The role of positive boundary data in the clamped plate equation, perturbation results and other generalizations

Hans-Christoph Grunau
Fachgruppe Mathematik
Universität Bayreuth
D-95440 Bayreuth
Germany

Guido Sweers
Vakgroep Algemene Wiskunde
Postbus 5031
Technische Universiteit Delft
NL-2600 GA Delft
The Netherlands

1 Introduction

Like Boggio [Bo1] and Hadamard [Ha] (1901/08) one might conjecture that positive data $f \geq 0$, $\varphi \geq 0$, $\psi \geq 0$ in the clamped plate equation

$$(1) \quad \begin{cases} (-\Delta)^2 u = f \text{ in } \Omega, \\ u|\partial\Omega = \psi, \quad \left(\dfrac{\partial}{\partial \nu}\right) u|\partial\Omega = \varphi, \end{cases}$$

yield positive solutions $u \geq 0$. Here $\Omega \subset \mathbb{R}^n$ is the "shape of the plate" (physically relevant for $n = 2$), ν is the exterior unit normal at $\partial\Omega$, f is the (perpendicular) load, φ and ψ are the boundary data and u is the deflection of the "plate".

Most authors concentrated on the Green function $G_{2,n,\Omega}$ for the Dirichlet problem (1) in case of homogeneous boundary data $\varphi = \psi = 0$. Boggio [Bo2] could show by explicit calculation that the Green function $G_{m,n,\Omega}$ for any power $(-\Delta)^m$ is positive if $\Omega = B \subset \mathbb{R}^n$ is the (unit) ball. Numerous counterexamples ([Du], [Ga], [CD], [ST] and many others) have shown that this result actually does *not* hold in arbitrary domains Ω. A perturbation theory for Boggio's positivity result has been developed by the authors in [GS1] with respect to lower order terms of the differential operator and in [GS2] with respect to the domain and the highest order terms of the differential operator in two dimensions.

In the present note we focus on the role of the boundary data φ and ψ. Hence we may assume that $f = 0$. As we pointed out in [GS3] if $\psi = 0$ the positivity behaviour of the Dirichlet problem (1) with respect to the highest order datum is more or less the same as with respect to the right-hand side. But if also $\psi \geq 0$, $\psi \not\equiv 0$ is considered the situation becomes more involved. From [Nic, p. 34] we take

$$(2) \quad u(x) = \int_{\partial B} K_{2,n}(x,y)\psi \, d\omega(y) + \int_{\partial B} L_{2,n}(x,y)\varphi(y) \, d\omega(y), \quad x \in B,$$

where

$$(3) \quad K_{2,n}(x,y) = \frac{1}{2\omega_n} \frac{(1-|x|^2)^2}{|x-y|^{n+2}} \left\{2 + [n-4) x \cdot y - (n-2)|x|^2\right\},$$

$$(4)\qquad L_{2,n}(x,y)=\frac{1}{2\omega_n}\frac{(1-|x|^2)^2}{|x-y|^n},$$

$x\in B$, $y\in\partial B$, ω_n denotes the $(n-1)$-dimensional surface area of the unit ball. Evidently $L_{2,n}>0$ for any n, while $K_{2,n}>0$ only and $n\le 4$ and $K_{2,n}$ changes sign for $n\ge 5$. Moreover perturbation results like Theorem 7 below could have been shown only for $n\le 3$, see [GS3].

In the next section we will show that the Dirichlet problem (1) may be reformulated ina way such that we have a positivity result with respect to both boundary data in any dimension. Moreover for $n\le 3$ and in particular for $n=2$ the above mentioned result may be sharpened so that if $\psi(x_0)>0$ for some $x_0\in\partial B$, close to x_0 also negative values for φ are admissible.

In the last section we switch to polyharmonic Dirichlet problems of arbitrary order $2m$ and develop a perturbation theory of positivity with respect to the Dirichlet data of order $(m-1)$ and $(m-2)$, provided the other boundary data are preccribed homogeneously and the positivity assumption is posed in an adequate way.

2 The adequate positivity assumption for the clamped plate equation

In order to find the adequate positivity assumption on the boundary data in the Dirichlet problem

$$(5)\qquad \begin{cases} (-\Delta)^2u=0 \text{ in } B,\\ u|\partial B=\psi, \quad \left(\dfrac{\partial}{\partial\nu}\right)u|\partial B=\varphi, \end{cases}$$

the key observation is that the negative part of the kernel $K_{2,n}$ corresponding to ψ has the form as the kernel $L_{2,n}$ corresponding to the datum φ.

Lemma 1. *Let $s\in\mathbb{R}$, $s\ge\frac{1}{2}(n-4)$. Then for*

$$(6)\qquad \hat{K}_{2,n,s}(x,y):=K_{2,n}(x,y)+sL_{2,n}(x,y),\quad x\in B,\quad y\in\partial B,$$

we have

$$\hat{K}_{2,n,s}(x,y)>0.$$

Proof. We observe that for $x\in B$, $y\in\partial B$ (i.e. $|y|=1$) we have

$$\begin{aligned} K_{2,n}(x,y) &= \frac{1}{2\omega_n}\frac{(1-|x|^2)^2}{|x-y|^{n+2}}\left\{\frac{n}{2}(1-|x|^2)-\frac{n-4}{2}|x-y|^2\right\}\\ &= \frac{n}{4\omega_n}\frac{(1-|x|^2)^3}{|x-y|^{n+2}}-\frac{n-4}{2}L_{2,n}(x,y). \end{aligned}$$

