Existence of the principal eigenvalue for cooperative elliptic systems in a general domain

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1 Introduction

In this paper we shall study the existence of a principal eigenfunction for the vector-valued elliptic eigenvalue problem

$$\begin{cases} (\mathcal{L} - H)\,\Phi &= \lambda B\Phi \qquad \text{in } \Omega, \\ \Phi &= 0 \qquad \text{on } \partial\Omega, \end{cases} \tag{1}$$

and its relation with a maximum principle. The operator $\mathcal{L}$ is supposed to be a diagonal matrix of second order uniformly elliptic partial differential operators and H and B are cooperative matrices with entries in $C(\bar{\Omega})$. The domain $\Omega \subset R^N$ is bounded. We do not assume that the boundary satisfies a regularity condition.

Systems of elliptic and parabolic differential equations arise in studies and models related to population dynamics, combustion theory and nerve conduction. In this respect, see for instance the survey of Aronson and Weinberger [3] and the book of Bebernes and Eberly [5] .

Dealing with problems connected to existence, uniqueness and stability of solutions to these systems, the associated elliptic eigenvalue problem plays an important role (see [1],[8], [15],[19]). Closely related to the latter problem is the existence of a comparison principle or maximum principle.

In most of the literature on elliptic and parabolic systems (see for example [9],[10], [18],[23],[24],[26],[29],[30]) it is assumed that the boundary of the domain Ω is smooth.

Without this assumption one is faced with several difficulties even in the case of a single equation. For the single equation on domains without much regularity there exist at least two approaches. On domains that are regular in the sense of Perron (see [11]) solutions that are continuous up to the boundary can be defined by a limiting process. On general domains self-adjoint problems can be studied in weak sense by minimizing the associated energy functional.

Only recently Berestycki, Nirenberg and Varadhan in [6] (see also Ancona [2], Nussbaum and Pinchover [20] and Pinsky [22] for related results) studied the eigenvalue problem removing all regularity assumptions of the boundary without using the self-adjointness. In doing so they had to define in which sense a solution, which in general is not continuous at boundary points, satisfies the boundary condition. It comes out that a good definition can be obtained by an approximation procedure, see Definition below.

Vector-valued elliptic systems are in general not self-adjoint even if the second order differential operators are self-adjoint. Hence without regularity assumptions on the boundary $\partial\Omega$ the approach of Berestycki, Nirenberg and Varadhan for the scalar equation seems to be the most natural for vector-valued problems.

Elliptic systems make up a broad class of problems. A structural condition for these systems to have similar features as a scalar elliptic equation is that the coupling is weak and quasimonotone. Weak coupling means that there is no coupling by derivatives of the solution; quasimonotone, generally called cooperative for linear systems, gives a sign condition on the coupling (see [18],[23],[29]). In this paper we shall consider only systems which satisfy the cooperativity assumption.

Let us mention briefly some of the complications due to the possible lack of smoothness of $\partial\Omega$. A main tool that is used in [6] as well as in the present paper is the Krein-Rutman Theorem. For applying the Krein-Rutman Theorem it is sufficient to have a strongly positive and compact solution operator (see Amann [1]). Without the regularity at the boundary proving the compactness of the solution operator becomes rather complicated even for the scalar case. However recently Birindelli in [7] gave an alternative and simple proof of the compactness in the case of second order scalar equations. Her method of proof applies as well as in the vector valued case, indeed we shall follow an argument taken from [7] in this paper. Regularity results for domains that are smooth except for a finite number of corners become very involved, the interested reader may refer to Grisvard [12] and [13] for appropriate Sobolev type spaces.

In general, in the vector-valued case the strong positivity of the solution operator does not hold. However, it follows from a theorem of De Pagter [21] that the strong positivity and the compactness property of the solution operator that is used in the Krein-Rutman Theorem, may be replaced by positivity, irreducibility and compactness. The cooperative assumption on the coupling matrix together with the maximum principle for the scalar case implies the positivity preserving property. The condition which we call *fully coupled* implies together with the maximum principle that the solution operator is irreducible.

The notion in which sense the boundary condition is satisfied was introduced by Berestycki, Nirenberg and Varadhan in [6]. In their approach one starts with constructing a function u_o by the following limit process.

Let $\{\Omega_j\}_{j=1}^{\infty}$ be a sequence of smooth domains that approximates Ω from inside:

$$\Omega_j \subset \bar{\Omega}_j \subset \Omega_{j+1} \subset \ldots \subset \Omega \text{ and } \bigcup_{j\in N} \Omega_j = \Omega \tag{2}$$

and let $\tilde{c}$ be such that $L1 + \tilde{c} \geq 0$. Finally let u_j denote the solution of

$$\begin{cases} (L+\tilde{c})\, u_j &= 1 \quad \text{in } \Omega_j, \\ u_j &= 0 \quad \text{on } \partial\Omega_j. \end{cases}$$

Defining u_o by $u_o(x) = \lim_{j\to\infty} u_j(x)$ for $x \in \Omega$, it follows (see [6]) that u_o in $W^{2,p}_{loc}(\Omega)$ for any p and that $u_j \to u_o$ in $C^1_{loc}(\Omega)$.

Definition (Berestycki-Nirenberg-Varadhan) *Let u_o be as above. A solution u of the elliptic equation $Lu = f$ (with appropriate assumptions on $L = -\sum a_{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} + \sum b_i \frac{\partial}{\partial x_i} + c$ and f) satisfies the zero Dirichlet boundary condition on $\partial\Omega$ in the BNV-sense, denoted by*

$$u \overset{u_o}{=} 0 \quad \text{on } \partial\Omega,$$

if $\lim_{j\to\infty} u(x_j) = 0$ for every sequence $x_j \to \partial\Omega$, such that $u_o(x_j) \to 0$.

