\pi i-i\psi}\mathrm{Ln}\left(\frac{1-x_1e^{\frac{2}{3}\pi i+i\psi}}{1-x_1^3e^{\frac{2}{3}\pi i+i\psi}}\right) - e^{\frac{2}{3}\pi i+i\psi}\mathrm{Ln}\left(\frac{1-x_1e^{-\frac{2}{3}\pi i-i\psi}}{1-x_1^3e^{-\frac{2}{3}\pi i-i\psi}}\right)\right).
\end{aligned}
$$

$$= \frac{1}{2x_1^2}\left(\frac{2i\sin\psi}{1-2x_1^{-1}\cos\psi+x_1^{-2}}+\frac{2i\sin\left(\psi-\frac{2}{3}\pi\right)}{1-2x_1^{-1}\cos\left(\psi-\frac{2}{3}\pi\right)+x_1^{-2}}+\frac{2i\sin\left(\psi+\frac{2}{3}\pi\right)}{1-2x_1^{-1}\cos\left(\psi+\frac{2}{3}\pi\right)+x_1^{-2}}\right)(1-x_1^2)+$$

$$+\ \frac{1}{2x_1^2}\Bigg(-2i\sin\psi\ln\sqrt{1-2x_1\cos\psi+x_1^2}-2i\cos\psi\arctan\left(\frac{x_1\sin\psi}{1-x_1\cos\psi}\right)\ +$$

$$+\ 2i\sin\psi\ln\sqrt{1-2x_1^3\cos\psi+x_1^6}+2i\cos\psi\arctan\left(\frac{x_1^3\sin\psi}{1-x_1^3\cos\psi}\right)+$$

$$-\ 2i\sin\left(\psi-\tfrac{2}{3}\pi\right)\ln\sqrt{1-2x_1\cos\left(\psi-\tfrac{2}{3}\pi\right)+x_1^2}-2i\cos\left(\psi-\tfrac{2}{3}\pi\right)\arctan\left(\frac{x_1\sin\left(\psi-\frac{2}{3}\pi\right)}{1-x_1\cos\left(\psi-\frac{2}{3}\pi\right)}\right)+$$

$$+\ 2i\sin\left(\psi-\tfrac{2}{3}\pi\right)\ln\sqrt{1-2x_1^3\cos\left(\psi-\tfrac{2}{3}\pi\right)+x_1^6}+2i\cos\left(\psi-\frac{2}{3}\pi\right)\arctan\left(\frac{x_1^3\sin\left(\psi-\frac{2}{3}\pi\right)}{1-x_1^3\cos\left(\psi-\frac{2}{3}\pi\right)}\right)+$$

$$-\ 2i\sin\left(\psi+\tfrac{2}{3}\pi\right)\ln\sqrt{1-2x_1\cos\left(\psi+\tfrac{2}{3}\pi\right)+x_1^2}-2i\cos\left(\psi+\tfrac{2}{3}\pi\right)\arctan\left(\frac{x_1\sin\left(\psi+\frac{2}{3}\pi\right)}{1-x_1\cos\left(\psi+\frac{2}{3}\pi\right)}\right)+$$

$$+\ 2i\sin\left(\psi+\tfrac{2}{3}\pi\right)\ln\sqrt{1-2x_1^3\cos\left(\psi+\tfrac{2}{3}\pi\right)+x_1^6}+2i\cos\left(\psi+\tfrac{2}{3}\pi\right)\arctan\left(\frac{x_1^3\sin\left(\psi+\frac{2}{3}\pi\right)}{1-x_1^3\cos\left(\psi+\frac{2}{3}\pi\right)}\right)\Bigg)$$

$$= i\left(\frac{\sin\psi}{x_1^2-2x_1\cos\psi+1}+\frac{\sin\left(\psi-\frac{2}{3}\pi\right)}{x_1^2-2x_1\cos\left(\psi-\frac{2}{3}\pi\right)+1}+\frac{\sin\left(\psi+\frac{2}{3}\pi\right)}{x_1^2-2x_1\cos\left(\psi+\frac{2}{3}\pi\right)+1}\right)(1-x_1^2)+$$

$$+\ \frac{i}{x_1^2}\Bigg(-\sin\psi\ln\sqrt{1-2x_1\cos\psi+x_1^2}-\cos\psi\arctan\left(\frac{x_1\sin\psi}{1-x_1\cos\psi}\right)\ +$$

$$+\ \sin\psi\ln\sqrt{1-2x_1^3\cos\psi+x_1^6}+\cos\psi\arctan\left(\frac{x_1^3\sin\psi}{1-x_1^3\cos\psi}\right)+$$

$$-\ \sin\left(\psi-\tfrac{2}{3}\pi\right)\ln\sqrt{1-2x_1\cos\left(\psi-\tfrac{2}{3}\pi\right)+x_1^2}-\cos\left(\psi-\tfrac{2}{3}\pi\right)\arctan\left(\frac{x_1\sin\left(\psi-\frac{2}{3}\pi\right)}{1-x_1\cos\left(\psi-\frac{2}{3}\pi\right)}\right)+$$

$$+\ \sin\left(\psi-\tfrac{2}{3}\pi\right)\ln\sqrt{1-2x_1^3\cos\left(\psi-\tfrac{2}{3}\pi\right)+x_1^6}+\cos\left(\psi-\tfrac{2}{3}\pi\right)\arctan\left(\frac{x_1^3\sin\left(\psi-\frac{2}{3}\pi\right)}{1-x_1^3\cos\left(\psi-\frac{2}{3}\pi\right)}\right)+$$

$$-\ \sin\left(\psi+\tfrac{2}{3}\pi\right)\ln\sqrt{1-2x_1\cos\left(\psi+\tfrac{2}{3}\pi\right)+x_1^2}-\cos\left(\psi+\tfrac{2}{3}\pi\right)\arctan\left(\frac{x_1\sin\left(\psi+\frac{2}{3}\pi\right)}{1-x_1\cos\left(\psi+\frac{2}{3}\pi\right)}\right)+$$

$$+\ \sin\left(\psi+\tfrac{2}{3}\pi\right)\ln\sqrt{1-2x_1^3\cos\left(\psi+\tfrac{2}{3}\pi\right)+x_1^6}+\cos\left(\psi+\tfrac{2}{3}\pi\right)\arctan\left(\frac{x_1^3\sin\left(\psi+\frac{2}{3}\pi\right)}{1-x_1^3\cos\left(\psi+\frac{2}{3}\pi\right)}\right)\Bigg)$$

$$
\begin{aligned}
&= i\left(1-x_1^2\right)\sum_{k=0}^{2}\frac{\sin\left(\psi+\frac{2}{3}k\pi\right)}{x_1^2-2x_1\cos\left(\psi+\frac{2}{3}k\pi\right)+1}+\\
&+\quad \frac{i}{x_1^2}\sum_{k=0}^{2}\Bigg(-\sin\left(\psi+\tfrac{2}{3}k\pi\right)\ln\sqrt{1-2x_1\cos\left(\psi+\tfrac{2}{3}k\pi\right)+x_1^2}+\\
&\qquad -\cos\left(\psi+\tfrac{2}{3}k\pi\right)\arctan\left(\frac{x_1\sin\left(\psi+\frac{2}{3}k\pi\right)}{1-x_1\cos\left(\psi+\frac{2}{3}k\pi\right)}\right)+\\
&\qquad +\sin\left(\psi+\tfrac{2}{3}k\pi\right)\ln\sqrt{1-2x_1^3\cos\left(\psi+\tfrac{2}{3}k\pi\right)+x_1^6}+\\
&\qquad +\cos\left(\psi+\tfrac{2}{3}k\pi\right)\arctan\left(\frac{x_1^3\sin\left(\psi+\frac{2}{3}k\pi\right)}{1-x_1^3\cos\left(\psi+\frac{2}{3}k\pi\right)}\right)\Bigg)
\end{aligned}
\tag{61}
$$

5.5. Conclusion of case 2)

In the present case the enumerator of (5) becomes

$$
\int_{z\in S}\lim_{x_2\downarrow 0}\frac{G_S(x,z)}{x_2}\lim_{|y|\uparrow 1}\frac{G_S(z,y)}{1-|y|^2}dz=\frac{2\pi i}{8\pi^2}\left(\#_{(59)}+\#_{(61)}\right)\;=
$$

$$
\begin{aligned}
=\;&\frac{1}{4\pi}\Bigg(\sin\psi\ln\sqrt{1-2x_1\cos\psi+x_1^2}+\cos\psi\arctan\left(\frac{x_1\sin\psi}{1-x_1\cos\psi}\right)+\\
&+\sin\left(\psi-\tfrac{2}{3}\pi\right)\ln\sqrt{1-2x_1\cos\left(\psi-\tfrac{2}{3}\pi\right)+x_1^2}+\cos\left(\psi-\tfrac{2}{3}\pi\right)\arctan\left(\frac{x_1\sin\left(\psi-\frac{2}{3}\pi\right)}{1-x_1\cos\left(\psi-\frac{2}{3}\pi\right)}\right)+\\
&+\sin\left(\psi+\tfrac{2}{3}\pi\right)\ln\sqrt{1-2x_1\cos\left(\psi+\tfrac{2}{3}\pi\right)+x_1^2}+\cos\left(\psi+\tfrac{2}{3}\pi\right)\arctan\left(\frac{x_1\sin\left(\psi+\frac{2}{3}\pi\right)}{1-x_1\cos\left(\psi+\frac{2}{3}\pi\right)}\right)+\\
&-\frac{\sin\psi}{x_1^2}\ln\sqrt{1-2x_1^3\cos\psi+x_1^6}-\frac{\cos\psi}{x_1^2}\arctan\left(\frac{x_1^3\sin\psi}{1-x_1^3\cos\psi}\right)+\\
&-\frac{\sin\left(\psi-\frac{2}{3}\pi\right)}{x_1^2}\ln\sqrt{1-2x_1^3\cos\left(\psi-\tfrac{2}{3}\pi\right)+x_1^6}+\frac{\cos\left(\psi-\frac{2}{3}\pi\right)}{x_1^2}\arctan\left(\frac{x_1^3\sin\left(\psi-\frac{2}{3}\pi\right)}{1-x_1^3\cos\left(\psi-\frac{2}{3}\pi\right)}\right)+\\
&-\frac{\sin\left(\psi+\frac{2}{3}\pi\right)}{x_1^2}\ln\sqrt{1-2x_1^3\cos\left(\psi+\tfrac{2}{3}\pi\right)+x_1^6}+\frac{\cos\left(\psi+\frac{2}{3}\pi\right)}{x_1^2}\arctan\left(\frac{x_1^3\sin\left(\psi+\frac{2}{3}\pi\right)}{1-x_1^3\cos\left(\psi+\frac{2}{3}\pi\right)}\right)\Bigg)+\\
&+\frac{1}{4\pi}\left(\frac{\sin\psi}{x_1^2-2x_1\cos\psi+1}+\frac{\sin\left(\psi-\frac{2}{3}\pi\right)}{x_1^2-2x_1\cos\left(\psi-\frac{2}{3}\pi\right)+1}+\frac{\sin\left(\psi+\frac{2}{3}\pi\right)}{x_1^2-2x_1\cos\left(\psi+\frac{2}{3}\pi\right)+1}\right)\left(1-x_1^2\right)+\\
&+\frac{1}{4\pi x_1^2}\Bigg(-\sin\psi\ln\sqrt{1-2x_1\cos\psi+x_1^2}-\cos\psi\arctan\left(\frac{x_1\sin\psi}{1-x_1\cos\psi}\right)+\\
&+\sin\psi\ln\sqrt{1-2x_1^3\cos\psi+x_1^6}+\cos\psi\arctan\left(\frac{x_1^3\sin\psi}{1-x_1^3\cos\psi}\right)+
\end{aligned}
$$

$$
\begin{aligned}
&- \sin\left(\psi - \tfrac{2}{3}\pi\right) \ln \sqrt{1 - 2x_1 \cos\left(\psi - \tfrac{2}{3}\pi\right) + x_1^2} - \cos\left(\psi - \tfrac{2}{3}\pi\right) \arctan\left(\frac{x_1 \sin\left(\psi - \frac{2}{3}\pi\right)}{1 - x_1 \cos\left(\psi - \frac{2}{3}\pi\right)}\right) + \\
&+ \sin\left(\psi - \tfrac{2}{3}\pi\right) \ln \sqrt{1 - 2x_1^3 \cos\left(\psi - \tfrac{2}{3}\pi\right) + x_1^6} + \cos\left(\psi - \tfrac{2}{3}\pi\right) \arctan\left(\frac{x_1^3 \sin\left(\psi - \frac{2}{3}\pi\right)}{1 - x_1^3 \cos\left(\psi - \frac{2}{3}\pi\right)}\right) + \\
&- \sin\left(\psi + \tfrac{2}{3}\pi\right) \ln \sqrt{1 - 2x_1 \cos\left(\psi + \tfrac{2}{3}\pi\right) + x_1^2} - \cos\left(\psi + \tfrac{2}{3}\pi\right) \arctan\left(\frac{x_1 \sin\left(\psi + \frac{2}{3}\pi\right)}{1 - x_1 \cos\left(\psi + \frac{2}{3}\pi\right)}\right) + \\
&+ \sin\left(\psi + \tfrac{2}{3}\pi\right) \ln \sqrt{1 - 2x_1^3 \cos\left(\psi + \tfrac{2}{3}\pi\right) + x_1^6} + \cos\left(\psi + \tfrac{2}{3}\pi\right) \arctan\left(\frac{x_1^3 \sin\left(\psi + \frac{2}{3}\pi\right)}{1 - x_1^3 \cos\left(\psi + \frac{2}{3}\pi\right)}\right)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{1 - x_1^{-2}}{\pi} \sin\psi \ln \sqrt{1 - 2x_1 \cos\psi + x_1^2} + \frac{1 - x_1^{-2}}{\pi} \cos\psi \arctan\left(\frac{x_1 \sin\psi}{1 - x_1 \cos\psi}\right) + \\
&+ \frac{1 - x_1^{-2}}{\pi} \sin\left(\psi - \tfrac{2}{3}\pi\right) \ln \sqrt{1 - 2x_1 \cos\left(\psi - \tfrac{2}{3}\pi\right) + x_1^2} + \\
&+ \frac{1 - x_1^{-2}}{\pi} \cos\left(\psi - \tfrac{2}{3}\pi\right) \arctan\left(\frac{x_1 \sin\left(\psi - \frac{2}{3}\pi\right)}{1 - x_1 \cos\left(\psi - \frac{2}{3}\pi\right)}\right) + \\
&+ \frac{1 - x_1^{-2}}{\pi} \sin\left(\psi + \tfrac{2}{3}\pi\right) \ln \sqrt{1 - 2x_1 \cos\left(\psi + \tfrac{2}{3}\pi\right) + x_1^2} + \\
&+ \frac{1 - x_1^{-2}}{\pi} \cos\left(\psi + \tfrac{2}{3}\pi\right) \arctan\left(\frac{x_1 \sin\left(\psi + \frac{2}{3}\pi\right)}{1 - x_1 \cos\left(\psi + \frac{2}{3}\pi\right)}\right) + \\
&+ \frac{1 - x_1^2}{\pi} \frac{\sin\psi}{x_1^2 - 2x_1 \cos\psi + 1} + \frac{1 - x_1^2}{\pi} \frac{\sin\left(\psi - \frac{2}{3}\pi\right)}{x_1^2 - 2x_1 \cos\left(\psi - \frac{2}{3}\pi\right) + 1} + \frac{1 - x_1^2}{\pi} \frac{\sin\left(\psi + \frac{2}{3}\pi\right)}{x_1^2 - 2x_1 \cos\left(\psi + \frac{2}{3}\pi\right) + 1}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & - \frac{1 - x_1^2}{\pi} \frac{\cos\psi \arctan\left(\frac{x_1 \sin\psi}{1 - x_1 \cos\psi}\right) + \sin\psi \ln \sqrt{1 - 2x_1 \cos\psi + x_1^2}}{x_1^2} + \\
&- \frac{1 - x_1^2}{\pi} \frac{\cos\left(\psi - \frac{2}{3}\pi\right) \arctan\left(\frac{x_1 \sin\left(\psi - \frac{2}{3}\pi\right)}{1 - x_1 \cos\left(\psi - \frac{2}{3}\pi\right)}\right) + \sin\left(\psi - \frac{2}{3}\pi\right) \ln \sqrt{1 - 2x_1 \cos\left(\psi - \frac{2}{3}\pi\right) + x_1^2}}{x_1^2} + \\
&- \frac{1 - x_1^2}{\pi} \frac{\cos\left(\psi + \frac{2}{3}\pi\right) \arctan\left(\frac{x_1 \sin\left(\psi + \frac{2}{3}\pi\right)}{1 - x_1 \cos\left(\psi + \frac{2}{3}\pi\right)}\right) + \sin\left(\psi + \frac{2}{3}\pi\right) \ln \sqrt{1 - 2x_1 \cos\left(\psi + \frac{2}{3}\pi\right) + x_1^2}}{x_1^2} + \\
&+ \frac{1 - x_1^2}{\pi} \left(\frac{\sin\psi}{x_1^2 - 2x_1 \cos\psi + 1} + \frac{\sin\left(\psi - \frac{2}{3}\pi\right)}{x_1^2 - 2x_1 \cos\left(\psi - \frac{2}{3}\pi\right) + 1} + \frac{\sin\left(\psi + \frac{2}{3}\pi\right)}{x_1^2 - 2x_1 \cos\left(\psi + \frac{2}{3}\pi\right) + 1}\right)
\end{aligned}
$$

$$
\begin{aligned}
= - \frac{1 - x_1^2}{\pi x_1^2} & \sum_{k=0}^{2} \cos\left(\psi + \tfrac{2}{3}k\pi\right) \arctan\left(\frac{x_1 \sin\left(\psi + \frac{2}{3}k\pi\right)}{1 - x_1 \cos\left(\psi + \frac{2}{3}k\pi\right)}\right) + \\
+ \frac{1 - x_1^2}{\pi x_1^2} & \sum_{k=0}^{2} \sin\left(\psi + \tfrac{2}{3}k\pi\right) \ln \sqrt{1 - 2x_1 \cos\left(\psi + \tfrac{2}{3}k\pi\right) + x_1^2} + \\
+ \frac{1 - x_1^2}{\pi} & \sum_{k=0}^{2} \frac{\sin\left(\psi + \frac{2}{3}k\pi\right)}{x_1^2 - 2x_1 \cos\left(\psi + \frac{2}{3}k\pi\right) + 1} \ .
\end{aligned} \qquad (62)
$$

Together with the denominator (41) we find

$$T_{12}(x_1,\psi) = \frac{\#_{(62)}}{\#_{(41)}} =$$

$$= \frac{\frac{\sin\psi}{x_1^2-2x_1\cos\psi+1} + \frac{\sin\left(\psi-\frac{2}{3}\pi\right)}{x_1^2-2x_1\cos\left(\psi-\frac{2}{3}\pi\right)+1} + \frac{\sin\left(\psi+\frac{2}{3}\pi\right)}{x_1^2-2x_1\cos\left(\psi+\frac{2}{3}\pi\right)+1}}{4\left(\frac{\sin\psi}{\left(x_1^2-2x_1\cos\psi+1\right)^2} + \frac{\sin\left(\psi-\frac{2}{3}\pi\right)}{\left(x_1^2-2x_1\cos\left(\psi-\frac{2}{3}\pi\right)+1\right)^2} + \frac{\sin\left(\psi+\frac{2}{3}\pi\right)}{\left(x_1^2-2x_1\cos\left(\psi+\frac{2}{3}\pi\right)+1\right)^2}\right)} +$$

$$-\frac{\cos\psi\arctan\left(\frac{x_1\sin\psi}{1-x_1\cos\psi}\right) + \sin\psi\ln\sqrt{1-2x_1\cos\psi+x_1^2}}{4x_1^2\left(\frac{\sin\psi}{\left(x_1^2-2x_1\cos\psi+1\right)^2} + \frac{\sin\left(\psi-\frac{2}{3}\pi\right)}{\left(x_1^2-2x_1\cos\left(\psi-\frac{2}{3}\pi\right)+1\right)^2} + \frac{\sin\left(\psi+\frac{2}{3}\pi\right)}{\left(x_1^2-2x_1\cos\left(\psi+\frac{2}{3}\pi\right)+1\right)^2}\right)} +$$

$$-\frac{\cos\left(\psi-\frac{2}{3}\pi\right)\arctan\left(\frac{x_1\sin\left(\psi-\frac{2}{3}\pi\right)}{1-x_1\cos\left(\psi-\frac{2}{3}\pi\right)}\right) + \sin\left(\psi-\frac{2}{3}\pi\right)\ln\sqrt{1-2x_1\cos\left(\psi-\frac{2}{3}\pi\right)+x_1^2}}{4x_1^2\left(\frac{\sin\psi}{\left(x_1^2-2x_1\cos\psi+1\right)^2} + \frac{\sin\left(\psi-\frac{2}{3}\pi\right)}{\left(x_1^2-2x_1\cos\left(\psi-\frac{2}{3}\pi\right)+1\right)^2} + \frac{\sin\left(\psi+\frac{2}{3}\pi\right)}{\left(x_1^2-2x_1\cos\left(\psi+\frac{2}{3}\pi\right)+1\right)^2}\right)} +$$

$$-\frac{\cos\left(\psi+\frac{2}{3}\pi\right)\arctan\left(\frac{x_1\sin\left(\psi+\frac{2}{3}\pi\right)}{1-x_1\cos\left(\psi+\frac{2}{3}\pi\right)}\right) + \sin\left(\psi+\frac{2}{3}\pi\right)\ln\sqrt{1-2x_1\cos\left(\psi+\frac{2}{3}\pi\right)+x_1^2}}{4x_1^2\left(\frac{\sin\psi}{\left(x_1^2-2x_1\cos\psi+1\right)^2} + \frac{\sin\left(\psi-\frac{2}{3}\pi\right)}{\left(x_1^2-2x_1\cos\left(\psi-\frac{2}{3}\pi\right)+1\right)^2} + \frac{\sin\left(\psi+\frac{2}{3}\pi\right)}{\left(x_1^2-2x_1\cos\left(\psi+\frac{2}{3}\pi\right)+1\right)^2}\right)}.$$

and a careful analysis shows that

$$\sup_{\substack{0<x_1<1\\0<\psi<\frac{1}{3}\pi}} T_{12}(x_1,\psi) = \lim_{x_1\downarrow 0} T_{12}(x_1,\alpha) = \frac{1}{16}$$

holds for all $\alpha \in \left(0,\frac{1}{3}\pi\right)$. Aside from this global maximum T_{12} has a local maximum at $x_1 = 1$ and $\psi = \frac{1}{3}\pi$, but

$$T_{12}\left(1,\tfrac{1}{3}\pi\right) = \frac{5}{27} - \frac{4}{243}\sqrt{3}\pi - \frac{4}{81}\ln 2 \ < \ \frac{1}{16}.$$

See Figure 2 on page 51.

6. The case that $x, y \in \Gamma_2$

The third section is concerned with $x \to (\cos\phi, sin\phi)$ and $y \to (\cos\psi, \sin\psi)$. We study

$$T_{22}(\phi,\psi) \ := \ \frac{\displaystyle\int_S \lim_{|x|\uparrow 1}\frac{G_S(x,z)}{1-|x|^2}\lim_{|y|\uparrow 1}\frac{G_S(z,y)}{1-|y|^2}dz}{\displaystyle\lim_{\substack{|x|\uparrow 1\\|y|\uparrow 1}}\frac{G_S(x,y)}{(1-|x|^2)(1-|y|^2)}}\ . \tag{63}$$

6.1. Limit of the Green function

By (7) the denominator becomes

$$\lim_{\substack{y\to(\cos\psi,\sin\psi)\\x\to(\cos\phi,\sin\phi)}} \frac{G_S(x,y)}{\left(1-|y|^2\right)\left(1-|x|^2\right)} =$$

$$= \ \frac{1}{8\pi}\sum_{k=0}^{2}\left(\frac{1}{1-\cos\left(\psi+\phi+\frac{2}{3}k\pi\right)} - \frac{1}{1-\cos\left(\psi-\phi+\frac{2}{3}k\pi\right)}\right), \tag{64}$$

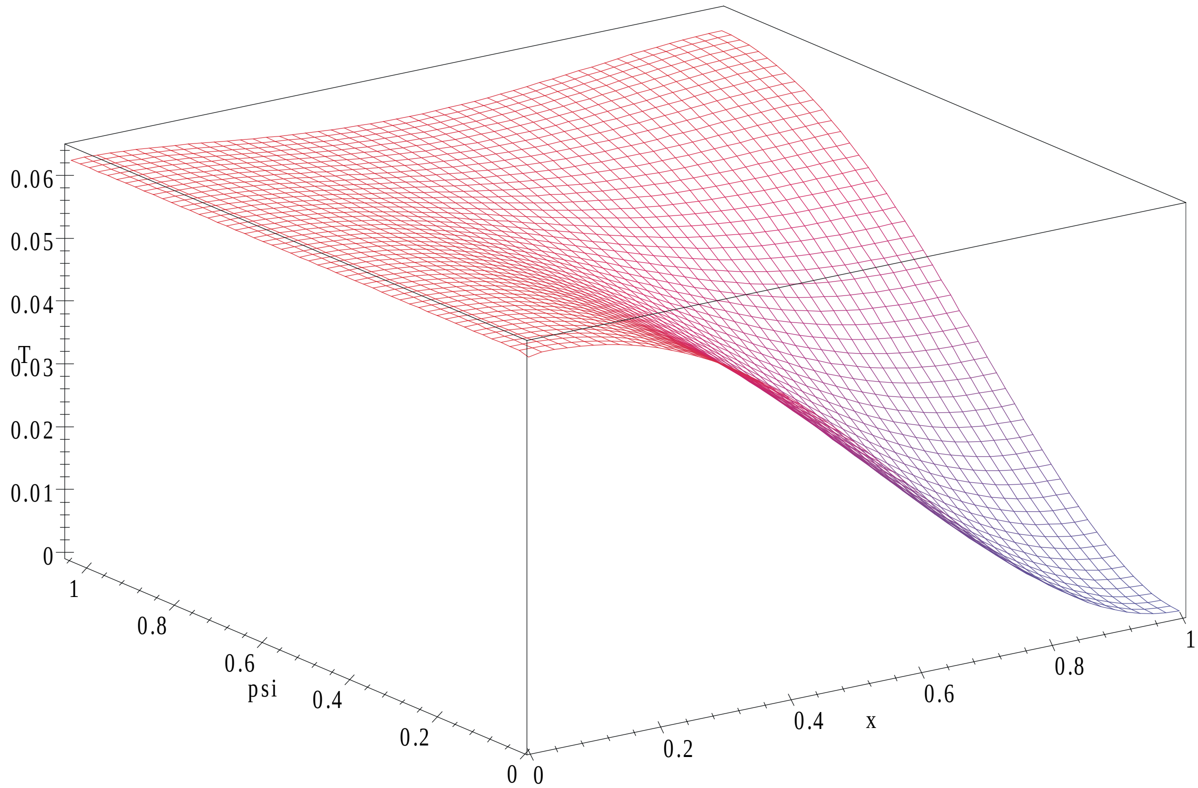


Figure 2: $x \in \Gamma_1$ and $y \in \Gamma_2$: $T_{12}(x_1, \psi)$.