□

Proposition 2. *Let $\varphi \in C^0(\partial B)$, $\psi \in C^1(\partial B)$. We assume that for some number $s \geq \frac{1}{2}(n-4)$ there holds*

$$\psi(x) \geq 0 \quad \text{and} \quad \varphi(x) \geq s\psi(x) \quad \text{for } x \in \partial B.$$

Then the uniquely determined solution $u \in C^4(B) \cap C^1(\overline{B})$ of the Dirichlet problem (5) is positive:

$$u \geq 0 \quad \text{in } B.$$

Proof. From (2) and (6) there follows:

$$\begin{aligned} u(x) &= \int_{\partial B} K_{2,n}(x,y)\psi \, d\omega(y) + \int_{\partial B} L_{2,n}(x,y)\varphi(y)\, d\omega(y) \\ &= \int_{\partial B} \hat{K}_{2,n,s}(x,y)\psi \, d\omega(y) + \int_{\partial B} L_{2,n}(x,y)\Big(\varphi(y) - s\,\psi(y)\Big)\, d\omega(y). \end{aligned}$$

□

Remark. For $n = 1, 2, 3$, also negative values for s are admissible.

We are interested in whether this positivity result remains under perturbations of the prototype problem(5). As in higher order Dirichlet problems quite similar phenomena can be observed, we develop the perturbation theory for the biharmonic Dirichlet problem (5) as a special case of the perturbation theory for the polyharmonic Dirichlet problem (7) below. The latter is subject of the following section.

3 Higher order equations. Perturbations

In what follows we always assume $m \geq 2$.

First we consider the polyharmonic prototype problem:

$$(7) \quad \begin{cases} (-\Delta)^m u = 0 & \text{in } B, \\ \left(-\frac{\partial}{\partial\nu}\right)^j u = 0 & \text{on } \partial B \text{ for } j = 0, \ldots, m-3, \\ \left(-\frac{\partial}{\partial\nu}\right)^{m-2} u = \psi & \text{on } \partial B, \\ \left(-\frac{\partial}{\partial\nu}\right)^{m-1} u = \varphi & \text{on } \partial B. \end{cases}$$

Except in the radial case $u = u(|x|)$ (see [Sor, Proposition 1, Remark 9]) no positivity result can be expected with respect to the boundary data of order $0, \ldots, m-3$, so we prescribe them homogeneously.

After some elementary calculations we find from [Ed] or directly from (2) that for $\varphi \in C^0(\partial B)$, $\psi \in C^1(\partial B)$ the solution $u \in C^{2m}(B) \cap C^{m-1}(\overline{B})$ to the Dirichlet problem (7) is given by

$$(8) \quad u(x) = \int_{\partial B} K_{m,n}(x,y)\psi \, d\omega(y) + \int_{\partial B} L_{m,n}(x,y)\varphi(y)\, d\omega(y), \quad x \in B,$$

where

$$(9) \quad K_{m,n}(x,y) = \frac{1}{2^m\,(m-2)!\,\omega_n} \frac{(1-|x|^2)^m}{|x-y|^{n+2}} \left\{ n(1-|x|^2) - (n-2-m)|x-y|^2 \right\},$$

(10) $L_{m,n}(x,y) = \dfrac{1}{2^{m-1}\,(m-1)!\,\omega_n}\,\dfrac{(1-|x|^2)^m}{|x-y|^n},$

$x \in B$, $y \in \partial B$. As in Section 2 we have

Lemma 3. *Let $s \in \mathbb{R}$, $s \geq \frac{1}{2}(n-2-m)(m-1)$. Then for*

(11) $\hat{K}_{m,n,s}(x,y) := K_{m,n}(x,y) + sL_{m,n}(x,y), \quad x \in B, \quad y \in \partial B,$

we have

$$\hat{K}_{m,n,s}(x,y) > 0.$$

Proposition 4. *Let $\varphi \in C^0(\partial B)$, $\psi \in C^1(\partial B)$. We assume that for some number $s \geq \frac{1}{2}(n-2-m)(m-1)$ there holds*

$$\psi(x) \geq 0 \quad \text{and} \quad \varphi(x) \geq s\psi(x) \quad \text{for } x \in \partial B.$$

Then the uniquely determined solution $u \in C^{2m}(B) \cap C^{m-1}(\overline{B})$ of the Dirichlet problem (5) is positive:

$$u \geq 0 \quad \text{in } B.$$

As a starting point for the peturbation theory of Proposition 4 we describe the essential properties ot the integral kernels $\hat{K}_{m,n,s}$ and $L_{m,n}$.

For brevity we introduce a notation for the boundary distance

(12) $d(x) := 1 - |x|\,, \qquad x \in B.$

Lemma 5. *a) Let $s \geq \frac{1}{2}(n-2-m)(m-1)$. On $B \times \partial B$ (i.e. for $x \in B$, $y \in \partial B$) we have*

(13) $\hat{K}_{m,n,s}(x,y) \begin{cases} \preceq |x-y|^{-n-1} d(x)^m, \\ \succeq |x-y|^{-n-2} d(x)^{m+1}, \end{cases}$

(14) $L_{m,n}(x,y) \sim |x-y|^{-n} d(x)^m.$

b) If we assume additionally that $s > \frac{1}{2}(n-2-m)(m-1)$ then we have on $B \times \partial B$:

(15) $\hat{K}_{m,n,s}(x,y) \begin{cases} \preceq |x-y|^{-n-1} d(x)^m, \\ \succeq |x-y|^{-n} d(x)^m, \end{cases}$