In this paper we shall assume that the boundary condition appearing in (1) is satisfied in the appropriate BNV-sense (see Definition 4 below).

From the remark in [6, page 73] it follows that $u_o \overset{v}{=} 0$ on $\partial\Omega$ for any $v \in W^{2,N}_{loc}(\Omega)$ with $v > 0$ and $Lv \geq 0$ in Ω. Hence the choice of u_o is not restrictive.

This paper is organized as follows. In the next section we introduce the necessary definitions and state the main results. The third section contains a necessary tool that will be used to establish the maximum principle. In the fourth section we prove the main theorem for the system in (1) when B equals the identity matrix. Finally, using the result of the two previous sections, the main theorem is proven in the section five. In the final section, for sake of completeness, we will state a Krein-Rutman-De Pagter type theorem that is suitable for our purposes.

2 Definitions and main result

The general assumptions that we will make throughout the paper are as follows.

- The set Ω is a bounded, open and connected subset of R

- The operator $\mathcal{L}$ is a diagonal $k \times k$ matrix of elliptic operators L_μ $(1 \le \mu \le k)$

$$L_\mu := -\sum_{i,j=1}^{N} a_{ij}^{\mu}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i^{\mu}(x)\frac{\partial}{\partial x_i}(x), (3)$$

satisfying, for some positive constants c_o, C_o, and b, and for all $x \in \Omega$, $\xi \in R^k$ the following

$$c_o|\xi|^2 \le \sum_{i,j=1}^{N} a_{ij}^{\mu}(x)\,\xi_i\xi_j \le C_o|\xi|^2, \tag{4}$$

$$\begin{gathered} a_{ij}^{\mu} \in C(\Omega), \quad b_i^{\mu},\ c^{\mu} \in L^{\infty}, \\ \left(\sum_{i=1}^{k} (b_i^{\mu}(x))^2\right)^{\frac{1}{2}} \le b,\ |c^{\mu}(x)| \le b. \end{gathered} \tag{5}$$

- The $k \times k$ matrices H and B have $C\left(\bar{\Omega}\right)$ entries.

Note that we do not assume any regularity hypothesis on $\partial\Omega$.

Let $p \in (1, \infty)$ and let $L^p(\Omega)$ be the usual Lebesgue space. Notice that $(L^p(\Omega))^k$ can be identified with $L^p(\omega)$ where

$$\omega = (\underbrace{\Omega, \Omega, \ldots, \Omega}_{k \text{ copies}}). \tag{6}$$

Definition 1 (inequalities) *Let $D \subset R^M$ be a bounded open set. A function $w \in L^p(D)$ is said to satisfy*

1. $w > 0$ *if* $w \ge 0$ *a.e. in D and not $w = 0$ a.e. in D ;*
2. $w \gg 0$ *if* $w_{|D^*} > 0$ *on every open set $D^* \subset D$.*

Definition 2 (matrices) *A $k \times k$ matrix A with $A_{ij} \in C(\bar{\Omega})$ is called*

1. *positive* *if* $A_{ij}(x) \geq 0$ *for all* $i, j \in \{1, \ldots, k\}$ *and* $x \in \bar{\Omega}$;
2. *cooperative* *if* $A_{ij}(x) \geq 0$ *for all* $i, j \in \{1, \ldots, k\}$ *with* $i \neq j$, *and* $x \in \bar{\Omega}$.

A cooperative matrix A is called

3. *fully coupled* *if* *the matrix $\tilde{A} + I$ is irreducible*

where the entries of $\tilde{A}$ are defined by $\tilde{A}_{ij} = ||A_{ij}||_\infty$.

In the literature cooperativity is also known as essential positivity or quasimonotonicity ([24],[28]).

Definition 3 (halfsolutions) *A function $w \in \left(W^{2,p}_{loc}(\Omega) \cap L^\infty(\Omega)\right)^k$, with $w > 0$ and $(\mathcal{L} - H)\, w \in \left(L^N(\Omega)\right)^k$ is called*

1. *a supersolution for the operator $\mathcal{L} - H$* *if* $(\mathcal{L} - H)w \geq 0$;
2. *a strict supersolution for $\mathcal{L} - H$* *if* $(\mathcal{L} - H)w > 0$;
3. *a strong supersolution for $\mathcal{L} - H$* *if* $(\mathcal{L} - H)w \gg 0$.

Next we will define the boundary condition in an appropriate BNV-sense. Let the sequence $\{\Omega_j\}$ consists of smooth domains that approximate Ω from inside as in (2) and set $M_\mu = L_\mu - c^\mu$. Let u^μ_o be the limit of the functions u^μ_j that solve $M_\mu u^\mu = 1$ in Ω_j and $u^\mu = 0$ on $\partial\Omega_j$.

Definition 4 (Dirichlet boundary condition) *Let u_o be as above. A function $u \in (C(\Omega))^k$ is said to satisfy the Dirichlet boundary condition in BNV-sense, $u \overset{u_o}{=} 0$ if, for every $\mu \in \{1, 2, \ldots, k\}$ and for every sequence $\{x^j\}_{j \in N} \subset \Omega$ with Ω, it follows that*

$$\lim_{j\to\infty} u^\mu_o\left(x^j\right) = 0 \quad \textit{implies} \quad \lim_{j\to\infty} u_\mu\left(x^j\right) = 0.$$

We are now in the position to formulate the eigenvalue problem that we shall study in this paper. We say that

$$\Phi \in \left(W_{loc}^{2,N}(\Omega) \cap L^{\infty}(\Omega)\right)^k$$

is an eigenfunction of (1) with corresponding eigenvalue λ if

$$\begin{cases} (\mathcal{L} - H)\,\Phi &= \lambda B \Phi \quad \text{in } \Omega, \\ \Phi &\overset{u_o}{=} 0 \quad \text{on } \partial\Omega. \end{cases} \tag{7}$$

As in [6] we set

$$\lambda_0 = \sup \left\{ \lambda ; \exists w \in \left(W_{loc}^{2,N}(\Omega)\right)^k : \begin{array}{c} (\mathcal{L} - H)w \geq \lambda B w \\ \text{and} \quad w \gg 0 \end{array} \right\}. \tag{8}$$

For smooth domains, under appropriate conditions of the operators involved and for $B = I$ it is known (see [26]) that λ_0 is the first eigenvalue in the usual sense.