while for the enumerator we have as in case 2, denoting $z = (r\cos\theta, r\sin\theta)$,

$$\begin{aligned}
&\lim_{y \to (\cos\psi, \sin\psi)} \frac{G_S(z,y)}{1-|y|^2} = \\
= \quad & \frac{1-r^2}{4\pi}\left(\frac{1}{r^2-2r\cos(\theta+\psi)+1} - \frac{1}{r^2-2r\cos(\theta-\psi)+1}\right) \\
&+ \frac{1-r^2}{4\pi}\left(\frac{1}{r^2-2r\cos\left(\theta+\psi+\frac{2}{3}\pi\right)+1} - \frac{1}{r^2-2r\cos\left(\theta-\psi+\frac{2}{3}\pi\right)+1}\right) + \\
&+ \frac{1-r^2}{4\pi}\left(\frac{1}{r^2-2r\cos\left(\theta+\psi-\frac{2}{3}\pi\right)+1} - \frac{1}{r^2-2r\cos\left(\theta-\psi-\frac{2}{3}\pi\right)+1}\right),
\end{aligned}$$

and by using symmetry

$$\begin{aligned}
&\lim_{x \to (\cos\phi, \sin\phi)} \frac{G_S(x,z)}{1-|x|^2} = \\
= \quad & \frac{1-r^2}{4\pi}\left(\frac{1}{r^2-2r\cos(\theta+\phi)+1} - \frac{1}{r^2-2r\cos(\theta-\phi)+1}\right) \\
&+ \frac{1-r^2}{4\pi}\left(\frac{1}{r^2-2r\cos\left(\theta+\phi+\frac{2}{3}\pi\right)+1} - \frac{1}{r^2-2r\cos\left(\theta-\phi+\frac{2}{3}\pi\right)+1}\right) + \\
&+ \frac{1-r^2}{4\pi}\left(\frac{1}{r^2-2r\cos\left(\theta+\phi-\frac{2}{3}\pi\right)+1} - \frac{1}{r^2-2r\cos\left(\theta-\phi-\frac{2}{3}\pi\right)+1}\right).
\end{aligned}$$

6.2. Derivation of a contour integral

We want to compute

$$\pi^2 \int_{z\in S} \lim_{x\to(\cos\phi,\sin\phi)} \frac{G_S(x,z)}{1-|x|^2} \lim_{\substack{y\to(\cos\psi,\sin\psi)\\ y\in S}} \frac{G_S(z,y)}{1-|y|^2} dz =$$

$$= \int_{r=0}^{1} \int_{\theta=0}^{\frac{1}{3}\pi} \sum_{k=0}^{2} g\left(\theta + k\tfrac{2}{3}\pi\right) \sum_{m=0}^{2} f\left(\theta + m\tfrac{2}{3}\pi\right) d\theta\, r dr \tag{65}$$

with f as in (42) and

$$g(\theta) = \frac{1-r^2}{4}\left(\frac{1}{r^2 - 2r\cos(\theta+\phi)+1} - \frac{1}{r^2-2r\cos(\theta-\phi)+1}\right). \tag{66}$$

Again we find

$$\int_{\theta=0}^{\frac{1}{3}\pi} \sum_{k=0}^{2} g\left(\theta + k\frac{2}{3}\pi\right) \sum_{m=0}^{2} f\left(\theta + m\tfrac{2}{3}\pi\right) d\theta =$$

$$= \frac{1}{6}\int_{\theta=0}^{2\pi} \sum_{k=0}^{2} g\left(\theta + k\frac{2}{3}\pi\right) \sum_{m=0}^{2} f\left(\theta + m\tfrac{2}{3}\pi\right) d\theta =$$

$$= \frac{1}{2}\int_{\theta=0}^{2\pi} \sum_{k=0}^{2} g\left(\theta + k\frac{2}{3}\pi\right) f(\theta)\, d\theta =$$

$$= \frac{1}{2}\int_{\theta=0}^{2\pi} \sum_{k=0}^{2} g\left(\theta + k\frac{2}{3}\pi\right) \Big(f_\psi(\theta) - f_{-\psi}(\theta)\Big) d\theta,$$

which means that with $w = re^{i\theta}$ as before

$$\begin{aligned}
f_\psi(\theta) &= \frac{1}{4}\left(\frac{r^2e^{-i\psi}}{w-r^2e^{-i\psi}} - \frac{e^{-i\psi}}{w-e^{-i\psi}}\right),\\
g(\theta) &= \frac{1}{4}\left(\frac{r^2e^{-i\phi}}{w-r^2e^{-i\phi}} - \frac{e^{-i\phi}}{w-e^{-i\phi}} - \frac{r^2e^{i\phi}}{w-r^2e^{i\phi}} + \frac{e^{i\phi}}{w-e^{i\phi}}\right),\\
g\left(\theta - \tfrac{2}{3}\pi\right) &= \frac{1}{4}\left(\frac{r^2e^{-i\phi-\frac{2}{3}\pi i}}{w-r^2e^{-i\phi-\frac{2}{3}\pi i}} - \frac{e^{-i\phi-\frac{2}{3}\pi i}}{w-e^{-i\phi-\frac{2}{3}\pi i}} - \frac{r^2e^{i\phi-\frac{2}{3}\pi i}}{w-r^2e^{i\phi-\frac{2}{3}\pi i}} + \frac{e^{i\phi-\frac{2}{3}\pi i}}{w-e^{i\phi-\frac{2}{3}\pi i}}\right),\\
g\left(\theta + \tfrac{2}{3}\pi\right) &= \frac{1}{4}\left(\frac{r^2e^{-i\phi+\frac{2}{3}\pi i}}{w-r^2e^{-i\phi+\frac{2}{3}\pi i}} - \frac{e^{-i\phi+\frac{2}{3}\pi i}}{w-e^{-i\phi+\frac{2}{3}\pi i}} - \frac{r^2e^{i\phi+\frac{2}{3}\pi i}}{w-r^2e^{i\phi+\frac{2}{3}\pi i}} + \frac{e^{i\phi+\frac{2}{3}\pi i}}{w-e^{i\phi+\frac{2}{3}\pi i}}\right).
\end{aligned}$$

Then

$$\pi^2 \int_{z\in S} \lim_{\substack{x\to(\cos\phi,\sin\phi)\\ x\in S}} \frac{G_S(x,z)}{1-|x|^2} \lim_{\substack{y\to(\cos\psi,\sin\psi)\\ y\in S}} \frac{G_S(z,y)}{1-|y|^2} dz =$$

$$\begin{aligned}
= \ & \frac{1}{2}\int_{r=0}^{1}\oint_{|w|=r}\left(\frac{1}{4}\left(\frac{r^2e^{-i\phi}}{w-r^2e^{-i\phi}}-\frac{e^{-i\phi}}{w-e^{-i\phi}}-\frac{r^2e^{i\phi}}{w-r^2e^{i\phi}}+\frac{e^{i\phi}}{w-e^{i\phi}}\right)\right.\\
& \frac{1}{4}\left(\frac{r^2e^{-i\phi-\frac{2}{3}\pi i}}{w-r^2e^{-i\phi-\frac{2}{3}\pi i}}-\frac{e^{-i\phi-\frac{2}{3}\pi i}}{w-e^{-i\phi-\frac{2}{3}\pi i}}-\frac{r^2e^{i\phi-\frac{2}{3}\pi i}}{w-r^2e^{i\phi-\frac{2}{3}\pi i}}+\frac{e^{i\phi-\frac{2}{3}\pi i}}{w-e^{i\phi-\frac{2}{3}\pi i}}\right)\\
& \left.\frac{1}{4}\left(\frac{r^2e^{-i\phi+\frac{2}{3}\pi i}}{w-r^2e^{-i\phi+\frac{2}{3}\pi i}}-\frac{e^{-i\phi+\frac{2}{3}\pi i}}{w-e^{-i\phi+\frac{2}{3}\pi i}}-\frac{r^2e^{i\phi+\frac{2}{3}\pi i}}{w-r^2e^{i\phi+\frac{2}{3}\pi i}}+\frac{e^{i\phi+\frac{2}{3}\pi i}}{w-e^{i\phi+\frac{2}{3}\pi i}}\right)\right)\times\\
& \times\frac{1}{4}\left(\frac{r^2e^{-i\psi}}{w-r^2e^{-i\psi}}-\frac{e^{-i\psi}}{w-e^{-i\psi}}-\frac{r^2e^{i\psi}}{w-r^2e^{i\psi}}+\frac{e^{i\psi}}{w-e^{i\psi}}\right)\frac{dw}{iw}rdr.
\end{aligned}$$

6.3. Computation of the contour integral

Let us first consider

$$\frac{1}{2}\int_{\theta=0}^{2\pi}\sum_{k=0}^{2} g\left(\theta+k\tfrac{2}{3}\pi\right) f_\psi(\theta)\, d\theta.$$

The integrand has the following poles within the unit circle:

$$P=\left\{r^2e^{-i\phi},r^2e^{i\phi},r^2e^{-i\phi-\frac{2}{3}\pi i},r^2e^{i\phi-\frac{2}{3}\pi i},r^2e^{-i\phi+\frac{2}{3}\pi i},r^2e^{i\phi+\frac{2}{3}\pi i}\right\} \text{ and } \left\{0,r^2e^{-i\psi}\right\}.$$

With the exception of 0 all these poles satisfy $|w| = r^2 < r$. Hence, independently of r, we find

$$\begin{aligned}
& \frac{1}{32i}\oint_{|w|=r}\left(\frac{r^2e^{-i\phi}}{w-r^2e^{-i\phi}}-\frac{e^{-i\phi}}{w-e^{-i\phi}}-\frac{r^2e^{i\phi}}{w-r^2e^{i\phi}}+\frac{e^{i\phi}}{w-e^{i\phi}}+\right.\\
& +\frac{r^2e^{-i\phi-\frac{2}{3}\pi i}}{w-r^2e^{-i\phi-\frac{2}{3}\pi i}}-\frac{e^{-i\phi-\frac{2}{3}\pi i}}{w-e^{-i\phi-\frac{2}{3}\pi i}}-\frac{r^2e^{i\phi-\frac{2}{3}\pi i}}{w-r^2e^{i\phi-\frac{2}{3}\pi i}}+\frac{e^{i\phi-\frac{2}{3}\pi i}}{w-e^{i\phi-\frac{2}{3}\pi i}}+\\
& \left.+\frac{r^2e^{-i\phi+\frac{2}{3}\pi i}}{w-r^2e^{-i\phi+\frac{2}{3}\pi i}}-\frac{e^{-i\phi+\frac{2}{3}\pi i}}{w-e^{-i\phi+\frac{2}{3}\pi i}}-\frac{r^2e^{i\phi+\frac{2}{3}\pi i}}{w-r^2e^{i\phi+\frac{2}{3}\pi i}}+\frac{e^{i\phi+\frac{2}{3}\pi i}}{w-e^{i\phi+\frac{2}{3}\pi i}}\right)\times\\
& \times\left(\frac{r^2e^{-i\psi}}{w-r^2e^{-i\psi}}-\frac{e^{-i\psi}}{w-e^{-i\psi}}\right)\frac{1}{w}dw.\\
= \ & \frac{\pi}{16}\sum_{w_i\in P\cup\{0,r^2e^{-i\psi}\}}\operatorname{Res}\left\{F_\psi(w)\right\}_{w=w_i}
\end{aligned}$$

with F_ψ the integrand. Again the contribution by $w_1=0$ cancels and we find

$$
\begin{aligned}
&\sum_{w_i\in P\cup\{r^2e^{-i\psi}\}} \operatorname{Res}\left\{F_\psi\left(w\right)\right\}_{w=w_i} = \\
=\ & \left(\frac{r^2e^{-i\psi}}{w-r^2e^{-i\psi}}-\frac{e^{-i\psi}}{w-e^{-i\psi}}\right)_{w=r^2e^{-i\phi}}-\left(\frac{r^2e^{-i\psi}}{w-r^2e^{-i\psi}}-\frac{e^{-i\psi}}{w-e^{-i\psi}}\right)_{w=r^2e^{i\phi}}+ \\
&+\left(\frac{r^2e^{-i\psi}}{w-r^2e^{-i\psi}}-\frac{e^{-i\psi}}{w-e^{-i\psi}}\right)_{w=r^2e^{-i\phi-\frac{2}{3}\pi i}}-\left(\frac{r^2e^{-i\psi}}{w-r^2e^{-i\psi}}-\frac{e^{-i\psi}}{w-e^{-i\psi}}\right)_{w=r^2e^{i\phi-\frac{2}{3}\pi i}}+ \\
&+\left(\frac{r^2e^{-i\psi}}{w-r^2e^{-i\psi}}-\frac{e^{-i\psi}}{w-e^{-i\psi}}\right)_{w=r^2e^{-i\phi+\frac{2}{3}\pi i}}-\left(\frac{r^2e^{-i\psi}}{w-r^2e^{-i\psi}}-\frac{e^{-i\psi}}{w-e^{-i\psi}}\right)_{w=r^2e^{i\phi+\frac{2}{3}\pi i}}+ \\
&+\left(\frac{r^2e^{-i\phi}}{w-r^2e^{-i\phi}}-\frac{e^{-i\phi}}{w-e^{-i\phi}}-\frac{r^2e^{i\phi}}{w-r^2e^{i\phi}}+\frac{e^{i\phi}}{w-e^{i\phi}}+\right. \\
&+\frac{r^2e^{-i\phi-\frac{2}{3}\pi i}}{w-r^2e^{-i\phi-\frac{2}{3}\pi i}}-\frac{e^{-i\phi-\frac{2}{3}\pi i}}{w-e^{-i\phi-\frac{2}{3}\pi i}}-\frac{r^2e^{i\phi-\frac{2}{3}\pi i}}{w-r^2e^{i\phi-\frac{2}{3}\pi i}}+\frac{e^{i\phi-\frac{2}{3}\pi i}}{w-e^{i\phi-\frac{2}{3}\pi i}}+ \\
&\left.+\frac{r^2e^{-i\phi+\frac{2}{3}\pi i}}{w-r^2e^{-i\phi+\frac{2}{3}\pi i}}-\frac{e^{-i\phi+\frac{2}{3}\pi i}}{w-e^{-i\phi+\frac{2}{3}\pi i}}-\frac{r^2e^{i\phi+\frac{2}{3}\pi i}}{w-r^2e^{i\phi+\frac{2}{3}\pi i}}+\frac{e^{i\phi+\frac{2}{3}\pi i}}{w-e^{i\phi+\frac{2}{3}\pi i}}\right)_{w=r^2e^{-i\psi}} \\
=\ & \frac{r^2e^{-i\psi}}{r^2e^{-i\phi}-r^2e^{-i\psi}}-\frac{e^{-i\psi}}{r^2e^{-i\phi}-e^{-i\psi}}-\frac{r^2e^{-i\psi}}{r^2e^{i\phi}-r^2e^{-i\psi}}+\frac{e^{-i\psi}}{r^2e^{i\phi}-e^{-i\psi}}+ \\
&+\frac{r^2e^{-i\psi}}{r^2e^{-i\phi-\frac{2}{3}\pi i}-r^2e^{-i\psi}}-\frac{e^{-i\psi}}{r^2e^{-i\phi-\frac{2}{3}\pi i}-e^{-i\psi}}-\frac{r^2e^{-i\psi}}{r^2e^{i\phi-\frac{2}{3}\pi i}-r^2e^{-i\psi}}+\frac{e^{-i\psi}}{r^2e^{i\phi-\frac{2}{3}\pi i}-e^{-i\psi}}+ \\
&+\frac{r^2e^{-i\psi}}{r^2e^{-i\phi+\frac{2}{3}\pi i}-r^2e^{-i\psi}}-\frac{e^{-i\psi}}{r^2e^{-i\phi+\frac{2}{3}\pi i}-e^{-i\psi}}-\frac{r^2e^{-i\psi}}{r^2e^{i\phi+\frac{2}{3}\pi i}-r^2e^{-i\psi}}+\frac{e^{-i\psi}}{r^2e^{i\phi+\frac{2}{3}\pi i}-e^{-i\psi}}+ \\
&+\frac{r^2e^{-i\phi}}{r^2e^{-i\psi}-r^2e^{-i\phi}}-\frac{e^{-i\phi}}{r^2e^{-i\psi}-e^{-i\phi}}-\frac{r^2e^{i\phi}}{r^2e^{-i\psi}-r^2e^{i\phi}}+\frac{e^{i\phi}}{r^2e^{-i\psi}-e^{i\phi}}+ \\
&+\frac{r^2e^{-i\phi-\frac{2}{3}\pi i}}{r^2e^{-i\psi}-r^2e^{-i\phi-\frac{2}{3}\pi i}}-\frac{e^{-i\phi-\frac{2}{3}\pi i}}{r^2e^{-i\psi}-e^{-i\phi-\frac{2}{3}\pi i}}-\frac{r^2e^{i\phi-\frac{2}{3}\pi i}}{r^2e^{-i\psi}-r^2e^{i\phi-\frac{2}{3}\pi i}}+\frac{e^{i\phi-\frac{2}{3}\pi i}}{r^2e^{-i\psi}-e^{i\phi-\frac{2}{3}\pi i}}+ \\
&+\frac{r^2e^{-i\phi+\frac{2}{3}\pi i}}{r^2e^{-i\psi}-r^2e^{-i\phi+\frac{2}{3}\pi i}}-\frac{e^{-i\phi+\frac{2}{3}\pi i}}{r^2e^{-i\psi}-e^{-i\phi+\frac{2}{3}\pi i}}-\frac{r^2e^{i\phi+\frac{2}{3}\pi i}}{r^2e^{-i\psi}-r^2e^{i\phi+\frac{2}{3}\pi i}}+\frac{e^{i\phi+\frac{2}{3}\pi i}}{r^2e^{-i\psi}-e^{i\phi+\frac{2}{3}\pi i}} \\
=\ & \frac{e^{-i\psi}}{e^{-i\phi}-e^{-i\psi}}-\frac{e^{-i\psi}}{r^2e^{-i\phi}-e^{-i\psi}}-\frac{e^{-i\psi}}{e^{i\phi}-e^{-i\psi}}+\frac{e^{-i\psi}}{r^2e^{i\phi}-e^{-i\psi}}+ \\
&+\frac{e^{-i\psi}}{e^{-i\phi-\frac{2}{3}\pi i}-e^{-i\psi}}-\frac{e^{-i\psi}}{r^2e^{-i\phi-\frac{2}{3}\pi i}-e^{-i\psi}}-\frac{e^{-i\psi}}{e^{i\phi-\frac{2}{3}\pi i}-e^{-i\psi}}+\frac{e^{-i\psi}}{r^2e^{i\phi-\frac{2}{3}\pi i}-e^{-i\psi}}+ \\
&+\frac{e^{-i\psi}}{e^{-i\phi+\frac{2}{3}\pi i}-e^{-i\psi}}-\frac{e^{-i\psi}}{r^2e^{-i\phi+\frac{2}{3}\pi i}-e^{-i\psi}}-\frac{e^{-i\psi}}{e^{i\phi+\frac{2}{3}\pi i}-e^{-i\psi}}+\frac{e^{-i\psi}}{r^2e^{i\phi+\frac{2}{3}\pi i}-e^{-i\psi}}+ \\
&+\frac{e^{-i\phi}}{e^{-i\psi}-e^{-i\phi}}-\frac{e^{-i\phi}}{r^2e^{-i\psi}-e^{-i\phi}}-\frac{e^{i\phi}}{e^{-i\psi}-e^{i\phi}}+\frac{e^{i\phi}}{r^2e^{-i\psi}-e^{i\phi}}+ \\
&+\frac{e^{-i\phi-\frac{2}{3}\pi i}}{e^{-i\psi}-e^{-i\phi-\frac{2}{3}\pi i}}-\frac{e^{-i\phi-\frac{2}{3}\pi i}}{r^2e^{-i\psi}-e^{-i\phi-\frac{2}{3}\pi i}}-\frac{e^{i\phi-\frac{2}{3}\pi i}}{e^{-i\psi}-e^{i\phi-\frac{2}{3}\pi i}}+\frac{e^{i\phi-\frac{2}{3}\pi i}}{r^2e^{-i\psi}-e^{i\phi-\frac{2}{3}\pi i}}+ \\
&+\frac{e^{-i\phi+\frac{2}{3}\pi i}}{e^{-i\psi}-e^{-i\phi+\frac{2}{3}\pi i}}-\frac{e^{-i\phi+\frac{2}{3}\pi i}}{r^2e^{-i\psi}-e^{-i\phi+\frac{2}{3}\pi i}}-\frac{e^{i\phi+\frac{2}{3}\pi i}}{e^{-i\psi}-e^{i\phi+\frac{2}{3}\pi i}}+\frac{e^{i\phi+\frac{2}{3}\pi i}}{r^2e^{-i\psi}-e^{i\phi+\frac{2}{3}\pi i}}
\end{aligned}
$$

$$\begin{aligned}
= \quad & \frac{e^{-i\psi}}{e^{-i\phi}-e^{-i\psi}} - \frac{e^{-i\psi}}{e^{i\phi}-e^{-i\psi}} + \frac{e^{-i\phi}}{e^{-i\psi}-e^{-i\phi}} - \frac{e^{i\phi}}{e^{-i\psi}-e^{i\phi}} + \\
& + \frac{e^{-i\psi}}{e^{-i\phi-\frac{2}{3}\pi i}-e^{-i\psi}} - \frac{e^{-i\psi}}{e^{i\phi-\frac{2}{3}\pi i}-e^{-i\psi}} + \frac{e^{-i\psi}}{e^{-i\phi+\frac{2}{3}\pi i}-e^{-i\psi}} - \frac{e^{-i\psi}}{e^{i\phi+\frac{2}{3}\pi i}-e^{-i\psi}} + \\
& + \frac{e^{-i\phi-\frac{2}{3}\pi i}}{e^{-i\psi}-e^{-i\phi-\frac{2}{3}\pi i}} - \frac{e^{i\phi-\frac{2}{3}\pi i}}{e^{-i\psi}-e^{i\phi-\frac{2}{3}\pi i}} + \frac{e^{-i\phi+\frac{2}{3}\pi i}}{e^{-i\psi}-e^{-i\phi+\frac{2}{3}\pi i}} - \frac{e^{i\phi+\frac{2}{3}\pi i}}{e^{-i\psi}-e^{i\phi+\frac{2}{3}\pi i}} + \\
& - \frac{e^{i\phi-i\psi}}{r^2-e^{i\phi-i\psi}} + \frac{e^{-i\phi-i\psi}}{r^2-e^{-i\phi-i\psi}} - \frac{e^{i\phi-i\psi+\frac{2}{3}\pi i}}{r^2-e^{i\phi-i\psi+\frac{2}{3}\pi i}} + \frac{e^{-i\phi-i\psi+\frac{2}{3}\pi i}}{r^2-e^{-i\phi-i\psi+\frac{2}{3}\pi i}} + \\
& - \frac{e^{i\phi-i\psi-\frac{2}{3}\pi i}}{r^2-e^{i\phi-i\psi-\frac{2}{3}\pi i}} + \frac{e^{-i\phi-i\psi-\frac{2}{3}\pi i}}{r^2-e^{-i\phi-i\psi-\frac{2}{3}\pi i}} - \frac{e^{-i\phi+i\psi}}{r^2-e^{-i\phi+i\psi}} + \frac{e^{i\phi+i\psi}}{r^2-e^{i\phi+i\psi}} + \\
& - \frac{e^{-i\phi+i\psi-\frac{2}{3}\pi i}}{r^2-e^{-i\phi+i\psi-\frac{2}{3}\pi i}} + \frac{e^{i\phi+i\psi-\frac{2}{3}\pi i}}{r^2-e^{i\phi+i\psi-\frac{2}{3}\pi i}} - \frac{e^{-i\phi+i\psi+\frac{2}{3}\pi i}}{r^2-e^{-i\phi+i\psi+\frac{2}{3}\pi i}} + \frac{e^{i\phi+i\psi+\frac{2}{3}\pi i}}{r^2-e^{i\phi+i\psi+\frac{2}{3}\pi i}} \\
= \quad & - \frac{e^{i\phi-i\psi}}{r^2-e^{i\phi-i\psi}} + \frac{e^{-i\phi-i\psi}}{r^2-e^{-i\phi-i\psi}} - \frac{e^{i\phi-i\psi+\frac{2}{3}\pi i}}{r^2-e^{i\phi-i\psi+\frac{2}{3}\pi i}} + \frac{e^{-i\phi-i\psi+\frac{2}{3}\pi i}}{r^2-e^{-i\phi-i\psi+\frac{2}{3}\pi i}} + \\
& - \frac{e^{i\phi-i\psi-\frac{2}{3}\pi i}}{r^2-e^{i\phi-i\psi-\frac{2}{3}\pi i}} + \frac{e^{-i\phi-i\psi-\frac{2}{3}\pi i}}{r^2-e^{-i\phi-i\psi-\frac{2}{3}\pi i}} - \frac{e^{-i\phi+i\psi}}{r^2-e^{-i\phi+i\psi}} + \frac{e^{i\phi+i\psi}}{r^2-e^{i\phi+i\psi}} + \\
& - \frac{e^{-i\phi+i\psi-\frac{2}{3}\pi i}}{r^2-e^{-i\phi+i\psi-\frac{2}{3}\pi i}} + \frac{e^{i\phi+i\psi-\frac{2}{3}\pi i}}{r^2-e^{i\phi+i\psi-\frac{2}{3}\pi i}} - \frac{e^{-i\phi+i\psi+\frac{2}{3}\pi i}}{r^2-e^{-i\phi+i\psi+\frac{2}{3}\pi i}} + \frac{e^{i\phi+i\psi+\frac{2}{3}\pi i}}{r^2-e^{i\phi+i\psi+\frac{2}{3}\pi i}}. \qquad (67)
\end{aligned}$$

6.4. Integration in the radial direction

Remember that

$$\begin{aligned}
& \int_{z\in S} \lim_{\substack{x\to(\cos\phi,\sin\phi)\\ x\in S}} \frac{G_S(x,z)}{1-|x|^2} \lim_{\substack{y\to(\cos\psi,\sin\psi)\\ y\in S}} \frac{G_S(z,y)}{1-|y|^2} dz = \\
= \quad & \frac{1}{\pi^2}\frac{\pi}{16}\int_{r=0}^{1}\left(\sum_{w_i\in P\cup\{r^2e^{-i\psi}\}} \operatorname{Res}\left\{F_\psi(w)\right\}_{w=w_i} - \sum_{w_i\in P\cup\{r^2e^{i\psi}\}} \operatorname{Res}\left\{F_{-\psi}(w)\right\}_{w=w_i}\right) rdr \qquad (68)
\end{aligned}$$