Here for $f, g : M \subset \mathbb{R}^k \to \mathbb{R}^+$ we used the notation:

$$f \sim g \quad \Leftrightarrow \quad \exists C > 0\ \forall x \in M: \ \frac{1}{C} f(x) \geq g(x) \leq C f(x),$$
$$f \preceq g \quad \Leftrightarrow \quad \exists C > 0\ \forall x \in M: \ f(x) \geq C g(x).$$

Proof of Lemma 5. The claim follows from $1 - |x|^2 \sim d(x)$, $d(x) \leq |x-y|$ and

$$K_{m,n}(x,y) \;=\; \frac{n}{2^m\,(m-2)!\,\omega_n}\,\frac{(1-|x|^2)^{m+1}}{|x-y|^{n+2}} - \frac{1}{2}(n-2-m)(m-1)\,L_{m,n}(x,y).$$

$\square$

Remarks. 1) The estimation constants in (15) depend stronly on s.
2) If $s = \frac{1}{2}(n-2-m)(m-1)$ then we have $\hat{K}_{m,n,s}(x,y) \sim |x-y|^{-n-2} d(x)^{m+1}$, i.e. for $x \to \partial B$ we have a zero of order m. We would have expected and actually need in order to prove perturbation results a zero of order m. Consequently in what follows we have to assume $s > \frac{1}{2}(n-2-m)(m-1)$. The estimate (15) is more appropriate. But as $\hat{K}_{m,n,s}(x,y) \not\sim |x-y|^{-n-1} d(x)^m$ our perturbation result Theorem 7 below is (necessarily ?) less general than the corresponding results in [GS1] and [GS2]. In particular we are not yet able to consider domain perturbations.

For our purposes the following "3-G-type" estimates are essential. We use the multiindex notation $D^\alpha = \prod_{i=1}^n \left(\frac{\partial}{\partial x_i}\right)^{\alpha_i}$ for $\alpha \in \mathbb{N}_0^n$; $|\alpha| = \sum_{i=1}^n \alpha_i$. We recall that $G_{m,n}$ denotes the Dirichlet Green function for $(-\Delta)^m$ in the unit ball $B \subset \mathbb{R}^n$.

Lemma 6. *Let $s > \frac{1}{2}(n-2-m)(m-1)$, $\alpha \in \mathbb{N}_0^n$ Then on $B \times \partial B \times B$ (i.e. for $x \in B$, $y \in \partial B$, $z \in B$) we have the following.*

$$\frac{|D_z^\alpha G_{m,n}(x,z)|\, \hat{K}_{m,n,s}(z,y)}{\hat{K}_{m,n,s}(x,y)} \preceq \tag{16}$$
$$\preceq \begin{cases} 1, & \text{if } |\alpha| < 2m-n, \\ |x-z|^{2m-1-n-|\alpha|} + |y-z|^{2m-1-n-|\alpha|}, & \text{if } |\alpha| \geq 2m-n. \end{cases}$$

$$\frac{|D_z^\alpha G_{m,n}(x,z)|\, L_{m,n}(z,y)}{L_{m,n}(x,y)} \preceq \tag{17}$$
$$\preceq \begin{cases} 1, & \text{if } |\alpha| < 2m-n \text{ and } n \text{ odd}, \\ & \text{or if } |\alpha| < 2m-n, \\ \log\left(\dfrac{3}{|x-z|}\right), & \text{if } |\alpha| = 2m-n \text{ and } n \text{ even}, \\ |x-z|^{2m-n-|\alpha|} + |y-z|^{2m-n-|\alpha|}, & \text{if } |\alpha| > 2m-n. \end{cases}$$

Proof. We repeatedly employ $d(x) \leq |x-y|$ $(y \in \partial B)$ without mention.
a) Proof of (16): We use estimate (15) of Lemma 5
The case: $|\alpha| \leq 2m-n$ and n odd, or $|\alpha| < 2m-n$.
Here we use Corollary 10.

$$\begin{aligned}
\frac{|D_z^\alpha G_{m,n}(x,z)|\, \hat{K}_{m,n,s}(z,y)}{\hat{K}_{m,n,s}(x,y)} &\preceq \frac{d(x)^{m-\frac{n}{2}} d(z)^{m-\frac{n}{2}-|\alpha|} \min\left\{1, \frac{d(x)^{n/2} d(z)^{n/2}}{|x-z|^n}\right\} \frac{d(z)^m}{|z-y|^{n+1}}}{\frac{d(x)^m}{|x-y|^n}} \\
&\preceq d(x)^{-\frac{n}{2}} d(z)^{2m-\frac{n}{2}-|\alpha|} \min\left\{1, \frac{d(x)^{\frac{n}{2}} d(z)^{\frac{n}{2}}}{|x-z|^n}\right\} |y-z|^{-n-1} \left(|x-z|^n + |y-z|^n\right) \\
&\preceq d(z)^{2m-|\alpha|} |y-z|^{-n-1} + d(x)^{-\frac{n}{2}} d(z)^{2m-\frac{n}{2}-|\alpha|} \left(\frac{d(x)}{d(z)}\right)^{\frac{n}{2}} |y-z|^{-1} \\
&= d(z)^{2m-|\alpha|} |y-z|^{-n-1} + d(z)^{2m-n-|\alpha|} |y-z|^{-1} \preceq |y-z|^{2m-1-n-|\alpha|}.
\end{aligned}$$

The case: $|\alpha| = 2m-n$ and n even.
We use Lemma 9.