Furthermore, we note that if B satisfies $\sum_{j=1}^{k} B_{ij}(x) > 0$ for all $1 \leq i \leq k$, then the definition in (8) is closely related with Barta ([4]) type inequalities. Namely for all $w \in \left(C^2(\Omega)\right)^k$ with $w \gg 0$ one has

$$\lambda_0 \geq \inf_{\substack{1 \leq i \leq k \\ x \in \Omega}} \frac{\left((\mathcal{L} - H)\, w\right)_i (x)}{(Bw)_i\,(x)}. \tag{9}$$

The main results of this paper are contained in the following two theorems.

Theorem 5 *Let $\Omega, \mathcal{L}, H, B$ satisfy the assumptions above and let λ_0 be defined in (8). If*

a. there exists a positive strong supersolution of $\mathcal{L} - H$;

b. H is cooperative;

c. H is fully coupled;

d. B is cooperative;

e. $B_{ii}(x) > 0$ for some $i \in \{1, \ldots, k\}$ and $x \in \Omega$;

then

i. λ_0 *is a positive eigenvalue with a strongly positive associated eigenfunction;*

ii. λ_0 *is the only eigenvalue with a positive eigenfunction and its algebraic multiplicity is one;*

iii. *there are no eigenvalues in* $[0, \lambda_0)$.

Next we consider the boundary value problem

$$\begin{cases} (\mathcal{L} - H)\, u &= \lambda B u + f \qquad \text{in } \Omega, \\ u &\overset{u_o}{\equiv} 0 \qquad\qquad \text{on } \partial\Omega. \end{cases} \tag{10}$$

Theorem 6 *Let the assumptions of Theorem 5 be satisfied. Let* $f \in (L^p(\Omega))^k$ *with* $f > 0$ *and let* λ_0 *be defined in (8). Then the following holds:*

i. *if* $0 \leq \lambda < \lambda_0$*, then there exists a solution* $W^{2,N}_{loc}(\Omega) \cap L^\infty(\Omega)^k$ *of (10) and* $u \gg 0$.

If B *is positive we have:*

ii. *if* $\lambda = \lambda_0$*, then (10) has no solutions for any* $f > 0$*;*

iii. *if* $\lambda > \lambda_0$*, then (10) has no positive solutions for any* $f > 0$.

Remark 6.1 If in addition to the assumptions made in Theorem 5 we suppose that B is a positive diagonal matrix ($B_{ij} \equiv 0$ for $i \neq j$ and $B_{ii} \geq 0$) then there is no eigenvalue in $(-\infty, \lambda_0)$ and Theorem 6.*i.* holds for all $\lambda < \lambda_0$.

Remark 6.2 Since we do not assume any sign condition on c^μ, we may replace c^μ by $c^\mu - \gamma$ and $H_{\mu\mu}$ by $H_{\mu\mu} + \gamma$. Therefore without loss of generality we may assume that $H_{\mu\mu} \geq 0$ (H positive) or even

Corollary 7 *Suppose that the conditions of Theorem 5 are satisfied and that* $\sum i \leq k$*. Then*

$$\lambda_0 = \sup_{w \in (C^2(\Omega))^k} \; \inf_{1 \leq i \leq k} \frac{((\mathcal{L} - H)\, w)_i\,(x)}{(Bw)_i\,(x)} \tag{11}$$

holds.

Barta in [4] obtained such a result for the Laplace operator. For more general second order elliptic equations see [23], [27] and for systems see [26].

Proof. Let us denote the expression at the right hand side of (11) by λ_{Barta}. From (9) we find that $\lambda_0 \geq \lambda_{\text{Barta}}$. Using the first eigenfunction that is guaranteed to exist by Theorem 5 we find $\lambda_0 \leq \lambda_{\text{Barta}}$.

3 The maximum principle, subdomains and nonzero boundary values

For elliptic equations it is well known that if the solution operator for the Dirichlet problem is positive on Ω, then it is so for the Dirichlet problem on any subdomain $\Omega_s \subseteq \Omega$. A similar result also holds for cooperative systems.

Proposition 8 *Let $\Omega, \mathcal{L}, H$ satisfy the assumptions above and suppose that the conditions b. and c. of Theorem 5 are satisfied. If there exists $u_* \in \left(W^{2,N}_{loc}(\Omega) \cap L^\infty(\Omega)\right)^k$ such that*

$$\begin{cases} (\mathcal{L} - H)\, u_* \geq 0 & in\ \Omega, \\ u_* \overset{u_{o,\Omega}}{=} 0 & on\ \partial\Omega, \end{cases} \tag{12}$$

and $u_* \geq 0$ in Ω, then the following statement holds.
For every open set $\Omega_s \subseteq \Omega$, with $\partial\Omega_s$ smooth, and $u \in \left(W^{2,N}(\Omega_s)\right)^k$ such that

$$\begin{cases} (\mathcal{L} - H)\, u \geq 0 & in\ \Omega_s, \\ u \geq 0 & on\ \partial\Omega_s, \end{cases} \tag{13}$$

we have $u \geq 0$ in Ω_s.

Remark. In general the assumption that H fully coupled on Ω doesn't imply that $H_{|\Omega_s}$ is fully coupled on Ω_s. This explain why we find $u \geq 0$ for (13). However if H is fully coupled on Ω_s then we can improve the last statement above to $u \equiv 0$ or $u \gg 0$ in Ω_s.