Let us first compute the left half:

$$\begin{aligned}
& \frac{1}{\pi^2}\frac{\pi}{16}\int_{r=0}^{1}\left(\sum_{w_i\in P\cup\{r^2e^{-i\psi}\}} \operatorname{Res}\left\{F_\psi(w)\right\}_{w=w_i}\right) rdr = \\
= \quad & \frac{1}{16\pi}\int_{r=0}^{1}\left(- \frac{e^{i\phi-i\psi}}{r^2-e^{i\phi-i\psi}} + \frac{e^{-i\phi-i\psi}}{r^2-e^{-i\phi-i\psi}} - \frac{e^{i\phi-i\psi+\frac{2}{3}\pi i}}{r^2-e^{i\phi-i\psi+\frac{2}{3}\pi i}} + \frac{e^{-i\phi-i\psi+\frac{2}{3}\pi i}}{r^2-e^{-i\phi-i\psi+\frac{2}{3}\pi i}} + \right.\\
& - \frac{e^{i\phi-i\psi-\frac{2}{3}\pi i}}{r^2-e^{i\phi-i\psi-\frac{2}{3}\pi i}} + \frac{e^{-i\phi-i\psi-\frac{2}{3}\pi i}}{r^2-e^{-i\phi-i\psi-\frac{2}{3}\pi i}} - \frac{e^{-i\phi+i\psi}}{r^2-e^{-i\phi+i\psi}} + \frac{e^{i\phi+i\psi}}{r^2-e^{i\phi+i\psi}} + \\
& \left. - \frac{e^{-i\phi+i\psi-\frac{2}{3}\pi i}}{r^2-e^{-i\phi+i\psi-\frac{2}{3}\pi i}} + \frac{e^{i\phi+i\psi-\frac{2}{3}\pi i}}{r^2-e^{i\phi+i\psi-\frac{2}{3}\pi i}} - \frac{e^{-i\phi+i\psi+\frac{2}{3}\pi i}}{r^2-e^{-i\phi+i\psi+\frac{2}{3}\pi i}} + \frac{e^{i\phi+i\psi+\frac{2}{3}\pi i}}{r^2-e^{i\phi+i\psi+\frac{2}{3}\pi i}}\right) rdr
\end{aligned}$$

$$\begin{aligned}
= \quad & \frac{1}{32\pi}\Bigg[-e^{i\phi-i\psi}\mathrm{Ln}\left(1-r^2e^{-i\phi+i\psi}\right)+e^{-i\phi-i\psi}\mathrm{Ln}\left(1-r^2e^{i\phi+i\psi}\right)+\\
& -e^{i\phi-i\psi+\frac{2}{3}\pi i}\mathrm{Ln}\left(1-r^2e^{-i\phi+i\psi-\frac{2}{3}\pi i}\right)+e^{-i\phi-i\psi+\frac{2}{3}\pi i}\mathrm{Ln}\left(1-r^2e^{i\phi+i\psi-\frac{2}{3}\pi i}\right)+\\
& -e^{i\phi-i\psi-\frac{2}{3}\pi i}\mathrm{Ln}\left(1-r^2e^{-i\phi+i\psi+\frac{2}{3}\pi i}\right)+e^{-i\phi-i\psi-\frac{2}{3}\pi i}\mathrm{Ln}\left(1-r^2e^{i\phi+i\psi+\frac{2}{3}\pi i}\right)+\\
& -e^{-i\phi+i\psi}\mathrm{Ln}\left(1-r^2e^{i\phi-i\psi}\right)+e^{i\phi+i\psi}\mathrm{Ln}\left(1-r^2e^{-i\phi-i\psi}\right)+\\
& -e^{-i\phi+i\psi-\frac{2}{3}\pi i}\mathrm{Ln}\left(1-r^2e^{i\phi-i\psi+\frac{2}{3}\pi i}\right)+e^{i\phi+i\psi-\frac{2}{3}\pi i}\mathrm{Ln}\left(1-r^2e^{-i\phi-i\psi+\frac{2}{3}\pi i}\right)+\\
& -e^{-i\phi+i\psi+\frac{2}{3}\pi i}\mathrm{Ln}\left(1-r^2e^{i\phi-i\psi-\frac{2}{3}\pi i}\right)+e^{i\phi+i\psi+\frac{2}{3}\pi i}\mathrm{Ln}\left(1-r^2e^{-i\phi-i\psi-\frac{2}{3}\pi i}\right)\Bigg]_{r=0}^{1}\\
= \quad & \frac{1}{32\pi}\Bigg(-e^{i\phi-i\psi}\mathrm{Ln}\left(1-e^{-i\phi+i\psi}\right)+e^{-i\phi-i\psi}\mathrm{Ln}\left(1-e^{i\phi+i\psi}\right)+\\
& -e^{i\phi-i\psi+\frac{2}{3}\pi i}\mathrm{Ln}\left(1-e^{-i\phi+i\psi-\frac{2}{3}\pi i}\right)+e^{-i\phi-i\psi+\frac{2}{3}\pi i}\mathrm{Ln}\left(1-e^{i\phi+i\psi-\frac{2}{3}\pi i}\right)+\\
& -e^{i\phi-i\psi-\frac{2}{3}\pi i}\mathrm{Ln}\left(1-e^{-i\phi+i\psi+\frac{2}{3}\pi i}\right)+e^{-i\phi-i\psi-\frac{2}{3}\pi i}\mathrm{Ln}\left(1-e^{i\phi+i\psi+\frac{2}{3}\pi i}\right)+\\
& -e^{-i\phi+i\psi}\mathrm{Ln}\left(1-e^{i\phi-i\psi}\right)+e^{i\phi+i\psi}\mathrm{Ln}\left(1-e^{-i\phi-i\psi}\right)+\\
& -e^{-i\phi+i\psi-\frac{2}{3}\pi i}\mathrm{Ln}\left(1-e^{i\phi-i\psi+\frac{2}{3}\pi i}\right)+e^{i\phi+i\psi-\frac{2}{3}\pi i}\mathrm{Ln}\left(1-e^{-i\phi-i\psi+\frac{2}{3}\pi i}\right)+\\
& -e^{-i\phi+i\psi+\frac{2}{3}\pi i}\mathrm{Ln}\left(1-e^{i\phi-i\psi-\frac{2}{3}\pi i}\right)+e^{i\phi+i\psi+\frac{2}{3}\pi i}\mathrm{Ln}\left(1-e^{-i\phi-i\psi-\frac{2}{3}\pi i}\right)\Bigg)\\
= \quad & \frac{1}{32\pi}\Bigg(-e^{i\phi-i\psi}\mathrm{Ln}\left(1-e^{-i\phi+i\psi}\right)-e^{-i\phi+i\psi}\mathrm{Ln}\left(1-e^{i\phi-i\psi}\right)+\\
& -e^{i\phi-i\psi+\frac{2}{3}\pi i}\mathrm{Ln}\left(1-e^{-i\phi+i\psi-\frac{2}{3}\pi i}\right)-e^{-i\phi+i\psi-\frac{2}{3}\pi i}\mathrm{Ln}\left(1-e^{i\phi-i\psi+\frac{2}{3}\pi i}\right)+\\
& -e^{-i\phi+i\psi+\frac{2}{3}\pi i}\mathrm{Ln}\left(1-e^{i\phi-i\psi-\frac{2}{3}\pi i}\right)-e^{i\phi-i\psi-\frac{2}{3}\pi i}\mathrm{Ln}\left(1-e^{-i\phi+i\psi+\frac{2}{3}\pi i}\right)+\\
& +e^{i\phi+i\psi}\mathrm{Ln}\left(1-e^{-i\phi-i\psi}\right)+e^{-i\phi-i\psi}\mathrm{Ln}\left(1-e^{i\phi+i\psi}\right)+\\
& +e^{-i\phi-i\psi+\frac{2}{3}\pi i}\mathrm{Ln}\left(1-e^{i\phi+i\psi-\frac{2}{3}\pi i}\right)+e^{i\phi+i\psi-\frac{2}{3}\pi i}\mathrm{Ln}\left(1-e^{-i\phi-i\psi+\frac{2}{3}\pi i}\right)+\\
& +e^{i\phi+i\psi+\frac{2}{3}\pi i}\mathrm{Ln}\left(1-e^{-i\phi-i\psi-\frac{2}{3}\pi i}\right)+e^{-i\phi-i\psi-\frac{2}{3}\pi i}\mathrm{Ln}\left(1-e^{i\phi+i\psi+\frac{2}{3}\pi i}\right)\Bigg). \qquad (69)
\end{aligned}$$

Using that for $\alpha \in (0, 2\pi)$

$$\begin{aligned}
& e^{i\alpha}\mathrm{Ln}\left(1-e^{-i\alpha}\right)+e^{-i\alpha}\mathrm{Ln}\left(1-e^{i\alpha}\right)= \\
= \;& \left(e^{i\alpha}+e^{-i\alpha}\right)\ln\left|1-e^{i\alpha}\right|+i\left(e^{-i\alpha}-e^{i\alpha}\right)\mathrm{Arg}\left(1-e^{i\alpha}\right) \\
= \;& 2\frac{e^{i\alpha}+e^{-i\alpha}}{2}\ln\sqrt{1-e^{i\alpha}-e^{-i\alpha}+e^{i\alpha}e^{-i\alpha}}+2\frac{e^{i\alpha}-e^{-i\alpha}}{2i}\mathrm{Arg}\left(1-e^{i\alpha}\right) \\
= \;& 2\left(\cos\alpha\right)\ln\sqrt{2-2\cos\alpha}+2\left(\sin\alpha\right)\frac{\alpha-\pi}{2} \\
= \;& \cos\alpha\,\ln\left(2-2\cos\alpha\right)+\left(\alpha-\pi\right)\,\sin\alpha
\end{aligned}$$

and assuming $|\psi| < \phi < \frac{1}{3}\pi$ we continue (69) by

$$\begin{aligned}
= \;& \frac{1}{32\pi}\Bigg(-\Big(\cos\left(\phi-\psi\right)\,\ln\left(2-2\cos\left(\phi-\psi\right)\right)+\left(\phi-\psi-\pi\right)\,\sin\left(\phi-\psi\right)\Big)+ \\
& -\Big(\cos\left(\phi-\psi+\tfrac{2}{3}\pi\right)\,\ln\left(2-2\cos\left(\phi-\psi+\tfrac{2}{3}\pi\right)\right)+\left(\phi-\psi+\tfrac{2}{3}\pi-\pi\right)\,\sin\left(\phi-\psi+\tfrac{2}{3}\pi\right)\Big)+ \\
& -\Big(\cos\left(-\phi+\psi+\tfrac{2}{3}\pi\right)\,\ln\left(2-2\cos\left(-\phi+\psi+\tfrac{2}{3}\pi\right)\right)+\left(-\phi+\psi+\tfrac{2}{3}\pi-\pi\right)\,\sin\left(-\phi+\psi+\tfrac{2}{3}\pi\right)\Big)+ \\
& +\Big(\cos\left(\phi+\psi\right)\,\ln\left(2-2\cos\left(\phi+\psi\right)\right)+\left(\phi+\psi-\pi\right)\,\sin\left(\phi+\psi\right)\Big)+ \\
& +\Big(\cos\left(-\phi-\psi+\tfrac{2}{3}\pi\right)\,\ln\left(2-2\cos\left(-\phi-\psi+\tfrac{2}{3}\pi\right)\right)+\left(-\phi-\psi+\tfrac{2}{3}\pi-\pi\right)\,\sin\left(-\phi-\psi+\tfrac{2}{3}\pi\right)\Big)+ \\
& +\Big(\cos\left(\phi+\psi+\tfrac{2}{3}\pi\right)\,\ln\left(2-2\cos\left(\phi+\psi+\tfrac{2}{3}\pi\right)\right)+\left(\phi+\psi+\tfrac{2}{3}\pi-\pi\right)\,\sin\left(\phi+\psi+\tfrac{2}{3}\pi\right)\Big)\Bigg). \qquad (70)
\end{aligned}$$

Since $\cos(\alpha) + \cos\left(\alpha + \frac{2}{3}\pi\right) + \cos\left(\alpha - \frac{2}{3}\pi\right) = 0$ and a similar identity with sin one proceeds by

$$\begin{aligned}
= \;& \frac{1}{32\pi}\Bigg(-\Big(\cos\left(\phi-\psi\right)\,\ln\left(1-\cos\left(\phi-\psi\right)\right)-\pi\,\sin\left(\phi-\psi\right)\Big)+ \\
& -\Big(\cos\left(\phi-\psi+\tfrac{2}{3}\pi\right)\,\ln\left(1-\cos\left(\phi-\psi+\tfrac{2}{3}\pi\right)\right)-\frac{1}{3}\pi\,\sin\left(\phi-\psi+\tfrac{2}{3}\pi\right)\Big)+ \\
& -\Big(\cos\left(\phi-\psi-\tfrac{2}{3}\pi\right)\,\ln\left(1-\cos\left(\phi-\psi-\tfrac{2}{3}\pi\right)\right)+\frac{1}{3}\pi\,\sin\left(\phi-\psi-\tfrac{2}{3}\pi\right)\Big)+ \\
& +\Big(\cos\left(\phi+\psi\right)\,\ln\left(1-\cos\left(\phi+\psi\right)\right)-\pi\,\sin\left(\phi+\psi\right)\Big)+ \\
& +\Big(\cos\left(\phi+\psi-\tfrac{2}{3}\pi\right)\,\ln\left(1-\cos\left(\phi+\psi-\tfrac{2}{3}\pi\right)\right)+\frac{1}{3}\pi\,\sin\left(\phi+\psi-\tfrac{2}{3}\pi\right)\Big)+ \\
& +\Big(\cos\left(\phi+\psi+\tfrac{2}{3}\pi\right)\,\ln\left(1-\cos\left(\phi+\psi+\tfrac{2}{3}\pi\right)\right)-\frac{1}{3}\pi\,\sin\left(\phi+\psi+\tfrac{2}{3}\pi\right)\Big)\Bigg) \qquad (71)
\end{aligned}$$

$$\begin{aligned} = \quad & \frac{1}{32\pi}\Bigg(-\cos(\phi-\psi)\,\ln(1-\cos(\phi-\psi)) + \pi\,\sin(\phi-\psi) + \\ & -\cos\left(\phi-\psi+\tfrac{2}{3}\pi\right)\,\ln\left(1-\cos\left(\phi-\psi+\tfrac{2}{3}\pi\right)\right) + \frac{1}{3}\pi\,\sin\left(\phi-\psi+\tfrac{2}{3}\pi\right) + \\ & -\cos\left(\phi-\psi-\tfrac{2}{3}\pi\right)\,\ln\left(1-\cos\left(\phi-\psi-\tfrac{2}{3}\pi\right)\right) - \frac{1}{3}\pi\,\sin\left(\phi-\psi-\tfrac{2}{3}\pi\right) + \\ & +\cos(\phi+\psi)\,\ln(1-\cos(\phi+\psi)) - \pi\,\sin(\phi+\psi) + \\ & +\cos\left(\phi+\psi-\tfrac{2}{3}\pi\right)\,\ln\left(1-\cos\left(\phi+\psi-\tfrac{2}{3}\pi\right)\right) + \frac{1}{3}\pi\,\sin\left(\phi+\psi-\tfrac{2}{3}\pi\right) + \\ & +\cos\left(\phi+\psi+\tfrac{2}{3}\pi\right)\,\ln\left(1-\cos\left(\phi+\psi+\tfrac{2}{3}\pi\right)\right) - \frac{1}{3}\pi\,\sin\left(\phi+\psi+\tfrac{2}{3}\pi\right)\Bigg) \end{aligned} \tag{72}$$

$$\begin{aligned} = \quad & \frac{1}{32}\Bigg(\sin(\phi-\psi) - \sin(\phi+\psi) \\ & + \frac{1}{3}\sin\left(\phi-\psi+\tfrac{2}{3}\pi\right) - \frac{1}{3}\sin\left(\phi+\psi+\tfrac{2}{3}\pi\right) + \\ & - \frac{1}{3}\sin\left(\phi-\psi-\tfrac{2}{3}\pi\right) + \frac{1}{3}\sin\left(\phi+\psi-\tfrac{2}{3}\pi\right)\Bigg) + \\ & + \frac{1}{32\pi}\Bigg(-\cos(\phi-\psi)\,\ln(1-\cos(\phi-\psi)) + \\ & -\cos\left(\phi-\psi+\tfrac{2}{3}\pi\right)\,\ln\left(1-\cos\left(\phi-\psi+\tfrac{2}{3}\pi\right)\right) + \\ & -\cos\left(\phi-\psi-\tfrac{2}{3}\pi\right)\,\ln\left(1-\cos\left(\phi-\psi-\tfrac{2}{3}\pi\right)\right) + \\ & +\cos(\phi+\psi)\,\ln(1-\cos(\phi+\psi)) + \\ & +\cos\left(\phi+\psi-\tfrac{2}{3}\pi\right)\,\ln\left(1-\cos\left(\phi+\psi-\tfrac{2}{3}\pi\right)\right) + \\ & +\cos\left(\phi+\psi+\tfrac{2}{3}\pi\right)\,\ln\left(1-\cos\left(\phi+\psi+\tfrac{2}{3}\pi\right)\right)\Bigg) \end{aligned} \tag{73}$$

$$\begin{aligned} = \quad & \frac{1}{32}\sin\psi\left(-2\cos\phi - \tfrac{2}{3}\cos\left(\phi+\tfrac{2}{3}\pi\right) + \tfrac{2}{3}\cos\left(\phi-\tfrac{2}{3}\pi\right)\right) + \\ & + \frac{1}{32\pi}\Bigg(-\cos(\phi-\psi)\,\ln(1-\cos(\phi-\psi)) + \\ & -\cos\left(\phi-\psi+\tfrac{2}{3}\pi\right)\,\ln\left(1-\cos\left(\phi-\psi+\tfrac{2}{3}\pi\right)\right) + \\ & -\cos\left(\phi-\psi-\tfrac{2}{3}\pi\right)\,\ln\left(1-\cos\left(\phi-\psi-\tfrac{2}{3}\pi\right)\right) + \\ & +\cos(\phi+\psi)\,\ln(1-\cos(\phi+\psi)) + \\ & +\cos\left(\phi+\psi-\tfrac{2}{3}\pi\right)\,\ln\left(1-\cos\left(\phi+\psi-\tfrac{2}{3}\pi\right)\right) + \\ & +\cos\left(\phi+\psi+\tfrac{2}{3}\pi\right)\,\ln\left(1-\cos\left(\phi+\psi+\tfrac{2}{3}\pi\right)\right)\Bigg) \end{aligned} \tag{74}$$

$$\begin{aligned} = & \frac{1}{16}\sin\psi\left(-\cos\phi+\frac{1}{3}\sqrt{3}\sin\phi\right)+ \\ & +\frac{1}{32\pi}\Bigg(-\cos(\phi-\psi)\,\ln(1-\cos(\phi-\psi))+ \\ & -\ \cos\left(\phi-\psi+\tfrac{2}{3}\pi\right)\,\ln\left(1-\cos\left(\phi-\psi+\tfrac{2}{3}\pi\right)\right)+ \\ & -\ \cos\left(\phi-\psi-\tfrac{2}{3}\pi\right)\,\ln\left(1-\cos\left(\phi-\psi-\tfrac{2}{3}\pi\right)\right)+ \\ & +\ \cos(\phi+\psi)\,\ln(1-\cos(\phi+\psi))\ + \\ & +\ \cos\left(\phi+\psi-\tfrac{2}{3}\pi\right)\,\ln\left(1-\cos\left(\phi+\psi-\tfrac{2}{3}\pi\right)\right)+ \\ & +\ \cos\left(\phi+\psi+\tfrac{2}{3}\pi\right)\,\ln\left(1-\cos\left(\phi+\psi+\tfrac{2}{3}\pi\right)\right)\Bigg). \end{aligned} \tag{75}$$

Since we still have to subtract in (68) the expression for $-\psi$ and since (75) is odd in ψ, (75) doubles.

6.5. Conclusion of Case 3)

The enumerator turns out to be (for $\psi < \phi$)

$$\begin{aligned} & \int_{z\in S}\lim_{x\to(\cos\phi,\sin\phi)}\frac{G_S(x,z)}{1-|x|^2}\lim_{y\to(\cos\psi,\sin\psi)}\frac{G_S(z,y)}{1-|y|^2}dz\ = \\ = & \ \frac{1}{8}\ \sin\psi\left(-\cos\phi+\tfrac{1}{3}\sqrt{3}\sin\phi\right)\ + \\ & -\frac{1}{16\pi}\sum_{k=0}^{2}\cos\left(\phi-\psi+\tfrac{2}{3}k\pi\right)\,\ln\left(1-\cos\left(\phi-\psi+\tfrac{2}{3}k\pi\right)\right)\ + \\ & +\frac{1}{16\pi}\sum_{k=0}^{2}\cos\left(\phi+\psi+\tfrac{2}{3}k\pi\right)\,\ln\left(1-\cos\left(\phi+\psi+\tfrac{2}{3}k\pi\right)\right)\ . \end{aligned} \tag{76}$$

For $\psi > \phi$, by symmetry $T_{22}(\phi,\psi) := T_{22}(\psi,\phi)$. Again we find

$$\sup_{\phi,\psi\in\left(0,\frac{1}{3}\pi\right)} T_{22}(\phi,\psi) = \lim_{\phi\uparrow\frac{1}{3}\pi,\ \psi\downarrow 0} T_{22}(\phi,\psi) = \frac{4}{243}\pi\sqrt{3}-\frac{5}{27}+\frac{4}{81}\ln 2.$$

See Figure 3 on p. 60.

7. The case that $x \in \Gamma_1,\ y \in \Gamma_3$

We set $y = (\rho\cos\psi, \rho\sin\psi)$ and we are interested in

$$T_{13}(x_1,\rho)\ =\ \frac{\displaystyle\int_S \lim_{x_2\downarrow 0}\frac{G_S(x,z)}{x_2}\ \lim_{\psi\uparrow\frac{1}{3}\pi}\frac{G_S(z,y)}{\rho\sin\left(\frac{1}{3}\pi-\psi\right)}\ dz}{\displaystyle\lim_{x_2\downarrow 0,\ \psi\uparrow\frac{1}{3}\pi}\frac{G_S(x,y)}{x_2\,\rho\sin\left(\frac{1}{3}\pi-\psi\right)}}\ . \tag{77}$$

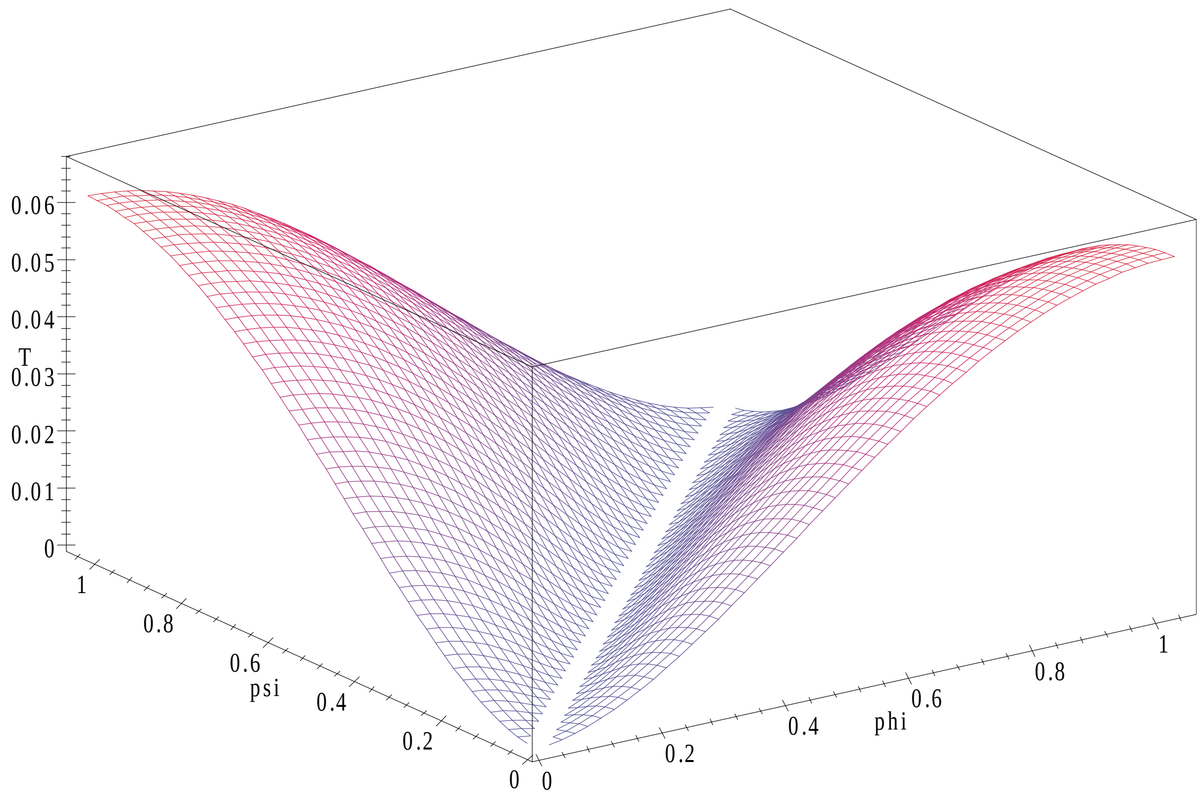


Figure 3: x and y both on Γ_2: $T_{22}(\phi, \psi)$

7.1. Limit of the Green function

First we look at the factors in the enumerator: As before we have with $z = (r\cos\theta, r\sin\theta)$ that

$$\begin{aligned}
&\lim_{x_2 \downarrow 0} \frac{G_S(x,z)}{x_2} = \\
= \;& \frac{1}{\pi}\left(\frac{r\sin\theta}{x_1^2 - 2x_1 r\cos\theta + r^2} - \frac{r\sin\theta}{1 - 2x_1 r\cos\theta + x_1^2 r^2}\right) + \\
&+ \frac{1}{\pi}\left(\frac{r\sin\left(\theta - \frac{2}{3}\pi\right)}{x_1^2 - 2x_1 r\cos\left(\theta - \frac{2}{3}\pi\right) + r^2} - \frac{r\sin\left(\theta - \frac{2}{3}\pi\right)}{1 - 2x_1 r\cos\left(\theta - \frac{2}{3}\pi\right) + x_1^2 r^2}\right) + \\
&+ \frac{1}{\pi}\left(\frac{r\sin\left(\theta + \frac{2}{3}\pi\right)}{x_1^2 - 2x_1 r\cos\left(\theta + \frac{2}{3}\pi\right) + r^2} - \frac{r\sin\left(\theta + \frac{2}{3}\pi\right)}{1 - 2x_1 r\cos\left(\theta + \frac{2}{3}\pi\right) + x_1^2 r^2}\right).
\end{aligned}$$

By symmetry we find

$$\begin{aligned}
&\lim_{\substack{\psi \uparrow \frac{1}{3}\pi \\ y = (\rho\cos\psi, \rho\sin\psi)}} \frac{G_S(z,y)}{\rho\sin\left(\frac{1}{3}\pi - \psi\right)} = \\
= \;& \frac{1}{\pi}\left(\frac{r\sin\left(\frac{1}{3}\pi - \theta\right)}{\rho^2 - 2\rho\, r\cos\left(\frac{1}{3}\pi - \theta\right) + r^2} - \frac{r\sin\left(\frac{1}{3}\pi - \theta\right)}{1 - 2\rho\, r\cos\left(\frac{1}{3}\pi - \theta\right) + \rho^2 r^2}\right) + \\
&+ \frac{1}{\pi}\left(\frac{r\sin\left(-\frac{1}{3}\pi - \theta\right)}{\rho^2 - 2\rho\, r\cos\left(-\frac{1}{3}\pi - \theta\right) + r^2} - \frac{r\sin\left(-\frac{1}{3}\pi - \theta\right)}{1 - 2\rho\, r\cos\left(-\frac{1}{3}\pi - \theta\right) + \rho^2 r^2}\right) + \\
&+ \frac{1}{\pi}\left(\frac{r\sin(\pi - \theta)}{\rho^2 - 2\rho\, r\cos(\pi - \theta) + r^2} - \frac{r\sin(\pi - \theta)}{1 - 2\rho\, r\cos(\pi - \theta) + \rho^2 r^2}\right).
\end{aligned}$$

$$\begin{aligned}
= \quad & \frac{1}{\pi}\left(\frac{r\sin\left(\theta-\frac{1}{3}\pi\right)}{1-2\rho\, r\cos\left(\theta-\frac{1}{3}\pi\right)+\rho^2r^2}-\frac{r\sin\left(\theta-\frac{1}{3}\pi\right)}{\rho^2-2\rho\, r\cos\left(\theta-\frac{1}{3}\pi\right)+r^2}\right)+\\
& +\frac{1}{\pi}\left(\frac{r\sin\left(\theta+\frac{1}{3}\pi\right)}{1-2\rho\, r\cos\left(\theta+\frac{1}{3}\pi\right)+\rho^2r^2}-\frac{r\sin\left(\theta+\frac{1}{3}\pi\right)}{\rho^2-2\rho\, r\cos\left(\theta+\frac{1}{3}\pi\right)+r^2}\right)+\\
& +\frac{1}{\pi}\left(\frac{r\sin\left(\theta-\pi\right)}{1-2\rho\, r\cos\left(\theta-\pi\right)+\rho^2r^2}-\frac{r\sin\left(\theta-\pi\right)}{\rho^2-2\rho\, r\cos\left(\theta-\pi\right)+r^2}\right).
\end{aligned}$$