$$
\begin{aligned}
&\frac{|D_z^\alpha G_{m,n}(x,z)|\,\hat{K}_{m,n,s}(z,y)}{K_{m,n,s}(x,y)}\\
\preceq\;& \frac{\log\left(2+\frac{d(z)}{|x-z|}\right)\min\left\{1,\frac{d(x)}{|x-z|}\right\}^m \min\left\{1,\frac{d(z)}{|x-z|}\right\}^{\max\{m-|\alpha|,0\}}\frac{d(z)^m}{|z-y|^{n+1}}}{\frac{d(x)^m}{|x-y|^n}}\\
\preceq\;& \left(1+\frac{d(x)}{|x-z|}\right) d(x)^{-m} d(z)^m \min\left\{1,\frac{d(x)}{|x-z|}\right\}^m\\
&\quad \cdot\min\left\{1,\frac{d(z)}{|x-z|}\right\}^{\max\{m-|\alpha|,0\}} |y-z|^{-n-1}\left(|x-z|^n+|y-z|^n\right)\\
=\;& \Big\{d(x)^{-m}d(z)^m|y-z|^{-n-1}|x-z|^n + d(x)^{-m}d(z)^m|y-z|^{-1}\\
&\quad +d(x)^{-m}d(z)^{m+1}|y-z|^{-n-1}+d(x)^{-m}d(z)^{m+1}|y-z|^{-1}|x-z|^{-1}\Big\}\\
&\quad\quad \cdot\min\left\{1,\frac{d(x)}{|x-z|}\right\}^m \min\left\{1,\frac{d(z)}{|x-z|}\right\}^{\max\{m-|\alpha|,0\}}\\
\preceq\;& d(x)^{-m}d(z)^m|y-z|^{-n-1}|x-z|^n\left(\frac{d(x)}{|x-z|}\right)^{\min\{m,n\}}\\
&\quad \cdot\left(\frac{d(x)}{d(z)}\right)^{\max\{m-n,0\}}\left(\frac{d(z)}{|x-z|}\right)^{\max\{m-|\alpha|,0\}}\\
&+d(x)^{-m}d(z)^m|y-z|^{-1}\left(\frac{d(x)}{d(z)}\right)^m\\
&d(x)^{-m}d(z)^{m+1}|y-z|^{-n-1}|x-z|^{n-1}\left(\frac{d(x)}{|x-z|}\right)^{\min\{m,n\}-1}\\
&\quad \cdot\left(\frac{d(x)}{d(z)}\right)^{1+\max\{m-n,0\}}\left(\frac{d(z)}{|x-z|}\right)^{\max\{m-|\alpha|,0\}}\\
&+d(x)^{-m}d(z)^{m+1}|y-z|^{-1}|x-z|^{-1}\left(\frac{d(x)}{d(z)}\right)^m\\
\sim\;& d(z)^n|y-z|^{-n-1}+|y-z|^{-1}+d(z)|y-z|^{-1}|x-z|^{-1}\\
\preceq\;& |x-z|^{-1}+|y-z|^{-1}.
\end{aligned}
$$

The case: $|\alpha| > 2m-n$ and $|\alpha| \le m$.
We use Lemma 9.

$$
\begin{aligned}
&\frac{|D_z^\alpha G_{m,n}(x,z)|\,\hat{K}_{m,n,s}(z,y)}{\hat{K}_{m,n,s}(x,y)}\\
\preceq\;& \frac{|x-z|^{2m-n-|\alpha|}\min\left\{1,\frac{d(x)}{|x-z|}\right\}^m \min\left\{1,\frac{d(z)}{|x-z|}\right\}^{m-|\alpha|}\frac{d(z)^m}{|z-y|^{n+1}}}{\frac{d(x)^m}{|x-y|^n}}
\end{aligned}
$$

$$
\begin{aligned}
&\preceq d(x)^{-m} d(z)^m |x-z|^{2m-n-|\alpha|} |y-z|^{-n-1} \\
&\quad \cdot \min\left\{1, \frac{d(x)}{|x-z|}\right\}^m \min\left\{1, \frac{d(z)}{|x-z|}\right\}^{m-|\alpha|} (|x-z|^n + |y-z|^n) \\
&\preceq d(x)^{-m} d(z)^m |x-z|^{2m-|\alpha|} |y-z|^{-n-1} \left(\frac{d(x)}{|x-z|}\right)^m \left(\frac{d(z)}{|x-z|}\right)^{m-|\alpha|} \\
&\quad + d(x)^{-m} d(z)^m |x-z|^{2m-n-|\alpha|} |y-z|^{-1} \left(\frac{d(x)}{d(z)}\right)^m \\
&\preceq d(z)^{2m-|\alpha|} |y-z|^{1-n} + |x-z|^{2m-n-|\alpha|} |y-z|^{-1} \\
&\preceq |x-z|^{2m-1-n-|\alpha|} + |y-z|^{2m-1-n-|\alpha|}.
\end{aligned}
$$

The case: $|\alpha| > 2m-n$ *and* $|\alpha| > m$.