Proof. First we observe that Kato's inequality [16], which can be used for $u_\mu \in W^{2,1}_{loc}(\Omega)$, and the cooperativity of H imply that in a distributional sense we have

$$((\mathcal{L}-H)\min(u,0))_\mu = \mathcal{L}_\mu\left(\chi_{[u_\mu<0]}u_\mu\right) - \sum_j H_{\mu j}\left(\chi_{[u_j<0]}u_j\right) \geq$$

$$\geq \chi_{[u_\mu<0]}\mathcal{L}_\mu u_\mu - \sum_j H_{\mu j}\chi_{[u_j<0]}u_j =$$

$$= \chi_{[u_\mu<0]}\left((\mathcal{L}-H)u\right)_\mu + \sum_j H_{\mu j}\left(\chi_{[u_\mu<0]} - \chi_{[u_j<0]}\right)u_j \geq$$

$$\geq \sum_j H_{\mu j}\left(\chi_{[u_\mu<0]} - \chi_{[u_j<0]}\right)u_j =$$

$$= \sum_{j\neq\mu} H_{\mu j}\chi_{[u_\mu<0]}\chi_{[u_j\geq 0]}u_j - \sum_{j\neq\mu} H_{\mu j}\chi_{[u_\mu\geq 0]}\chi_{[u_j<0]}u_j \geq 0.$$

Hence $(\mathcal{L}-H)\min(u,0) \geq 0$ in Ω_s and $\min(u,0)=0$ on $\partial\Omega_s$. Since u_* is a strongly positive supersolution on Ω_s for $\mathcal{L}-H$ it follows from [26] that on every fully coupled subset of $\{1,\ldots,k\}$ we have $\min(u,0)\geq 0$ in Ω_s. Hence $u \geq 0$ in Ω_s.

Remark. Proposition 8 will be used for the proof of Theorems 5 and 6. Once these theorems have been proved we may use them and prove Proposition 8 for $u \in \left(W^{2,N}_{loc}(\Omega_s)\cap L^\infty(\Omega_s)\right)^k$ without assuming that $\partial\Omega_s$ is smooth. In doing so one needs to replace the boundary condition $u \geq 0$ on $\partial\Omega_s$ by $\min(u,0) \overset{u_{o,\Omega}}{=} 0$ on $\partial\Omega_s$.

4 The case $B = I$

In this section we shall study a special case of problem (1) namely $B = I$ (the identity matrix). Throughout this section we will suppose that the conditions *a.*, *b.* and *c.* of Theorem 5 are satisfied. Let

$$\kappa \geq \sup_{\mu,x}\left(\sum_{j=1}^{k} H_{\mu j}(x) - c^\mu(x)\right) \tag{14}$$

and consider $\mathcal{L}-H+\kappa I$.

We have the following.

Lemma 9 *Let* $\mathbf{e} = (1,\ldots,1)^{\mathsf{T}} \in R^k$, $k \geq 1$ *and let* $\kappa > 0$ *satisfy (14). If* u_o *is as in Definition 4 then there exists* $u_e = (u_e^1, ...u_e^k) \in \left(W_{loc}^{2,N}(\Omega)\right)^k$ *such that*

$$\begin{cases} (\mathcal{L} - H + \kappa I)u_e &= \mathbf{e} \quad \text{in } \Omega, \\ u_e &\overset{u_o}{=} 0 \quad \text{on } \partial\Omega, \\ u_e &\gg 0 \quad \text{in } \Omega. \end{cases}$$

Moreover, we have that $u_o \overset{u_e}{=} 0$ *on* $\partial\Omega$.

Remark: A consequence of the above lemma is that the statements $u \overset{u_e}{=} 0$ on $\partial\Omega$ and u

Proof: Let $\{\Omega_i\}_{i\in N}$ be a sequence of smooth domains that approximate Ω from inside as in (2) and let $u_{e,i} = (u_{e,i}^1, ...u_{e,i}^k)$ be the solution of

$$\begin{cases} (\mathcal{L} - H + \kappa I)u_{e,i} &= \mathbf{e} \quad \text{in } \Omega_i, \\ u_{e,i} &= 0 \quad \text{on } \partial\Omega_i. \end{cases} \tag{15}$$

Since H is fully coupled on Ω, it follows that H is fully coupled on all Ω_i for all i large enough. This implies that we may apply [26, Theorem 1.1]. As a consequence we have $u_{e,i} \gg 0$ for all i large. Since Ω is bounded we may suppose that Ω lies in the half space $\left\{x \in R^N; x_1 > 0\right\}$. Let d_μ be defined by

$$d_\mu = \sup_x (c^\mu - \sum_{i=1}^k H_{\mu i} + \kappa)$$

and consider $\sigma \in R$ such that:

$$\sigma > \sup_{\mu,x} \frac{b_1^\mu + \sqrt{(b_1^\mu)^2 + 4a_{11}^\mu(1 + d_\mu)}}{2a_{11}^\mu}$$

and $v(x) := (e^{\sigma d} - e^{\sigma x_1})\mathbf{e}$, where d is the diameter of Ω.

We have

$$((\mathcal{L} - H + \kappa I)\, v)_\mu =$$

$$= \left(\sigma^2 a_{11}^\mu - b_1^\mu \sigma\right) e^{\sigma x_1} + \left(c^\mu - \sum_{j=1}^k H_{\mu j} + \kappa\right) \left(e^{\sigma d} - e^{\sigma x_1}\right) \geq$$

$$\geq \left(\sigma^2 a_{11}^\mu - b_1^\mu \sigma - d_\mu - 1\right) e^{\sigma x_1} + \left(c^\mu - \sum_{j=1}^k H_{\mu j} + \kappa\right) e^{\sigma d} + 1,$$

hence $(\mathcal{L} - H + \kappa I)\, v \gg 1$ and therefore

$$(\mathcal{L} - H + \kappa I)(u_{e,i} - v) \ll 0.$$

By using the maximum principle of [26, Theorem 1.1] it follows that

$$0 < u_{e,i} < v.$$

As a consequence, by using a standard argument, we find that u_e. Indeed, since $\{u_{e,i}\,(x)\}_{i \in N}$, $x \in \Omega$, is a bounded and increasing sequence, it converges.