The denominator becomes

$$\lim_{\substack{x_2\downarrow 0 \text{ and } \psi\uparrow\frac{1}{3}\pi \\ y=(\rho\cos\psi,\rho\sin\psi)}}\frac{G_S\left(x,y\right)}{x_2\,\rho\sin\left(\frac{1}{3}\pi-\psi\right)}=$$

$$\begin{aligned}
= \quad & \lim_{\theta\downarrow 0}\frac{1}{\pi x_1\sin\theta}\left(\frac{x_1\sin\left(\frac{1}{3}\pi-\theta\right)}{\rho^2-2\rho\, x_1\cos\left(\frac{1}{3}\pi-\theta\right)+x_1^2}-\frac{x_1\sin\left(\frac{1}{3}\pi-\theta\right)}{1-2\rho\, x_1\cos\left(\frac{1}{3}\pi-\theta\right)+\rho^2x_1^2}+\right.\\
& +\frac{x_1\sin\left(-\frac{1}{3}\pi-\theta\right)}{\rho^2-2\rho\, x_1\cos\left(-\frac{1}{3}\pi-\theta\right)+x_1^2}-\frac{x_1\sin\left(-\frac{1}{3}\pi-\theta\right)}{1-2\rho\, x_1\cos\left(-\frac{1}{3}\pi-\theta\right)+\rho^2x_1^2}+\\
& \left.+\frac{x_1\sin\left(\pi-\theta\right)}{\rho^2-2\rho\, x_1\cos\left(\pi-\theta\right)+x_1^2}-\frac{x_1\sin\left(\pi-\theta\right)}{1-2\rho\, x_1\cos\left(\pi-\theta\right)+\rho^2x_1^2}\right)
\end{aligned}$$

$$\begin{aligned}
= \quad & \lim_{\theta\downarrow 0}\frac{1}{\pi\sin\theta}\left(\frac{\sin\left(\frac{1}{3}\pi-\theta\right)}{\rho^2-2\rho x_1\cos\left(\frac{1}{3}\pi-\theta\right)+x_1^2}+\frac{\sin\left(-\frac{1}{3}\pi-\theta\right)}{\rho^2-2\rho\, x_1\cos\left(-\frac{1}{3}\pi-\theta\right)+x_1^2}+\right.\\
& -\frac{\sin\left(\frac{1}{3}\pi-\theta\right)}{1-2\rho x_1\cos\left(\frac{1}{3}\pi-\theta\right)+\rho^2x_1^2}-\frac{\sin\left(-\frac{1}{3}\pi-\theta\right)}{1-2\rho\, x_1\cos\left(-\frac{1}{3}\pi-\theta\right)+\rho^2x_1^2}+\\
& \left.+\frac{\sin\left(\pi-\theta\right)}{\rho^2-2\rho\, x_1\cos\left(\pi-\theta\right)+x_1^2}-\frac{\sin\left(\pi-\theta\right)}{1-2\rho\, x_1\cos\left(\pi-\theta\right)+\rho^2x_1^2}\right)
\end{aligned}$$

$$\begin{aligned}
= \quad & \lim_{\theta\downarrow 0}\frac{1}{\pi\sin\theta}\left(\frac{\sin\left(\frac{1}{3}\pi\right)\cos\theta-\cos\left(\frac{1}{3}\pi\right)\sin\theta}{\rho^2-2\rho x_1\cos\left(\frac{1}{3}\pi-\theta\right)+x_1^2}-\frac{\sin\left(\frac{1}{3}\pi\right)\cos\theta+\cos\left(\frac{1}{3}\pi\right)\sin\theta}{\rho^2-2\rho\, x_1\cos\left(\frac{1}{3}\pi+\theta\right)+x_1^2}+\right.\\
& -\frac{\sin\left(\frac{1}{3}\pi\right)\cos\theta-\cos\left(\frac{1}{3}\pi\right)\sin\theta}{1-2\rho x_1\cos\left(\frac{1}{3}\pi-\theta\right)+\rho^2x_1^2}+\frac{\sin\left(\frac{1}{3}\pi\right)\cos\theta+\cos\left(\frac{1}{3}\pi\right)\sin\theta}{1-2\rho\, x_1\cos\left(\frac{1}{3}\pi+\theta\right)+\rho^2x_1^2}+\\
& \left.+\frac{\sin\theta}{\rho^2+2\rho\, x_1\cos\theta+x_1^2}-\frac{\sin\theta}{1+2\rho\, x_1\cos\theta+\rho^2x_1^2}\right)
\end{aligned}$$

$$
\begin{aligned}
= \quad & \lim_{\theta \downarrow 0} \frac{1}{\pi \sin\theta} \left(\frac{\sin\left(\frac{1}{3}\pi\right)\cos\theta}{\rho^2 - 2\rho x_1 \cos\left(\frac{1}{3}\pi - \theta\right) + x_1^2} - \frac{\cos\left(\frac{1}{3}\pi\right)\sin\theta}{\rho^2 - 2\rho x_1 \cos\left(\frac{1}{3}\pi - \theta\right) + x_1^2} + \right. \\
& - \frac{\sin\left(\frac{1}{3}\pi\right)\cos\theta}{\rho^2 - 2\rho\, x_1 \cos\left(\frac{1}{3}\pi + \theta\right) + x_1^2} - \frac{\cos\left(\frac{1}{3}\pi\right)\sin\theta}{\rho^2 - 2\rho\, x_1 \cos\left(\frac{1}{3}\pi + \theta\right) + x_1^2} + \\
& - \frac{\sin\left(\frac{1}{3}\pi\right)\cos\theta}{1 - 2\rho x_1 \cos\left(\frac{1}{3}\pi - \theta\right) + \rho^2 x_1^2} + \frac{\cos\left(\frac{1}{3}\pi\right)\sin\theta}{1 - 2\rho x_1 \cos\left(\frac{1}{3}\pi - \theta\right) + \rho^2 x_1^2} + \\
& + \frac{\sin\left(\frac{1}{3}\pi\right)\cos\theta}{1 - 2\rho\, x_1 \cos\left(\frac{1}{3}\pi + \theta\right) + \rho^2 x_1^2} + \frac{\cos\left(\frac{1}{3}\pi\right)\sin\theta}{1 - 2\rho\, x_1 \cos\left(\frac{1}{3}\pi + \theta\right) + \rho^2 x_1^2} + \\
& \left. + \frac{\sin\theta}{\rho^2 + 2\rho\, x_1 \cos\theta + x_1^2} - \frac{\sin\theta}{1 + 2\rho\, x_1 \cos\theta + \rho^2 x_1^2} \right)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \lim_{\theta \downarrow 0} \frac{1}{\pi \sin\theta} \left(\frac{\sin\left(\frac{1}{3}\pi\right)\cos\theta}{\rho^2 - 2\rho x_1 \cos\left(\frac{1}{3}\pi - \theta\right) + x_1^2} - \frac{\sin\left(\frac{1}{3}\pi\right)\cos\theta}{\rho^2 - 2\rho\, x_1 \cos\left(\frac{1}{3}\pi + \theta\right) + x_1^2} + \right. \\
& \left. + \frac{\sin\left(\frac{1}{3}\pi\right)\cos\theta}{1 - 2\rho\, x_1 \cos\left(\frac{1}{3}\pi + \theta\right) + \rho^2 x_1^2} - \frac{\sin\left(\frac{1}{3}\pi\right)\cos\theta}{1 - 2\rho x_1 \cos\left(\frac{1}{3}\pi - \theta\right) + \rho^2 x_1^2} \right) + \\
& + \lim_{\theta \downarrow 0} \frac{1}{\pi \sin\theta} \left(\frac{\sin\theta}{\rho^2 + 2\rho\, x_1 \cos\theta + x_1^2} - \frac{\sin\theta}{1 + 2\rho\, x_1 \cos\theta + \rho^2 x_1^2} + \right. \\
& - \frac{\cos\left(\frac{1}{3}\pi\right)\sin\theta}{\rho^2 - 2\rho\, x_1 \cos\left(\frac{1}{3}\pi + \theta\right) + x_1^2} - \frac{\cos\left(\frac{1}{3}\pi\right)\sin\theta}{\rho^2 - 2\rho x_1 \cos\left(\frac{1}{3}\pi - \theta\right) + x_1^2} + \\
& \left. + \frac{\cos\left(\frac{1}{3}\pi\right)\sin\theta}{1 - 2\rho x_1 \cos\left(\frac{1}{3}\pi - \theta\right) + \rho^2 x_1^2} + \frac{\cos\left(\frac{1}{3}\pi\right)\sin\theta}{1 - 2\rho\, x_1 \cos\left(\frac{1}{3}\pi + \theta\right) + \rho^2 x_1^2} \right)
\end{aligned}
$$

$$= \lim_{\theta \downarrow 0} \frac{\frac{1}{2}\sqrt{3}\cos\theta}{\pi \sin\theta} \left(\frac{\left(\rho^2 - 2\rho x_1 \cos\left(\frac{1}{3}\pi + \theta\right) + x_1^2\right) - \left(\rho^2 - 2\rho x_1 \cos\left(\frac{1}{3}\pi - \theta\right) + x_1^2\right) 1}{\left(\rho^2 - 2\rho x_1 \cos\left(\frac{1}{3}\pi - \theta\right) + x_1^2\right)\left(\rho^2 - 2\rho x_1 \cos\left(\frac{1}{3}\pi + \theta\right) + x_1^2\right)} + \right.$$

$$\left. + \frac{\left(1 - 2\rho x_1 \cos\left(\frac{1}{3}\pi - \theta\right) + \rho^2 x_1^2\right) - \left(1 - 2\rho x_1 \cos\left(\frac{1}{3}\pi + \theta\right) + \rho^2 x_1^2\right)}{\left(1 - 2\rho x_1 \cos\left(\frac{1}{3}\pi + \theta\right) + \rho^2 x_1^2\right)\left(1 - 2\rho x_1 \cos\left(\frac{1}{3}\pi - \theta\right) + \rho^2 x_1^2\right)} \right) +$$

$$+ \lim_{\theta \downarrow 0} \frac{1}{\pi} \left(\frac{1}{\rho^2 + 2\rho x_1 \cos\theta + x_1^2} - \frac{1}{1 + 2\rho x_1 \cos\theta + \rho^2 x_1^2} + \right.$$

$$- \frac{\frac{1}{2}}{\rho^2 - 2\rho x_1 \cos\left(\frac{1}{3}\pi + \theta\right) + x_1^2} - \frac{\frac{1}{2}}{\rho^2 - 2\rho x_1 \cos\left(\frac{1}{3}\pi - \theta\right) + x_1^2} +$$

$$\left. + \frac{\frac{1}{2}}{1 - 2\rho x_1 \cos\left(\frac{1}{3}\pi - \theta\right) + \rho^2 x_1^2} + \frac{\frac{1}{2}}{1 - 2\rho x_1 \cos\left(\frac{1}{3}\pi + \theta\right) + \rho^2 x_1^2} \right)$$

$$= \lim_{\theta \downarrow 0} \frac{\frac{1}{2}\sqrt{3} 2\rho x_1 \cos\theta}{\pi \sin\theta} \left(\frac{\cos\left(\frac{1}{3}\pi - \theta\right) - \cos\left(\frac{1}{3}\pi + \theta\right)}{\left(\rho^2 - 2\rho x_1 \cos\left(\frac{1}{3}\pi - \theta\right) + x_1^2\right)\left(\rho^2 - 2\rho x_1 \cos\left(\frac{1}{3}\pi + \theta\right) + x_1^2\right)} + \right.$$

$$\left. + \frac{\cos\left(\frac{1}{3}\pi + \theta\right) - \cos\left(\frac{1}{3}\pi - \theta\right)}{\left(1 - 2\rho x_1 \cos\left(\frac{1}{3}\pi + \theta\right) + \rho^2 x_1^2\right)\left(1 - 2\rho x_1 \cos\left(\frac{1}{3}\pi - \theta\right) + \rho^2 x_1^2\right)} \right) +$$

$$+ \frac{1}{\pi} \left(\frac{1}{\rho^2 + 2\rho x_1 + x_1^2} - \frac{1}{1 + 2\rho x_1 + \rho^2 x_1^2} + \right.$$

$$\left. - \frac{\frac{1}{2}}{\rho^2 - \rho x_1 + x_1^2} - \frac{\frac{1}{2}}{\rho^2 - \rho x_1 + x_1^2} + \frac{\frac{1}{2}}{1 - \rho x_1 + \rho^2 x_1^2} + \frac{\frac{1}{2}}{1 - \rho x_1 + \rho^2 x_1^2} \right)$$

$$= \lim_{\theta \downarrow 0} \frac{\sqrt{3}\rho x_1}{\pi \sin\theta} \left(\frac{2\sin\left(\frac{1}{3}\pi\right)\sin\theta}{\left(\rho^2 - 2\rho x_1 \cos\left(\frac{1}{3}\pi - \theta\right) + x_1^2\right)\left(\rho^2 - 2\rho x_1 \cos\left(\frac{1}{3}\pi + \theta\right) + x_1^2\right)} + \right.$$

$$\left. - \frac{2\sin\left(\frac{1}{3}\pi\right)\sin\theta}{\left(1 - 2\rho x_1 \cos\left(\frac{1}{3}\pi + \theta\right) + \rho^2 x_1^2\right)\left(1 - 2\rho x_1 \cos\left(\frac{1}{3}\pi - \theta\right) + \rho^2 x_1^2\right)} \right) +$$

$$+ \frac{1}{\pi} \left(\frac{1}{\rho^2 + 2\rho x_1 + x_1^2} - \frac{1}{1 + 2\rho x_1 + \rho^2 x_1^2} - \frac{1}{\rho^2 - \rho x_1 + x_1^2} + \frac{1}{1 - \rho x_1 + \rho^2 x_1^2} \right)$$

$$= \lim_{\theta \downarrow 0} \frac{\rho x_1}{\pi} \left(\frac{3}{\left(\rho^2 - 2\rho x_1 \cos\left(\frac{1}{3}\pi - \theta\right) + x_1^2\right)\left(\rho^2 - 2\rho\, x_1 \cos\left(\frac{1}{3}\pi + \theta\right) + x_1^2\right)} + \right.$$

$$\left. - \frac{3}{\left(1 - 2\rho\, x_1 \cos\left(\frac{1}{3}\pi + \theta\right) + \rho^2 x_1^2\right)\left(1 - 2\rho x_1 \cos\left(\frac{1}{3}\pi - \theta\right) + \rho^2 x_1^2\right)} \right) +$$

$$+ \frac{1}{\pi}\left(\frac{1}{\rho^2 + 2\rho\, x_1 + x_1^2} - \frac{1}{1 + 2\rho\, x_1 + \rho^2 x_1^2} - \frac{1}{\rho^2 - \rho\, x_1 + x_1^2} + \frac{1}{1 - \rho\, x_1 + \rho^2 x_1^2} \right)$$

$$= \frac{\rho x_1}{\pi}\left(\frac{3}{(\rho^2 - \rho\, x_1 + x_1^2)(\rho^2 - \rho\, x_1 + x_1^2)} - \frac{3}{(1 - \rho\, x_1 + \rho^2 x_1^2)(1 - \rho x_1 + \rho^2 x_1^2)} \right) +$$
$$+ \frac{1}{\pi}\left(\frac{1}{\rho^2 + 2\rho\, x_1 + x_1^2} - \frac{1}{1 + 2\rho\, x_1 + \rho^2 x_1^2} - \frac{1}{\rho^2 - \rho\, x_1 + x_1^2} + \frac{1}{1 - \rho\, x_1 + \rho^2 x_1^2} \right)$$

$$= \frac{1}{\pi}\left(\frac{3\rho x_1}{(\rho^2 - \rho\, x_1 + x_1^2)^2} - \frac{3\rho x_1}{(1 - \rho\, x_1 + \rho^2 x_1^2)^2} + \frac{1}{\rho^2 + 2\rho\, x_1 + x_1^2} + \right.$$

$$\left. - \frac{1}{1 + 2\rho\, x_1 + \rho^2 x_1^2} - \frac{1}{\rho^2 - \rho\, x_1 + x_1^2} + \frac{1}{1 - \rho\, x_1 + \rho^2 x_1^2} \right).$$

7.2. Derivation of a contour integral

As before

$$\lim_{x_2 \downarrow 0} \frac{G_S(x, z)}{x_2} =$$
$$= \frac{1}{\pi}\left(\frac{r \sin\theta}{x_1^2 - 2x_1 r\cos\theta + r^2} - \frac{r\sin\theta}{1 - 2x_1 r\cos\theta + x_1^2 r^2} \right) +$$
$$+ \frac{1}{\pi}\left(\frac{r\sin\left(\theta - \frac{2}{3}\pi\right)}{x_1^2 - 2x_1 r\cos\left(\theta - \frac{2}{3}\pi\right) + r^2} - \frac{r\sin\left(\theta - \frac{2}{3}\pi\right)}{1 - 2x_1 r\cos\left(\theta - \frac{2}{3}\pi\right) + x_1^2 r^2} \right) +$$
$$+ \frac{1}{\pi}\left(\frac{r\sin\left(\theta + \frac{2}{3}\pi\right)}{x_1^2 - 2x_1 r\cos\left(\theta + \frac{2}{3}\pi\right) + r^2} - \frac{r\sin\left(\theta + \frac{2}{3}\pi\right)}{1 - 2x_1 r\cos\left(\theta + \frac{2}{3}\pi\right) + x_1^2 r^2} \right)$$

By symmetry we find

$$\lim_{\substack{\psi \uparrow \frac{1}{3}\pi \\ y = (\rho\cos\psi, \rho\sin\psi)}} \frac{G_S(z, y)}{\rho\sin\left(\frac{1}{3}\pi - \psi\right)} =$$
$$= \frac{1}{\pi}\left(\frac{r\sin\left(\frac{1}{3}\pi - \theta\right)}{\rho^2 - 2\rho\, r\cos\left(\frac{1}{3}\pi - \theta\right) + r^2} - \frac{r\sin\left(\frac{1}{3}\pi - \theta\right)}{1 - 2\rho\, r\cos\left(\frac{1}{3}\pi - \theta\right) + \rho^2 r^2} \right) +$$
$$+ \frac{1}{\pi}\left(\frac{r\sin\left(-\frac{1}{3}\pi - \theta\right)}{\rho^2 - 2\rho\, r\cos\left(-\frac{1}{3}\pi - \theta\right) + r^2} - \frac{r\sin\left(-\frac{1}{3}\pi - \theta\right)}{1 - 2\rho\, r\cos\left(-\frac{1}{3}\pi - \theta\right) + \rho^2 r^2} \right) +$$
$$+ \frac{1}{\pi}\left(\frac{r\sin(\pi - \theta)}{\rho^2 - 2\rho\, r\cos(\pi - \theta) + r^2} - \frac{r\sin(\pi - \theta)}{1 - 2\rho\, r\cos(\pi - \theta) + \rho^2 r^2} \right).$$

The expression we want to compute explicitly is

$$\pi^2 \int_{z\in S} \left(\lim_{x_2 \downarrow 0} \frac{G_S(x,z)}{x_2} \right) \left(\lim_{\substack{\psi \uparrow \frac{1}{3}\pi \\ y=(\rho\cos\psi, \rho\sin\psi)}} \frac{G_S(z,y)}{\rho \sin\left(\frac{1}{3}\pi - \psi\right)} \right) dz =$$

$$= \int_{r=0}^{1} \int_{\theta=0}^{\frac{1}{3}\pi} \sum_{k=-1}^{1} \sum_{m=-1}^{1} h\left(\theta + k\tfrac{2}{3}\pi\right) \ell\left(\pi - \theta + m\tfrac{2}{3}\pi\right) d\theta \, r\, dr. \tag{78}$$

As before with $w = re^{i\theta}$ we have $m = 1$

$$h\left(\theta + k\tfrac{2}{3}\pi\right) = \frac{1}{2i}\left(\frac{r^2 e^{-k\frac{2}{3}\pi i}}{w - x_1 r^2 e^{-k\frac{2}{3}\pi i}} + \frac{x_1^{-2} e^{-k\frac{2}{3}\pi i}}{w - x_1^{-1} e^{-k\frac{2}{3}\pi i}} - \frac{x_1^{-2} r^2 e^{-k\frac{2}{3}\pi i}}{w - x_1^{-1} r^2 e^{-k\frac{2}{3}\pi i}} - \frac{e^{-k\frac{2}{3}\pi i}}{w - x_1 e^{-k\frac{2}{3}\pi i}} \right),$$

$$\ell\left(\pi - \theta + m\tfrac{2}{3}\pi\right) =$$

$$= \frac{1}{2i}\left(\frac{r^2 e^{-m\frac{2}{3}\pi i}}{-\bar{w} - \rho r^2 e^{-m\frac{2}{3}\pi i}} + \frac{\rho^{-2} e^{-m\frac{2}{3}\pi i}}{-\bar{w} - \rho^{-1} e^{-m\frac{2}{3}\pi i}} - \frac{\rho^{-2} r^2 e^{-m\frac{2}{3}\pi i}}{-\bar{w} - \rho^{-1} r^2 e^{-m\frac{2}{3}\pi i}} - \frac{e^{-m\frac{2}{3}\pi i}}{-\bar{w} - \rho e^{-m\frac{2}{3}\pi i}} \right)$$

$$= \frac{1}{2i}\left(\frac{w r^2 e^{-m\frac{2}{3}\pi i}}{-r^2 - w\rho r^2 e^{-m\frac{2}{3}\pi i}} + \frac{w\rho^{-2} e^{-m\frac{2}{3}\pi i}}{-r^2 - w\rho^{-1} e^{-m\frac{2}{3}\pi i}} - \frac{w r^2 \rho^{-2} e^{-m\frac{2}{3}\pi i}}{-r^2 - w\rho^{-1} r^2 e^{-m\frac{2}{3}\pi i}} - \frac{w e^{-m\frac{2}{3}\pi i}}{-r^2 - w\rho e^{-m\frac{2}{3}\pi i}} \right)$$

$$= \frac{1}{2i\rho}\left(-\frac{w}{w + \rho^{-1} e^{m\frac{2}{3}\pi i}} - \frac{w}{w + r^2 \rho e^{m\frac{2}{3}\pi i}} + \frac{w}{w + \rho e^{m\frac{2}{3}\pi i}} + \frac{w}{w + r^2 \rho^{-1} e^{m\frac{2}{3}\pi i}} \right)$$

$$= \frac{1}{2i\rho}\left(\frac{\rho^{-1} e^{m\frac{2}{3}\pi i}}{w + \rho^{-1} e^{m\frac{2}{3}\pi i}} + \frac{r^2 \rho e^{m\frac{2}{3}\pi i}}{w + r^2 \rho e^{m\frac{2}{3}\pi i}} - \frac{\rho e^{m\frac{2}{3}\pi i}}{w + \rho e^{m\frac{2}{3}\pi i}} - \frac{r^2 \rho^{-1} e^{m\frac{2}{3}\pi i}}{w + r^2 \rho^{-1} e^{m\frac{2}{3}\pi i}} \right).$$

Again the inner integral in (78) equals

$$\int_{\theta=0}^{\frac{1}{3}\pi} \sum_{k=-1}^{1} \sum_{m=-1}^{1} h\left(\theta + k\tfrac{2}{3}\pi\right) \ell\left(\pi - \theta + m\tfrac{2}{3}\pi\right) d\theta =$$

$$= \frac{1}{2} \int_{\theta=0}^{2\pi} \sum_{k=-1}^{1} h\left(\theta + k\frac{2}{3}\pi\right) \ell(\pi - \theta)\, d\theta$$

$$= \frac{i}{8\rho x_1} \int_{\theta=0}^{2\pi} \left(\frac{x_1 r^2}{w - x_1 r^2} + \frac{x_1^{-1}}{w - x_1^{-1}} - \frac{x_1^{-1} r^2}{w - x_1^{-1} r^2} - \frac{x_1}{w - x_1} + \right.$$

$$+ \frac{x_1 r^2 e^{-\frac{2}{3}\pi i}}{w - x_1 r^2 e^{-\frac{2}{3}\pi i}} + \frac{x_1^{-1} e^{-\frac{2}{3}\pi i}}{w - x_1^{-1} e^{-\frac{2}{3}\pi i}} - \frac{x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}}{w - x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}} - \frac{x_1 e^{-\frac{2}{3}\pi i}}{w - x_1 e^{-\frac{2}{3}\pi i}} +$$

$$\left. + \frac{x_1 r^2 e^{\frac{2}{3}\pi i}}{w - x_1 r^2 e^{\frac{2}{3}\pi i}} + \frac{x_1^{-1} e^{\frac{2}{3}\pi i}}{w - x_1^{-1} e^{\frac{2}{3}\pi i}} - \frac{x_1^{-1} r^2 e^{\frac{2}{3}\pi i}}{w - x_1^{-1} r^2 e^{\frac{2}{3}\pi i}} - \frac{x_1 e^{\frac{2}{3}\pi i}}{w - x_1 e^{\frac{2}{3}\pi i}} \right) \times$$

$$\times \left(\frac{\rho^{-1}}{w + \rho^{-1}} + \frac{r^2 \rho}{w + r^2 \rho} - \frac{\rho}{w + \rho} - \frac{r^2 \rho^{-1}}{w + r^2 \rho^{-1}} \right) \frac{dw}{w}$$

leaving us with three different cases with each having 6 poles. Note that a pole in 0 does not contribute.