$$
\begin{aligned}
&\frac{|D_z^\alpha G_{m,n}(x,z)|\, \hat{K}_{m,n,s}(z,y)}{\hat{K}_{m,n,s}(x,y)} \\
&\preceq \frac{|x-z|^{2m-n-|\alpha|} \min\left\{1, \frac{d(x)}{|x-z|}\right\}^m \frac{d(z)^m}{|z-y|^{n+1}}}{\frac{d(x)^m}{|x-y|^n}} \\
&\preceq d(x)^{-m} d(z)^m |x-z|^{2m-n-|\alpha|} |y-z|^{-n-1} \\
&\quad \cdot \min\left\{1, \frac{d(x)}{|x-z|}\right\}^m (|x-z|^n + |y-z|^n) \\
&\preceq d(x)^{-m} d(z)^m |x-z|^{2m-|\alpha|} |y-z|^{-n-1} \left(\frac{d(x)}{|x-z|}\right)^{\min\{m,n\}} \left(\frac{d(x)}{d(z)}\right)^{\max\{m-n,0\}} \\
&\quad + d(x)^{-m} d(z)^m |x-z|^{2m-n-|\alpha|} |y-z|^{-1} \left(\frac{d(x)}{d(z)}\right)^m \\
&\preceq d(z)^{\min\{m,n\}} |x-z|^{m-|\alpha|+\max\{m-n,0\}} |y-z|^{-n-1} + |x-z|^{2m-n-|\alpha|} |y-z|^{-1} \\
&\preceq |x-z|^{m-|\alpha|+\max\{m-n,0\}} |y-z|^{\min\{m-n,0\}-1} + |x-z|^{2m-n-|\alpha|} |y-z|^{-1} \\
&\preceq |x-z|^{2m-1-n-|\alpha|} + |y-z|^{2m-1-n-|\alpha|} \text{ by Hölder's inequality.}
\end{aligned}
$$

b) The proof of (17) is almost analogous to the above reasoning with the obvious simplifications: In the numerator $\frac{1}{|y-z|^{n+1}}$ has to be replaced by $\frac{1}{|y-z|^n}$. Only the case $|\alpha| = 2m-n$ and n even is different and will be carried out here.

The case: $|\alpha| = 2m-n$ *and* n *even.*

We use Lemma 9.

$$
\begin{aligned}
&\frac{|D_z^\alpha G_{m,n}(x,z)|\, \hat{K}_{m,n,s}(z,y)}{K_{m,n,s}(x,y)} \\
&\preceq \frac{\log\left(2 + \frac{d(z)}{|x-z|}\right) \min\left\{1, \frac{d(x)}{|x-z|}\right\}^m \min\left\{1, \frac{d(z)}{|x-z|}\right\}^{\max\{m-|\alpha|,0\}} \frac{d(z)^m}{|z-y|^n}}{\frac{d(x)^m}{|x-y|^n}} \\
&\preceq \log\left(\frac{3}{|x-z|}\right) d(x)^{-m} d(z)^m |y-z|^{-n} (|x-z|^n + |y-z|^n)
\end{aligned}
$$

$$
\begin{aligned}
&\quad \cdot \min\left\{1, \frac{d(x)}{|x-z|}\right\}^m \min\left\{1, \frac{d(z)}{|x-z|}\right\}^{\max\{m-|\alpha|,0\}} \\
&\preceq \ \log\left(\frac{3}{|x-z|}\right) d(x)^{-m} d(z)^m |y-z|^{-n}|x-z|^n \\
&\quad \cdot \left(\frac{d(x)}{|x-z|}\right)^{\min\{m,n\}} \left(\right)^{\max\{m-n,0\}} \left(\frac{d(z)}{|x-z|}\right)^{\max\{m-|\alpha|,0\}} \\
&\quad + \log\left(\frac{3}{|x-z|}\right) d(x)^{-m} d(z)^m \left(\frac{d(x)}{d(z)}\right)^m \\
&\preceq \ \log\left(\frac{3}{|x-z|}\right) d(z)^n |y-z|^{-n} + \log\left(\frac{3}{|x-z|}\right) \preceq \log\left(\frac{3}{|x-z|}\right).
\end{aligned}
$$

□

The estimates (16) and (17) in the Lemma above are integrable with respect to $z \in B$ uniformly in $x \in B$, $y \in \partial B$, if $|\alpha| \le 2m-2$. Our main result is a direct consequence of this fact.

Theorem 7. *Let $s > \frac{1}{2}(n-2-m)(m-1)$. Then there exists $\varepsilon_0 = \varepsilon_0(m,n,s) > 0$ such that the following holds.*

If $\|b_\alpha\|_{C^{|\alpha|}(\overline{B})} \le \varepsilon_0$ for $|\alpha| \le 2m-2$, then for every $\varphi \in C^0(\partial B)$, $\psi \in C^1(\partial B)$ with

$$
\left.\begin{array}{l} \psi \ge 0 \\ \varphi \ge s\psi \end{array}\right\} \text{ on } \partial B,\ \psi \not\equiv 0 \text{ or } \varphi \not\equiv 0,
$$

the Dirichlet problem

$$
(18) \quad \begin{cases}
(-\Delta)^m u + \sum |\alpha| \le 2m-2 b_\alpha(x) D^\alpha u = 0 & \text{in } B, \\
\left(-\frac{\partial}{\partial\nu}\right)^j u = 0 & \text{on } \partial B \text{ for } j = 0,\dots,m-3, \\
\left(-\frac{\partial}{\partial\nu}\right)^{m-2} u = \psi & \text{on } \partial B, \\
\left(-\frac{\partial}{\partial\nu}\right)^{m-1} u = \varphi & \text{on } \partial B.
\end{cases}
$$

has a strictly positive solution $u \in W^{2m,p}_{\mathrm{loc}}(B) \cap C^{m-1}(\overline{B})$ ($p > 1$ arbitrary):

$$
u > 0 \quad \text{in } B.
$$

Proof. For existence and regularity we refer to [ADN] and [Ag]. First we assume additionally $\psi \in C^{m+2,\gamma}(\partial B)$, $\varphi \in C^{m+1,\gamma}(\partial B)$. We denote $\hat\varphi_s = \varphi - s\psi$. Let $p > 1$ be arbitrary. The operator