By choosing $c^* \in R$ such that

$$M_\mu u_e^\mu = (-\kappa - c^\mu)\, u_e^\mu + \sum\nolimits_{j=1}^k H_{\mu i} u_e^j + 1 \leq c^* 1$$

it follows that $0 \leq u_e^\mu \leq c^* u_o^\mu$ and hence $u_e \overset{u_o}{=} 0$ on $\partial\Omega$.

By the result in [6, page 73] this implies $u_o \overset{u_e}{=} 0$ on $\partial\Omega$.

Lemma 10 *Let $f \in (L^\infty(\Omega))^k$.*
Then there exists a unique $u \in \left(L^\infty(\Omega) \cap W^{2,N}_{loc}(\Omega)\right)^k$ such that

$$\begin{cases} (\mathcal{L} - H + \kappa I)u = f & \text{in } \Omega, \\ u \overset{u_o}{=} 0 & \text{on } \partial\Omega. \end{cases}$$

Furthermore there exists $C \in R$ (independent of u and f) such that

$$\|u\|_{L^\infty} \leq C \|f\|_{L^\infty}. \tag{16}$$

Proof. Consider a sequence of open domains Ω_i such that $\Omega_i \subset \overline{\Omega}_i \subset \Omega_{i+1}$ and $\Omega = \cup_{i=1}^\infty \Omega_i$. Let u_i, $i = 1, 2, \ldots$, be the solution of

$$\begin{cases} (\mathcal{L} - H + \kappa I)u_i = f & \text{in } \Omega_i, \\ u_i = 0 & \text{on } \partial\Omega_i. \end{cases}$$

A comparison argument shows that

$$-z \leq u_i \leq z. \tag{17}$$

where $z := u_e \|f\|_{L^\infty}$. Therefore, as in the previous proof, it follows that $u_i \to u$, where u is a solution of

$$\begin{cases} (\mathcal{L} - H + \kappa I)u = f & \text{in } \Omega, \\ u \overset{u_o}{=} 0 & \text{on } \partial\Omega. \end{cases}$$

We complete the proof by noticing that the boundary condition $u \overset{u_o}{=} 0$ on $\partial\Omega$ and (16) are satisfied by virtue of (17). Uniqueness follows from (16).

Proposition 11 *Let* $f \in \left(L^N(\Omega)\right)^k$.
Then there exists a unique $u \in \left(W^{2,N}_{loc}(\Omega) \cap L^\infty(\Omega)\right)^k$ *such that*

$$\begin{cases} (\mathcal{L} - H + \kappa I)u = f & \text{in } \Omega, \\ u \overset{u_o}{=} 0 & \text{on } \partial\Omega. \end{cases} \tag{18}$$

Furthermore there exists C *(independent of* u *and* f*) such that*

$$\|u\|_{L^\infty} \leq C\|f\|_{L^N} \tag{19}$$

and $f > 0$ *implies* $u \gg 0$.

The proof will proceed following the ideas of the proof of [26, Theorem 1.1].

Proof. First we observe that $(\mathcal{L} + \kappa I)$ is a diagonal matrix of uniformly elliptic operators. From our choice of κ it follows that the hypothesis of [6, Theorem 1.2] are met. Therefore, the operator $(\mathcal{L} + \kappa I)^{-1}$ subject to the boundary condition in BNV-sense (Definition 4), is well defined in

Let $A := (\mathcal{L} + \kappa I)^{-1}H$ be the solution operator of the problem

$$\begin{cases} (\mathcal{L} + \kappa I)u = Hf & \text{in } \Omega, \\ u \overset{u_o}{=} 0 & \text{on } \partial\Omega, \end{cases}$$

i.e. $A(f) = u$.

• $A := (\mathcal{L} + \kappa I)^{-1}H$ *is compact and irreducible in* $(L^N(\Omega))^k$.

Indeed, by [7, Proposition 1.1] it follows that $(\mathcal{L} + \kappa I)^{-1}$ is a compact operator. Therefore, since H is bounded, the compactness of A follows.

Let us prove that A is irreducible. First, for easy reference, we recall that A on $(L^N(\Omega))^k$ is irreducible if for any measurable set $\mathsf{K} \subset \omega$, with $\mu(\mathsf{K}) > 0$ and $\mu(\omega \backslash \mathsf{K}) > 0$, the set

$$\left\{f \in (L^N(\Omega))^k; f_i(x) = 0 \text{ for all } x \in \mathsf{K}_i,\ 1 \leq i \leq k\right\}$$

is not invariant with respect to A.

In our situation, since the maximum principle holds (see [6]), it follows that every component of $(\mathcal{L}+\kappa I)^{-1}$ is irreducible. Using the fact that H is fully coupled (recall that we have assumed $H_{ii} \geq 0$) we find that A is irreducible and positive (see [26, proof of Lemma 1.3]).

Now, by using a result of De Pagter (see Theorem 18) it follows that $r(A) > 0$.

- *The operator* $(I-A)^{-1}$ *is well defined and* $(I-A)^{-1} = \sum_{\nu=0}^{\infty} A^{\nu}$.

By Theorem 17 (Krein-Rutman) it follows that $r\,(A)\,(>0)$ is an eigenvalue of A and its adjoint A^{*}. We shall denote by ϕ and ψ (respectively) the corresponding positive eigenfunctions .