7.3. Computation of the contour integral

Let us assume that $x_1 < \rho$. Then according to the size of r the integrand has different sets of poles. In the following table we give a scheme in which we denote how we will split the integral (and which range of r corresponds with which poles):

poles due to:		h for $k=0$		h for $k=-1$		h for $k=1$		ℓ	
range:		$a_1.$	$a_2.$	$b_1.$	$b_2.$	$c_1.$	$c_2.$	$d_1.$	$d_2.$
$r \in (0, x_1)$	I.	$x_1 r^2$	$x_1^{-1} r^2$	$x_1 r^2 e^{-\frac{2}{3}\pi i}$	$x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}$	$x_1 r^2 e^{\frac{2}{3}\pi i}$	$x_1^{-1} r^2 e^{\frac{2}{3}\pi i}$	$-r^2\rho$	$-r^2\rho^{-1}$
$r \in (x_1, \rho)$	II.	$x_1 r^2$	x_1	$x_1 r^2 e^{-\frac{2}{3}\pi i}$	$x_1 e^{-\frac{2}{3}\pi i}$	$x_1 r^2 e^{\frac{2}{3}\pi i}$	$x_1 e^{\frac{2}{3}\pi i}$	$-r^2\rho$	$-r^2\rho^{-1}$
$r \in (\rho, 1)$	III.	$x_1 r^2$	x_1	$x_1 r^2 e^{-\frac{2}{3}\pi i}$	$x_1 e^{-\frac{2}{3}\pi i}$	$x_1 r^2 e^{\frac{2}{3}\pi i}$	$x_1 e^{\frac{2}{3}\pi i}$	$-r^2\rho$	$-\rho$

We consider the contribution in the following integral by each of the poles separately:

$$
\begin{aligned}
&\frac{i}{8\rho x_1}\int_{\theta=0}^{2\pi}\Bigg(\frac{x_1 r^2}{w - x_1 r^2} + \frac{x_1^{-1}}{w - x_1^{-1}} - \frac{x_1^{-1} r^2}{w - x_1^{-1} r^2} - \frac{x_1}{w - x_1} + \\
&+ \frac{x_1 r^2 e^{-\frac{2}{3}\pi i}}{w - x_1 r^2 e^{-\frac{2}{3}\pi i}} + \frac{x_1^{-1} e^{-\frac{2}{3}\pi i}}{w - x_1^{-1} e^{-\frac{2}{3}\pi i}} - \frac{x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}}{w - x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}} - \frac{x_1 e^{-\frac{2}{3}\pi i}}{w - x_1 e^{-\frac{2}{3}\pi i}} + \\
&+ \frac{x_1 r^2 e^{\frac{2}{3}\pi i}}{w - x_1 r^2 e^{\frac{2}{3}\pi i}} + \frac{x_1^{-1} e^{\frac{2}{3}\pi i}}{w - x_1^{-1} e^{\frac{2}{3}\pi i}} - \frac{x_1^{-1} r^2 e^{\frac{2}{3}\pi i}}{w - x_1^{-1} r^2 e^{\frac{2}{3}\pi i}} - \frac{x_1 e^{\frac{2}{3}\pi i}}{w - x_1 e^{\frac{2}{3}\pi i}}\Bigg) \times \\
&\times \left(\frac{\rho^{-1}}{w + \rho^{-1}} + \frac{r^2\rho}{w + r^2\rho} - \frac{\rho}{w + \rho} - \frac{r^2\rho^{-1}}{w + r^2\rho^{-1}}\right)\frac{dw}{w}
\end{aligned}
$$

I, II and III, a_1**:** pole at $w = x_1 r^2$.

$$
\begin{aligned}
&\frac{-\pi}{4x_1\rho}\left[\frac{\rho^{-1}}{w + \rho^{-1}} + \frac{r^2\rho}{w + r^2\rho} - \frac{\rho}{w + \rho} - \frac{r^2\rho^{-1}}{w + r^2\rho^{-1}}\right]_{w = x_1 r^2} = \\
= &\frac{-\pi}{4x_1\rho}\left(\frac{\rho^{-1}}{x_1 r^2 + \rho^{-1}} + \frac{r^2\rho}{x_1 r^2 + r^2\rho} - \frac{\rho}{x_1 r^2 + \rho} - \frac{r^2\rho^{-1}}{x_1 r^2 + r^2\rho^{-1}}\right) \\
= &\frac{-\pi}{4x_1\rho}\left(\frac{\rho^{-1}}{x_1 r^2 + \rho^{-1}} + \frac{\rho}{x_1 + \rho} - \frac{\rho}{x_1 r^2 + \rho} - \frac{\rho^{-1}}{x_1 + \rho^{-1}}\right) \\
= &\frac{\pi}{4x_1\rho}\left(\frac{1}{x_1\rho + 1} - \frac{\rho}{x_1 + \rho} + \frac{x_1^{-1}\rho}{r^2 + x_1^{-1}\rho} - \frac{x_1^{-1}\rho^{-1}}{r^2 + x_1^{-1}\rho^{-1}}\right) \qquad (79)
\end{aligned}
$$

I, a_2**:** pole at $w = x_1^{-1} r^2$.

$$
\begin{aligned}
&-\frac{-\pi}{4x_1\rho}\left[\frac{\rho^{-1}}{w + \rho^{-1}} + \frac{r^2\rho}{w + r^2\rho} - \frac{\rho}{w + \rho} - \frac{r^2\rho^{-1}}{w + r^2\rho^{-1}}\right]_{w = x_1^{-1} r^2} = \\
= &\frac{\pi}{4x_1\rho}\left(\frac{\rho^{-1}}{x_1^{-1} r^2 + \rho^{-1}} + \frac{r^2\rho}{x_1^{-1} r^2 + r^2\rho} - \frac{\rho}{x_1^{-1} r^2 + \rho} - \frac{r^2\rho^{-1}}{x_1^{-1} r^2 + r^2\rho^{-1}}\right) \\
= &\frac{\pi}{4x_1\rho}\left(\frac{x_1\rho^{-1}}{r^2 + x_1\rho^{-1}} + \frac{x_1\rho}{1 + x_1\rho} - \frac{x_1\rho}{r^2 + x_1\rho} - \frac{x_1}{\rho + x_1}\right) \\
= &\frac{\pi}{4x_1\rho}\left(\frac{x_1\rho}{1 + x_1\rho} - \frac{x_1}{\rho + x_1} + \frac{x_1\rho^{-1}}{r^2 + x_1\rho^{-1}} - \frac{x_1\rho}{r^2 + x_1\rho}\right) \qquad (80)
\end{aligned}
$$

II and III, a_2: pole at $w = x_1$.

$$
\begin{aligned}
&- \frac{-\pi}{4x_1\rho}\left[\frac{\rho^{-1}}{w+\rho^{-1}} + \frac{r^2\rho}{w+r^2\rho} - \frac{\rho}{w+\rho} - \frac{r^2\rho^{-1}}{w+r^2\rho^{-1}}\right]_{w=x_1} = \\
&= \frac{\pi}{4x_1\rho}\left(\frac{\rho^{-1}}{x_1+\rho^{-1}} + \frac{r^2\rho}{x_1+r^2\rho} - \frac{\rho}{x_1+\rho} - \frac{r^2\rho^{-1}}{x_1+r^2\rho^{-1}}\right) \\
&= \frac{\pi}{4x_1\rho}\left(\frac{1}{x_1\rho+1} + 1 - \frac{x_1}{x_1+r^2\rho} - \frac{\rho}{x_1+\rho} - 1 + \frac{x_1}{x_1+r^2\rho^{-1}}\right) \\
&= \frac{\pi}{4x_1\rho}\left(\frac{1}{x_1\rho+1} - \frac{\rho}{x_1+\rho} + \frac{x_1\rho}{r^2+x_1\rho} - \frac{x_1\rho^{-1}}{r^2+x_1\rho^{-1}}\right) \qquad (81)
\end{aligned}
$$

I, II and III, b_1: pole at $w = x_1 r^2 e^{-\frac{2}{3}\pi i}$.

$$
\begin{aligned}
&\frac{-\pi}{4x_1\rho}\left[\frac{\rho^{-1}}{w+\rho^{-1}} + \frac{r^2\rho}{w+r^2\rho} - \frac{\rho}{w+\rho} - \frac{r^2\rho^{-1}}{w+r^2\rho^{-1}}\right]_{w=x_1 r^2 e^{-\frac{2}{3}\pi i}} = \\
&= \frac{-\pi}{4x_1\rho}\left(\frac{\rho^{-1}}{x_1 r^2 e^{-\frac{2}{3}\pi i}+\rho^{-1}} + \frac{r^2\rho}{x_1 r^2 e^{-\frac{2}{3}\pi i}+r^2\rho} - \frac{\rho}{x_1 r^2 e^{-\frac{2}{3}\pi i}+\rho} - \frac{r^2\rho^{-1}}{x_1 r^2 e^{-\frac{2}{3}\pi i}+r^2\rho^{-1}}\right) \\
&= \frac{-\pi}{4x_1\rho}\left(\frac{x_1^{-1}\rho^{-1}e^{\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho^{-1}e^{\frac{2}{3}\pi i}} + \frac{\rho}{x_1 e^{-\frac{2}{3}\pi i}+\rho} - \frac{x_1^{-1}\rho e^{\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho e^{\frac{2}{3}\pi i}} - \frac{1}{x_1\rho e^{-\frac{2}{3}\pi i}+1}\right) \\
&= \frac{\pi}{4x_1\rho}\left(\frac{1}{x_1\rho e^{-\frac{2}{3}\pi i}+1} - \frac{\rho}{x_1 e^{-\frac{2}{3}\pi i}+\rho} + \frac{x_1^{-1}\rho e^{\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho e^{\frac{2}{3}\pi i}} - \frac{x_1^{-1}\rho^{-1}e^{\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho^{-1}e^{\frac{2}{3}\pi i}}\right) \qquad (82)
\end{aligned}
$$

I, b_2: pole at $w = x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}$.

$$
\begin{aligned}
&- \frac{-\pi}{4x_1\rho}\left[\frac{\rho^{-1}}{w+\rho^{-1}} + \frac{r^2\rho}{w+r^2\rho} - \frac{\rho}{w+\rho} - \frac{r^2\rho^{-1}}{w+r^2\rho^{-1}}\right]_{w=x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}} = \\
&= \frac{\pi}{4x_1\rho}\left(\frac{\rho^{-1}}{x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}+\rho^{-1}} + \frac{r^2\rho}{x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}+r^2\rho} - \frac{\rho}{x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}+\rho} - \frac{r^2\rho^{-1}}{x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}+r^2\rho^{-1}}\right) \\
&= \frac{\pi}{4x_1\rho}\left(\frac{x_1\rho^{-1}e^{\frac{2}{3}\pi i}}{r^2+x_1\rho^{-1}e^{\frac{2}{3}\pi i}} + \frac{x_1\rho e^{\frac{2}{3}\pi i}}{1+x_1\rho e^{\frac{2}{3}\pi i}} - \frac{x_1\rho e^{\frac{2}{3}\pi i}}{r^2+x_1\rho e^{\frac{2}{3}\pi i}} - \frac{x_1}{\rho e^{-\frac{2}{3}\pi i}+x_1}\right) \\
&= \frac{\pi}{4x_1\rho}\left(\frac{x_1\rho e^{\frac{2}{3}\pi i}}{1+x_1\rho e^{\frac{2}{3}\pi i}} - \frac{x_1 e^{\frac{2}{3}\pi i}}{\rho+x_1 e^{\frac{2}{3}\pi i}} + \frac{x_1\rho^{-1}e^{\frac{2}{3}\pi i}}{r^2+x_1\rho^{-1}e^{\frac{2}{3}\pi i}} - \frac{x_1\rho e^{\frac{2}{3}\pi i}}{r^2+x_1\rho e^{\frac{2}{3}\pi i}}\right) \qquad (83)
\end{aligned}
$$

II and III, b_2: pole at $w = x_1 e^{-\frac{2}{3}\pi i}$.

$$
\begin{aligned}
& -\frac{-\pi}{4x_1\rho}\left[\frac{\rho^{-1}}{w+\rho^{-1}}+\frac{r^2\rho}{w+r^2\rho}-\frac{\rho}{w+\rho}-\frac{r^2\rho^{-1}}{w+r^2\rho^{-1}}\right]_{w=x_1e^{-\frac{2}{3}\pi i}} = \\
= \quad & \frac{\pi}{4x_1\rho}\left(\frac{\rho^{-1}}{x_1e^{-\frac{2}{3}\pi i}+\rho^{-1}}+\frac{r^2\rho}{x_1e^{-\frac{2}{3}\pi i}+r^2\rho}-\frac{\rho}{x_1e^{-\frac{2}{3}\pi i}+\rho}-\frac{r^2\rho^{-1}}{x_1e^{-\frac{2}{3}\pi i}+r^2\rho^{-1}}\right) \\
= \quad & \frac{\pi}{4x_1\rho}\left(\frac{\rho^{-1}}{x_1e^{-\frac{2}{3}\pi i}+\rho^{-1}}+1-\frac{x_1e^{-\frac{2}{3}\pi i}}{x_1e^{-\frac{2}{3}\pi i}+r^2\rho}-\frac{\rho}{x_1e^{-\frac{2}{3}\pi i}+\rho}-1+\frac{x_1e^{-\frac{2}{3}\pi i}}{x_1e^{-\frac{2}{3}\pi i}+r^2\rho^{-1}}\right) \\
= \quad & \frac{\pi}{4x_1\rho}\left(\frac{1}{x_1\rho e^{-\frac{2}{3}\pi i}+1}-\frac{\rho}{x_1e^{-\frac{2}{3}\pi i}+\rho}+\frac{x_1\rho e^{-\frac{2}{3}\pi i}}{r^2+x_1\rho e^{-\frac{2}{3}\pi i}}-\frac{x_1\rho^{-1}e^{-\frac{2}{3}\pi i}}{r^2+x_1\rho^{-1}e^{-\frac{2}{3}\pi i}}\right)
\end{aligned} \tag{84}
$$

I, II and III, c_1: pole at $w = x_1 r^2 e^{\frac{2}{3}\pi i}$.

$$
\begin{aligned}
& \frac{-\pi}{4x_1\rho}\left[\frac{\rho^{-1}}{w+\rho^{-1}}+\frac{r^2\rho}{w+r^2\rho}-\frac{\rho}{w+\rho}-\frac{r^2\rho^{-1}}{w+r^2\rho^{-1}}\right]_{w=x_1r^2e^{\frac{2}{3}\pi i}} = \\
= \quad & \frac{-\pi}{4x_1\rho}\left(\frac{\rho^{-1}}{x_1r^2e^{\frac{2}{3}\pi i}+\rho^{-1}}+\frac{r^2\rho}{x_1r^2e^{\frac{2}{3}\pi i}+r^2\rho}-\frac{\rho}{x_1r^2e^{\frac{2}{3}\pi i}+\rho}-\frac{r^2\rho^{-1}}{x_1r^2e^{\frac{2}{3}\pi i}+r^2\rho^{-1}}\right) \\
= \quad & \frac{\pi}{4x_1\rho}\left(\frac{1}{x_1\rho e^{\frac{2}{3}\pi i}+1}-\frac{\rho}{x_1e^{\frac{2}{3}\pi i}+\rho}+\frac{x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}-\frac{x_1^{-1}\rho^{-1}e^{-\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho^{-1}e^{-\frac{2}{3}\pi i}}\right)
\end{aligned} \tag{85}
$$

I, c_2: pole at $w = x_1^{-1} r^2 e^{\frac{2}{3}\pi i}$.

$$
\begin{aligned}
& -\frac{-\pi}{4x_1\rho}\left[\frac{\rho^{-1}}{w+\rho^{-1}}+\frac{r^2\rho}{w+r^2\rho}-\frac{\rho}{w+\rho}-\frac{r^2\rho^{-1}}{w+r^2\rho^{-1}}\right]_{w=x_1^{-1}r^2e^{\frac{2}{3}\pi i}} = \\
= \quad & \frac{\pi}{4x_1\rho}\left(\frac{\rho^{-1}}{x_1^{-1}r^2e^{\frac{2}{3}\pi i}+\rho^{-1}}+\frac{r^2\rho}{x_1^{-1}r^2e^{\frac{2}{3}\pi i}+r^2\rho}-\frac{\rho}{x_1^{-1}r^2e^{\frac{2}{3}\pi i}+\rho}-\frac{r^2\rho^{-1}}{x_1^{-1}r^2e^{\frac{2}{3}\pi i}+r^2\rho^{-1}}\right) \\
= \quad & \frac{\pi}{4x_1\rho}\left(\frac{x_1\rho e^{-\frac{2}{3}\pi i}}{1+x_1\rho e^{-\frac{2}{3}\pi i}}-\frac{x_1e^{-\frac{2}{3}\pi i}}{\rho+x_1e^{-\frac{2}{3}\pi i}}+\frac{x_1\rho^{-1}e^{-\frac{2}{3}\pi i}}{r^2+x_1\rho^{-1}e^{-\frac{2}{3}\pi i}}-\frac{x_1\rho e^{-\frac{2}{3}\pi i}}{r^2+x_1\rho e^{-\frac{2}{3}\pi i}}\right)
\end{aligned} \tag{86}
$$

II and III, c_2: pole at $w = x_1 e^{\frac{2}{3}\pi i}$.

$$
\begin{aligned}
& -\frac{-\pi}{4x_1\rho}\left[\frac{\rho^{-1}}{w+\rho^{-1}}+\frac{r^2\rho}{w+r^2\rho}-\frac{\rho}{w+\rho}-\frac{r^2\rho^{-1}}{w+r^2\rho^{-1}}\right]_{w=x_1e^{\frac{2}{3}\pi i}} = \\
= \quad & \frac{\pi}{4x_1\rho}\left(\frac{\rho^{-1}}{x_1e^{\frac{2}{3}\pi i}+\rho^{-1}}+\frac{r^2\rho}{x_1e^{\frac{2}{3}\pi i}+r^2\rho}-\frac{\rho}{x_1e^{\frac{2}{3}\pi i}+\rho}-\frac{r^2\rho^{-1}}{x_1e^{\frac{2}{3}\pi i}+r^2\rho^{-1}}\right) \\
= \quad & \frac{\pi}{4x_1\rho}\left(\frac{1}{x_1\rho e^{\frac{2}{3}\pi i}+1}+1-\frac{x_1e^{\frac{2}{3}\pi i}}{x_1e^{\frac{2}{3}\pi i}+r^2\rho}-\frac{\rho}{x_1e^{\frac{2}{3}\pi i}+\rho}-1+\frac{x_1e^{\frac{2}{3}\pi i}}{x_1e^{\frac{2}{3}\pi i}+r^2\rho^{-1}}\right) \\
= \quad & \frac{\pi}{4x_1\rho}\left(\frac{1}{x_1\rho e^{\frac{2}{3}\pi i}+1}-\frac{\rho}{x_1e^{\frac{2}{3}\pi i}+\rho}+\frac{x_1\rho e^{\frac{2}{3}\pi i}}{r^2+x_1\rho e^{\frac{2}{3}\pi i}}-\frac{x_1\rho^{-1}e^{\frac{2}{3}\pi i}}{r^2+x_1\rho^{-1}e^{\frac{2}{3}\pi i}}\right).
\end{aligned} \tag{87}
$$

I, II and III, d_1: pole at $w = -r^2\rho$.

$$\frac{\pi}{4\rho x_1}\left[\frac{x_1 r^2}{w - x_1 r^2} + \frac{x_1^{-1}}{w - x_1^{-1}} - \frac{x_1^{-1} r^2}{w - x_1^{-1} r^2} - \frac{x_1}{w - x_1}\right]_{w=-r^2\rho} +$$
$$+ \frac{\pi}{4\rho x_1}\left[\frac{x_1 r^2 e^{-\frac{2}{3}\pi i}}{w - x_1 r^2 e^{-\frac{2}{3}\pi i}} + \frac{x_1^{-1} e^{-\frac{2}{3}\pi i}}{w - x_1^{-1} e^{-\frac{2}{3}\pi i}} - \frac{x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}}{w - x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}} - \frac{x_1 e^{-\frac{2}{3}\pi i}}{w - x_1 e^{-\frac{2}{3}\pi i}}\right]_{w=-r^2\rho} +$$
$$+ \frac{\pi}{4\rho x_1}\left[\frac{x_1 r^2 e^{\frac{2}{3}\pi i}}{w - x_1 r^2 e^{\frac{2}{3}\pi i}} + \frac{x_1^{-1} e^{\frac{2}{3}\pi i}}{w - x_1^{-1} e^{\frac{2}{3}\pi i}} - \frac{x_1^{-1} r^2 e^{\frac{2}{3}\pi i}}{w - x_1^{-1} r^2 e^{\frac{2}{3}\pi i}} - \frac{x_1 e^{\frac{2}{3}\pi i}}{w - x_1 e^{\frac{2}{3}\pi i}}\right]_{w=-r^2\rho} =$$

$$= \frac{\pi}{4\rho x_1}\left(\frac{x_1 r^2}{-r^2\rho - x_1 r^2} + \frac{x_1^{-1}}{-r^2\rho - x_1^{-1}} - \frac{x_1^{-1} r^2}{-r^2\rho - x_1^{-1} r^2} - \frac{x_1}{-r^2\rho - x_1}\right) +$$
$$+ \frac{\pi}{4\rho x_1}\left(\frac{x_1 r^2 e^{-\frac{2}{3}\pi i}}{-r^2\rho - x_1 r^2 e^{-\frac{2}{3}\pi i}} + \frac{x_1^{-1} e^{-\frac{2}{3}\pi i}}{-r^2\rho - x_1^{-1} e^{-\frac{2}{3}\pi i}} - \frac{x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}}{-r^2\rho - x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}} - \frac{x_1 e^{-\frac{2}{3}\pi i}}{-r^2\rho - x_1 e^{-\frac{2}{3}\pi i}}\right) +$$
$$+ \frac{\pi}{4\rho x_1}\left(\frac{x_1 r^2 e^{\frac{2}{3}\pi i}}{-r^2\rho - x_1 r^2 e^{\frac{2}{3}\pi i}} + \frac{x_1^{-1} e^{\frac{2}{3}\pi i}}{-r^2\rho - x_1^{-1} e^{\frac{2}{3}\pi i}} - \frac{x_1^{-1} r^2 e^{\frac{2}{3}\pi i}}{-r^2\rho - x_1^{-1} r^2 e^{\frac{2}{3}\pi i}} - \frac{x_1 e^{\frac{2}{3}\pi i}}{-r^2\rho - x_1 e^{\frac{2}{3}\pi i}}\right)$$

$$= \frac{\pi}{4\rho x_1}\left(-\frac{x_1}{\rho + x_1} - \frac{x_1^{-1}\rho^{-1}}{r^2 + x_1^{-1}\rho^{-1}} + \frac{1}{x_1\rho + 1} + \frac{x_1\rho^{-1}}{r^2 + x_1\rho^{-1}}\right) +$$
$$+ \frac{\pi}{4\rho x_1}\left(-\frac{x_1 e^{-\frac{2}{3}\pi i}}{\rho + x_1 e^{-\frac{2}{3}\pi i}} - \frac{x_1^{-1}\rho^{-1} e^{-\frac{2}{3}\pi i}}{r^2 + x_1^{-1}\rho^{-1} e^{-\frac{2}{3}\pi i}} + \frac{1}{1 + x_1\rho e^{\frac{2}{3}\pi i}} + \frac{x_1\rho^{-1} e^{-\frac{2}{3}\pi i}}{r^2 + x_1\rho^{-1} e^{-\frac{2}{3}\pi i}}\right) +$$
$$+ \frac{\pi}{4\rho x_1}\left(-\frac{x_1 e^{\frac{2}{3}\pi i}}{\rho + x_1 e^{\frac{2}{3}\pi i}} - \frac{x_1^{-1}\rho^{-1} e^{\frac{2}{3}\pi i}}{r^2 + x_1^{-1}\rho^{-1} e^{\frac{2}{3}\pi i}} + \frac{1}{x_1\rho e^{-\frac{2}{3}\pi i} + 1} + \frac{x_1\rho^{-1} e^{\frac{2}{3}\pi i}}{r^2 + x_1\rho^{-1} e^{\frac{2}{3}\pi i}}\right)$$

$$= \frac{\pi}{4\rho x_1}\left(\frac{1}{1 + x_1\rho} - \frac{x_1}{\rho + x_1}\right) +$$
$$+ \frac{\pi}{4\rho x_1}\left(\frac{1}{1 + x_1\rho e^{\frac{2}{3}\pi i}} + \frac{1}{1 + x_1\rho e^{-\frac{2}{3}\pi i}} - \frac{x_1 e^{-\frac{2}{3}\pi i}}{\rho + x_1 e^{-\frac{2}{3}\pi i}} - \frac{x_1 e^{\frac{2}{3}\pi i}}{\rho + x_1 e^{\frac{2}{3}\pi i}}\right)$$
$$+ \frac{\pi}{4\rho x_1}\left(\frac{x_1\rho^{-1}}{r^2 + x_1\rho^{-1}} - \frac{x_1^{-1}\rho^{-1}}{r^2 + x_1^{-1}\rho^{-1}}\right) +$$
$$+ \frac{\pi}{4\rho x_1}\left(\frac{x_1\rho^{-1} e^{-\frac{2}{3}\pi i}}{r^2 + x_1\rho^{-1} e^{-\frac{2}{3}\pi i}} - \frac{x_1^{-1}\rho^{-1} e^{-\frac{2}{3}\pi i}}{r^2 + x_1^{-1}\rho^{-1} e^{-\frac{2}{3}\pi i}}\right) +$$
$$+ \frac{\pi}{4\rho x_1}\left(\frac{x_1\rho^{-1} e^{\frac{2}{3}\pi i}}{r^2 + x_1\rho^{-1} e^{\frac{2}{3}\pi i}} - \frac{x_1^{-1}\rho^{-1} e^{\frac{2}{3}\pi i}}{r^2 + x_1^{-1}\rho^{-1} e^{\frac{2}{3}\pi i}}\right) \tag{88}$$

I and II, d_2: pole at $w = -r^2\rho^{-1}$.