$$
\mathcal{L}_{m,n}\hat\varphi_s(x) := \int_{\partial B} L_{m,n}(x,y)\hat\varphi_s(y)\, d\omega(y)
$$

maps $\mathcal{L}_{m,n}: \ C^{m+1}(\partial B) \to C^{2m,\gamma}(\overline{B}) \hookrightarrow W^{2m,p}(B)$,

$$
\hat{\mathcal{K}}_{m,n,s}\psi(x) := \int_{\partial B} \hat K_{m,n,s}(x,y)\psi(y)\, d\omega(y)
$$

maps $\hat{\mathcal{K}}_{m,n,s}: \ C^{m+2}(\partial B) \to C^{2m,\gamma}(\overline{B}) \hookrightarrow W^{2m,p}(B)$, while the Green operator

$$\mathcal{G}_{m,n}f(x) := \int_B G_{m,n}(x,y)f(y)\,dy$$

maps $\mathcal{G}_{m,n}: \ L^p(B) \to W^{2m,p}(B)$, see [ADN]. We write $\mathcal{A} := \sum_{|\alpha|\leq 2m-2} b_\alpha(\,.\,)D^\alpha$. The solution of (18) is given by $u = -\mathcal{G}_{m,n}\mathcal{A}u + \hat{\mathcal{K}}_{m,n,s}\psi + \mathcal{L}_{m,n}\hat{\varphi}_s$ or $(\mathcal{I}+\mathcal{G}_{m,n}\mathcal{A})\,u = \hat{\mathcal{K}}_{m,n,s}\psi + \mathcal{L}_{m,n}\hat{\varphi}_s$. Here $\mathcal{I}+\mathcal{G}_{m,n}\mathcal{A}$ is a bounded linear operator in $W^{2m,p}(B)$, which for sufficiently small ε_0 is invertible. Hence

$$\begin{aligned} u &= (\mathcal{I}+\mathcal{G}_{m,n}\mathcal{A})^{-1}\,\hat{\mathcal{K}}_{m,n,s}\psi + (\mathcal{I}+\mathcal{G}_{m,n}\mathcal{A})^{-1}\,\mathcal{L}_{m,n}\hat{\varphi}_s \\ &= \hat{\mathcal{K}}_{m,n,s}\psi + \sum_{i=1}^{\infty}(-\mathcal{G}_{m,n}\mathcal{A})^i\,\hat{\mathcal{K}}_{m,n,s}\psi + \mathcal{L}_{m,n}\hat{\varphi}_s + \sum_{i=1}^{\infty}(-\mathcal{G}_{m,n}\mathcal{A})^i\,\mathcal{L}_{m,n}\hat{\varphi}_s. \end{aligned}$$

We only show how to deal with the first series containing $\hat{\mathcal{K}}_{m,n,s}$, the second series containing $\mathcal{L}_{m,n}$ is treated in the same way with some obvious simplifications. For $i \geq 1$ we integrate by parts. As $\mathcal{A}$ is of order $\leq 2m-2$ and $\hat{\mathcal{K}}_{m,n,s}\psi$ vanishes on ∂B of order $m-2$ no additional boundary integrals arise. By means of Fubini-Tonelli we obtain for $x \in B$:

$$\begin{aligned} &(-)^i\,\hat{\mathcal{K}}_{m,n,s}\psi(x) = (-1)^i \int_{z_1\in B} G_{2,n}(x,z_1)\mathcal{A}_{\ddagger_\infty}\int_{\ddagger_\in\in\mathcal{B}}\mathcal{G}_{\in,\backslash}(\ddagger_\infty,\ddagger_\in) \\ &\qquad \ldots\cdot\mathcal{A}_{z_{i-1}}\int_{z_i\in B} G_{2,n}(z_{i-1},z_i)\mathcal{A}_{z_i}\int_{y\in\partial B} K_n(z_i,y)\psi(y)\,d\omega(y)dz_i\ldots dz_1 \\ &= (-1)^i\int_{z_1\in B}\left(\mathcal{A}^*_{z_1}G_{2,n}(x,z_1)\right)\int_{z_2\in B}\left(\mathcal{A}^*_{z_2}G_{2,n}(z_1,z_2)\right)\ldots \\ &\qquad \ldots\int_{z_i\in B}\left(\mathcal{A}^*_{z_i}G_{2,n}(z_{i-1},z_i)\right)\int_{y\in\partial B}K_n(z_i,y)\psi(y)d\omega(y)dz_i\ldots dz_1 \\ &= (-1)^i\int_B\cdots\int_B\int_{\partial B}\left(\mathcal{A}^*_{z_1}G_{2,n}(x,z_1)\right)\left(\mathcal{A}^*_{z_2}G_{2,n}(z_1,z_2)\right)\cdot\ldots \\ &\qquad \ldots\cdot\left(\mathcal{A}^*_{z_i}G_{2,n}(z_{i-1},z_i)\right)K_n(z_i,y)\psi(y)d\omega(y)d(z_1,\ldots,z_i). \end{aligned}$$