Next we consider the function u_e given by Lemma 9. We recall that 0. We have

$$(\mathcal{L}+\kappa I)u_e = Hu_e + \mathbf{e} \gg Hu_e$$

and then

$$u_e \gg (\mathcal{L}+\kappa I)^{-1}Hu_e.$$

As a consequence

$$\langle \psi, u_e\rangle > \langle \psi, Au_e\rangle = \langle A^{*}\psi, u_e\rangle = r(A)\,\langle \psi, u_e\rangle$$

and $\langle \psi, u_e\rangle > 0$. This proves that $r(A) < 1$ and the claim follows.

In order to complete the proof, let us observe that, as a consequence of the preceding claims, we know that for any $f \in (L^N(\Omega))^k$ there exists u such that

$$u = (I-A)^{-1}(\mathcal{L}+\kappa I)^{-1}f.$$

This in turn is equivalent to

$$u - (\mathcal{L}+\kappa I)^{-1}Hu = (\mathcal{L}+\kappa I)^{-1}f$$

i.e.

$$\begin{cases} (\mathcal{L}+\kappa I)u - Hu = f & \text{in } \Omega, \\ u \overset{u_o}{=} 0 & \text{on } \partial\Omega. \end{cases} \tag{20}$$

It remains to prove (19). From Lemma 10 and $r\,(A) < 1$ it follows that

$$\|u\|_{\infty} \leq \frac{M}{1-r(A)} \left\| (\mathcal{L}+\kappa I)^{-1} f \right\|_{\infty}.$$

On the other hand, by the generalized version of the Alexandrov-Bakelman-Pucci Theorem (see [11, Theorem 9.1] and [6]) we obtain

$$\left\|(\mathcal{L}+\kappa I)^{-1} f\right\|_\infty \leq C \|f\|_{L^N}.$$

Finally, by noticing that H is fully coupled, if $f > 0$ it follows that

$$(I-A)^{-1}(\mathcal{L}+\kappa I)^{-1} f \geq A^k(\mathcal{L}+\kappa I)^{-1} f \gg 0.$$

This completes the proof.

Corollary 12 *There exists a positive eigenvalue λ_1 of the problem*

$$\begin{cases} (\mathcal{L}-H)\phi = \lambda_1\phi & \text{in } \Omega, \\ \phi \overset{u_o}{=} 0 & \text{on } \partial\Omega, \end{cases}$$

with corresponding eigenfunction $\phi \in \left(W^{2,N}_{loc}(\Omega) \cap L^\infty(\Omega)\right)^k$ satisfying $\phi \gg 0$.

Proof. Let $S_\kappa := (\mathcal{L} - H + \kappa I)^{-1}$ in $\left(L^N(\Omega)\right)^k$ be the inverse subject to the boundary condition in BNV-sense (Definition 4). As before it follows that S_κ is positive and irreducible, while by standard Sobolev embedding and inequality (19) it is compact. Therefore the elements of the spectrum of S_κ are isolated eigenvalues and at most converging to 0. An application of Theorem 16 (see below) gives the existence of a principal eigenvalue μ of S_κ with its corresponding eigenfunction $\phi > 0$.

Hence $\lambda_1 = \frac{1}{\mu} - \kappa$ is the principal eigenvalue of $(\mathcal{L}-H)$ with corresponding eigenfunction ϕ such that $\phi \gg 0$. Let us prove that $\lambda_1 > 0$.

By assumption there exists a strong positive supersolution w of $(\mathcal{L} - H)$, that is

$$(\mathcal{L} - H + \kappa I)\, w > \kappa w > 0. \tag{21}$$

By Proposition 11 the function defined by

$$\tilde{w} = S_\kappa\,(\mathcal{L} - H + \kappa I)\, w$$

satisfies $\tilde{w} \in \left(W^{2,N}_{loc}(\Omega) \cap L^\infty(\Omega)\right)^k$ and $\tilde{w} \gg 0$. Next we consider a sequence $\{\Omega_i\}$ of smooth domains contained in Ω and satisfying (2). Let $S_{\kappa,i} :=$

$(\mathcal{L} - H + \kappa I)^{-1}$ be the resolvent operator of problem (20) considered on Ω_i. By Proposition 8 it follows that

$$w \geq S_{\kappa,i} (\mathcal{L} - H + \kappa I) \, w \text{ on } \Omega_i.$$

Since $S_{\kappa,i} (\mathcal{L} - H + \kappa I) \, w = S_{\kappa,i} (\mathcal{L} - H + \kappa I) \, \tilde{w} \to \tilde{w}$ for ∞ we find

$$\tilde{w} \leq w \text{ on } \Omega$$

and hence

$$\tilde{w} = S_\kappa (\mathcal{L} - H + \kappa I) \, w > \kappa S_\kappa w \geq \kappa S_\kappa \tilde{w} \text{ on } \Omega.$$

Let μ denote the principal eigenvalue of S_κ. Since μ is also the principal eigenvalue of the adjoint operator S_κ^* with corresponding eigenfunction ψ it follows that

$$\begin{aligned} 0 < \kappa \, \langle \psi, S_\kappa \tilde{w} \rangle < \langle \psi, \tilde{w} \rangle = \\ = \frac{1}{\mu} \langle S_\kappa^* \psi, \tilde{w} \rangle = \frac{1}{\mu} \langle \psi, S_\kappa \tilde{w} \rangle \, , \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes pairing of $\left(L^N (\Omega) \right)^k$ with its dual. As a consequence we find $\lambda_1 = \frac{1}{\mu} - \kappa > 0$.

Corollary 13 *Let $\lambda < \lambda_1$ and $f \in (L^N(\Omega))^k$.*
Then there exists a unique $u \in \left(W^{2,N}_{loc}(\Omega) \cap L^\infty(\Omega) \right)^k$ such that

$$\begin{cases} (\mathcal{L} - H)u = \lambda u + f & \text{in } \Omega, \\ u \overset{u_o}{=} 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover if $f > 0$, then $u \gg 0$.