$$\frac{-\pi}{4\rho x_1}\left[\frac{x_1 r^2}{w - x_1 r^2} + \frac{x_1^{-1}}{w - x_1^{-1}} - \frac{x_1^{-1} r^2}{w - x_1^{-1} r^2} - \frac{x_1}{w - x_1}\right]_{w=-r^2\rho^{-1}} +$$

$$\frac{-\pi}{4\rho x_1}\left[\frac{x_1 r^2 e^{-\frac{2}{3}\pi i}}{w - x_1 r^2 e^{-\frac{2}{3}\pi i}} + \frac{x_1^{-1} e^{-\frac{2}{3}\pi i}}{w - x_1^{-1} e^{-\frac{2}{3}\pi i}} - \frac{x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}}{w - x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}} - \frac{x_1 e^{-\frac{2}{3}\pi i}}{w - x_1 e^{-\frac{2}{3}\pi i}}\right]_{w=-r^2\rho^{-1}} +$$

$$\frac{-\pi}{4\rho x_1}\left[\frac{x_1 r^2 e^{\frac{2}{3}\pi i}}{w - x_1 r^2 e^{\frac{2}{3}\pi i}} + \frac{x_1^{-1} e^{\frac{2}{3}\pi i}}{w - x_1^{-1} e^{\frac{2}{3}\pi i}} - \frac{x_1^{-1} r^2 e^{\frac{2}{3}\pi i}}{w - x_1^{-1} r^2 e^{\frac{2}{3}\pi i}} - \frac{x_1 e^{\frac{2}{3}\pi i}}{w - x_1 e^{\frac{2}{3}\pi i}}\right]_{w=-r^2\rho^{-1}} =$$

$$= \frac{-\pi}{4\rho x_1}\left(\frac{x_1 r^2}{-r^2\rho^{-1} - x_1 r^2} + \frac{x_1^{-1}}{-r^2\rho^{-1} - x_1^{-1}} - \frac{x_1^{-1} r^2}{-r^2\rho^{-1} - x_1^{-1} r^2} - \frac{x_1}{-r^2\rho^{-1} - x_1}\right) +$$

$$+ \frac{-\pi}{4\rho x_1}\left(\frac{x_1 r^2 e^{-\frac{2}{3}\pi i}}{-r^2\rho^{-1} - x_1 r^2 e^{-\frac{2}{3}\pi i}} + \frac{x_1^{-1} e^{-\frac{2}{3}\pi i}}{-r^2\rho^{-1} - x_1^{-1} e^{-\frac{2}{3}\pi i}} - \frac{x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}}{-r^2\rho^{-1} - x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}} - \frac{x_1 e^{-\frac{2}{3}\pi i}}{-r^2\rho^{-1} - x_1 e^{-\frac{2}{3}\pi i}}\right) +$$

$$+ \frac{-\pi}{4\rho x_1}\left(\frac{x_1 r^2 e^{\frac{2}{3}\pi i}}{-r^2\rho^{-1} - x_1 r^2 e^{\frac{2}{3}\pi i}} + \frac{x_1^{-1} e^{\frac{2}{3}\pi i}}{-r^2\rho^{-1} - x_1^{-1} e^{\frac{2}{3}\pi i}} - \frac{x_1^{-1} r^2 e^{\frac{2}{3}\pi i}}{-r^2\rho^{-1} - x_1^{-1} r^2 e^{\frac{2}{3}\pi i}} - \frac{x_1 e^{\frac{2}{3}\pi i}}{-r^2\rho^{-1} - x_1 e^{\frac{2}{3}\pi i}}\right)$$

$$= \frac{\pi}{4\rho x_1}\left(\frac{\rho}{x_1 + \rho} - \frac{x_1\rho}{1 + x_1\rho} + \frac{x_1\rho}{r^2 + x_1\rho} - \frac{x_1^{-1}\rho}{r^2 + x_1^{-1}\rho}\right) +$$

$$+ \frac{\pi}{4\rho x_1}\left(\frac{\rho e^{-\frac{2}{3}\pi i}}{x_1 + \rho e^{-\frac{2}{3}\pi i}} - \frac{x_1\rho e^{-\frac{2}{3}\pi i}}{1 + x_1\rho e^{-\frac{2}{3}\pi i}} + \frac{x_1\rho e^{-\frac{2}{3}\pi i}}{r^2 + x_1\rho e^{-\frac{2}{3}\pi i}} - \frac{x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}{r^2 + x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}\right) +$$

$$+ \frac{\pi}{4\rho x_1}\left(\frac{\rho e^{\frac{2}{3}\pi i}}{x_1 + \rho e^{\frac{2}{3}\pi i}} - \frac{x_1\rho e^{\frac{2}{3}\pi i}}{1 + x_1\rho e^{\frac{2}{3}\pi i}} + \frac{x_1\rho e^{\frac{2}{3}\pi i}}{r^2 + x_1\rho e^{\frac{2}{3}\pi i}} - \frac{x_1^{-1}\rho e^{\frac{2}{3}\pi i}}{r^2 + x_1^{-1}\rho e^{\frac{2}{3}\pi i}}\right)$$

$$= \frac{\pi}{4\rho x_1}\left(\frac{x_1\rho}{1 + x_1\rho} - \frac{\rho}{x_1 + \rho} + \frac{x_1\rho e^{-\frac{2}{3}\pi i}}{1 + x_1\rho e^{-\frac{2}{3}\pi i}} + \frac{x_1\rho e^{\frac{2}{3}\pi i}}{1 + x_1\rho e^{\frac{2}{3}\pi i}} - \frac{\rho e^{-\frac{2}{3}\pi i}}{x_1 + \rho e^{-\frac{2}{3}\pi i}} - \frac{\rho e^{\frac{2}{3}\pi i}}{x_1 + \rho e^{\frac{2}{3}\pi i}}\right) +$$

$$+ \frac{\pi}{4\rho x_1}\left(\frac{x_1^{-1}\rho}{r^2 + x_1^{-1}\rho} - \frac{x_1\rho}{r^2 + x_1\rho}\right) +$$

$$+ \frac{\pi}{4\rho x_1}\left(\frac{x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}{r^2 + x_1^{-1}\rho e^{-\frac{2}{3}\pi i}} - \frac{x_1\rho e^{-\frac{2}{3}\pi i}}{r^2 + x_1\rho e^{-\frac{2}{3}\pi i}}\right) +$$

$$+ \frac{\pi}{4\rho x_1}\left(\frac{x_1^{-1}\rho e^{\frac{2}{3}\pi i}}{r^2 + x_1^{-1}\rho e^{\frac{2}{3}\pi i}} - \frac{x_1\rho e^{\frac{2}{3}\pi i}}{r^2 + x_1\rho e^{\frac{2}{3}\pi i}}\right)$$

$$\begin{aligned}
= \quad & \frac{\pi}{4\rho x_1}\left(\frac{x_1\rho}{1+x_1\rho}-\frac{\rho}{x_1+\rho}+\frac{x_1\rho\,(2x_1\rho-1)}{1-x_1\rho+x_1^2\rho^2}-\frac{\rho\,(2\rho-x_1)}{x_1^2-x_1\rho+\rho^2}\right)+\\
&+\frac{\pi}{4\rho x_1}\left(\frac{x_1^{-1}\rho}{r^2+x_1^{-1}\rho}-\frac{x_1\rho}{r^2+x_1\rho}\right)+\\
&+\frac{\pi}{4\rho x_1}\left(\frac{x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}-\frac{x_1\rho e^{-\frac{2}{3}\pi i}}{r^2+x_1\rho e^{-\frac{2}{3}\pi i}}\right)+\\
&+\frac{\pi}{4\rho x_1}\left(\frac{x_1^{-1}\rho e^{\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho e^{\frac{2}{3}\pi i}}-\frac{x_1\rho e^{\frac{2}{3}\pi i}}{r^2+x_1\rho e^{\frac{2}{3}\pi i}}\right)
\end{aligned} \tag{89}$$

III, d_2: pole at $w=-\rho$.

$$\begin{aligned}
&\frac{-\pi}{4\rho x_1}\left[\frac{x_1r^2}{w-x_1r^2}+\frac{x_1^{-1}}{w-x_1^{-1}}-\frac{x_1^{-1}r^2}{w-x_1^{-1}r^2}-\frac{x_1}{w-x_1}\right]_{w=-\rho}+\\
&\frac{-\pi}{4\rho x_1}\left[\frac{x_1r^2e^{-\frac{2}{3}\pi i}}{w-x_1r^2e^{-\frac{2}{3}\pi i}}+\frac{x_1^{-1}e^{-\frac{2}{3}\pi i}}{w-x_1^{-1}e^{-\frac{2}{3}\pi i}}-\frac{x_1^{-1}r^2e^{-\frac{2}{3}\pi i}}{w-x_1^{-1}r^2e^{-\frac{2}{3}\pi i}}-\frac{x_1e^{-\frac{2}{3}\pi i}}{w-x_1e^{-\frac{2}{3}\pi i}}\right]_{w=-\rho}+\\
&\frac{-\pi}{4\rho x_1}\left[\frac{x_1r^2e^{\frac{2}{3}\pi i}}{w-x_1r^2e^{\frac{2}{3}\pi i}}+\frac{x_1^{-1}e^{\frac{2}{3}\pi i}}{w-x_1^{-1}e^{\frac{2}{3}\pi i}}-\frac{x_1^{-1}r^2e^{\frac{2}{3}\pi i}}{w-x_1^{-1}r^2e^{\frac{2}{3}\pi i}}-\frac{x_1e^{\frac{2}{3}\pi i}}{w-x_1e^{\frac{2}{3}\pi i}}\right]_{w=-\rho}=
\end{aligned}$$

$$\begin{aligned}
= \quad & \frac{-\pi}{4\rho x_1}\left(\frac{x_1r^2}{-\rho-x_1r^2}+\frac{x_1^{-1}}{-\rho-x_1^{-1}}-\frac{x_1^{-1}r^2}{-\rho-x_1^{-1}r^2}-\frac{x_1}{-\rho-x_1}\right)+\\
&+\frac{-\pi}{4\rho x_1}\left(\frac{x_1r^2e^{-\frac{2}{3}\pi i}}{-\rho-x_1r^2e^{-\frac{2}{3}\pi i}}+\frac{x_1^{-1}e^{-\frac{2}{3}\pi i}}{-\rho-x_1^{-1}e^{-\frac{2}{3}\pi i}}-\frac{x_1^{-1}r^2e^{-\frac{2}{3}\pi i}}{-\rho-x_1^{-1}r^2e^{-\frac{2}{3}\pi i}}-\frac{x_1e^{-\frac{2}{3}\pi i}}{-\rho-x_1e^{-\frac{2}{3}\pi i}}\right)+\\
&+\frac{-\pi}{4\rho x_1}\left(\frac{x_1r^2e^{\frac{2}{3}\pi i}}{-\rho-x_1r^2e^{\frac{2}{3}\pi i}}+\frac{x_1^{-1}e^{\frac{2}{3}\pi i}}{-\rho-x_1^{-1}e^{\frac{2}{3}\pi i}}-\frac{x_1^{-1}r^2e^{\frac{2}{3}\pi i}}{-\rho-x_1^{-1}r^2e^{\frac{2}{3}\pi i}}-\frac{x_1e^{\frac{2}{3}\pi i}}{-\rho-x_1e^{\frac{2}{3}\pi i}}\right)
\end{aligned}$$

$$\begin{aligned}
= \quad & \frac{-\pi}{4\rho x_1}\left(-1+\frac{\rho}{\rho+x_1r^2}-\frac{x_1^{-1}}{\rho+x_1^{-1}}+1-\frac{\rho}{\rho+x_1^{-1}r^2}+\frac{x_1}{\rho+x_1}\right)+\\
&+\frac{-\pi}{4\rho x_1}\left(-1+\frac{\rho}{\rho+x_1r^2e^{-\frac{2}{3}\pi i}}-\frac{x_1^{-1}e^{-\frac{2}{3}\pi i}}{\rho+x_1^{-1}e^{-\frac{2}{3}\pi i}}+1-\frac{\rho}{\rho+x_1^{-1}r^2e^{-\frac{2}{3}\pi i}}+\frac{x_1e^{-\frac{2}{3}\pi i}}{\rho+x_1e^{-\frac{2}{3}\pi i}}\right)+\\
&+\frac{-\pi}{4\rho x_1}\left(-1+\frac{\rho}{\rho+x_1r^2e^{\frac{2}{3}\pi i}}-\frac{x_1^{-1}e^{\frac{2}{3}\pi i}}{\rho+x_1^{-1}e^{\frac{2}{3}\pi i}}+1-\frac{\rho}{\rho+x_1^{-1}r^2e^{\frac{2}{3}\pi i}}+\frac{x_1e^{\frac{2}{3}\pi i}}{\rho+x_1e^{\frac{2}{3}\pi i}}\right)
\end{aligned}$$

$$\begin{aligned}
= \quad & \frac{-\pi}{4\rho x_1}\left(\frac{x_1}{\rho+x_1}-\frac{1}{x_1\rho+1}+\frac{x_1^{-1}\rho}{r^2+x_1^{-1}\rho}-\frac{x_1\rho}{r^2+x_1\rho}\right)+\\
&+\frac{-\pi}{4\rho x_1}\left(\frac{x_1e^{-\frac{2}{3}\pi i}}{\rho+x_1e^{-\frac{2}{3}\pi i}}-\frac{1}{1+x_1\rho e^{\frac{2}{3}\pi i}}+\frac{x_1^{-1}\rho e^{\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho e^{\frac{2}{3}\pi i}}-\frac{x_1\rho e^{\frac{2}{3}\pi i}}{r^2+x_1\rho e^{\frac{2}{3}\pi i}}\right)+\\
&+\frac{-\pi}{4\rho x_1}\left(\frac{x_1e^{\frac{2}{3}\pi i}}{\rho+x_1e^{\frac{2}{3}\pi i}}-\frac{1}{1+x_1\rho e^{-\frac{2}{3}\pi i}}+\frac{x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}-\frac{x_1\rho e^{-\frac{2}{3}\pi i}}{r^2+x_1\rho e^{-\frac{2}{3}\pi i}}\right).
\end{aligned} \tag{90}$$

$$
\begin{aligned}
= \quad & \frac{-\pi}{4\rho x_1}\left(\frac{x_1}{\rho+x_1}-\frac{1}{x_1\rho+1}-\frac{1}{1+x_1\rho e^{\frac{2}{3}\pi i}}-\frac{1}{1+x_1\rho e^{-\frac{2}{3}\pi i}}+\frac{x_1 e^{-\frac{2}{3}\pi i}}{\rho+x_1 e^{-\frac{2}{3}\pi i}}+\frac{x_1 e^{\frac{2}{3}\pi i}}{\rho+x_1 e^{\frac{2}{3}\pi i}}\right)+\\
& +\frac{-\pi}{4\rho x_1}\left(\frac{x_1^{-1}\rho}{r^2+x_1^{-1}\rho}-\frac{x_1\rho}{r^2+x_1\rho}\right)+\\
& +\frac{-\pi}{4\rho x_1}\left(\frac{x_1^{-1}\rho e^{\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho e^{\frac{2}{3}\pi i}}-\frac{x_1\rho e^{\frac{2}{3}\pi i}}{r^2+x_1\rho e^{\frac{2}{3}\pi i}}\right)+\\
& +\frac{-\pi}{4\rho x_1}\left(\frac{x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}-\frac{x_1\rho e^{-\frac{2}{3}\pi i}}{r^2+x_1\rho e^{-\frac{2}{3}\pi i}}\right).
\end{aligned} \tag{91}
$$

$$
\begin{aligned}
= \quad & \frac{\pi}{4\rho x_1}\left(\frac{1}{x_1\rho+1}-\frac{x_1}{\rho+x_1}+\frac{2-x_1\rho}{1-x_1\rho+x_1^2\rho^2}+\frac{(\rho-2x_1)\,x_1}{x_1^2-x_1\rho+\rho^2}\right)+\\
& +\frac{\pi}{4\rho x_1}\left(\frac{x_1\rho}{r^2+x_1\rho}-\frac{x_1^{-1}\rho}{r^2+x_1^{-1}\rho}\right)+\\
& +\frac{\pi}{4\rho x_1}\left(\frac{x_1\rho e^{\frac{2}{3}\pi i}}{r^2+x_1\rho e^{\frac{2}{3}\pi i}}-\frac{x_1^{-1}\rho e^{\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho e^{\frac{2}{3}\pi i}}\right)+\\
& +\frac{\pi}{4\rho x_1}\left(\frac{x_1\rho e^{-\frac{2}{3}\pi i}}{r^2+x_1\rho e^{-\frac{2}{3}\pi i}}-\frac{x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}\right).
\end{aligned} \tag{92}
$$

7.4. Integration in the radial direction

It remains to combine the appropriate residues and integrate them with respect to r. We will still assume $x_1 < \rho$.

• From 0 to 1 :

The part that is integrated without distinction in formula from 0 to 1 we denote

$$\int_{r=0}^{1} I_6\left(x_1,\rho;r\right)\, r\, dr, \tag{93}$$

where

$$I_6\left(x_1,\rho;r\right)=\#_{(79)}+\#_{(82)}+\#_{(85)}+\#_{(88)}.$$

We have

$$\int_{r=0}^{1} I_6\left(x_1,\rho;r\right)\,r\,dr =$$

$$= \frac{\pi}{4x_1\rho}\int_{r=0}^{1}\Bigg(\frac{1}{x_1\rho+1}-\frac{\rho}{x_1+\rho}+\frac{x_1^{-1}\rho}{r^2+x_1^{-1}\rho}-\frac{x_1^{-1}\rho^{-1}}{r^2+x_1^{-1}\rho^{-1}}+$$

$$+\frac{1}{x_1\rho e^{-\frac{2}{3}\pi i}+1}-\frac{\rho}{x_1e^{-\frac{2}{3}\pi i}+\rho}+\frac{x_1^{-1}\rho e^{\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho e^{\frac{2}{3}\pi i}}-\frac{x_1^{-1}\rho^{-1}e^{\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho^{-1}e^{\frac{2}{3}\pi i}}+$$

$$+\frac{1}{x_1\rho e^{\frac{2}{3}\pi i}+1}-\frac{\rho}{x_1e^{\frac{2}{3}\pi i}+\rho}+\frac{x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}-\frac{x_1^{-1}\rho^{-1}e^{-\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho^{-1}e^{-\frac{2}{3}\pi i}}+$$

$$+\frac{1}{x_1\rho+1}-\frac{x_1}{\rho+x_1}+$$

$$+\frac{1}{1+x_1\rho e^{\frac{2}{3}\pi i}}-\frac{x_1e^{-\frac{2}{3}\pi i}}{\rho+x_1e^{-\frac{2}{3}\pi i}}+\frac{1}{x_1\rho e^{-\frac{2}{3}\pi i}+1}-\frac{x_1e^{\frac{2}{3}\pi i}}{\rho+x_1e^{\frac{2}{3}\pi i}}+$$

$$+\frac{x_1\rho^{-1}}{r^2+x_1\rho^{-1}}-\frac{x_1^{-1}\rho^{-1}}{r^2+x_1^{-1}\rho^{-1}}+$$

$$+\frac{x_1\rho^{-1}e^{-\frac{2}{3}\pi i}}{r^2+x_1\rho^{-1}e^{-\frac{2}{3}\pi i}}-\frac{x_1^{-1}\rho^{-1}e^{-\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho^{-1}e^{-\frac{2}{3}\pi i}}+$$

$$+\frac{x_1\rho^{-1}e^{\frac{2}{3}\pi i}}{r^2+x_1\rho^{-1}e^{\frac{2}{3}\pi i}}-\frac{x_1^{-1}\rho^{-1}e^{\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho^{-1}e^{\frac{2}{3}\pi i}}\Bigg)\,r\,dr$$

$$= \frac{\pi}{4x_1\rho}\int_{r=0}^{1}\Bigg(\frac{1}{x_1\rho+1}+\frac{1}{x_1\rho+1}-\frac{x_1}{\rho+x_1}-\frac{\rho}{x_1+\rho}+$$

$$+\frac{1}{x_1\rho e^{-\frac{2}{3}\pi i}+1}+\frac{1}{x_1\rho e^{\frac{2}{3}\pi i}+1}+\frac{1}{1+x_1\rho e^{\frac{2}{3}\pi i}}+\frac{1}{x_1\rho e^{-\frac{2}{3}\pi i}+1}+$$

$$-\frac{\rho}{x_1e^{-\frac{2}{3}\pi i}+\rho}-\frac{\rho}{x_1e^{\frac{2}{3}\pi i}+\rho}-\frac{x_1e^{-\frac{2}{3}\pi i}}{\rho+x_1e^{-\frac{2}{3}\pi i}}-\frac{x_1e^{\frac{2}{3}\pi i}}{\rho+x_1e^{\frac{2}{3}\pi i}}+$$

$$+\frac{x_1^{-1}\rho}{r^2+x_1^{-1}\rho}-\frac{x_1^{-1}\rho^{-1}}{r^2+x_1^{-1}\rho^{-1}}+\frac{x_1\rho^{-1}}{r^2+x_1\rho^{-1}}-\frac{x_1^{-1}\rho^{-1}}{r^2+x_1^{-1}\rho^{-1}}+$$

$$+\frac{x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}+\frac{x_1^{-1}\rho e^{\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho e^{\frac{2}{3}\pi i}}+\frac{x_1\rho^{-1}e^{\frac{2}{3}\pi i}}{r^2+x_1\rho^{-1}e^{\frac{2}{3}\pi i}}+\frac{x_1\rho^{-1}e^{-\frac{2}{3}\pi i}}{r^2+x_1\rho^{-1}e^{-\frac{2}{3}\pi i}}+$$

$$-\frac{x_1^{-1}\rho^{-1}e^{-\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho^{-1}e^{-\frac{2}{3}\pi i}}-\frac{x_1^{-1}\rho^{-1}e^{\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho^{-1}e^{\frac{2}{3}\pi i}}-\frac{x_1^{-1}\rho^{-1}e^{-\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho^{-1}e^{-\frac{2}{3}\pi i}}-\frac{x_1^{-1}\rho^{-1}e^{\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho^{-1}e^{\frac{2}{3}\pi i}}\Bigg)\,r\,dr$$

$$
\begin{aligned}
= \quad & \frac{\pi}{4x_1\rho}\int_{r=0}^{1}\left(\frac{2}{x_1\rho+1}+\frac{2}{1+x_1\rho e^{-\frac{2}{3}\pi i}}+\frac{2}{1+x_1\rho e^{\frac{2}{3}\pi i}}-3+\right.\\
& +\frac{x_1^{-1}\rho}{r^2+x_1^{-1}\rho}+\frac{x_1\rho^{-1}}{r^2+x_1\rho^{-1}}-2\frac{x_1^{-1}\rho^{-1}}{r^2+x_1^{-1}\rho^{-1}}+\\
& +\frac{x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}+\frac{x_1^{-1}\rho e^{\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho e^{\frac{2}{3}\pi i}}+\frac{x_1\rho^{-1}e^{\frac{2}{3}\pi i}}{r^2+x_1\rho^{-1}e^{\frac{2}{3}\pi i}}+\frac{x_1\rho^{-1}e^{-\frac{2}{3}\pi i}}{r^2+x_1\rho^{-1}e^{-\frac{2}{3}\pi i}}+\\
& \left.-2\frac{x_1^{-1}\rho^{-1}e^{-\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho^{-1}e^{-\frac{2}{3}\pi i}}-2\frac{x_1^{-1}\rho^{-1}e^{\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho^{-1}e^{\frac{2}{3}\pi i}}\right) r\,dr
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{\pi}{8x_1\rho}\left(\frac{2}{x_1\rho+1}+\frac{2}{1+x_1\rho e^{-\frac{2}{3}\pi i}}+\frac{2}{1+x_1\rho e^{\frac{2}{3}\pi i}}-3+\right.\\
& +x_1^{-1}\rho\mathrm{Ln}\left(\frac{1+x_1^{-1}\rho}{x_1^{-1}\rho}\right)+x_1\rho^{-1}\mathrm{Ln}\left(\frac{1+x_1\rho^{-1}}{x_1\rho^{-1}}\right)-2x_1^{-1}\rho^{-1}\mathrm{Ln}\left(\frac{1+x_1^{-1}\rho^{-1}}{x_1^{-1}\rho^{-1}}\right)+\\
& +x_1^{-1}\rho e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{1+x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}{x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}\right)+x_1^{-1}\rho e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{1+x_1^{-1}\rho e^{\frac{2}{3}\pi i}}{x_1^{-1}\rho e^{\frac{2}{3}\pi i}}\right)+\\
& +x_1\rho^{-1}e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{1+x_1\rho^{-1}e^{\frac{2}{3}\pi i}}{x_1\rho^{-1}e^{\frac{2}{3}\pi i}}\right)+x_1\rho^{-1}e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{1+x_1\rho^{-1}e^{-\frac{2}{3}\pi i}}{x_1\rho^{-1}e^{-\frac{2}{3}\pi i}}\right)+\\
& \left.-2x_1^{-1}\rho^{-1}e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{1+x_1^{-1}\rho^{-1}e^{-\frac{2}{3}\pi i}}{x_1^{-1}\rho^{-1}e^{-\frac{2}{3}\pi i}}\right)-2x_1^{-1}\rho^{-1}e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{1+x_1^{-1}\rho^{-1}e^{\frac{2}{3}\pi i}}{x_1^{-1}\rho^{-1}e^{\frac{2}{3}\pi i}}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{\pi}{8x_1\rho}\left(\frac{1-x_1\rho}{1+x_1\rho}+\frac{1-x_1\rho e^{-\frac{2}{3}\pi i}}{1+x_1\rho e^{-\frac{2}{3}\pi i}}+\frac{1-x_1\rho e^{\frac{2}{3}\pi i}}{1+x_1\rho e^{\frac{2}{3}\pi i}}+\right.\\
& +x_1^{-1}\rho\mathrm{Ln}\left(1+x_1\rho^{-1}\right)+x_1\rho^{-1}\mathrm{Ln}\left(1+x_1^{-1}\rho\right)-2x_1^{-1}\rho^{-1}\mathrm{Ln}\left(1+x_1\rho\right)+\\
& +x_1^{-1}\rho e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(1+x_1\rho^{-1}e^{\frac{2}{3}\pi i}\right)+x_1^{-1}\rho e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(1+x_1\rho^{-1}e^{-\frac{2}{3}\pi i}\right)+\\
& +x_1\rho^{-1}e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(1+x_1^{-1}\rho e^{-\frac{2}{3}\pi i}\right)+x_1\rho^{-1}e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(1+x_1^{-1}\rho e^{\frac{2}{3}\pi i}\right)+\\
& \left.-2x_1^{-1}\rho^{-1}e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(1+x_1\rho e^{\frac{2}{3}\pi i}\right)-2x_1^{-1}\rho^{-1}e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(1+x_1\rho e^{-\frac{2}{3}\pi i}\right)\right)
\end{aligned}
\tag{94}
$$

- From 0 to ρ :

The part that is integrated without distinction in formula from 0 to ρ we denote

$$\int_{r=0}^{\rho} I_7\left(x_1,\rho;r\right)\ r\,dr, \tag{95}$$

where

$$I_7\left(x_1,\rho;r\right)=\#_{(89)}.$$

Hence

$$
\begin{aligned}
&\int_{r=0}^{\rho} I_7\left(x_1,\rho;r\right)\, r\, dr = \\
= \quad & \int_{r=0}^{\rho} \left(\frac{\pi}{4\rho x_1}\left(\frac{x_1\rho}{1+x_1\rho} - \frac{\rho}{x_1+\rho} + \frac{x_1\rho\left(2x_1\rho-1\right)}{1-x_1\rho+x_1^2\rho^2} - \frac{\rho\left(2\rho-x_1\right)}{x_1^2-x_1\rho+\rho^2}\right) + \right.\\
&+ \frac{\pi}{4\rho x_1}\left(\frac{x_1^{-1}\rho}{r^2+x_1^{-1}\rho} - \frac{x_1\rho}{r^2+x_1\rho}\right) + \\
&+ \frac{\pi}{4\rho x_1}\left(\frac{x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho e^{-\frac{2}{3}\pi i}} - \frac{x_1\rho e^{-\frac{2}{3}\pi i}}{r^2+x_1\rho e^{-\frac{2}{3}\pi i}}\right) + \\
&\left. + \frac{\pi}{4\rho x_1}\left(\frac{x_1^{-1}\rho e^{\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho e^{\frac{2}{3}\pi i}} - \frac{x_1\rho e^{\frac{2}{3}\pi i}}{r^2+x_1\rho e^{\frac{2}{3}\pi i}}\right)\right) r\, dr \\
= \quad & \frac{\pi}{8\rho x_1}\left[\left(\frac{x_1\rho}{1+x_1\rho} - \frac{\rho}{x_1+\rho} + \frac{x_1\rho\left(2x_1\rho-1\right)}{1-x_1\rho+x_1^2\rho^2} - \frac{\rho\left(2\rho-x_1\right)}{x_1^2-x_1\rho+\rho^2}\right) r^2 + \right.\\
&+ x_1^{-1}\rho \mathrm{Ln}\left(r^2+x_1^{-1}\rho\right) - x_1\rho \mathrm{Ln}\left(r^2+x_1\rho\right) + \\
&+ x_1^{-1}\rho e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(r^2+x_1^{-1}\rho e^{-\frac{2}{3}\pi i}\right) - x_1\rho e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(r^2+x_1\rho e^{-\frac{2}{3}\pi i}\right) + \\
&\left. + x_1^{-1}\rho e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(r^2+x_1^{-1}\rho e^{\frac{2}{3}\pi i}\right) - x_1\rho e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(r^2+x_1\rho e^{\frac{2}{3}\pi i}\right)\right]_{r=0}^{\rho} \\
= \quad & \frac{\pi}{8\rho x_1}\left(\left(\frac{x_1\rho}{1+x_1\rho} - \frac{\rho}{x_1+\rho} + \frac{x_1\rho\left(2x_1\rho-1\right)}{1-x_1\rho+x_1^2\rho^2} - \frac{\rho\left(2\rho-x_1\right)}{x_1^2-x_1\rho+\rho^2}\right)\rho^2 + \right.\\
&+ x_1^{-1}\rho \mathrm{Ln}\left(\frac{\rho^2+x_1^{-1}\rho}{x_1^{-1}\rho}\right) - x_1\rho \mathrm{Ln}\left(\frac{\rho^2+x_1\rho}{x_1\rho}\right) + \\
&+ x_1^{-1}\rho e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{\rho^2+x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}{x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}\right) - x_1\rho e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{\rho^2+x_1\rho e^{-\frac{2}{3}\pi i}}{x_1\rho e^{-\frac{2}{3}\pi i}}\right) + \\
&\left. + x_1^{-1}\rho e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{\rho^2+x_1^{-1}\rho e^{\frac{2}{3}\pi i}}{x_1^{-1}\rho e^{\frac{2}{3}\pi i}}\right) - x_1\rho e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{\rho^2+x_1\rho e^{\frac{2}{3}\pi i}}{x_1\rho e^{\frac{2}{3}\pi i}}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{\pi}{8\rho x_1}\Bigg(\left(\frac{x_1\rho}{1+x_1\rho}-\frac{\rho}{x_1+\rho}-\frac{\rho\,(2\rho-x_1)}{x_1^2-x_1\rho+\rho^2}+\frac{x_1\rho\,(2x_1\rho-1)}{1-x_1\rho+x_1^2\rho^2}\right)\rho^2+ \\
& -x_1\rho\mathrm{Ln}\left(1+x_1^{-1}\rho\right)+x_1^{-1}\rho\mathrm{Ln}\left(1+x_1\rho\right)+ \\
& -x_1\rho e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(1+x_1^{-1}\rho e^{\frac{2}{3}\pi i}\right)+x_1^{-1}\rho e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(1+x_1\rho e^{\frac{2}{3}\pi i}\right)+ \\
& -x_1\rho e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(1+x_1^{-1}\rho e^{-\frac{2}{3}\pi i}\right)+x_1^{-1}\rho e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(1+x_1\rho e^{-\frac{2}{3}\pi i}\right)\Bigg) \qquad (96)
\end{aligned}
$$