Here $\mathcal{A}^*\,. = \sum_{|\alpha|\leq 2m-2}(-1)^{|\alpha|}D^\alpha(b_\alpha\,.\,)$ is the (formally) adjoint operator of the perturbation $\mathcal{A}$. By virtue of Lemma 6 we find:

$$\begin{aligned} \left|(-\mathcal{G}_{m,n}\mathcal{A})^i\,\hat{\mathcal{K}}_{m,n,s}\psi(x)\right| &\leq \int_{\partial B}\int_B\cdots\int_B K_n(x,y)\frac{\left|\mathcal{A}^*_{z_1}G_{2,n}(x,z_1)\right|\,K_n(z_1,y)}{K_n(x,y)} \\ &\quad \cdot\frac{\left|\mathcal{A}^*_{z_2}G_{2,n}(z_1,z_2)\right|\,K_n(z_2,y)}{K_n(z_1,y)}\cdot\ldots \\ &\quad \ldots\cdot\frac{\left|\mathcal{A}^*_{z_i}G_{2,n}(z_{i-1},z_i)\right|\,K_n(z_i,y)}{K_n(z_{i-1},y)}\psi(y)\,d(z_1,\ldots,z_i)d\omega(y) \\ &\leq (C_0\varepsilon_0)^i\int_{\partial B}K_n(x,y)\psi(y)\,d\omega(y) = (C_0\varepsilon_0)^i\left(\hat{\mathcal{K}}_{m,n,s}\psi\right)(x). \end{aligned}$$

Analogously we have:

$$\left|(-\mathcal{G}_{m,n}\mathcal{A})^i\,\mathcal{L}_{m,n}\hat{\varphi}_s(x)\right| \leq \left(\hat{C}_0\varepsilon_0\right)^i(\mathcal{L}_{m,n}\hat{\varphi})\,(x).$$

The constants $C_0 = C_0(m,n,s)$, $\hat{C}_0 = \hat{C}_0(m,n)$ do not depend on i. If $\varepsilon_0 = \varepsilon_0(m,n,s) > 0$ is chosen sufficiently small, we come up with

$$(19)\quad u \geq \frac{1}{C}\hat{\mathcal{K}}_{m,n,s}\psi + \frac{1}{C}\mathcal{L}_{m,n}\hat{\varphi}_s.$$

The general case $\varphi \in C^0(\partial B)$, $\psi \in C^1(\partial B)$ follows from (19) with help of approximation, the maximum estimates of [Ag] and local L^p-estimates [ADN]. $\square$.

A Appendix

For the reader's convenience here we collect a technical lemma and the Green function estimates from [GS1].

Lemma 8. *On B^2 (for $x, y \in B$) we have with $p, q \geq 0$ fixed:*

$$\begin{aligned}
&i. && \min\left\{1, \frac{d(y)}{|x-y|}\right\} \sim \min\left\{1, \frac{d(y)}{d(x)}, \frac{d(y)}{|x-y|}\right\},\\
&ii. && \min\left\{1, \frac{d(x)\,d(y)}{|x-y|^2}\right\} \sim \min\left\{\frac{d(y)}{d(x)}, \frac{d(x)}{d(y)}, \frac{d(x)\,d(y)}{|x-y|^2}\right\},\\
&iii. && \min\left\{1, \frac{d(x)^p\,d(y)^q}{|x-y|^{p+q}}\right\} \sim \min\left\{1, \frac{d(x)^p}{|x-y|^p}, \frac{d(y)^q}{|x-y|^q}, \frac{d(x)^p\,d(y)^q}{|x-y|^{p+q}}\right\},\\
&iv. && \min\left\{1, \frac{d(x)^p\,d(y)^q}{|x-y|^{p+q}}\right\} \sim \min\left\{1, \frac{d(x)}{|x-y|}\right\}^p \min\left\{1, \frac{d(y)}{|x-y|}\right\}^q.
\end{aligned}$$

On B^2 (for $x, y \in B$) we have with $p, q \geq 0$ and $p + q > 0$ fixed:

$$v. \quad \log\left(1 + \frac{d(x)^p\,d(y)^q}{|x-y|^{p+q}}\right) \sim \log\left(2 + \frac{d(y)}{|x-y|}\right)\min\left\{1, \frac{d(x)^p\,d(y)^q}{|x-y|^{p+q}}\right\}.$$

Lemma 9. *Let $\alpha \in \mathbb{N}^n$. Then on B^2 we have the following.*

1. *For $|\alpha| \geq 2m-n$ and n odd, or, $|\alpha| > 2m-n$ and n even:*

   (a) *if $|\alpha| \leq m$ then*

   $$|D_x^\alpha G_{m,n}(x,y)| \preceq |x-y|^{2m-n-|\alpha|}\,\min\left\{1, \frac{d(x)^{m-|\alpha|}\,d(y)^m}{|x-y|^{2m-|\alpha|}}\right\}\;;$$

   (b) *if $|\alpha| \geq m$ then*

   $$|D_x^\alpha G_{m,n}(x,y)| \preceq |x-y|^{2m-n-|\alpha|}\,\min\left\{1, \frac{d(y)^m}{|x-y|^m}\right\}.$$

2. *For $|\alpha| = 2m-n$ and n even:*

(a) *if* $|\alpha| \le m$ *(that is* $m \le n$*) then*

$$|D_x^\alpha G_{m,n}(x,y)| \preceq \log\left(2 + \frac{d(y)}{|x-y|}\right) \ \min\left\{1 \wedge \frac{d(x)^{m-|\alpha|}\, d(y)^m}{|x-y|^{2m-|\alpha|}}\right\} \ ;$$

(b) *if* $|\alpha| \ge m$ *(that is* $m \ge n$*) then*

$$|D_x^\alpha G_{m,n}(x,y)| \preceq \log\left(2 + \frac{d(y)}{|x-y|}\right) \ \min\left\{1 \wedge \frac{d(y)^m}{|x-y|^m}\right\}.$$