Proof. For $\lambda = -\kappa$ the result follows from Proposition 11. The only restriction for κ is (14) and hence the result holds for all $\lambda \leq \kappa$. For $\lambda \in (-\kappa, \lambda_1)$ we can proceed as in the proof of Proposition 11. Indeed observe that $\nu \left((\mathcal{L} - H + \kappa)^{-1} (\kappa + \lambda) \right) = (\lambda_1 + \kappa)^{-1} (\kappa + \lambda) < 1$ and $(\mathcal{L} - H + \kappa)^{-1}$ is strictly positive, hence the following function

$$u = \sum_{k=0}^{\infty} \left((\mathcal{L} - H + \kappa)^{-1} (\kappa + \lambda) \right)^k (\mathcal{L} - H + \kappa)^{-1} f$$

is well defined and it is the desired solution for our problem.

5 Hess' lemma, cooperative B

In this section we shall study the existence of a positive eigenvalue λ_B with positive eigenfunction Φ of the problem

$$\begin{cases} (\mathcal{L} - H)\Phi = \lambda_B \ B\Phi & \text{in } \Omega, \\ \Phi \overset{u_o}{=} 0 & \text{on } \partial\Omega. \end{cases} \tag{22}$$

Our result is the following.

Proposition 14 *Let the assumptions a.-e. of Theorem 5 be satisfied. Then there exists* $\lambda_B > 0$ *and* $\Phi \in \left(W^{2,N}_{loc}(\Omega) \cap L^\infty(\Omega)\right)^k$ *with* $\Phi \gg 0$ *such that (22) holds.*

The proof is organized by partially following the ideas contained in [15] and [14].

Proof. First we observe that without loss of generality we may assume that $B_{ii} > -1$ for all $i \in \{1, \dots, k\}$. Consider now $K_\alpha : \left(L^N(\Omega)\right)^k \to \left(L^N(\Omega)\right)^k$ defined by

$$K_\alpha = (\mathcal{L} - H + \alpha I)^{-1} (B + I),$$

where $\alpha \geq 0$ and $(\mathcal{L} - H + \alpha I)^{-1}$ is subjected to the Dirichlet condition as in Definition 4. From Corollary 13 it follows that for any $\alpha \geq 0$ the operator $compact, positive and irreducible.$

A useful property shared by K_α is contained in the following lemma.

Lemma 15 *There exists* $\alpha > 0$ *and* $w \in \left(W^{2,N}_{loc}(\Omega) \cap L^\infty(\Omega)\right)^k$ *with* $w \gg 0$ *such that* $\alpha K_\alpha w \geq w$.

Proof: Let $i \in \{1, \dots, k\}$, $\sigma > 0$ and let us choose $\delta > 0$ and $x_0 \in \Omega$ such that $\mathrm{B}, x_0 \subset \Omega$ and $B_{ii}(x) \geq \sigma$ for B_{δ,x_0}. Here, as usual, we have denoted by B_{δ,x_0} the set $\left\{x \in R^N; |x - x_0| < \delta\right\}$.

Consider the eigenvalue problem

$$\begin{cases} (L_i - H_{ii})\, v = \lambda v & \text{in } \mathrm{B}_{\delta,x_0}, \\ v = 0 & \text{on } \partial\mathrm{B}_{\delta,x_0}. \end{cases} \tag{23}$$

A direct computation shows that $\left((\mathcal{L}-H)^{-1}\mathbf{e}\right)_i$ is a positive strict supersolution of (23the operator $T_i = \left((L_i - H_{ii})_{|\mathrm{B},x_0}\right)^{-1}$, subject to zero Dirichlet boundary condition on B_{δ,x_0}, is positive, compact and irreducible (by the strong maximum principle). As a consequence there exists a first positive eigenvalue $\tilde{\lambda}$ with eigenfunction $\tilde{\phi}$ of T_i. Let us extend $\tilde{\phi}$ equal to 0 outside of B_{δ,x_0}. Consider

$$\tilde{\Phi} = (0,\ldots,\underset{i^{th}\text{-entry}}{\tilde{\phi}},\ldots,0)^{\mathsf{T}}$$

and set $w = \alpha K_\alpha \tilde{\Phi}$ with $\alpha = \tilde{\lambda}$

We claim that $\tilde{\Phi} \leq \alpha K_\alpha \tilde{\Phi}$. Indeed, since $\tilde{\Phi} > 0$ we find that $w \gg 0$. Hence $(B+I)\,\tilde{\Phi}\sigma + 1\tilde{\Phi}$ and

$$(\mathcal{L}-H+\alpha I)\left(w-\tilde{\Phi}\right) = \alpha\,(B+I)\,\tilde{\Phi} - \left(\tilde{\lambda}+\alpha\right)\tilde{\Phi} > 0 \quad \text{on } \mathrm{B}_{\delta,x_0}. \tag{24}$$

Observe that although the strong maximum might fail for the system on B_{δ,x_0} (the system is not necessarily fully coupled on a subdomain of Ω) the component-wise strong maximum principle still holds (see Proposition 8). Hence (24) and $w - \tilde{\Phi} \gg 0$ on $\partial\mathrm{B}_{\delta,x_0}$ imply that $w - \tilde{\Phi} \gg 0$ on B_{δ,x_0}. Moreover from $w \gg 0 = \tilde{\Phi}$ on $\Omega\backslash\mathrm{B}_{\delta,x_0}$ it follows that $\alpha K_\alpha\tilde{\Phi} \gg \tilde{\Phi}$ on Ω. In addition, using the fact that αK_α is a strongly positive operator we find that

$$\alpha K_\alpha w = (\alpha K_\alpha)^2\,\tilde{\Phi} \gg \alpha K_\alpha \tilde{\Phi} = w \gg \tilde{\Phi} > 0.$$

This completes the proof.