- From 0 to x_1 :

The part of the formula which only concerns the interval $(0, x_1)$ we denote as

$$\int_{r=0}^{x_1} I_8\left(x_1,\rho;r\right)\,r\,dr, \qquad (97)$$

where

$$I_8\left(x_1,\rho;r\right)=\#_{(80)}+\#_{(83)}+\#_{(86)}.$$

$$
\begin{aligned}
& \int_{r=0}^{x_1} I_8\left(x_1,\rho;r\right)\,r\,dr = \\
= \quad & \frac{\pi}{4x_1\rho}\int_{r=0}^{x_1}\left(\frac{x_1\rho}{1+x_1\rho}-\frac{x_1}{\rho+x_1}+\frac{x_1\rho^{-1}}{r^2+x_1\rho^{-1}}-\frac{x_1\rho}{r^2+x_1\rho}\right)r\,dr+ \\
& +\frac{\pi}{4x_1\rho}\int_{r=0}^{x_1}\left(\frac{x_1\rho e^{\frac{2}{3}\pi i}}{1+x_1\rho e^{\frac{2}{3}\pi i}}-\frac{x_1}{\rho e^{-\frac{2}{3}\pi i}+x_1}+\frac{x_1\rho^{-1}e^{\frac{2}{3}\pi i}}{r^2+x_1\rho^{-1}e^{\frac{2}{3}\pi i}}-\frac{x_1\rho e^{\frac{2}{3}\pi i}}{r^2+x_1\rho e^{\frac{2}{3}\pi i}}\right)r\,dr+ \\
& +\frac{\pi}{4x_1\rho}\int_{r=0}^{x_1}\left(\frac{x_1\rho}{e^{\frac{2}{3}\pi i}+x_1\rho}-\frac{x_1}{\rho e^{\frac{2}{3}\pi i}+x_1}+\frac{x_1\rho^{-1}e^{-\frac{2}{3}\pi i}}{r^2+x_1\rho^{-1}e^{-\frac{2}{3}\pi i}}-\frac{x_1\rho e^{-\frac{2}{3}\pi i}}{r^2+x_1\rho e^{-\frac{2}{3}\pi i}}\right)r\,dr
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{\pi}{4x_1\rho}\int_{r=0}^{x_1}\left(\frac{x_1\rho}{1+x_1\rho}-\frac{x_1}{\rho+x_1}+\frac{x_1\rho e^{\frac{2}{3}\pi i}}{1+x_1\rho e^{\frac{2}{3}\pi i}}+\frac{x_1\rho e^{-\frac{2}{3}\pi i}}{1+x_1\rho e^{-\frac{2}{3}\pi i}}-\frac{x_1}{\rho e^{-\frac{2}{3}\pi i}+x_1}-\frac{x_1}{\rho e^{\frac{2}{3}\pi i}+x_1}\right)r\,dr+ \\
& +\frac{\pi}{4x_1\rho}\int_{r=0}^{x_1}\left(\frac{x_1\rho^{-1}}{r^2+x_1\rho^{-1}}-\frac{x_1\rho}{r^2+x_1\rho}\right)r\,dr+ \\
& +\frac{\pi}{4x_1\rho}\int_{r=0}^{x_1}\left(\frac{x_1\rho^{-1}e^{\frac{2}{3}\pi i}}{r^2+x_1\rho^{-1}e^{\frac{2}{3}\pi i}}-\frac{x_1\rho e^{\frac{2}{3}\pi i}}{r^2+x_1\rho e^{\frac{2}{3}\pi i}}\right)r\,dr+ \\
& +\frac{\pi}{4x_1\rho}\int_{r=0}^{x_1}\left(\frac{x_1\rho^{-1}e^{-\frac{2}{3}\pi i}}{r^2+x_1\rho^{-1}e^{-\frac{2}{3}\pi i}}-\frac{x_1\rho e^{-\frac{2}{3}\pi i}}{r^2+x_1\rho e^{-\frac{2}{3}\pi i}}\right)r\,dr
\end{aligned}
$$

$$\begin{aligned}
= & \frac{\pi}{8x_1\rho}\left(\frac{x_1\rho}{1+x_1\rho}-\frac{x_1}{\rho+x_1}+\frac{(2x_1\rho-1)\,x_1\rho}{1-x_1\rho+x_1^2\rho^2}-\frac{(2x_1-\rho)\,x_1}{\rho^2-x_1\rho+x_1^2}\right)x_1^2+\\
&+\frac{\pi}{8x_1\rho}\left(x_1\rho^{-1}\mathrm{Ln}\left(\frac{x_1^2+x_1\rho^{-1}}{x_1\rho^{-1}}\right)-x_1\rho\mathrm{Ln}\left(\frac{x_1^2+x_1\rho}{x_1\rho}\right)\right)+\\
&+\frac{\pi}{8x_1\rho}\left(x_1\rho^{-1}e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{x_1^2+x_1\rho^{-1}e^{\frac{2}{3}\pi i}}{x_1\rho^{-1}e^{\frac{2}{3}\pi i}}\right)-x_1\rho e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{x_1^2+x_1\rho e^{\frac{2}{3}\pi i}}{x_1\rho e^{\frac{2}{3}\pi i}}\right)\right)+\\
&+\frac{\pi}{8x_1\rho}\left(x_1\rho^{-1}e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{x_1^2+x_1\rho^{-1}e^{-\frac{2}{3}\pi i}}{x_1\rho^{-1}e^{-\frac{2}{3}\pi i}}\right)-x_1\rho e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{x_1^2+x_1\rho e^{-\frac{2}{3}\pi i}}{x_1\rho e^{-\frac{2}{3}\pi i}}\right)\right)
\end{aligned}$$

$$\begin{aligned}
= & \frac{\pi}{8x_1\rho}\left(\frac{x_1\rho}{1+x_1\rho}-\frac{x_1}{\rho+x_1}+\frac{(2x_1\rho-1)\,x_1\rho}{1-x_1\rho+x_1^2\rho^2}-\frac{(2x_1-\rho)\,x_1}{\rho^2-x_1\rho+x_1^2}\right)x_1^2+\\
&+\frac{\pi}{8x_1\rho}\left(x_1\rho^{-1}\mathrm{Ln}\left(1+x_1\rho\right)-x_1\rho\mathrm{Ln}\left(1+x_1\rho^{-1}\right)\right)+\\
&+\frac{\pi}{8x_1\rho}\left(x_1\rho^{-1}e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(1+x_1\rho e^{-\frac{2}{3}\pi i}\right)-x_1\rho e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(1+x_1\rho^{-1}e^{-\frac{2}{3}\pi i}\right)\right)+\\
&+\frac{\pi}{8x_1\rho}\left(x_1\rho^{-1}e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(1+x_1\rho e^{\frac{2}{3}\pi i}\right)-x_1\rho e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(1+x_1\rho^{-1}e^{\frac{2}{3}\pi i}\right)\right)
\end{aligned}$$

$$\begin{aligned}
= & \frac{\pi}{8x_1\rho}x_1^2\left(\frac{x_1\rho}{1+x_1\rho}-\frac{x_1}{\rho+x_1}+\frac{(2x_1\rho-1)\,x_1\rho}{1-x_1\rho+x_1^2\rho^2}-\frac{(2x_1-\rho)\,x_1}{\rho^2-x_1\rho+x_1^2}\right)+\\
&+\frac{\pi}{8x_1\rho}x_1\rho^{-1}\left(\mathrm{Ln}\left(1+x_1\rho\right)+e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(1+x_1\rho e^{-\frac{2}{3}\pi i}\right)+e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(1+x_1\rho e^{\frac{2}{3}\pi i}\right)\right)+\\
&-\frac{\pi}{8x_1\rho}x_1\rho\left(\mathrm{Ln}\left(1+x_1\rho^{-1}\right)+e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(1+x_1\rho^{-1}e^{-\frac{2}{3}\pi i}\right)+e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(1+x_1\rho^{-1}e^{\frac{2}{3}\pi i}\right)\right). \qquad (98)
\end{aligned}$$

• From x_1 to 1 :

The part that is integrated without distinction in formula from x_1 to 1 we denote as

$$\int_{r=x_1}^{1} I_9\left(x_1,\rho;r\right)\,r\,dr, \qquad (99)$$

where

$$I_9\left(x_1,\rho;r\right)=\#_{(81)}+\#_{(84)}+\#_{(87)}.$$

$$\begin{aligned}
&\int_{r=x_1}^{1} I_9\left(x_1,\rho;r\right)\,r\,dr=\\
= & \frac{\pi}{4x_1\rho}\int_{r=x_1}^{1}\left(\frac{1}{x_1\rho+1}-\frac{\rho}{x_1+\rho}+\frac{x_1\rho}{r^2+x_1\rho}-\frac{x_1\rho^{-1}}{r^2+x_1\rho^{-1}}\right)r\,dr+\\
&+\frac{\pi}{4x_1\rho}\int_{r=x_1}^{1}\left(\frac{1}{x_1\rho e^{-\frac{2}{3}\pi i}+1}-\frac{\rho}{x_1e^{-\frac{2}{3}\pi i}+\rho}+\frac{x_1\rho e^{-\frac{2}{3}\pi i}}{r^2+x_1\rho e^{-\frac{2}{3}\pi i}}-\frac{x_1\rho^{-1}e^{-\frac{2}{3}\pi i}}{r^2+x_1\rho^{-1}e^{-\frac{2}{3}\pi i}}\right)r\,dr+\\
&+\frac{\pi}{4x_1\rho}\int_{r=x_1}^{1}\left(\frac{1}{x_1\rho e^{\frac{2}{3}\pi i}+1}-\frac{\rho}{x_1e^{\frac{2}{3}\pi i}+\rho}+\frac{x_1\rho e^{\frac{2}{3}\pi i}}{r^2+x_1\rho e^{\frac{2}{3}\pi i}}-\frac{x_1\rho^{-1}e^{\frac{2}{3}\pi i}}{r^2+x_1\rho^{-1}e^{\frac{2}{3}\pi i}}\right)r\,dr
\end{aligned}$$

$$= \frac{\pi}{4x_1\rho}\int_{r=x_1}^{1}\left(\frac{1}{x_1\rho+1}-\frac{\rho}{x_1+\rho}+\frac{1}{x_1\rho e^{-\frac{2}{3}\pi i}+1}+\frac{1}{1+x_1\rho e^{\frac{2}{3}\pi i}}-\frac{\rho}{x_1e^{-\frac{2}{3}\pi i}+\rho}-\frac{\rho}{x_1e^{\frac{2}{3}\pi i}+\rho}\right) r\,dr +$$
$$+\frac{\pi}{4x_1\rho}\int_{r=x_1}^{1}\left(\frac{x_1\rho}{r^2+x_1\rho}-\frac{x_1\rho^{-1}}{r^2+x_1\rho^{-1}}\right) r\,dr +$$
$$+\frac{\pi}{4x_1\rho}\int_{r=x_1}^{1}\left(+\frac{x_1\rho e^{-\frac{2}{3}\pi i}}{r^2+x_1\rho e^{-\frac{2}{3}\pi i}}-\frac{x_1\rho^{-1}e^{-\frac{2}{3}\pi i}}{r^2+x_1\rho^{-1}e^{-\frac{2}{3}\pi i}}\right) r\,dr +$$
$$+\frac{\pi}{4x_1\rho}\int_{r=x_1}^{1}\left(+\frac{x_1\rho e^{\frac{2}{3}\pi i}}{r^2+x_1\rho e^{\frac{2}{3}\pi i}}-\frac{x_1\rho^{-1}e^{\frac{2}{3}\pi i}}{r^2+x_1\rho^{-1}e^{\frac{2}{3}\pi i}}\right) r\,dr$$

$$= \frac{\pi}{8x_1\rho}\left(\frac{1}{x_1\rho+1}-\frac{\rho}{x_1+\rho}+\frac{2-x_1\rho}{1-x_1\rho+x_1^2\rho^2}-\frac{(2\rho-x_1)\,\rho}{x_1^2-x_1\rho+\rho^2}\right)\left(1-x_1^2\right)+$$
$$+\frac{\pi}{8x_1\rho}\left(x_1\rho\mathrm{Ln}\left(\frac{1+x_1\rho}{x_1^2+x_1\rho}\right)-x_1\rho^{-1}\mathrm{Ln}\left(\frac{1+x_1\rho^{-1}}{x_1^2+x_1\rho^{-1}}\right)\right)+$$
$$+\frac{\pi}{8x_1\rho}\left(x_1\rho e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{1+x_1\rho e^{-\frac{2}{3}\pi i}}{x_1^2+x_1\rho e^{-\frac{2}{3}\pi i}}\right)-x_1\rho^{-1}e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{1+x_1\rho^{-1}e^{-\frac{2}{3}\pi i}}{x_1^2+x_1\rho^{-1}e^{-\frac{2}{3}\pi i}}\right)\right)+$$
$$+\frac{\pi}{8x_1\rho}\left(x_1\rho e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{1+x_1\rho e^{\frac{2}{3}\pi i}}{x_1^2+x_1\rho e^{\frac{2}{3}\pi i}}\right)-x_1\rho^{-1}e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{1+x_1\rho^{-1}e^{\frac{2}{3}\pi i}}{x_1^2+x_1\rho^{-1}e^{\frac{2}{3}\pi i}}\right)\right). \tag{100}$$

• From ρ to 1 :

The last remaining part is

$$\int_{r=\rho}^{1} I_{10}\left(x_1,\rho;r\right)\, r\,dr, \tag{101}$$

where

$$I_{10}\left(x_1,\rho;r\right)=\#_{(92)}=$$

$$= \frac{-\pi}{4\rho x_1}\left(\frac{x_1}{\rho+x_1}-\frac{1}{x_1\rho+1}-\frac{1}{1+x_1\rho e^{\frac{2}{3}\pi i}}-\frac{1}{1+x_1\rho e^{-\frac{2}{3}\pi i}}+\frac{x_1e^{-\frac{2}{3}\pi i}}{\rho+x_1e^{-\frac{2}{3}\pi i}}+\frac{x_1e^{\frac{2}{3}\pi i}}{\rho+x_1e^{\frac{2}{3}\pi i}}\right)+$$
$$+\frac{-\pi}{4\rho x_1}\left(\frac{x_1^{-1}\rho}{r^2+x_1^{-1}\rho}-\frac{x_1\rho}{r^2+x_1\rho}\right)$$
$$+\frac{-\pi}{4\rho x_1}\left(\frac{x_1^{-1}\rho e^{\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho e^{\frac{2}{3}\pi i}}-\frac{x_1\rho e^{\frac{2}{3}\pi i}}{r^2+x_1\rho e^{\frac{2}{3}\pi i}}\right)+$$
$$+\frac{-\pi}{4\rho x_1}\left(\frac{x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}-\frac{x_1\rho e^{-\frac{2}{3}\pi i}}{r^2+x_1\rho e^{-\frac{2}{3}\pi i}}\right).$$

We obtain

$$\int_{r=\rho}^{1} I_{10}\left(x_1,\rho;r\right)\, r\, dr =$$

$$= \frac{\pi}{4\rho x_1}\int_{r=\rho}^{1} \left(\frac{1}{x_1\rho+1} - \frac{x_1}{\rho+x_1} + \frac{2-x_1\rho}{1-x_1\rho+x_1^2\rho^2} + \frac{(\rho-2x_1)\,x_1}{x_1^2-x_1\rho+\rho^2} + \right.$$
$$+ \frac{x_1\rho}{r^2+x_1\rho} - \frac{x_1^{-1}\rho}{r^2+x_1^{-1}\rho} + \frac{x_1\rho e^{\frac{2}{3}\pi i}}{r^2+x_1\rho e^{\frac{2}{3}\pi i}} +$$
$$\left. - \frac{x_1^{-1}\rho e^{\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho e^{\frac{2}{3}\pi i}} + \frac{x_1\rho e^{-\frac{2}{3}\pi i}}{r^2+x_1\rho e^{-\frac{2}{3}\pi i}} - \frac{x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho e^{-\frac{2}{3}\pi i}} \right) r\, dr$$

$$= \frac{\pi}{8\rho x_1}\left[\left(\frac{1}{x_1\rho+1} - \frac{x_1}{\rho+x_1} + \frac{2-x_1\rho}{1-x_1\rho+x_1^2\rho^2} + \frac{(\rho-2x_1)\,x_1}{x_1^2-x_1\rho+\rho^2} \right) r^2 + \right.$$
$$+ x_1\rho \mathrm{Ln}\left(r^2+x_1\rho\right) - x_1^{-1}\rho \mathrm{Ln}\left(r^2+x_1^{-1}\rho\right) +$$
$$+ x_1\rho e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(r^2+x_1\rho e^{\frac{2}{3}\pi i}\right) - x_1^{-1}\rho e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(r^2+x_1^{-1}\rho e^{\frac{2}{3}\pi i}\right) +$$
$$\left. + x_1\rho e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(r^2+x_1\rho e^{-\frac{2}{3}\pi i}\right) - x_1^{-1}\rho e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(r^2+x_1^{-1}\rho e^{-\frac{2}{3}\pi i}\right) \right]_{r=\rho}^{1}$$

$$= \frac{-\pi}{8\rho x_1}\left(\left(\frac{1}{x_1\rho+1} - \frac{x_1}{\rho+x_1} + \frac{2-x_1\rho}{1-x_1\rho+x_1^2\rho^2} + \frac{(\rho-2x_1)\,x_1}{x_1^2-x_1\rho+\rho^2} \right)\left(1-\rho^2\right) + \right.$$
$$+ x_1\rho \mathrm{Ln}\left(\frac{1+x_1\rho}{\rho^2+x_1\rho}\right) - x_1^{-1}\rho \mathrm{Ln}\left(\frac{1+x_1^{-1}\rho}{\rho^2+x_1^{-1}\rho}\right) +$$
$$+ x_1\rho e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{1+x_1\rho e^{\frac{2}{3}\pi i}}{\rho^2+x_1\rho e^{\frac{2}{3}\pi i}}\right) - x_1^{-1}\rho e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{1+x_1^{-1}\rho e^{\frac{2}{3}\pi i}}{\rho^2+x_1^{-1}\rho e^{\frac{2}{3}\pi i}}\right) +$$
$$\left. + x_1\rho e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{1+x_1\rho e^{-\frac{2}{3}\pi i}}{\rho^2+x_1\rho e^{-\frac{2}{3}\pi i}}\right) - x_1^{-1}\rho e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{1+x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}{\rho^2+x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}\right) \right)$$

$$= \frac{\pi}{8\rho x_1}\left(\left(\frac{1}{x_1\rho+1} - \frac{x_1}{\rho+x_1} + \frac{2-x_1\rho}{1-x_1\rho+x_1^2\rho^2} + \frac{(\rho-2x_1)\,x_1}{x_1^2-x_1\rho+\rho^2} \right)\left(1-\rho^2\right) + \right.$$
$$+ x_1\rho \mathrm{Ln}\left(\frac{1+x_1\rho}{(\rho+x_1)\,\rho}\right) - x_1^{-1}\rho \mathrm{Ln}\left(\frac{\rho+x_1}{(1+x_1\rho)\,\rho}\right) +$$
$$+ x_1\rho e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{1+x_1\rho e^{\frac{2}{3}\pi i}}{\left(\rho+x_1 e^{\frac{2}{3}\pi i}\right)\rho}\right) - x_1^{-1}\rho e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{\rho+x_1 e^{-\frac{2}{3}\pi i}}{\left(1+x_1\rho e^{-\frac{2}{3}\pi i}\right)\rho}\right) +$$
$$\left. + x_1\rho e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{1+x_1\rho e^{-\frac{2}{3}\pi i}}{\left(\rho+x_1 e^{-\frac{2}{3}\pi i}\right)\rho}\right) - x_1^{-1}\rho e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{\rho+x_1 e^{\frac{2}{3}\pi i}}{\left(1+x_1\rho e^{\frac{2}{3}\pi i}\right)\rho}\right) \right) \tag{102}$$

7.5. Conclusion of Case 4)

Adding the formulae in (94), (96), (98), (100) and (102) we obtain

$$\frac{8x_1\rho}{\pi}\pi^2 \int_S \left(\lim_{x_2\downarrow 0} \frac{G_S(x,z)}{x_2}\right)\left(\lim_{\substack{\psi\uparrow\frac13\pi \\ y=(\rho\cos\psi,\rho\sin\psi)}} \frac{G_S(z,y)}{\rho\sin\left(\frac13\pi-\psi\right)}\right)dz =$$

$$\begin{aligned}
= \quad & \frac{1-x_1\rho}{1+x_1\rho} + \frac{1-x_1\rho e^{-\frac23\pi i}}{1+x_1\rho e^{-\frac23\pi i}} + \frac{1-x_1\rho e^{\frac23\pi i}}{1+x_1\rho e^{\frac23\pi i}} + \\
& + x_1^{-1}\rho\mathrm{Ln}\left(1+x_1\rho^{-1}\right) + x_1\rho^{-1}\mathrm{Ln}\left(1+x_1^{-1}\rho\right) - 2x_1^{-1}\rho^{-1}\mathrm{Ln}\left(1+x_1\rho\right) + \\
& + x_1^{-1}\rho e^{-\frac23\pi i}\mathrm{Ln}\left(1+x_1\rho^{-1}e^{\frac23\pi i}\right) + x_1^{-1}\rho e^{\frac23\pi i}\mathrm{Ln}\left(1+x_1\rho^{-1}e^{-\frac23\pi i}\right) + \\
& + x_1\rho^{-1}e^{\frac23\pi i}\mathrm{Ln}\left(1+x_1^{-1}\rho e^{-\frac23\pi i}\right) + x_1\rho^{-1}e^{-\frac23\pi i}\mathrm{Ln}\left(1+x_1^{-1}\rho e^{\frac23\pi i}\right) + \\
& - 2x_1^{-1}\rho^{-1}e^{-\frac23\pi i}\mathrm{Ln}\left(1+x_1\rho e^{\frac23\pi i}\right) - 2x_1^{-1}\rho^{-1}e^{\frac23\pi i}\mathrm{Ln}\left(1+x_1\rho e^{-\frac23\pi i}\right) + \\
& + \left(-\frac{\rho}{x_1+\rho} + \frac{x_1\rho}{1+x_1\rho} - \frac{\rho\left(2\rho-x_1\right)}{x_1^2-x_1\rho+\rho^2} + \frac{x_1\rho\left(2x_1\rho-1\right)}{1-x_1\rho+x_1^2\rho^2}\right)\rho^2 + \\
& - x_1\rho\mathrm{Ln}\left(1+x_1^{-1}\rho\right) + x_1^{-1}\rho\mathrm{Ln}\left(1+x_1\rho\right) + \\
& - x_1\rho e^{-\frac23\pi i}\mathrm{Ln}\left(1+x_1^{-1}\rho e^{\frac23\pi i}\right) + x_1^{-1}\rho e^{-\frac23\pi i}\mathrm{Ln}\left(1+x_1\rho e^{\frac23\pi i}\right) + \\
& - x_1\rho e^{\frac23\pi i}\mathrm{Ln}\left(1+x_1^{-1}\rho e^{-\frac23\pi i}\right) + x_1^{-1}\rho e^{\frac23\pi i}\mathrm{Ln}\left(1+x_1\rho e^{-\frac23\pi i}\right) + \\
& + \frac{x_1^3\rho}{1+x_1\rho} - \frac{x_1^3}{\rho+x_1} + \frac{\left(2x_1\rho-1\right)x_1^3\rho}{1-x_1\rho+x_1^2\rho^2} - \frac{\left(2x_1-\rho\right)x_1^3}{\rho^2-x_1\rho+x_1^2} + \\
& + x_1\rho^{-1}\mathrm{Ln}\left(1+x_1\rho\right) + x_1\rho^{-1}e^{\frac23\pi i}\mathrm{Ln}\left(1+x_1\rho e^{-\frac23\pi i}\right) + x_1\rho^{-1}e^{-\frac23\pi i}\mathrm{Ln}\left(1+x_1\rho e^{\frac23\pi i}\right) + \\
& - x_1\rho\mathrm{Ln}\left(1+x_1\rho^{-1}\right) - x_1\rho e^{\frac23\pi i}\mathrm{Ln}\left(1+x_1\rho^{-1}e^{-\frac23\pi i}\right) - x_1\rho e^{-\frac23\pi i}\mathrm{Ln}\left(1+x_1\rho^{-1}e^{\frac23\pi i}\right) + \\
& + \left(\frac{1}{x_1\rho+1} - \frac{\rho}{x_1+\rho} + \frac{2-x_1\rho}{1-x_1\rho+x_1^2\rho^2} - \frac{\left(2\rho-x_1\right)\rho}{x_1^2-x_1\rho+\rho^2}\right)\left(1-x_1^2\right) + \\
& + x_1\rho\mathrm{Ln}\left(\frac{1+x_1\rho}{x_1^2+x_1\rho}\right) - x_1\rho^{-1}\mathrm{Ln}\left(\frac{1+x_1\rho^{-1}}{x_1^2+x_1\rho^{-1}}\right) + \\
& + x_1\rho e^{-\frac23\pi i}\mathrm{Ln}\left(\frac{1+x_1\rho e^{-\frac23\pi i}}{x_1^2+x_1\rho e^{-\frac23\pi i}}\right) - x_1\rho^{-1}e^{-\frac23\pi i}\mathrm{Ln}\left(\frac{1+x_1\rho^{-1}e^{-\frac23\pi i}}{x_1^2+x_1\rho^{-1}e^{-\frac23\pi i}}\right) + \\
& + x_1\rho e^{\frac23\pi i}\mathrm{Ln}\left(\frac{1+x_1\rho e^{\frac23\pi i}}{x_1^2+x_1\rho e^{\frac23\pi i}}\right) - x_1\rho^{-1}e^{\frac23\pi i}\mathrm{Ln}\left(\frac{1+x_1\rho^{-1}e^{\frac23\pi i}}{x_1^2+x_1\rho^{-1}e^{\frac23\pi i}}\right) +
\end{aligned}$$