3. *For* $|\alpha| \le 2m-n$ *and* n *odd, or,* $|k| < 2m-n$ *and* n *even:*

(a) *if* $|\alpha| \le m - \frac{1}{2}n$ *then*

$$|D_x^\alpha G_{m,n}(x,y)| \preceq d(x)^{m-\frac{1}{2}n-|\alpha|}\, d(y)^{m-\frac{1}{2}n} \ \min\left\{1, \frac{d(x)^{\frac{1}{2}n}\, d(y)^{\frac{1}{2}n}}{|x-y|^n}\right\};$$

(b) *if* $m - \frac{1}{2}n \le |\alpha| \le m$ *then*

$$|D_x^\alpha G_{m,n}(x,y)| \preceq d(y)^{2m-n-|\alpha|} \min\left\{1, \frac{d(x)^{m-|\alpha|}\, d(y)^{n-m+|\alpha|}}{|x-y|^n}\right\};$$

(c) *if* $m \le |\alpha|$ *then*

$$|D_x^\alpha G_{m,n}(x,y)| \preceq d(y)^{2m-n-|\alpha|} \ \min\left\{1, \frac{d(y)^{n-m+|\alpha|}}{|x-y|^{n-m+|\alpha|}}\right\}.$$

In general the following estimate is weaker than Part iii. of Lemma 9 but still appropriate and more convenient for our purposes.

Corollary 10. *For* $|\alpha| \le 2m-n$ *and* n *odd, or,* $|\alpha| < 2m-n$ *and* n *even we have:*

$$|D_x^\alpha G_{m,n}(x,y)| \preceq d(x)^{m-\frac{1}{2}n-|\alpha|}\, d(y)^{m-\frac{1}{2}n} \ \min\left\{1, \frac{d(x)^{\frac{1}{2}n}\, d(y)^{\frac{1}{2}n}}{|x-y|^n}\right\}.$$

References

[Ag] Sh. Agmon, Maximum theorems for solutions of higher order elliptic equations, *Bull. Amer. Math. Soc.* **66** (1960), 77-80.

[ADN] Sh. Agmon, A. Douglis, L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I, *Commun. Pure Appl. Math.* **12** (1959), 623-727.

[Bo1] T. Boggio, Sull'equilibrio delle piastre elastiche incastrate, *Rend. Acc. Lincei* **10** (1901), 197-205.

[Bo2] T. Boggio, Sulle funzioni di Green d'ordine m, *Rend. Circ. Mat. Palermo* **20** (1905), 97-135.

[Cof] C. V. Coffman, On the structure of solutions to $\Delta^2 u = \lambda u$ which satisfy the clamped plate conditions on a right angle, *SIAM J. Math. Anal.* **13**, 746-757 (1982).

[CD] C. V. Coffman, R. J. Duffin, On the structure of biharmonic functions satisfying the clamped plate conditions on a right angle, *Adv. Appl. Math.* **1** (1980), 373-389.

[Du] R. J. Duffin, On a question of Hadamard concerning super-biharmonic functions, *J. Math. Phys.* **27** (1949), 253-258.

[Ed] J. Edenhofer, Eine Integraldarstellung der Lösung der Dirichletschen Aufgabe bei der Polypotentialgleichung im Falle einer Hyperkugel, *Math. Nachr.* **69** (1975), 149-162.

[Ga] P. R. Garabedian, A partial differential equation arising in conformal mapping, *Pacific J. Math.* **1** (1951), 485-524.

[GS1] H.-Ch. Grunau, G. Sweers, Positivity for equations involving polyharmonic operators with Dirichlet boundary conditions, *Math. Ann.*, **307** (1997), 589-626.

[GS2] H.-Ch. Grunau, G. Sweers, Positivity for perturbations of polyharmonic operators with Dirichlet boundary conditions in two dimensions, *Math. Nachr.* **179** (1996), 89-102.

[GS3] H.-Ch. Grunau, G. Sweers, Positivity properties of elliptic boundary value problems of higher order, *Proc. Second World Congress of Nonlinear Analysts*, to appear.

[Ha] J. Hadamard, Mémoire sur le problème d'analyse relatif à l'équilibre des plaques élastiques encastrées, in: *Œuvres de Jaques Hadamard*, Tome II, pp. 515-641, Centre National de la Recherche Scientifique: Paris, 1968, Nachdruck von: *Mémoires présentés par divers savants a l'Académie des Sciences* **33** (1908), 1-128.

[KKM] V. A. Kozlov, V. A. Kondrat'ev, V. G. Maz'ya, On sign variation and the absence of "strong" zeros of solutions of elliptic equations, *Math. USSR Izvestiya* **34** (1990), 337-353.

[Loe] Ch. Loewner, On generation of solutions of the biharmonic equation in the plane by conformal mappings, *Pacific J. Math.* **3** (1953), 417-436.

[Nic] M. Nicolesco, *Les fonctions polyharmoniques*, Hermann: Paris, 1936.

[Osh] St. Osher, On Green's function for the biharmonic equation in a right angle wedge, *J. Math. Anal. Appl.* **43** (1973), 705-716.

[ST] H. S. Shapiro, M. Tegmark, An elementary proof that the biharmonic Green function of an eccentric ellipse changes sign, *SIAM Rev.* **36** (1994), 99-101.

[Sor] R. Soranzo, A priori estimates and existence of positive solutions of a superlinear polyharmonic equation, *Dyn. Syst. Appl.* **3** (1994), 465-487.

[Sz] G. Szegö, Remark on the preceding paper of Charles Loewner, *Pacific J. Math.* **3** (1953), 437-446.

[Ta] G. Talenti, On the first eigenvalue of the clamped plate, in: H. Brezis, J. L. Lions (eds.), *Nonlinear partial differential equations and their applications, Collège de France Seminar, Vol. VI, Pitman Research Notes Math.* **109** (1984), 309-313.