Proof of Proposition 14: Since K_α is compact, positive and irreducible it follows from Theorem 16 that there exist a first eigenvalue $\frac{1}{\alpha_1} > 0$ of K_α with corresponding eigenfunction Φ_1, that is

$$\Phi_1 = \alpha_1 K_\alpha \Phi_1.$$

Then, by Lemma 15, we find that there exists $w > 0$ satisfying $\alpha K_\alpha w \geq w$. Hence

$$\frac{1}{\alpha_1} = r\,(K_\alpha) \geq \frac{1}{\alpha},$$

where $r\,(K_\alpha)$ is the spectral radius of K_α.

Varying α we construct a sequence $(\alpha_n,\Phi_n)_{n\geq 1}$ with $\alpha_0 = \alpha$ such that for $n \geq 1$

$$\begin{cases} 0 < \alpha_n \leq \alpha_{n-1}; \\ \Phi_n = \alpha_n K_{\alpha_{n-1}}\Phi_n > 0 \text{ and } ||\Phi_n|| = 1. \end{cases}.$$

From the sequence $(\alpha_n, \Phi_n)_{n\geq 1}$ we can extract a subsequence, still denoted by (α_n, Φ_n), such that $\Phi_n \to \Phi$ and $\alpha_n \to \lambda > 0$ with

$$\Phi = \lambda K_\lambda \Phi.$$

It follows that

$$\Phi \overset{u_o}{=} 0 \text{ on } \partial\Omega$$

and

$$(\mathcal{L} - H + \lambda I)\,\Phi = \lambda\,(I + B)\,\Phi \;\Rightarrow\; (\mathcal{L} - H)\,\Phi = \lambda B \Phi$$

Hence $\lambda = \lambda_B$ and the proof is complete.

Remark. Using the existence of a positive supersolution in one step, we just get a value λ such that $\Phi = \lambda K_\alpha \Phi$ with $\lambda \leq \alpha$ and not necessarily equal.

Proof of Theorem 5 and 6. We have directly that $\lambda_B = \lambda_0$. Using the function Φ of Proposition 14 we find that for all $\lambda \in [0, \lambda_B)$ the conditions of Theorem 5 are satisfied for $(\mathcal{L} - H)$ replaced by $(\mathcal{L} - H - \lambda B)$. As a consequence the results of section 3 where $B = I$ apply thereby completing the proof.

6 The results of Krein-Rutman and De Pagter

A real vector space with a partial ordering, say $(E, \geq)$, is called a vector lattice if $f, g \in E$ implies that $f \vee g \in E$, where $least upper bound of \{f, g\}$. With a norm supplied $(E, \geq, \|\cdot\|)$ is called a Banach lattice if $(E, \|\cdot\|)$ is a Banach space and if $(E, \geq)$ is a vector lattice such that $|f| \leq |g|$ implies $\|f\| \leq \|g\|$. Here $|f| = f \vee (-f)$. The set $A \subseteq E$ is called a lattice ideal if $|f| \leq |g|$ and $g \in A$ imply $f \in A$. An positive operator $S \in L(E)$ is called irreducible if $\{0\}$ and E are the only closed lattice ideals that are invariant under S.

Theorem 16 *Let E be a Banach lattice with* $\dim(E) > 1$ *and let* $T \in L(E)$ *be a positive irreducible and compact operator. Then the spectral radius r of T satisfies $r > 0$ and there is $0 < \phi \in E$ such that $T\phi = r\phi$. Moreover, r is the unique eigenvalue with a positive eigenfunction and the eigenvalue is algebraically simple.*

This theorem is the combination of the famous Krein-Rutman [17] theorem and an important result of De Pagter [21] that replaces the positivity of the spectral radius of T by irreducibility. This last condition is in general much easier to check. In $E = L^p(\mathcal{W})$ (with the Lebesgue measure and $1 \leq p < \infty$), where $\mathcal{W}$ is an open set in R^n, the closed ideal are of the form $\{f \in L^p(\mathcal{W}); f = 0$ a.e. on $\mathsf{K}\}$ (see [25, page 158]). This implies that for a positive operator S on E, irreducibility is equivalent to $M_{\chi_{\mathcal{W}\backslash\mathsf{K}}} \circ S \circ M_{\chi_{\mathsf{K}}} \neq 0$ for every measurable set $\mathsf{K} \subset \mathcal{W}$ with $\mu(\mathsf{K}), \mu(\mathcal{W}\backslash\mathsf{K}) > 0$. Here the operator $M_{\chi_{\mathsf{K}}}$ is the multiplication with the characteristic function χ_{K} of K. For (compact) kernel operators the theorem above is known as the Theorem of Jentzsch.

In this paper we have used $E = (L^p(\Omega))^k$ for $p \in (1, \infty)$. Note that E can be identified with $L^p(\mathcal{W})$ where $\mathcal{W}$ is as in (6).

Theorem 17 (Krein-Rutman) *Let $T \in L(E)$ be a compact and positive operator with a strictly positive spectral radius r. Then there is $\varphi \in E$ with $\varphi > 0$ and $T\varphi = r\varphi$.*

Theorem 18 (De Pagter) *Let E be a Banach lattice with* $\dim(E) > 1$ *and let $T \in L(E)$ be a compact, positive and irreducible operator. Then it has a positive spectral radius r.*

The uniqueness in Theorem 16 remains to be shown. Since T is positive and compact, the adjoint $T' \in L(E')$ is positive and compact; it also has the same spectral radius $positive eigenfunction \phi \in E'$ with $T'\phi = r\phi$. By Theorem V.5.2. of Schaefer [25] it follows that φ is the only positive eigenfunction of T and moreover, the eigenvalue r for T is algebraically simple.

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