$$
\begin{aligned}
& + \left(\frac{1}{x_1\rho + 1} - \frac{x_1}{\rho + x_1} + \frac{2 - x_1\rho}{1 - x_1\rho + x_1^2\rho^2} + \frac{(\rho - 2x_1)\, x_1}{x_1^2 - x_1\rho + \rho^2} \right) \left(1 - \rho^2\right) + \\
& + x_1\rho \mathrm{Ln}\left(\frac{1 + x_1\rho}{(\rho + x_1)\,\rho} \right) - x_1^{-1}\rho \mathrm{Ln}\left(\frac{\rho + x_1}{(1 + x_1\rho)\,\rho} \right) + \\
& + x_1\rho e^{\frac{2}{3}\pi i} \mathrm{Ln}\left(\frac{1 + x_1\rho e^{\frac{2}{3}\pi i}}{\left(\rho + x_1 e^{\frac{2}{3}\pi i}\right)\rho} \right) - x_1^{-1}\rho e^{\frac{2}{3}\pi i} \mathrm{Ln}\left(\frac{\rho + x_1 e^{-\frac{2}{3}\pi i}}{\left(1 + x_1\rho e^{-\frac{2}{3}\pi i}\right)\rho} \right) + \\
& + x_1\rho e^{-\frac{2}{3}\pi i} \mathrm{Ln}\left(\frac{1 + x_1\rho e^{-\frac{2}{3}\pi i}}{\left(\rho + x_1 e^{-\frac{2}{3}\pi i}\right)\rho} \right) - x_1^{-1}\rho e^{-\frac{2}{3}\pi i} \mathrm{Ln}\left(\frac{\rho + x_1 e^{\frac{2}{3}\pi i}}{\left(1 + x_1\rho e^{\frac{2}{3}\pi i}\right)\rho} \right) \qquad (103)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & - \frac{\rho^3}{x_1 + \rho} - \frac{x_1^3}{\rho + x_1} - \frac{\rho\left(1 - x_1^2\right)}{x_1 + \rho} - \frac{x_1\left(1 - \rho^2\right)}{\rho + x_1} + \\
& + \frac{1 - x_1\rho}{1 + x_1\rho} + \frac{x_1\rho^3}{1 + x_1\rho} + \frac{x_1^3\rho}{1 + x_1\rho} + \frac{1 - x_1^2}{x_1\rho + 1} + \frac{1 - \rho^2}{x_1\rho + 1} + \\
& - \frac{(2x_1 - \rho)\, x_1^3}{\rho^2 - x_1\rho + x_1^2} - \frac{\rho^3\,(2\rho - x_1)}{x_1^2 - x_1\rho + \rho^2} - \frac{(2\rho - x_1)\,\rho\left(1 - x_1^2\right)}{x_1^2 - x_1\rho + \rho^2} + \frac{(\rho - 2x_1)\, x_1\left(1 - \rho^2\right)}{x_1^2 - x_1\rho + \rho^2} + \\
& + \frac{1 - x_1\rho e^{-\frac{2}{3}\pi i}}{1 + x_1\rho e^{-\frac{2}{3}\pi i}} + \frac{1 - x_1\rho e^{\frac{2}{3}\pi i}}{1 + x_1\rho e^{\frac{2}{3}\pi i}} + \frac{x_1\rho^3\,(2x_1\rho - 1)}{1 - x_1\rho + x_1^2\rho^2} + \\
& + \frac{(2x_1\rho - 1)\, x_1^3\rho}{1 - x_1\rho + x_1^2\rho^2} + \frac{(2 - x_1\rho)\left(1 - x_1^2\right)}{1 - x_1\rho + x_1^2\rho^2} + \frac{(2 - x_1\rho)\left(1 - \rho^2\right)}{1 - x_1\rho + x_1^2\rho^2} + \\
& - x_1\rho \mathrm{Ln}\left(1 + x_1^{-1}\rho\right) + x_1\rho \mathrm{Ln}\left(\frac{1 + x_1\rho}{x_1^2 + x_1\rho} \right) - x_1\rho \mathrm{Ln}\left(1 + x_1\rho^{-1}\right) + x_1\rho \mathrm{Ln}\left(\frac{1 + x_1\rho}{(\rho + x_1)\,\rho} \right) + \\
& - x_1\rho e^{\frac{2}{3}\pi i} \mathrm{Ln}\left(1 + x_1\rho^{-1} e^{-\frac{2}{3}\pi i}\right) - x_1\rho e^{\frac{2}{3}\pi i} \mathrm{Ln}\left(1 + x_1^{-1}\rho e^{-\frac{2}{3}\pi i}\right) + \\
& + x_1\rho e^{\frac{2}{3}\pi i} \mathrm{Ln}\left(\frac{1 + x_1\rho e^{\frac{2}{3}\pi i}}{x_1^2 + x_1\rho e^{\frac{2}{3}\pi i}} \right) + x_1\rho e^{\frac{2}{3}\pi i} \mathrm{Ln}\left(\frac{1 + x_1\rho e^{\frac{2}{3}\pi i}}{\left(\rho + x_1 e^{\frac{2}{3}\pi i}\right)\rho} \right) + \\
& - x_1\rho e^{-\frac{2}{3}\pi i} \mathrm{Ln}\left(1 + x_1\rho^{-1} e^{\frac{2}{3}\pi i}\right) - x_1\rho e^{-\frac{2}{3}\pi i} \mathrm{Ln}\left(1 + x_1^{-1}\rho e^{\frac{2}{3}\pi i}\right) + \\
& + x_1\rho e^{-\frac{2}{3}\pi i} \mathrm{Ln}\left(\frac{1 + x_1\rho e^{-\frac{2}{3}\pi i}}{x_1^2 + x_1\rho e^{-\frac{2}{3}\pi i}} \right) + x_1\rho e^{-\frac{2}{3}\pi i} \mathrm{Ln}\left(\frac{1 + x_1\rho e^{-\frac{2}{3}\pi i}}{\left(\rho + x_1 e^{-\frac{2}{3}\pi i}\right)\rho} \right) +
\end{aligned}
$$

$$
\begin{aligned}
&+ x_1^{-1}\rho \mathrm{Ln}\left(1+x_1\rho^{-1}\right) + x_1^{-1}\rho \mathrm{Ln}\left(1+x_1\rho\right) - x_1^{-1}\rho \mathrm{Ln}\left(\frac{\rho+x_1}{(1+x_1\rho)\,\rho}\right) + \\
&+ x_1^{-1}\rho e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(1+x_1\rho e^{-\frac{2}{3}\pi i}\right) + x_1^{-1}\rho e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(1+x_1\rho^{-1}e^{-\frac{2}{3}\pi i}\right) - x_1^{-1}\rho e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{\rho+x_1 e^{-\frac{2}{3}\pi i}}{\left(1+x_1\rho e^{-\frac{2}{3}\pi i}\right)\rho}\right) + \\
&+ x_1^{-1}\rho e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(1+x_1\rho e^{\frac{2}{3}\pi i}\right) + x_1^{-1}\rho e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(1+x_1\rho^{-1}e^{\frac{2}{3}\pi i}\right) - x_1^{-1}\rho e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{\rho+x_1 e^{\frac{2}{3}\pi i}}{\left(1+x_1\rho e^{\frac{2}{3}\pi i}\right)\rho}\right) + \\
&+ x_1\rho^{-1}\mathrm{Ln}\left(1+x_1^{-1}\rho\right) + x_1\rho^{-1}\mathrm{Ln}\left(1+x_1\rho\right) - x_1\rho^{-1}\mathrm{Ln}\left(\frac{1+x_1\rho^{-1}}{x_1^2+x_1\rho^{-1}}\right) + \\
&+ x_1\rho^{-1}e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(1+x_1^{-1}\rho e^{-\frac{2}{3}\pi i}\right) + x_1\rho^{-1}e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(1+x_1\rho e^{-\frac{2}{3}\pi i}\right) - x_1\rho^{-1}e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{1+x_1\rho^{-1}e^{\frac{2}{3}\pi i}}{x_1^2+x_1\rho^{-1}e^{\frac{2}{3}\pi i}}\right) + \\
&+ x_1\rho^{-1}e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(1+x_1^{-1}\rho e^{\frac{2}{3}\pi i}\right) + x_1\rho^{-1}e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(1+x_1\rho e^{\frac{2}{3}\pi i}\right) - x_1\rho^{-1}e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{1+x_1\rho^{-1}e^{-\frac{2}{3}\pi i}}{x_1^2+x_1\rho^{-1}e^{-\frac{2}{3}\pi i}}\right) + \\
&- 2x_1^{-1}\rho^{-1}\mathrm{Ln}\left(1+x_1\rho\right) - 2x_1^{-1}\rho^{-1}e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(1+x_1\rho e^{\frac{2}{3}\pi i}\right) - 2x_1^{-1}\rho^{-1}e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(1+x_1\rho e^{-\frac{2}{3}\pi i}\right)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & -\left(\rho-x_1\right)^2 + \frac{(1-x_1\rho)\left(2-\rho^2-x_1^2\right)}{1+x_1\rho} - 2\frac{(x_1-\rho)^2(x_1+\rho)^2}{\rho^2-x_1\rho+x_1^2} + 2\frac{(1+x_1\rho)(1-x_1\rho)\left(2-x_1^2-\rho^2\right)}{1-x_1\rho+x_1^2\rho^2} + \\
& + x_1\rho\mathrm{Ln}\left(\frac{1+x_1\rho}{x_1^2+x_1\rho}\frac{1+x_1\rho}{(\rho+x_1)\,\rho}\frac{1}{1+x_1^{-1}\rho}\frac{1}{1+x_1\rho^{-1}}\right) + \\
& + x_1\rho e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{1+x_1\rho e^{\frac{2}{3}\pi i}}{x_1^2+x_1\rho e^{\frac{2}{3}\pi i}}\frac{1+x_1\rho e^{\frac{2}{3}\pi i}}{\left(\rho+x_1 e^{\frac{2}{3}\pi i}\right)\rho}\frac{1}{1+x_1\rho^{-1}e^{-\frac{2}{3}\pi i}}\frac{1}{1+x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}\right) + \\
& + x_1\rho e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{1+x_1\rho e^{-\frac{2}{3}\pi i}}{x_1^2+x_1\rho e^{-\frac{2}{3}\pi i}}\frac{1+x_1\rho e^{-\frac{2}{3}\pi i}}{\left(\rho+x_1 e^{-\frac{2}{3}\pi i}\right)\rho}\frac{1}{1+x_1^{-1}\rho e^{\frac{2}{3}\pi i}}\frac{1}{1+x_1\rho^{-1}e^{\frac{2}{3}\pi i}}\right) + \\
& + x_1^{-1}\rho\mathrm{Ln}\left(\frac{\left(1+x_1\rho^{-1}\right)(1+x_1\rho)(1+x_1\rho)\,\rho}{\rho+x_1}\right) + \\
& + x_1^{-1}\rho e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{\left(1+x_1\rho e^{-\frac{2}{3}\pi i}\right)\left(1+x_1\rho^{-1}e^{-\frac{2}{3}\pi i}\right)\left(1+x_1\rho e^{-\frac{2}{3}\pi i}\right)\rho}{\rho+x_1 e^{-\frac{2}{3}\pi i}}\right) +
\end{aligned}
$$

$$
\begin{aligned}
&+ x_1^{-1}\rho e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{\left(1+x_1\rho e^{\frac{2}{3}\pi i}\right)\left(1+x_1\rho^{-1}e^{\frac{2}{3}\pi i}\right)\left(1+x_1\rho e^{\frac{2}{3}\pi i}\right)\rho}{\rho+x_1 e^{\frac{2}{3}\pi i}}\right)+\\
&+ x_1\rho^{-1}\mathrm{Ln}\left(\frac{\left(1+x_1^{-1}\rho\right)\left(1+x_1\rho\right)\left(x_1^2+x_1\rho^{-1}\right)}{1+x_1\rho^{-1}}\right)+\\
&+ x_1\rho^{-1}e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{\left(1+x_1^{-1}\rho e^{-\frac{2}{3}\pi i}\right)\left(1+x_1\rho e^{-\frac{2}{3}\pi i}\right)\left(x_1^2+x_1\rho^{-1}e^{\frac{2}{3}\pi i}\right)}{1+x_1\rho^{-1}e^{\frac{2}{3}\pi i}}\right)+\\
&+ x_1\rho^{-1}e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{\left(1+x_1^{-1}\rho e^{\frac{2}{3}\pi i}\right)\left(1+x_1\rho e^{\frac{2}{3}\pi i}\right)\left(x_1^2+x_1\rho^{-1}e^{-\frac{2}{3}\pi i}\right)}{1+x_1\rho^{-1}e^{-\frac{2}{3}\pi i}}\right)+\\
&- 2x_1^{-1}\rho^{-1}\mathrm{Ln}\left(1+x_1\rho\right) - 2x_1^{-1}\rho^{-1}e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(1+x_1\rho e^{\frac{2}{3}\pi i}\right) - 2x_1^{-1}\rho^{-1}e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(1+x_1\rho e^{-\frac{2}{3}\pi i}\right)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & -\left(\rho-x_1\right)^2 + \frac{\left(1-x_1\rho\right)\left(2-\rho^2-x_1^2\right)}{1+x_1\rho} - 2\frac{\left(x_1-\rho\right)^2\left(x_1+\rho\right)^2}{\rho^2-x_1\rho+x_1^2} + 2\frac{\left(1+x_1\rho\right)\left(1-x_1\rho\right)\left(2-x_1^2-\rho^2\right)}{1-x_1\rho+x_1^2\rho^2}+\\
&+ x_1\rho\mathrm{Ln}\left(\frac{\left(x_1\rho+1\right)^2}{\left(x_1+\rho\right)^4}\right)+\\
&+ x_1\rho e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{\left(1+x_1\rho e^{\frac{2}{3}\pi i}\right)^2}{\left(\rho+x_1e^{\frac{2}{3}\pi i}\right)^2\left(\rho+x_1e^{-\frac{2}{3}\pi i}\right)^2}\right)+\\
&+ x_1\rho e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(\frac{\left(1+x_1\rho e^{-\frac{2}{3}\pi i}\right)^2}{\left(\rho+x_1e^{\frac{2}{3}\pi i}\right)^2\left(\rho+x_1e^{-\frac{2}{3}\pi i}\right)^2}\right)+\\
&+ \left(x_1^{-1}\rho+x_1\rho^{-1}\right)\mathrm{Ln}\left(\left(1+x_1\rho\right)^2\right)+\\
&+ \left(x_1^{-1}\rho+x_1\rho^{-1}\right)e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(\left(1+x_1\rho e^{-\frac{2}{3}\pi i}\right)^2\right)+\\
&+ \left(x_1^{-1}\rho+x_1\rho^{-1}\right)e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(\left(1+x_1\rho e^{\frac{2}{3}\pi i}\right)^2\right)+\\
&- 2x_1^{-1}\rho^{-1}\mathrm{Ln}\left(1+x_1\rho\right) - 2x_1^{-1}\rho^{-1}e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(1+x_1\rho e^{\frac{2}{3}\pi i}\right) - 2x_1^{-1}\rho^{-1}e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(1+x_1\rho e^{-\frac{2}{3}\pi i}\right)
\end{aligned}
$$

$$
\begin{aligned}
= \; & -\left(\rho - x_1\right)^2 + \frac{\left(1 - x_1\rho\right)\left(2 - \rho^2 - x_1^2\right)}{1 + x_1\rho} - 2\frac{\left(x_1 - \rho\right)^2\left(x_1 + \rho\right)^2}{\rho^2 - x_1\rho + x_1^2} + 2\frac{\left(1 + x_1\rho\right)\left(1 - x_1\rho\right)\left(2 - x_1^2 - \rho^2\right)}{1 - x_1\rho + x_1^2\rho^2} + \\
& + 2x_1\rho \mathrm{Ln}\left(x_1\rho + 1\right) - 4x_1\rho \mathrm{Ln}\left(x_1 + \rho\right) + \\
& + 2x_1\rho\left(e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(1 + x_1\rho e^{\frac{2}{3}\pi i}\right) + e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(1 + x_1\rho e^{-\frac{2}{3}\pi i}\right)\right) + \\
& - 2x_1\rho\left(e^{-\frac{2}{3}\pi i} + e^{\frac{2}{3}\pi i}\right)\mathrm{Ln}\left(\left(\rho + x_1 e^{\frac{2}{3}\pi i}\right)\left(\rho + x_1 e^{-\frac{2}{3}\pi i}\right)\right) + \\
& + 2\left(x_1^{-1}\rho + x_1\rho^{-1}\right)\mathrm{Ln}\left(1 + x_1\rho\right) + \\
& + 2\left(x_1^{-1}\rho + x_1\rho^{-1}\right)\left(e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(1 + x_1\rho e^{-\frac{2}{3}\pi i}\right) + e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(1 + x_1\rho e^{\frac{2}{3}\pi i}\right)\right) + \\
& - 2x_1^{-1}\rho^{-1}\mathrm{Ln}\left(1 + x_1\rho\right) - 2x_1^{-1}\rho^{-1}\left(e^{-\frac{2}{3}\pi i}\mathrm{Ln}\left(1 + x_1\rho e^{\frac{2}{3}\pi i}\right) + e^{\frac{2}{3}\pi i}\mathrm{Ln}\left(1 + x_1\rho e^{-\frac{2}{3}\pi i}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
= \; & -\left(\rho - x_1\right)^2 + \frac{\left(1 - x_1\rho\right)\left(2 - \rho^2 - x_1^2\right)}{1 + x_1\rho} - 2\frac{\left(x_1 - \rho\right)^2\left(x_1 + \rho\right)^2}{\rho^2 - x_1\rho + x_1^2} + 2\frac{\left(1 + x_1\rho\right)\left(1 - x_1\rho\right)\left(2 - x_1^2 - \rho^2\right)}{1 - x_1\rho + x_1^2\rho^2} + \\
& + 2x_1\rho\ln\left(x_1\rho + 1\right) - 4x_1\rho\ln\left(x_1 + \rho\right) + \\
& + 2x_1\rho\left(\left(e^{\frac{2}{3}\pi i} + e^{-\frac{2}{3}\pi i}\right)\ln\left|1 + x_1\rho e^{\frac{2}{3}\pi i}\right| + i\left(e^{\frac{2}{3}\pi i} - e^{-\frac{2}{3}\pi i}\right)\arctan\left(\frac{x_1\rho\sin\frac{2}{3}\pi}{1 + x_1\rho\cos\frac{2}{3}\pi}\right)\right) + \\
& + 2x_1\rho\ln\left(\rho^2 - x_1\rho + x_1^2\right) + \\
& + 2\left(x_1^{-1}\rho + x_1\rho^{-1}\right)\ln\left(1 + x_1\rho\right) + \\
& + 2\left(x_1^{-1}\rho + x_1\rho^{-1}\right)\left(\left(e^{\frac{2}{3}\pi i} + e^{-\frac{2}{3}\pi i}\right)\ln\left|1 + x_1\rho e^{-\frac{2}{3}\pi i}\right| + i\left(e^{-\frac{2}{3}\pi i} - e^{\frac{2}{3}\pi i}\right)\arctan\left(\frac{x_1\rho\sin\frac{2}{3}\pi}{1 + x_1\rho\cos\frac{2}{3}\pi}\right)\right) + \\
& - 2x_1^{-1}\rho^{-1}\ln\left(1 + x_1\rho\right) + \\
& - 2x_1^{-1}\rho^{-1}\left(\left(e^{-\frac{2}{3}\pi i} + e^{\frac{2}{3}\pi i}\right)\ln\left|1 + x_1\rho e^{\frac{2}{3}\pi i}\right| + i\left(e^{-\frac{2}{3}\pi i} - e^{\frac{2}{3}\pi i}\right)\arctan\left(\frac{x_1\rho\sin\frac{2}{3}\pi}{1 + x_1\rho\cos\frac{2}{3}\pi}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
= \; & -\left(\rho - x_1\right)^2 + \frac{\left(1 - x_1\rho\right)\left(2 - \rho^2 - x_1^2\right)}{1 + x_1\rho} - 2\frac{\left(x_1 - \rho\right)^2\left(x_1 + \rho\right)^2}{\rho^2 - x_1\rho + x_1^2} + 2\frac{\left(1 + x_1\rho\right)\left(1 - x_1\rho\right)\left(2 - x_1^2 - \rho^2\right)}{1 - x_1\rho + x_1^2\rho^2} + \\
& + 2x_1\rho\ln\left(x_1\rho + 1\right) - 4x_1\rho\ln\left(x_1 + \rho\right) + \\
& + 2x_1\rho\left(-\ln\sqrt{\left(1 - x_1\rho + x_1^2\rho^2\right)} - \sqrt{3}\arctan\left(\frac{x_1\rho\sin\frac{2}{3}\pi}{1 + x_1\rho\cos\frac{2}{3}\pi}\right)\right) + \\
& + 2x_1\rho\ln\left(\rho^2 - x_1\rho + x_1^2\right) + \\
& + 2\left(x_1^{-1}\rho + x_1\rho^{-1} - x_1^{-1}\rho^{-1}\right)\ln\left(1 + x_1\rho\right) + \\
& + 2\left(x_1^{-1}\rho + x_1\rho^{-1}\right)\left(-\ln\sqrt{\left(1 - x_1\rho + x_1^2\rho^2\right)} + \sqrt{3}\arctan\left(\frac{x_1\rho\sin\frac{2}{3}\pi}{1 + x_1\rho\cos\frac{2}{3}\pi}\right)\right) + \\
& - 2x_1^{-1}\rho^{-1}\left(-\ln\sqrt{\left(1 - x_1\rho + x_1^2\rho^2\right)} + \sqrt{3}\arctan\left(\frac{x_1\rho\sin\frac{2}{3}\pi}{1 + x_1\rho\cos\frac{2}{3}\pi}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
= \; & -\left(\rho - x_1\right)^2 + \frac{\left(1 - x_1\rho\right)\left(2 - \rho^2 - x_1^2\right)}{1 + x_1\rho} - 2\frac{\left(x_1 - \rho\right)^2\left(x_1 + \rho\right)^2}{\rho^2 - x_1\rho + x_1^2} + 2\frac{\left(1 + x_1\rho\right)\left(1 - x_1\rho\right)\left(2 - x_1^2 - \rho^2\right)}{1 - x_1\rho + x_1^2\rho^2} + \\
& + 2x_1\rho\ln\left(\frac{\left(x_1\rho + 1\right)^2}{\left(x_1 + \rho\right)^2}\frac{\rho^2 - x_1\rho + x_1^2}{1 - x_1\rho + x_1^2\rho^2}\right) + \\
& + 2\left(x_1\rho - x_1^{-1}\rho - x_1\rho^{-1} + x_1^{-1}\rho^{-1}\right)\left(\ln\sqrt{\left(1 - x_1\rho + x_1^2\rho^2\right)} - \ln\left(1 + x_1\rho\right) - \sqrt{3}\arctan\left(\frac{x_1\rho\sqrt{3}}{2 - x_1\rho}\right)\right).
\end{aligned}
$$

After this very tedious calculation the enumerator of (77) can be listed as

$$\frac{1}{8x_1\rho\pi}\left(-(\rho-x_1)^2+\frac{(1-x_1\rho)\left(2-\rho^2-x_1^2\right)}{1+x_1\rho}-2\frac{(x_1-\rho)^2(x_1+\rho)^2}{\rho^2-x_1\rho+x_1^2}+\right.$$
$$+2\frac{\left(1-x_1^2\rho^2\right)\left(2-x_1^2-\rho^2\right)}{1-x_1\rho+x_1^2\rho^2}+2x_1\rho\ln\left(\frac{(x_1\rho+1)^2}{(x_1+\rho)^2}\frac{\rho^2-x_1\rho+x_1^2}{1-x_1\rho+x_1^2\rho^2}\right)+$$
$$\left.+\left(x_1\rho-\frac{\rho}{x_1}-\frac{x_1}{\rho}+\frac{1}{x_1\rho}\right)\left(\ln\frac{1-x_1\rho+x_1^2\rho^2}{(1+x_1\rho)^2}-2\sqrt{3}\arctan\left(\frac{x_1\rho\sqrt{3}}{2-x_1\rho}\right)\right)\right). \qquad (104)$$

Now an inspection of T_{13} reveals that it has three local maxima, namely at $(0,1)$, $(1,0)$ and $(1,1)$. See Figure 4. These points have already been considered in the previous cases. This concludes the proof of Theorem 1.

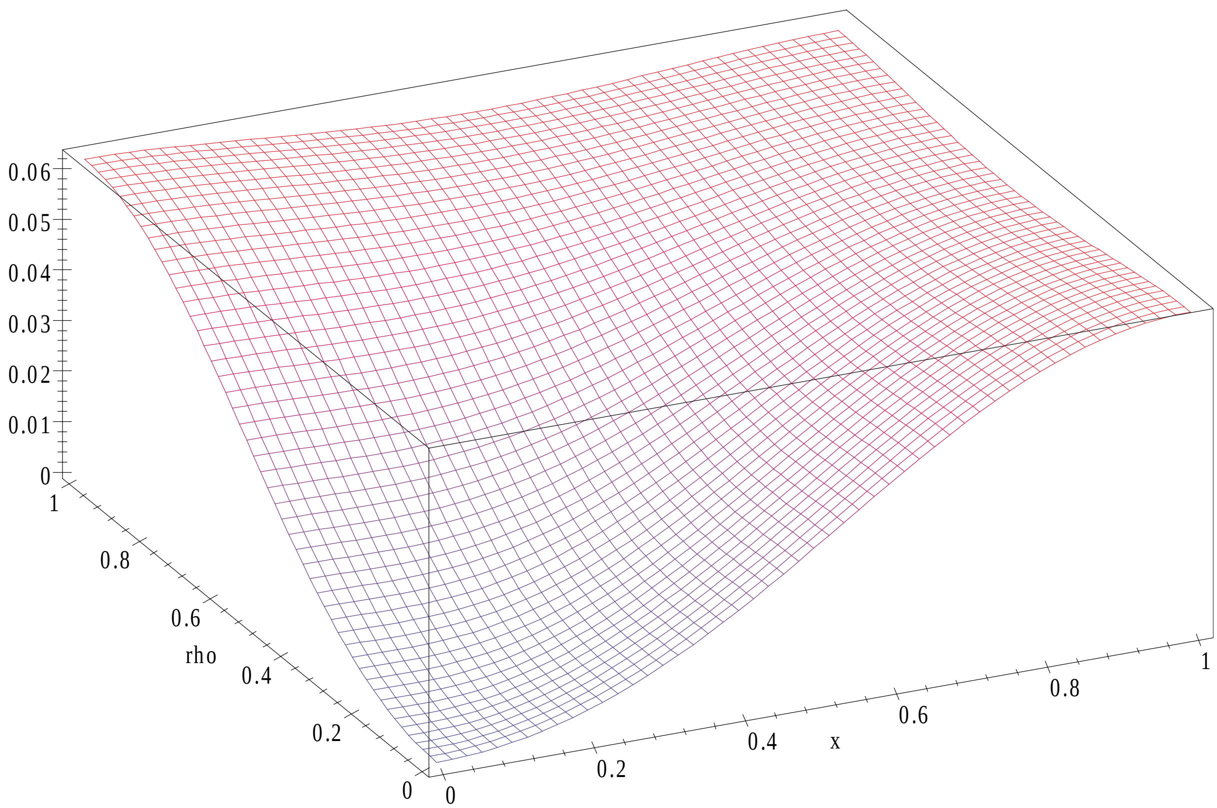


Figure 4: $x \in \Gamma_1$ and $y \in \Gamma_3$ with $|y| = \rho$: $T_{13}(x_1, \rho)$

Note added in proof: We would like to thank R. Bañuelos for pointing out that our result also provides a counterexample to a conjecture that recently appeared in [1].

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